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Natural and restricted Priestley duality for ternary algebras and their cousins

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Dedicated to the memory of Janusz Brzozowski (May 10, 1935–October 24, 2019)

Abstract. Up to term equivalence, there are three ways to assign a non-empty set C of constants to the three-element Kleene lattice, leading to ternary algebras ($C = \{0, d, 1\}$), Kleene algebras ($C = \{0, 1\}$), and don't know algebras ($C = \{d\}$). Our focus is on ternary algebras. We derive a strong, optimal natural duality and the restricted Priestley duality for ternary algebras and give axiomatisations of the dual categories. We apply these dualities in tandem to give straightforward and transparent proofs of some known results for ternary algebras. We also discuss, and in some cases prove, the corresponding dualities for Kleene lattices, Kleene algebras and don't know algebras.

1 Introduction

The algebraic formulation of two-valued logic is, of course, Boolean algebra. Once we go up to three values, there are competing logics and competing algebraic formulations. We are interested in a very natural three-valued logic

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which, along with *true* and *false*, allows an intermediate value corresponding to *don't know*. The logic was introduced in 1935 by G. C. Moisil [31], but was made famous by S. C. Kleene in his 1952 text *Introduction to Metamathematics* [29] and is referred to as Kleene's strong logic. For Boolean algebras we might as well introduce nullary operations 0 and 1 corresponding to false and true since they are term definable from $\wedge$ and $\neg$. But in an algebraic formulation of Kleene's strong logic, we have a choice as 0 and 1 are not term definable from $\wedge$, $\vee$ and $\neg$, nor is the intermediate value d corresponding to don't know: we can add none, some or all as nullaries depending on our point of view. All four possibilities will be considered, but the class of ternary algebras, where all three values are assigned names, is the focus of our attention.

Definition 1.1. An algebra $\mathbf{A} = \langle A; \vee, \wedge, \neg, 0, d, 1\rangle$ is a *ternary algebra* if

(L) $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice,

(M) $\mathbf{A}$ satisfies the *De Morgan laws* and the *double negation law*

$$\neg(a \vee b) = \neg a \wedge \neg b, \quad \neg(a \wedge b) = \neg a \vee \neg b, \quad \neg\neg a = a,$$

for all $a, b \in A$,

(K) $\mathbf{A}$ satisfies $a \vee \neg a \geqslant d$ and $a \wedge \neg a \leqslant d$, for all $a \in A$, and

(D) $\mathbf{A}$ satisfies $\neg d = d$.

In terms of truth values, we think of 0 and 1 as *false* and *true*, as usual. We interpret d as *don't know*, so that (D) corresponds to the fact that if we do not know if some statement is true, then we don't know whether its negation is true.

Conditions (L) and (M) say that the reduct $\mathbf{A}_{\mathrm{K}} := \langle A; \vee, \wedge, \neg, 0, 1\rangle$ is a De Morgan algebra. Condition (K) guarantees that $\mathbf{A}_{\mathrm{K}}$ satisfies

$$a \vee \neg a \geqslant b \wedge \neg b,$$

for all $a, b \in A$, so that (L), (M) and (K) together guarantee that $\mathbf{A}_{\mathrm{K}}$ is a *Kleene algebra*. Consequently, many results about Kleene algebras can be transported directly to corresponding results for ternary algebras. For example, since the only subdirectly irreducible Kleene algebras are the two-and three-element chains (Kalman [27]), it follows immediately that, up to isomorphism, the only subdirectly irreducible ternary algebra is the three-element algebra $\mathbf{T}$ shown in Figure 1. Consequently, in terms of the usual class operators, the variety $\mathcal{T}$ of ternary algebras can be written $\mathcal{T} = \mathsf{ISP}(\mathbf{T})$.

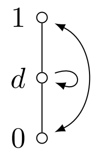

Figure 1 The three-element ternary algebra $\mathbf{T}$

Up to term equivalence, there are four different ways to assign a set of nullaries to the three-element algebra $\langle\{0, d, 1\}; \vee, \wedge, \neg\rangle$. The four possibilities, and the varieties they generate, are:

- *Kleene lattices*: $\mathcal{L} = \mathsf{Var}(\mathbf{L})$, where $\mathbf{L} := \langle\{0, d, 1\}; \vee, \wedge, \neg\rangle$;
- *Kleene algebras*: $\mathcal{K} = \mathsf{Var}(\mathbf{K})$, where $\mathbf{K} := \langle\{0, d, 1\}; \vee, \wedge, \neg, 0, 1\rangle$;
- *ternary algebras*: $\mathcal{T} = \mathsf{Var}(\mathbf{T})$, where $\mathbf{T} := \langle\{0, d, 1\}; \vee, \wedge, \neg, 0, d, 1\rangle$;
- *don't know algebras*: $\mathcal{D}\mathrm{k} = \mathsf{Var}(\mathbf{Dk})$, where $\mathbf{Dk} := \langle\{0, d, 1\}; \vee, \wedge, \neg, d\rangle$.

We include the variety of don't know algebras for completeness; it has not previously been studied.

Our primary aim is to derive a (strong and optimal) natural duality and the restricted Priestley duality for ternary algebras and to apply them to give straightforward and transparent proofs of some known results for ternary algebras. We will also present, and in some cases prove, the corresponding dualities for Kleene lattices, Kleene algebras and don't know algebras.

Ternary algebras and the corresponding three-valued logic have a number of applications in computer science where the constant d is variously interpreted as 'don't know', 'unknown', 'undefined', 'indefinite', 'transient', 'ambiguous', 'center', ... A good summary of applications of ternary algebras in computer science can be found in the 1995 text *Asynchronous circuits* by J. A. Brzozowski and C. H. Seger [9] and the 1999 paper by J. A. Brzozowski [6]. An application that is not mentioned in [9] and [6] is in the database language SQL. In order to deal with queries that involve a NULL value, SQL uses a logic based on the three-element ternary algebra; almost every text dedicated to SQL will include a section dedicated to three-valued logic with values true, false and unknown. Another example of an application outside the realm of circuit design is the three-valued attribute exploration created by Burmeister [10, 11] in the context of Formal Concept Analysis [40].

In Section 2, we trace some of the history of Kleene lattices, Kleene algebras and ternary algebras. In Section 3, we develop an optimal natural duality for each of the four varieties and in every case give an axiomatisation of the topological structures in the dual category. The restricted Priestley dualities for $\mathfrak{T}$ and $\mathfrak{D}$k are described in Section 4, and the restricted Priestley dualities for $\mathfrak{K}$ and $\mathfrak{L}$ are described in Section 5. The translation functors between the natural and restricted Priestley dual categories for $\mathfrak{T}$ are presented in detail in Section 6; the section ends with a brief description of the corresponding translation functors for $\mathfrak{D}$k, $\mathfrak{K}$ and $\mathfrak{L}$. Some applications of the dualities and the translation functors are described in Section 7—in particular, we will see that several results about ternary algebras, previously proved by purely algebraic means, fall out very easily from the dualities and the translations.

We will assume a familiarity with the basics of universal algebra and lattice theory. We refer to B. A. Davey and H. A. Priestley [21] for order-theoretic concepts and notation not defined here: for example, down-set, up-set, meet-irreducible, join-irreducible, linear sum, ${\downarrow}S$, ${\uparrow}S$, ...

An early draft of parts of this paper appeared in the report by the second author [30] on an AMSI Summer Vacation Research Project supervised by the first author.

2 What's in a name?

In this section we attempt to trace the history of the varieties of Kleene lattices, Kleene algebras and ternary algebras.

2.1 Kleene lattices In 1935, G. C. Moisil [31] considered *De Morgan lattices* (without naming them), he explicitly introduced the three-element Kleene lattice on pages 106–108. In place of our 0, d and 1, he used i,

p and c, respectively, and referred to them as *impossible*, *problématique* and *certaine*. Much later in 1957, and independently of Moisil, De Morgan lattices were studied by A. Białynicki-Birula and H. Rasiowa [4] and A. Białynicki-Birula [3] under the name *quasi-Boolean algebras*. The following year, again independently, J. A. Kalman [27] studied them under the name *distributive lattices with involution*. Kalman proved that the only subdirectly irreducible De Morgan lattices are the two-element Boolean lattice, the three-element Kleene lattice and the four-element non-Boolean De Morgan lattice.

2.2 Kleene algebras Kalman's results for De Morgan lattices apply directly to De Morgan algebras. Consequently, the only subdirectly irreducible De Morgan algebras are the two-element Boolean algebra, the three-element Kleene algebra and the four-element non-Boolean De Morgan algebra.

Kleene describes the three-element Kleene lattice/algebra on page 332 of his 1952 text [29], but he introduced it earlier in 1938 [28] in the context of partial recursive functions. In fact he introduced the corresponding logic which is referred to as Kleene's strong logic. He gave the truth tables for $\vee$, $\wedge$ and $\neg$ on $\{0, d, 1\}$—Kleene used the symbols $\mathfrak{f}$, $\mathfrak{t}$ and $\mathfrak{u}$ (for *undefined*). On page 333 of [29], Kleene remarks:

> *The third "truth value"* $\mathfrak{u}$ *is not on a par with the other two* $\mathfrak{t}$ *and* $\mathfrak{f}$ *in our theory.*

Thus it is reasonable to argue that the appropriate algebraic formulation of Kleene's strong logic is indeed Kleene algebra rather than ternary algebra.

Kleene algebras also arise naturally in the study of fuzzy switching functions, that is, the term functions of the algebra $\mathbf{I} := \langle [0, 1]; \vee, \wedge, \neg, 0, 1\rangle$ based on the unit interval: here $\vee$ and $\wedge$ are max and min, respectively, and $\neg x := 1 - x$. See the 1972 paper by F. P. Preparata and R. T. Yeh [37]. It is clear that $\mathbf{I}$ is a Kleene algebra and contains a subalgebra isomorphic to $\mathbf{K}$, from which it follows that the fuzzy switching functions are essentially the term functions of $\mathbf{K}$.

The name Kleene algebra seems to have been used first in print by D. Brignole and A. Monteiro [5] in 1967.

2.3 From Kleene lattices to ternary algebras The move from Kleene lattices with no constants to Kleene algebras with two constants to ternary algebras with three constants seems to have resulted from the application of three-valued logic to switching circuits/relay networks and later to logic circuits.

According to J. A. Brzozowski [6], the earliest work in this direction was in the late 1940s by M. Gotô [25, 26]. Like Kleene [28, 29], Gotô was more concerned with the logic rather than the algebra, and while he treated 0 and 1 as nullary operations, he did not treat d as a nullary.

In 1959, D. E. Muller [34] suggested the use of the three-element Kleene algebra for the 'treatment of transition signals in electronic switching circuits'. While he included 0 and 1 as constants, he did not formally name the intermediate value d.

In 1964, M. Yoeli and S. Rinon [41] introduced B-ternary operations on $\{0, d, 1\}$ and studied their application to the detection of static hazards in logic circuits (they use the older name switching circuits). Their B-ternary operations are precisely the term functions of the three-element Kleene algebra; they did not include the constant function with value d.

In 1972, M. Mukaidono [32] introduced the uncertainty order, which he referred to as *ambiguity* (see Figure 2), and proved that the term functions of the three-element Kleene algebra are precisely the functions that preserve the uncertainty order and the unary relation $B = \{0, 1\}$.

In 1986, M. Mukaidono [33] explicitly listed the three elements of $\{0, d, 1\}$ as constants and gave the axioms for ternary algebras, which he referred to as *Kleene algebras with center*. He characterised the term functions as those that preserve the uncertainty order (see our Theorem 7.1). This paper by Mukaidono appears to be the first devoted to what we now call ternary algebras.

The first use of the name *ternary algebra* appears to be in 1995 by J. A. Brzozowski and C. H. Seger in their text *Asynchronous Circuits* [9].

2.4 Don't-know algebras The remaining possibility is the variety $\mathsf{Var}(\mathbf{Dk})$ of *don't know algebras* which is generated by the algebra $\mathbf{Dk} = \langle\{0, d, 1\}; \vee, \wedge, \neg, d\rangle$ in which we name only the element d. This variety has not previously been studied.

3 Natural dualities for $\mathcal{L}$, $\mathcal{K}$, $\mathcal{T}$ and $\mathcal{D}$k

For the basics of the theory of natural dualities we refer to the text by Clark and Davey [13]. Here we sketch a restricted version tailored to ternary algebras and their cousins, Kleene algebras, Kleene lattices and don't know algebras.

3.1 Some general theory Let $\boldsymbol{\mathcal{A}} := \mathsf{ISP}(\mathbf{M})$ be the quasivariety generated by a finite algebra $\mathbf{M}$. We aim to build a category $\boldsymbol{\mathfrak{X}}$ of topological structures dual to $\boldsymbol{\mathcal{A}}$ (*qua* category). To this end, we search for topological structures $\mathbb{M} = \langle M; \mathcal{G}, \mathcal{R}, \mathcal{T}\rangle$, where $\mathcal{T}$ is the discrete topology on M, and $\mathcal{G}$ and $\mathcal{R}$ are, respectively, sets of finitary operations and relations on M that are *compatible with* $\mathbf{M}$, that is, the relations in $\mathcal{R}$ and the graphs of the operations in $\mathcal{G}$ are non-empty subuniverses of finite powers of $\mathbf{M}$. (In general, partial operations are allowed but we shall not need them. In fact, we will need only nullary operations.) The structure $\mathbb{M}$ is referred to as an *alter ego* of $\mathbf{M}$. The potential dual category for $\boldsymbol{\mathcal{A}}$ is the category $\boldsymbol{\mathfrak{X}} := \mathsf{IS_cP^+}(\mathbb{M})$ whose objects are isomorphic copies of topologically closed substructures of non-zero powers of $\mathbb{M}$. The compatibility between the operations and relations in $G \cup R$ and the algebra $\mathbf{M}$ guarantees that we have well-defined hom-functors

$$\begin{gathered}\mathrm{D}\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathfrak{X}} \text{ where } \mathrm{D}(\mathbf{A}) = \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}) \leqslant \mathbb{M}^A,\\ \mathrm{E}\colon \boldsymbol{\mathfrak{X}} \to \boldsymbol{\mathcal{A}} \text{ where } \mathrm{E}(\mathbb{X}) = \boldsymbol{\mathfrak{X}}(\mathbb{X}, \mathbb{M}) \leqslant \mathbf{M}^X,\end{gathered}$$

for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and all $\mathbb{X} \in \boldsymbol{\mathfrak{X}}$. The functors are defined on morphisms in the usual way via composition: for all $u \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{B})$ and all $\varphi \in \boldsymbol{\mathfrak{X}}(\mathbb{X}, \mathbb{Y})$,

$$\begin{gathered}\mathrm{D}(u)\colon \mathrm{D}(\mathbf{B}) \to \mathrm{D}(\mathbf{A}) \text{ is given by } \mathrm{D}(u)(y) := y \circ u, \text{ for all } y \in \boldsymbol{\mathcal{A}}(\mathbf{B}, \mathbf{M}),\\ \mathrm{E}(\varphi)\colon \mathrm{E}(\mathbb{Y}) \to \mathrm{E}(\mathbb{X}) \text{ is given by } \mathrm{E}(\varphi)(\alpha) := \alpha \circ \varphi, \text{ for all } \alpha \in \boldsymbol{\mathfrak{X}}(\mathbb{Y}, \mathbb{M}).\end{gathered}$$

We then have a dual adjunction $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon\rangle$ where the unit e and counit ε are both given by evaluation: the maps $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ and $\varepsilon_{\mathbb{X}} : \mathbb{X} \to \mathrm{DE}(\mathbb{X})$, which are easily seen to be embeddings, are defined, for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and all $\mathbb{X} \in \boldsymbol{\mathfrak{X}}$, by

$$\begin{gathered}e_{\mathbf{A}}(a)(x) := x(a), \text{ for all } a \in A \text{ and all } x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}),\\ \varepsilon_{\mathbb{X}}(x)(\alpha) := \alpha(x), \text{ for all } x \in X \text{ and all } \alpha \in \boldsymbol{\mathfrak{X}}(\mathbb{X}, \mathbb{M}).\end{gathered}$$

If $e_{\mathbf{A}}$ is an isomorphism for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, then $\boldsymbol{\mathcal{A}}$ is dually equivalent to a subcategory of $\boldsymbol{\mathcal{X}}$ and we say that $\mathbb{M}$ *yields a duality* on $\boldsymbol{\mathcal{A}}$. If, in addition, $\varepsilon_{\mathbb{X}}$ is an isomorphism for all $\mathbb{X} \in \boldsymbol{\mathcal{X}}$, then $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$ are dually equivalent categories and we say that $\mathbb{M}$ *yields a full duality* between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$. If $\mathbb{M}$ yields a full duality and $\mathbb{M}$ is injective in the category $\boldsymbol{\mathcal{X}}$ then we say that $\mathbb{M}$ *yields a strong duality* between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$. See Clark and Davey [13] for the missing details.

A duality is *optimal* if removing an operation or relation in $\mathcal{G} \cup \mathcal{R}$ would destroy it; that is, if an operation or relation in $\mathcal{G} \cup \mathcal{R}$ were removed, then we could find an algebra $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ such that $e_{\mathbf{A}} \colon \mathbf{A} \to \mathrm{DE}(\mathbf{A})$ is not an isomorphism.

An algebra $\mathbf{M}$ is called *lattice based* if there are binary term functions $\vee$ and $\wedge$ such that $\langle M; \vee, \wedge \rangle$ is a lattice. If $\mathbf{M}$ is a finite lattice-based algebra, we can use the NU Strong Duality Theorem [13, Theorem 3.3.8] to find a structure $\mathbb{M}$ that yields a strong duality on $\boldsymbol{\mathcal{A}}$. The theorem is more general than the version given here, but this special case is adequate for our needs.

Theorem 3.1 (**Special NU Strong Duality Theorem**). *Let* $\mathbf{M}$ *be a finite lattice-based algebra and let* $\boldsymbol{\mathcal{A}} = \mathsf{ISP}(\mathbf{M})$.

(1) *The structure* $\mathbb{M} = \langle M; \mathcal{R}^{(2)}_{\mathbf{M}}, \mathcal{T} \rangle$ *yields a duality on* $\boldsymbol{\mathcal{A}}$, *where* $\mathcal{R}^{(2)}_{\mathbf{M}}$ *is the set of all compatible binary relations on* $\mathbf{M}$ *and* $\mathcal{T}$ *is the discrete topology on* M.

(2) *Assume that every non-trivial subalgebra of* $\mathbf{M}$ *is subdirectly irreducible and has no non-constant homomorphisms into* $\mathbf{M}$ *other than the inclusion, and let* $C := \{c_1, \dots, c_n\}$ *be the set of all elements of* M *that form one-element subalgebras. Let* $\mathcal{R}$ *be a set of compatible finitary relations on* $\mathbf{M}$ *such that* $\mathbb{M} = \langle M; \mathcal{R}, \mathcal{T} \rangle$ *yields a duality on* $\boldsymbol{\mathcal{A}}$. *Then* $\mathbb{M}_C := \langle M; c_1, \dots, c_n, \mathcal{R}, \mathcal{T} \rangle$ *yields a strong and therefore full duality on* $\boldsymbol{\mathcal{A}} = \mathsf{ISP}(\mathbf{M})$.

Remark 3.2. Once we have applied Theorem 3.1(1), and know that the structure $\mathbb{M} = \langle M; \mathcal{R}^{(2)}_{\mathbf{M}}, \mathcal{T} \rangle$ yields a duality on $\boldsymbol{\mathcal{A}} = \mathsf{ISP}(\mathbf{M})$, we wish to simplify the set $\mathcal{R}^{(2)}_{\mathbf{M}}$, while retaining enough relations to yield a duality. That $\mathbb{M}$ yields a duality on $\boldsymbol{\mathcal{A}}$ is precisely the statement that, for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, the evaluation maps $e_{\mathbf{A}}(a) \colon \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}) \to M$ are the only continuous maps from $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$ to M that preserve the pointwise structure on $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$ inherited from $\mathbb{M}^A$. For all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, every map $\alpha \colon \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}) \to M$

- preserves the trivial relations $\Delta_M = \{\,(a,a) \mid a \in M\,\}$, M^2 and M,
- preserves an intersection $R \cap S$ of relations provided it preserves both R and S,
- preserves a binary relation R if and only if it preserves its converse $R^{\smile}$,
- preserves a product $R \times S$ of relations provided it preserves both R and S, and
- preserves the unary relation $\{a\}$ provided it preserves the nullary operation a.

A number of other constructs can be used to simplify $\mathcal{R}^{(2)}_{\mathbf{M}}$ (see [13, 2.4.5 and Section 9.2]) but the five listed above will be sufficient here.

Theorem 3.1 is very easy to apply to the classes of ternary algebras, Kleene algebras, Kleene lattices and don't know algebras. We will present a strong and optimal natural duality for each of the varieties as well as an axiomatisation of the dual category. The proofs of optimality and of the axiomatisations will be delayed until Subsections 3.6 and 3.7.

We start with Kleene lattices, as the generating algebra $\mathbf{L}$ has the most compatible binary relations, and work our way down to the other varieties.

Two relations, shown in Figure 2, will play a featured role in each of our dualities:

- the *uncertainty order* $\preccurlyeq\, = \{(0,0),(0,d),(d,d),(1,d),(1,1)\}$, and
- $\sim\, = \{0,d,1\}^2 \setminus \{(0,1),(1,0)\}$, the looped path of length 2.

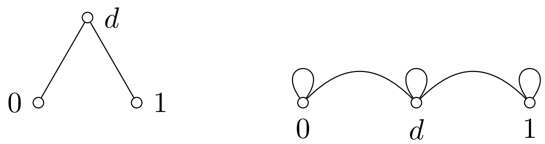

Figure 2 The uncertainty order $\preccurlyeq$ and looped path $\sim$ on $\{0,d,1\}$

Let $B = \{0,1\}$; the sixteen non-empty subuniverses of $\mathbf{L}^2$ in increasing size are

$$\begin{gathered}\{(d,d)\},\ \Delta_B,\ B\times\{d\},\ \{d\}\times B,\ \Delta_L,\ L\times\{d\},\ \{d\}\times L,\\ B^2,\ \preccurlyeq\cap(B\times L),\ \succcurlyeq\cap(L\times B),\ \preccurlyeq,\ \succcurlyeq,\ B\times L,\ L\times B,\ \sim,\ L^2.\end{gathered} \tag{$\mathcal{R}^{(2)}_{\mathbf{L}}$}$$

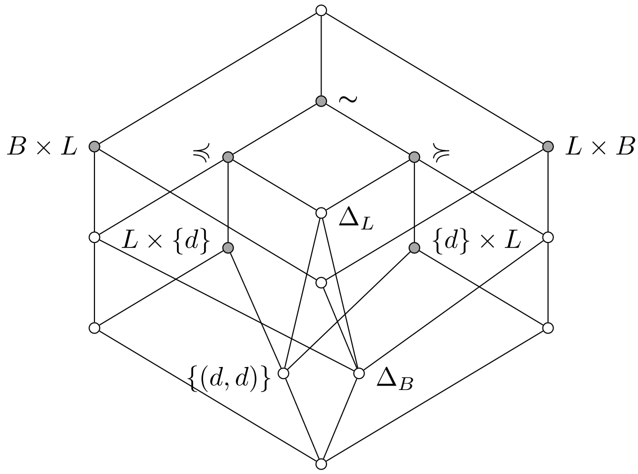

Figure 3 $\mathsf{Sub}(\mathbf{L}^2)$

The lattice $\mathsf{Sub}(\mathbf{L}^2)$ of subuniverses of $\mathbf{L}^2$ is shown in Figure 3; the meet-irreducibles are shaded and labelled.

Remark 3.3. It is very easy to use Figure 3 to find the lattices $\mathsf{Sub}(\mathbf{K}^2)$, $\mathsf{Sub}(\mathbf{T}^2)$ and $\mathsf{Sub}(\mathbf{Dk}^2)$ and their meet-irreducibles. Just use the following facts:

- let $\mathbf{M}$ be an algebra, let $B \subseteq M$ and let $\mathbf{M}_B$ be the algebra obtained from $\mathbf{M}$ by adding the elements of B as nullary operations; then $\mathsf{Sub}(\mathbf{M}_B) = {\uparrow}\overline{\mathbf{B}}$, where $\overline{\mathbf{B}}$ is the subuniverse of $\mathbf{M}_B$ generated by B;
- for every lattice $\mathbf{A}$ and all $a \in A$, the meet-irreducibles of the sublattice formed by the principal up-set ${\uparrow}a$ are precisely the meet-irreducibles in $\mathbf{A}$ that lie in ${\uparrow}a$.

We then find, modulo changing the name of the underlying set from L to K, T or Dk respectively, that $\mathsf{Sub}(\mathbf{K}^2) = {\uparrow}\Delta_B$, $\mathsf{Sub}(\mathbf{T}^2) = {\uparrow}\Delta_L$ and $\mathsf{Sub}(\mathbf{Dk}^2) = {\uparrow}\{(d,d)\}$, where each principal up-set is calculated in $\mathsf{Sub}(\mathbf{L}^2)$.

3.2 Kleene lattices By Theorem 3.1(1), the structure $\mathbb{L}_1 = \langle L; \mathcal{R}^{(2)}_{\mathbf{L}}, \mathcal{T}\rangle$ yields a duality on $\boldsymbol{\mathcal{L}}$. To optimise the set $\mathcal{R}^{(2)}_{\mathbf{L}}$, we apply the dot points in Remark 3.2. We first eliminate the trivial relations Δ_L and L^2, then eliminate all meet-reducible relations leaving

$$\{(d,d)\},\ L \times \{d\},\ \{d\} \times L,\ \preccurlyeq,\ \succcurlyeq,\ B \times L,\ L \times B,\ \text{and}\ \sim.$$

Next we eliminate one of each pair of mutually converse relations (such as $\preccurlyeq$ and $\succcurlyeq$), then replace all products by their unary-relation factors, then eliminate the trivial relation L; this leaves $\{d\}$, $\preccurlyeq$, $\sim$, and B. Finally we replace the one-element subuniverse $\{d\}$ by the corresponding nullary operation (this is needed to guarantee a full duality—see [13, Lemma 3.1.2]). Theorem 3.1(2), with $C = \{d\}$, yields the following strong duality; its optimality is proved in Subsection 3.6 below.

Theorem 3.4. *The structure* $\mathbb{L} = \langle\{0, d, 1\}; d, \preccurlyeq, \sim, B, \mathcal{T}\rangle$ *yields a strong* (*and therefore full*) *optimal duality between the variety* $\boldsymbol{\mathcal{L}} = \mathsf{ISP}(\mathbf{L})$ *of Kleene lattices and the category* $\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{L}}} := \mathsf{IS_cP}^+(\mathbb{L})$. *In particular,* $e_{\mathbf{A}} \colon \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ *is an isomorphism for every Kleene lattice* $\mathbf{A}$ *and* $\varepsilon_{\mathbb{X}} \colon \mathbb{X} \to \mathrm{DE}(\mathbb{X})$ *is an isomorphism for all* $\mathbb{X} \in \boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{L}}}$.

Strictly speaking, since the functors in this theorem depend upon the choice of the algebra $\mathbf{L}$ and the alter ego $\mathbb{L}$, we should denote them by $\mathrm{D}_{\mathbf{L}} \colon \boldsymbol{\mathcal{L}} \to \boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{L}}}$ and $\mathrm{E}_{\mathbb{L}} \colon \boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{L}}} \to \boldsymbol{\mathcal{L}}$. To avoid excessive subscripting, we shall drop the subscripts $\mathbf{L}$ and $\mathbb{L}$. The same comment applies to Theorems 3.6, 3.8 and 3.10.

We now state the characterisation of the objects in the dual category $\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{L}}}$. The proofs of all of the dual-category characterisations are given in Subsection 3.7. Recall that a topological structure $\langle X; \preccurlyeq, \mathcal{T}\rangle$ is a *Priestley space* if $\langle X; \mathcal{T}\rangle$ is a compact topological space, $\preccurlyeq$ is an order relation, and for all $x, y \in X$ with $x \not\preccurlyeq y$, there exists a clopen up-set U with $x \in U$ and $y \notin U$.

Theorem 3.5. *A topological structure* $\mathbb{X} = \langle X; d, \preccurlyeq, \sim, B, \mathcal{T}\rangle$ *in the signature of* $\mathbb{L}$ *belongs to* $\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{L}}}$ *if and only if*

(NL_1) $\langle X; \preccurlyeq, \mathcal{T}\rangle$ *is a Priestley space,*

(NL_2)
- $\sim$ *is a topologically closed binary relation on* X*,*
- B *is a topologically closed unary relation on* X*, and*

(NL_3) *the following universal axioms are satisfied:*

(a) $x \sim x$,
(b) $x \sim y \ \&\ y \preccurlyeq z \ \rightarrow\ z \sim x$,
(c) $x \sim y \ \&\ x \in B \ \rightarrow\ x \preccurlyeq y$,
(d) $d \notin B$,
(e) $x \preccurlyeq d$.

3.3 Kleene algebras Using Figure 3 and applying Remark 3.3, we see that $\mathbf{K}^2$ has eleven subuniverses five of which are meet-irreducible, namely

$$\preccurlyeq,\ \succcurlyeq,\ B \times K,\ K \times B,\ \text{and}\ \sim .$$

Applying the dot points in Remark 3.2, we find that $\preccurlyeq$, $\sim$ and B will yield a duality on the variety $\mathsf{Var}(\mathbf{K})$ of Kleene algebras. Then Theorem 3.1(2), with $C = \varnothing$, yields the following strong duality.

Theorem 3.6. *The structure* $\mathbb{K} = \langle \{0, d, 1\}; \preccurlyeq, \sim, B, \mathcal{T} \rangle$ *yields a strong* (*and therefore full*) *optimal duality between the variety* $\mathcal{K} = \mathsf{ISP}(\mathbf{K})$ *of Kleene algebras and the category* $\mathfrak{X}_{\mathcal{K}} := \mathsf{IS_cP^+}(\mathbb{K})$. *In particular,* $e_{\mathbf{A}} \colon \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ *is an isomorphism for every Kleene algebra* $\mathbf{A}$ *and* $\varepsilon_{\mathbb{X}} \colon \mathbb{X} \to \mathrm{DE}(\mathbb{X})$ *is an isomorphism for all* $\mathbb{X} \in \mathfrak{X}_{\mathcal{K}}$.

The axiomatisation of $\mathfrak{X}_{\mathcal{K}}$ is obtained from the axiomatisation of $\mathfrak{X}_{\mathcal{L}}$ by simply removing the axioms that no longer apply; we will see why in Subsection 3.7.

Theorem 3.7. *A topological structure* $\mathbb{X} = \langle X; \preccurlyeq, \sim, B, \mathcal{T} \rangle$ *in the signature of* $\mathbb{K}$ *belongs to* $\mathfrak{X}_{\mathcal{K}}$ *if and only if*

(NK_1) $\langle X; \preccurlyeq, \mathcal{T} \rangle$ *is a Priestley space,*

(NK_2)
- $\sim$ *is a topologically closed binary relation on* X,
- B *is a topologically closed unary relation on* X, *and*

(NK_3) *the following universal axioms are satisfied:*

(a) $x \sim x$,
(b) $x \sim y \ \&\ y \preccurlyeq z \ \rightarrow\ z \sim x$,
(c) $x \sim y \ \&\ x \in B \ \rightarrow\ x \preccurlyeq y$.

Theorems 3.6 (minus the optimality) and 3.7 were first proved by Davey and Werner [22, Section 2.11]; see also [13, Theorem 4.3.10]. The optimality of the duality was first proved in Davey and Priestley [20, Section 5]; see also [13, Section 8.4].

3.4 Ternary algebras Again, using Figure 3 and applying Remark 3.3 we see that $\mathbf{T}^2$ has five subuniverses three of which are meet-irreducible, namely

$$\preccurlyeq,\ \succcurlyeq,\ \text{and}\ \sim .$$

Since $\succcurlyeq$ can be safely removed, we find that $\preccurlyeq$ and $\sim$ yield a duality on the variety $\mathsf{Var}(\mathbf{T})$ of ternary algebras. Then Theorem 3.1(2), with $C = \varnothing$, yields all but the optimality in the following theorem.

Theorem 3.8. *The structure* $\mathbb{T} = \langle \{0, d, 1\}; \preccurlyeq, \sim, \mathcal{T}\rangle$ *yields a strong* (*and therefore full*) *optimal duality between the variety* $\mathfrak{T} = \mathsf{ISP}(\mathbf{T})$ *of ternary algebras and the category* $\mathfrak{X}_{\mathfrak{T}} := \mathsf{IS_cP}^{+}(\mathbb{T})$. *In particular,* $e_{\mathbf{A}} \colon \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ *is an isomorphism for every ternary algebra* $\mathbf{A}$ *and* $\varepsilon_{\mathbb{X}} \colon \mathbb{X} \to \mathrm{DE}(\mathbb{X})$ *is an isomorphism for all* $\mathbb{X} \in \mathfrak{X}_{\mathfrak{T}}$.

Again, the axiomatisation of $\mathfrak{X}_{\mathfrak{T}}$ is obtained from the axiomatisation of $\mathfrak{X}_{\mathfrak{L}}$ by simply removing the axioms that no longer apply—see Subsection 3.7.

Theorem 3.9. *A topological structure* $\mathbb{X} = \langle X; \preccurlyeq, \sim, \mathcal{T}\rangle$ *in the signature of* $\mathbb{T}$ *belongs to* $\mathfrak{X}_{\mathfrak{T}}$ *if and only if*

(NT_1) $\langle X; \preccurlyeq, \mathcal{T}\rangle$ *is a Priestley space,*

(NT_2) $\sim$ *is a topologically closed binary relation on* X, *and*

(NT_3) *the following universal axioms are satisfied:*

(a) $x \sim x$,
(b) $x \sim y \ \&\ y \preccurlyeq z \ \rightarrow\ z \sim x$.

3.5 Don't know algebras Once again, using Figure 3 and applying Remark 3.3 we see that $\mathbb{D}\mathrm{k}^2$ has eight subuniverses five of which are meet-irreducible, namely

$$T \times \{d\},\ \{d\} \times T,\ \preccurlyeq,\ \succcurlyeq,\ \text{and}\ \sim .$$

Applying the dot points in Remark 3.2, we find that $\{d\}$, $\preccurlyeq$ and $\sim$ yield a duality on the variety $\mathsf{Var}(\mathbf{Dk})$ of don't know algebras. Then Theorem 3.1(2), with $C = \{d\}$, yields the following strong duality.

Theorem 3.10. *The structure* $\mathbb{D}\mathrm{k} = \langle\{0, d, 1\}; d, \preccurlyeq, \sim, \mathcal{T}\rangle$ *yields a strong (and therefore full) optimal duality between the variety* $\boldsymbol{\mathfrak{D}}\mathrm{k} = \mathsf{ISP}(\mathbf{Dk})$ *of don't know algebras and the category* $\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathfrak{D}}\mathrm{k}} := \mathsf{IS}_{\mathrm{c}}\mathsf{P}^{+}(\mathbb{D}\mathrm{k})$*. In particular,* $e_{\mathbf{A}} \colon \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ *is an isomorphism for every don't know algebra* $\mathbf{A}$ *and* $\varepsilon_{\mathbb{X}} \colon \mathbb{X} \to \mathrm{DE}(\mathbb{X})$ *is an isomorphism for all* $\mathbb{X} \in \boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathfrak{D}}\mathrm{k}}$.

Once again, the axiomatisation of $\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathfrak{D}}\mathrm{k}}$ is the natural subset of the axioms for $\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{L}}}$.

Theorem 3.11. *A topological structure* $\mathbb{X} = \langle X; d, \preccurlyeq, \sim, \mathcal{T}\rangle$ *in the signature of* $\mathbb{D}\mathrm{k}$ *belongs to* $\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathfrak{D}}\mathrm{k}}$ *if and only if*

(NDk$_1$) $\langle X; \preccurlyeq, \mathcal{T}\rangle$ *is a Priestley space,*

(NDk$_2$) $\sim$ *is a topologically closed binary relation on* X*, and*

(NDk$_3$) *the following universal axioms are satisfied:*

(a) $x \sim x$,
(b) $x \sim y \ \&\ y \preccurlyeq z \ \to\ z \sim x$,
(c) $x \preccurlyeq d$.

We now provide the missing proofs. We begin with the proofs that the dualities in Theorems 3.4, 3.6, 3.8 and 3.10 are optimal.

3.6 The proofs that all four dualities are optimal We commence with the optimality of the duality for Kleene lattices, as the optimality of the other three dualities will follow almost immediately. To prove that we cannot remove $\preccurlyeq$ from the alter ego $\mathbb{L} = \langle\{0, d, 1\}; d, \preccurlyeq, \sim, B, \mathcal{T}\rangle$ without destroying the duality, we must find a Kleene lattice $\mathbf{A}$ and a map $\gamma \colon \boldsymbol{\mathcal{L}}(\mathbf{A}, \mathbf{L}) \to L$ such that γ preserves d, $\sim$ and B but not $\preccurlyeq$: such a map cannot be an evaluation $e_{\mathbf{A}}(a)$, for any $a \in A$, as the evaluation maps preserve every compatible relation on $\mathbf{L}$. The Test Algebra Lemma [13, 8.1.3] tells us that we may choose $\mathbf{A}$ to be the subalgebra $\mathbf{A}(\preccurlyeq)$ of $\mathbf{L}^2$ with underlying set $\preccurlyeq$.

It is very easy to see that $\mathcal{L}(\mathbf{A}(\preccurlyeq), \mathbf{L}) = \{\rho_1, \rho_2, \underline{d}\}$, where $\rho_i \colon \mathbf{A}(\preccurlyeq) \to \mathbf{L}$ is the ith projection and $\underline{d} \colon \mathbf{A}(\preccurlyeq) \to \mathbf{L}$ is the constant map onto $\{d\}$. Now define $\gamma \colon \mathcal{L}(\mathbf{A}(\preccurlyeq), \mathbf{L}) \to L$ by $\gamma(\rho_1) = d$, $\gamma(\rho_2) = 1$ and $\gamma(\underline{d}) = d$. Clearly γ preserves d and preserves B as the relation B is empty on $\mathcal{L}(\mathbf{A}(\preccurlyeq), \mathbf{L})$. Since $\{1, d\}^2 \subseteq \sim$, it is trivial that γ preserves $\sim$, and γ fails to preserve $\preccurlyeq$ as $\rho_1 \preccurlyeq \rho_2$ in $\mathcal{L}(\mathbf{A}(\preccurlyeq), \mathbf{L})$ but $\gamma(\rho_1) = d \not\preccurlyeq 1 = \gamma(\rho_2)$ in $\mathbb{L}$.

We turn now to the relation $\sim$. It is again easy to see that $\mathcal{L}(\mathbf{A}(\sim), \mathbf{L}) = \{\rho_1, \rho_2, \underline{d}\}$, where $\rho_i \colon \mathbf{A}(\sim) \to \mathbf{L}$ is the ith projection and $\underline{d} \colon \mathbf{A}(\sim) \to \mathbf{L}$ is the constant map onto $\{d\}$. In $\mathcal{L}(\mathbf{A}(\sim), \mathbf{L})$ we have $\rho_1 \sim \rho_2$, $\rho_1 \preccurlyeq \underline{d}$, $\rho_2 \preccurlyeq \underline{d}$ but $\rho_1 \not\preccurlyeq \rho_2$ and $\rho_2 \not\preccurlyeq \rho_1$, and the relation B is empty. Hence the function $\gamma \colon \mathcal{T}(\mathbf{A}(\sim), \mathbf{T}) \to T$ given by $\gamma(\rho_1) = 0$, $\gamma(\rho_2) = 1$ and $\gamma(\underline{d}) = d$ preserves d, $\preccurlyeq$ and B but not $\sim$.

Now consider the relation B and note that $\mathbf{A}(B) = \mathbf{B}$. We have $\mathcal{L}(\mathbf{B}, \mathbf{L}) = \{\rho_1, \underline{d}\}$, where $\rho_1 \colon \mathbf{B} \to \mathbf{L}$ is the inclusion map of B into L and $\underline{d} \colon \mathbf{B} \to \mathbf{L}$ is the constant map onto $\{d\}$. Define $\gamma \colon \mathcal{L}(\mathbf{B}, \mathbf{L}) \to L$ by $\gamma(\rho_1) = \gamma(\underline{d}) = d$. Then γ preserves d by definition, preserves $\preccurlyeq$ and $\sim$ since they both contain (d, d), and fails to preserve B as $\rho_1 \in B$ on $\mathcal{L}(\mathbf{B}, \mathbf{L})$ but $\gamma(\rho_1) = d \notin B$.

Finally, consider the relation $\{d\}$ corresponding to the nullary d. Then $\mathcal{L}(\mathbf{A}(\{d\}), \mathbf{L}) = \{\rho_1\}$, where $\rho_1(d) = d$. The map $\gamma \colon \mathcal{L}(\mathbf{A}(\{d\}), \mathbf{L}) \to L$ defined by $\gamma(\rho_1) = 1$ preserves $\preccurlyeq$, $\sim$ and B since $(1,1) \in \preccurlyeq$, $(1,1) \in \sim$ and $1 \in B$, but clearly does not preserve d.

This proves that the duality on $\mathcal{L}$ given by $\mathbb{L}$ is optimal. It follows almost immediately that the dualities for the other three varieties are also optimal. Indeed, let $\mathcal{C}$ be one of $\mathcal{K}$, $\mathcal{T}$ and $\mathcal{D}$k, denote its three-element generating algebra by $\mathbf{C}$, let R be one of d, $\preccurlyeq$, $\sim$ and B that occurs in the corresponding alter ego. Now let $\mathbf{R}$ be the algebra in $\mathcal{C}$ with underlying set R and, as above, let $\mathbf{A}(R)$ be the algebra in $\mathcal{L}$ with underlying set R. Thus $\mathbf{A}(R)$ is the $\mathcal{L}$-reduct of $\mathbf{R}$ obtained by deleting some of the nullary operations from the signature. Since $\mathcal{C}(\mathbf{R}, \mathbf{C}) \subseteq \mathcal{L}(\mathbf{A}(R), \mathbf{L})$, we may define $\gamma' \colon \mathcal{C}(\mathbf{R}, \mathbf{C}) \to C$ to be the restriction of the map $\gamma \colon \mathcal{L}(\mathbf{A}(R), \mathbf{L}) \to L$ defined in the proof above. Then γ' preserves every element of $\{d, \preccurlyeq, \sim, B\} \setminus \{R\}$ since γ does. Finally, γ' does not preserve R since the failure of γ to preserve R was witnessed in $\mathcal{L}(\mathbf{A}(R), \mathbf{L})$ by the projections ρ_i and they belong to the subset $\mathcal{C}(\mathbf{R}, \mathbf{C})$. Hence all four dualities are optimal. □

Remark 3.12. Some general theory provides an alternative proof that the dualities for $\mathfrak{K}$ and $\mathfrak{T}$ are optimal. The only $\mathfrak{K}$-homomorphisms from $\mathbf{A}(\sim)$ and from $\mathbf{A}(\preccurlyeq)$ to $\mathbf{K}$ are the two projections, and consequently the same is true of the $\mathfrak{T}$-homomorphisms. This tells us that the relations $\sim$ and $\preccurlyeq$ are *hom-minimal* in both $\mathfrak{K}$ and $\mathfrak{T}$. As both relations contain the diagonal and are meet-irreducible in $\mathsf{Sub}(\mathbf{T}^2)$, Proposition 8.2 in Craig, Davey and Haviar [16] tells us that they are *absolutely unavoidable within* $\mathsf{Sub}(\mathbf{T}^2)$, that is, if $\mathcal{R}$ is a subset of $\mathsf{Sub}(\mathbf{T}^2)$ such that $\langle T; \mathcal{R}, \mathcal{T}\rangle$ yields a duality on $\mathfrak{T}$, then $\mathcal{R}$ must contain $\sim$ and either $\preccurlyeq$ or $\succcurlyeq$. The same argument applies to $\mathsf{Sub}(\mathbf{K}^2)$. It follows at once that the duality for ternary algebras in Theorem 3.8 is optimal and, modulo proving that B cannot be removed from $\mathbb{K}$, that the duality for Kleene algebras in Theorem 3.6 is optimal.

3.7 The proofs of the axiomatisations The facts we require about Priestley spaces are contained in the following lemma; see Davey and Priestley [21].

Lemma 3.13. *Let $\mathbb{X} = \langle X; \preccurlyeq, \mathcal{T}\rangle$ be a Priestley space.*

(1) *Every open down-set in $\mathbb{X}$ is a union of clopen down-sets.*

(2) *Let U be a closed down-set in $\mathbb{X}$ and V be a closed up-set in $\mathbb{X}$ with $U \cap V = \varnothing$. Then there exists a clopen down-set W in $\mathbb{X}$ with $U \subseteq W$ and $W \cap V = \varnothing$.*

(3) *For all $x \in X$, the principal down-set ${\downarrow}x$ and the principal up-set ${\uparrow}x$ are closed in $\mathbb{X}$.*

(4) *The order $\preccurlyeq$ is a closed subset of $\mathbb{X}^2$.*

We begin with the proof of the axiomatisation of the dual category $\mathfrak{X}_{\mathcal{L}}$ for Kleene lattices given in Theorem 3.5.

The following consequences of axioms (a)–(e) in (NL_3) will be useful: quasi-equations (i)–(iii) follow from (a) and (b) along with the fact that $\preccurlyeq$ is reflexive, (iv) follows from (a)–(c) along with the fact that $\preccurlyeq$ is antisymmetric, while (v) follows from (a), (b) and (e).

(i) $x \sim y \ \rightarrow\ y \sim x$ (by (b) as $\preccurlyeq$ is reflexive),

(ii) $x \preccurlyeq y \ \rightarrow\ x \sim y$ (by (a) and (b) followed by (i)),

(iii) $x \preccurlyeq y \ \&\ x \preccurlyeq z \ \rightarrow\ y \sim z$ (by, in order, (ii), (i), (b) then (i)),

(iv) $y \preccurlyeq x \ \& \ x \in B \ \rightarrow \ y = x$ (by (ii) then (i) then (c)),

(v) $x \sim d$ (by (ii) and (e)).

We will use the following easily proved characterisation of $\mathcal{X}_{\mathcal{L}}$-morphisms.

Lemma 3.14. *Let $\mathbb{X} \in \mathcal{X}_{\mathcal{L}}$, let $U, V \subseteq X$ and define $\lambda_{UV} \colon X \to T$ by*

$$\lambda_{UV}(x) = \begin{cases} 0, & \text{if } x \in U, \\ 1, & \text{if } x \in V, \\ d, & \text{if } x \in X \setminus (U \cup V), \end{cases}$$

for all $x \in X$. Then λ_{UV} is an $\mathcal{X}_{\mathcal{L}}$-morphism if and only if

(1) *U and V are clopen down-sets,*

(2) $(U \times V) \cap \sim\, = \varnothing$,

(3) *$B \subseteq U \cup V$, and*

(4) $d \notin U \cup V$.

In particular, λ_{UV} is an $\mathcal{X}_{\mathcal{L}}$-morphism if V is empty and U is a proper clopen down-set containing B. Moreover, if U and V are non-empty, then (4) follows from (2).

We abbreviate the map $\lambda_{U\varnothing}$ to λ_U. We will also use without comment the fact that if Z is a closed subset of $\mathbb{X}^2$, then $\pi_1(Z)$ is closed in $\mathbb{X}$; a consequence of the compactness of $\mathbb{X}$.

Proof of Theorem 3.5. It is clear that the structure $\mathbb{L}$ satisfies (NL_1)–(NL_3). Since these properties are preserved under the formation of products and topologically closed subspaces, it follows that every structure $\mathbb{X} \in \mathcal{X}$ satisfies (NL_1)–(NL_3).

For the converse, assume that $\mathbb{X} = \langle X; d, \preccurlyeq, \sim, B, \mathcal{T} \rangle$ satisfies (NL_1)–(NL_3). To prove that $\mathbb{X}$ is isomorphic to a closed substructure of a power of $\mathbb{L}$, we call on the Separation Theorem [13, Thm 1.4.3]. We must prove the following:

(S_1) there is a morphism $\alpha \colon \mathbb{X} \to \mathbb{L}$,

(S_2) for all $x, y \in X$ with $x \not\preccurlyeq y$, there exists a morphism $\alpha \colon \mathbb{X} \to \mathbb{L}$ with $\alpha(x) \not\preccurlyeq \alpha(y)$, (S$_3$) for all $x, y \in X$ with $x \nleqslant y$, there exists a morphism $\alpha\colon \mathbb{X} \to \mathbb{L}$ with $\alpha(x) \nleqslant \alpha(y)$,

(S$_4$) for all $x \in X \backslash B$, there exists a morphism $\alpha\colon \mathbb{X} \to \mathbb{L}$ with $\alpha(x) \notin B$.

First assume that $\mathbb{X}$ is empty. Then the empty map from $\mathbb{X}$ to $\mathbb{L}$ is a morphism, whence (S$_1$) holds, and (S$_2$)–(S$_4$) hold vacuously. Now assume that $\mathbb{X}$ is non-empty.

(S$_1$): By assumption, B is topologically closed and by (iv) it is a down-set. Let $x \in X \setminus B$. (By (d), $x = d$ will suffice.) By Lemma 3.13(3), the set $\uparrow x$ is a closed up-set disjoint from B. Hence by Lemma 3.13(2), there exists a clopen down-set W that contains B but not x. It follows from Lemma 3.14 that the map λ_W is an $\mathbf{\mathfrak{X}}_{\mathcal{L}}$-morphism. Hence (S$_1$) holds.

(S$_2$): Let $x, y \in X$ with $x \not\sim y$. Since $\sim$ is closed in $\mathbb{X}^2$, the set

$$\{\, z \in X \mid z \sim x \,\} = \pi_1\big(\sim \cap\, (X \times \{x\})\big)$$

is closed in $\mathbb{X}$ and by (b) and (i) is an up-set. Hence $Y := \{\, z \in X \mid z \not\sim x \,\}$ is an open down-set containing y. By Lemma 3.13(1), there is a clopen down-set V containing y with $V \subseteq Y$. The set

$$\begin{aligned} W &:= \{\, w \in X \mid (\forall z \in V)\ w \not\sim z \,\} \\ &= X \setminus \{\, w \in X \mid (\exists z \in V)\ w \sim z \,\} \\ &= X \setminus \pi_1\big(\sim \cap\, (X \times V)\big), \end{aligned}$$

is a down-set as $\{\, w \in X \mid (\exists z \in V)\ w \sim z \,\}$ is an up-set, by (b) and (i), and is open in $\mathbb{X}$, as $\sim \cap\, (X \times V)$ is closed in $\mathbb{X}^2$. Since $V \subseteq Y$, we have $x \in W$, and by (c) and the fact that V is a down-set, we have $B \setminus V \subseteq W$; so $\downarrow x \cup (B \setminus V) \subseteq W$. By Lemma 3.13(3) and (iv), $\downarrow x \cup (B \setminus V)$ is a closed down-set. Hence, by Lemma 3.13(2), applied to $\downarrow x \cup (B \setminus V)$ and $X \setminus W$, there is a clopen down-set U containing $\downarrow x \cup (B \setminus V)$ with $U \subseteq W$. Thus $B \subseteq U \cup V$, and, by construction, we have $(U \times V) \cap \sim\, = \varnothing$, as $U \subseteq W$. As U and V are non-empty, Lemma 3.14 tells us that the map $\alpha = \lambda_{UV}$ is an $\mathbf{\mathfrak{X}}_{\mathcal{L}}$-morphism from $\mathbb{X}$ to $\mathbb{L}$ satisfying $\alpha(x) = 0 \not\sim 1 = \alpha(y)$. Hence (S$_2$) holds.

(S$_3$): Let $x, y \in X$ with $x \nleqslant y$. If $x \in B$, then $x \not\sim y$, by (c), and the map α constructed in the proof of (S$_2$) satisfies $\alpha(x) \not\sim \alpha(y)$ and so satisfies $\alpha(x) \nleqslant \alpha(y)$. If $x \notin B$, then the closed down-set $\downarrow y \cup B$ and the closed up-set $\uparrow x$ are disjoint and Lemma 3.13(2) yields a clopen down-set W with $\downarrow y \cup B \subseteq W$ and $W \cap \uparrow x = \varnothing$. Since $x \preccurlyeq d$, we have $d \notin W$. By Lemma 3.14, the map $\alpha = \lambda_W$ is an $\mathfrak{X}_{\mathcal{L}}$-morphism and satisfies $\alpha(x) \not\preccurlyeq \alpha(y)$ by construction. Hence (S_3) holds.

(S_4): Let $x \in X \setminus B$. The map $\alpha = \lambda_W$ constructed in the proof of (S_1) is an $\mathfrak{X}_{\mathcal{L}}$-morphism satisfying $\alpha(x) = d \notin B$. □

Proofs of Theorems 3.7, 3.9 and 3.11. We use the fact that the categories $\mathfrak{X}_{\mathcal{K}}$, $\mathfrak{X}_{\mathcal{T}}$ and $\mathfrak{X}_{\mathcal{D}\mathrm{k}}$ have natural enrichments in $\mathfrak{X}_{\mathcal{L}}$ (corresponding to the reducts of $\mathcal{K}$, $\mathcal{T}$ and $\mathcal{D}$k in $\mathcal{L}$). We start with Theorem 3.9 as $\mathbb{T}$ has the smallest signature.

Let $\mathbb{X} = \langle X; \preccurlyeq, \sim, \mathcal{T}\rangle$ be a topological structure in the signature of $\mathbb{T}$ and assume that $\mathbb{X}$ satisfies (NT_1)–(NT_3). Define the structure $\mathrm{S}(\mathbb{X}) := \langle X \dot\cup \{d\}; d, \preccurlyeq', \sim', B, \mathcal{T}\rangle$ where

- $\preccurlyeq' = \preccurlyeq \cup (X \times \{d\}) \cup \{(d, d)\}$,
- $\sim' = \sim \cup (X \times \{d\}) \cup (\{d\} \times X) \cup \{(d, d)\}$, and
- $B = \varnothing$.

It is easily seen that $\mathrm{S}(\mathbb{X})$ satisfies (NL_1)–(NL_3). Hence, by Theorem 3.5, $\mathrm{S}(\mathbb{X})$ belongs to $\mathfrak{X}_{\mathcal{L}}$ and so embeds into a power of $\mathbb{L}$. It follows at once that $\mathbb{X}$ embeds into a power of $\mathbb{T}$ and so belongs to $\mathfrak{X}_{\mathcal{T}}$. This proves Theorem 3.9. When restricted to $\mathcal{L}$, the map S yields a functor $\mathrm{S}\colon \mathfrak{X}_{\mathcal{L}} \to \mathfrak{X}_{\mathcal{T}}$ that is dual to the forgetful functor from $\mathcal{T}$ to $\mathcal{L}$.

Proofs of Theorems 3.7 and 3.11 are simple modifications of this argument. □

4 The Restricted Priestley dualities for $\mathcal{T}$ and $\mathcal{D}$k

4.1 The Restricted Priestley duality for $\mathcal{T}$ A general theory of restricted Priestley dualities is presented by B. A. Davey and A. Gair [18], but we will not require the general theory here. Priestley duality gives a dual equivalence between the category $\mathcal{D}$ of bounded distributive lattices and the category $\mathcal{P}$ of Priestley spaces with continuous order-preserving maps as morphisms (see Priestley [38, 39] and Davey and Priestley [21]). It is the strong duality that arises from the Special NU Strong Duality Theorem 3.1 applied to $\mathbf{D} = \langle \{0, 1\}; \wedge, \vee, 0, 1\rangle$, the two-element bounded lattice, and $\mathbb{D} = \langle \{0,1\}; \sqsubseteq, \mathcal{T}\rangle$, the two-element chain with the discrete topology. At the object level:

- the dual of a bounded distributive lattice $\mathbf{A}$ is $\mathrm{H}(\mathbf{A}) = \mathfrak{D}(\mathbf{A}, \mathbf{D})$, with its order and topology inherited from the power $\mathbb{D}^A$;
- the dual of a Priestley space $\mathbb{Y}$ is $\mathrm{K}(\mathbb{Y}) = \mathfrak{P}(\mathbb{Y}, \mathbb{D})$, a sublattice of $\mathbf{D}^Y$.

The functors are defined on morphisms via composition as described in Subsection 3.1.

Given a ternary algebra $\mathbf{A}$, let $\mathbf{A}^\flat = \langle A; \vee, \wedge, 0, 1\rangle$ be its bounded-distributive-lattice reduct. Since $\neg\colon A \to A$ is a homomorphism from $\mathbf{A}^\flat$ to $(\mathbf{A}^\flat)^\partial$, it is encoded in the Priestley dual $\mathrm{H}(\mathbf{A}^\flat)$ via a map $g\colon \mathrm{H}(\mathbf{A}^\flat) \to \mathrm{H}(\mathbf{A}^\flat)$ that is continuous and order-reversing. Formally, $g\colon \mathfrak{D}(\mathbf{A}^\flat, \mathbf{D}) \to \mathfrak{D}(\mathbf{A}^\flat, \mathbf{D})$ is given by

$$\big(\forall y \in \mathfrak{D}(\mathbf{A}^\flat, \mathbf{D})\big)\big(\forall a \in A\big) \quad g(y)(a) = 1 \iff y(\neg a) = 0, \tag{$\dagger$}$$

or equivalently by

$$\big(\forall y \in \mathfrak{D}(\mathbf{A}^\flat, \mathbf{D})\big) \quad g(y) = c \circ y \circ \neg, \tag{$\ddagger$}$$

where $c\colon \{0,1\} \to \{0,1\}$ is complementation. Our aim is to axiomatise the structures of the form $\langle \mathfrak{D}(\mathbf{A}^\flat, \mathbf{D}); g, \sqsubseteq, \mathcal{T}\rangle$, where $\mathbf{A}$ is a ternary algebra.

Definition 4.1. A topological structure $\mathbb{Y} = \langle Y; g, \sqsubseteq, \mathcal{T}\rangle$ is a *ternary Priestley space* if

(PT$_1$) $\mathbb{Y}^\flat := \langle Y; \sqsubseteq, \mathcal{T}\rangle$ is a Priestley space,

(PT$_2$) $g\colon Y \to Y$ is continuous and order-reversing,

(PT$_3$) $g \circ g = \mathrm{id}_Y$,

(PT$_4$) $y \sqsubseteq g(y)$ or $y \sqsupseteq g(y)$, for all $y \in Y$, and

(PT$_5$) $g(y) \neq y$, for all $y \in Y$.

The category of ternary Priestley spaces with continuous, order-preserving and g-preserving maps as morphisms is denoted by $\mathfrak{Y}_{\mathfrak{T}}$.

Every ternary Priestley space has a natural partition into a clopen down-set and a clopen up-set that are interchanged by g.

Lemma 4.2. *Let $\mathbb{Y} = \langle Y; g, \sqsubseteq, \mathcal{T}\rangle$ be a ternary Priestley space and define*

$$\mathbb{Y}^- = \{\, y \in Y \mid y \sqsubseteq g(y) \,\} \quad \textit{and} \quad \mathbb{Y}^+ = \{\, y \in Y \mid g(y) \sqsubseteq y \,\}.$$

(1) *$g(\mathbb{Y}^-) = \mathbb{Y}^+$, $g(\mathbb{Y}^+) = \mathbb{Y}^-$ and $Y = \mathbb{Y}^- \,\dot\cup\, \mathbb{Y}^+$,*

(2) *$\mathbb{Y}^-$ is a clopen down-set and $\mathbb{Y}^+$ is a clopen up-set.*

Proof. (1) Conditions (PT_4) and (PT_5) imply that $Y = \mathbb{Y}^- \,\dot\cup\, \mathbb{Y}^+$, while ($\mathrm{PT}_2$) and ($\mathrm{PT}_3$) imply that $g(\mathbb{Y}^-) = \mathbb{Y}^+$ and $g(\mathbb{Y}^+) = \mathbb{Y}^-$.

(2) As g is continuous, by (PT_2), $\mathrm{graph}(g)$ is closed in $\mathbb{Y}^2$, and by Lemma 3.13(4), $\sqsubseteq$ is closed in $\mathbb{Y}^2$, by (PT_1). It follows that both $\mathbb{Y}^-$ and $\mathbb{Y}^+$ are closed in $\mathbb{Y}$, whence, by (1), both are clopen. Let $y \in \mathbb{Y}^-$ and let $x \in Y$ with $x \sqsubseteq y$. Hence $y \sqsubseteq g(y)$ and $g(y) \sqsubseteq g(x)$, by (PT_2). Thus $x \sqsubseteq g(x)$, whence $x \in \mathbb{Y}^-$. Hence $\mathbb{Y}^-$ is a down-set, and similarly $\mathbb{Y}^+$ is an up-set. □

Given a ternary Priestley space $\mathbb{Y} = \langle Y; g, \sqsubseteq, \mathcal{T}\rangle$ with underlying Priestley space $\mathbb{Y}^\flat = \langle Y; \sqsubseteq, \mathcal{T}\rangle$, define $\neg\colon \boldsymbol{\mathcal{P}}(\mathbb{Y}^\flat, \mathbb{D}) \to \boldsymbol{\mathcal{P}}(\mathbb{Y}^\flat, \mathbb{D})$ by

$$\big(\forall \alpha \in \boldsymbol{\mathcal{P}}(\mathbb{Y}^\flat, \mathbb{D})\big)\big(\forall y \in Y\big) \quad \neg\alpha(y) = 1 \iff \alpha(g(y)) = 0, \tag{$*$}$$

or equivalently by

$$\big(\forall \alpha \in \boldsymbol{\mathcal{P}}(\mathbb{Y}^\flat, \mathbb{D})\big) \quad \neg\alpha = c \circ \alpha \circ g, \tag{$**$}$$

where $c\colon \{0,1\} \to \{0,1\}$ is complementation.

Theorem 4.3. *Let $\mathbf{A}$ be a ternary algebra and let $\mathbb{Y}$ be a ternary Priestley space.*

(1) *$\langle \boldsymbol{\mathcal{D}}(\mathbf{A}^\flat, \mathbf{D}); g, \sqsubseteq, \mathcal{T}\rangle$ is a ternary Priestley space.*

(2) *$\langle \boldsymbol{\mathcal{P}}(\mathbb{Y}^\flat, \mathbb{D}); \vee, \wedge, \neg, \underline{0}, d, \underline{1}\rangle$ is a ternary algebra, where $\underline{0}, \underline{1}\colon \mathbb{Y}^\flat \to \mathbb{D}$ are the constant maps and $d\colon \mathbb{Y}^\flat \to \mathbb{D}$ is defined by*

$$d(y) = 1 \iff y \sqsupseteq g(y). \tag{$\sharp$}$$

Proof. (1) As both $\neg$ and c are dual $\boldsymbol{\mathcal{D}}$-homomorphisms, $g(y) = c \circ y \circ \neg$ is a $\boldsymbol{\mathcal{D}}$-homomorphism, for all $y \in \boldsymbol{\mathcal{D}}(\mathbf{A}^\flat, \mathbf{D})$; hence g is well defined. We must prove (PT_2)–(PT_5) of Definition 4.1. The subbasic open sets in $\mathfrak{D}(\mathbf{A}^\flat, \mathbf{D})$ are of the form

$$U_{a,i} := \{\, y \in \mathfrak{D}(\mathbf{A}^\flat, \mathbf{D}) \mid y(a) = i \,\},$$

for $a \in A$ and $i \in \{0, 1\}$. We have $g^{-1}(U_{a,i}) = U_{\neg a, c(i)}$, by the definition of g. Hence g is continuous.

To prove that g is order-reversing, let $x, y \in \mathfrak{D}(\mathbf{A}^\flat, \mathbf{D})$ with $x \sqsubseteq y$. By (†), for all $a \in A$, we have

$$g(y)(a) = 1 \iff y(\neg a) = 0 \implies x(\neg a) = 0 \iff g(x)(a) = 1,$$

whence $g(x) \sqsupseteq g(y)$. Hence (PT_2) holds. That $g(g(y)) = y$ is an immediate consequence of (‡) and the fact that $\neg \circ \neg = \mathrm{id}_A$ and $c \circ c = \mathrm{id}_{\{0,1\}}$; hence ($\mathrm{PT}_3$) holds.

To prove (PT_4), let $y \in \mathfrak{D}(\mathbf{A}^\flat, \mathbf{D})$ and suppose that $y \not\sqsubseteq g(y)$ and $y \not\sqsupseteq g(y)$. Hence there exist $a, b \in A$ with

$$y(a) = 1 \;\;\&\;\; g(y)(a) = 0 \quad \text{and} \quad y(b) = 0 \;\;\&\;\; g(y)(b) = 1.$$

Thus, by (†), we have

$$\begin{aligned} & y(a) = 1 \;\;\&\;\; y(\neg a) = 1 \quad \text{and} \quad y(b) = 0 \;\;\&\;\; y(\neg b) = 0 \\ \implies & y(a \wedge \neg a) = 1 \quad \text{and} \quad y(b \vee \neg b) = 0 \\ \implies & a \wedge \neg a \nleqslant b \vee \neg b, \end{aligned}$$

which contradicts the fact that $\mathbf{A}$ satisfies $a \wedge \neg a \leqslant b \vee \neg b$, for all $a, b \in A$. Hence (PT_4) holds.

To see that $g(y) \neq y$ it suffices to note that, by (‡) and the fact that $\neg d = d$,

$$g(y)(d) = c(y(\neg d) = c(y(d)) \neq y(d).$$

Hence (PT_5) holds.

(2) It follows immediately from Lemma 4.2 that the map d is continuous and order-preserving and so belongs to $\mathfrak{P}(\mathbb{Y}^\flat, \mathbb{D})$, and moreover, d satisfies

$$d(y) = 0 \iff y \sqsubseteq g(y). \tag{♯♯}$$

We now prove conditions (M), (K) and (D) from Definition 1.1. Given the similarity between the definition of $\neg$ in ($*$) and ($**$) and the definition of g in (†) and (‡), the proof that $\neg$ is order-reversing and satisfies $\neg\neg\alpha = \alpha$, for all $\alpha \in \boldsymbol{\mathcal{P}}(\mathbb{Y}^\flat, \mathbb{D})$, is a simple symbol swap of the corresponding proof in (1). It follows at once that $\neg$ is a dual order-isomorphism (that is, $\neg$ is surjective and satisfies $\alpha \leqslant \beta \Leftrightarrow \neg\alpha \geqslant \neg\beta$) and so is a dual lattice automorphism. Hence (M) follows.

We turn now to condition (K). Let $\alpha \in \boldsymbol{\mathcal{P}}(\mathbb{Y}^\flat, \mathbb{D})$. To prove that $\alpha \vee \neg\alpha \geqslant d$ we must show that, for all $y \in Y$, if $d(y) = 1$ then $\alpha(y) \vee \neg\alpha(y) = 1$. Let $y \in Y$ with $d(y) = 1$. If $\alpha(y) = 1$, then we are done, so assume that $\alpha(y) = 0$. Since $d(y) = 1$, by (♯) we have $y \sqsupseteq g(y)$ and so $\alpha(y) \sqsupseteq \alpha(g(y))$. Thus,

$$\begin{aligned}\alpha(y) = 0 \implies \alpha(g(y)) = 0 &\implies c(\alpha(g(y))) = 1\\ &\implies \neg\alpha(y) = 1 \implies \alpha(y) \vee \neg\alpha(y) = 1,\end{aligned}$$

as required. The proof that $\alpha \wedge \neg\alpha \leqslant d$ is the order-theoretic dual of this argument but using (♯♯) rather than (♯).

Finally, we prove that $\neg d = d$. Let $y \in Y$. Using, in order, (∗∗), (♯♯), (PT_3) and (♯), we have

$$\begin{aligned}\neg d(y) = 1 \iff c(d(g(y))) = 1 &\iff d(g(y)) = 0\\ &\iff g(y) \sqsubseteq g(g(y)) \iff g(y) \sqsubseteq y \iff d(y) = 1.\end{aligned}$$

Hence $\neg d = d$. This completes the proof of (2). □

By restricting the domains and codomains of the Priestley-duality functors, with a slight abuse of notation we now have functors $\mathrm{H}\colon \boldsymbol{\mathcal{T}} \to \boldsymbol{\mathcal{Y}}_{\boldsymbol{\mathcal{T}}}$ and $\mathrm{K}\colon \boldsymbol{\mathcal{Y}}_{\boldsymbol{\mathcal{T}}} \to \boldsymbol{\mathcal{T}}$ given on objects by: for all $\mathbf{A} \in \boldsymbol{\mathcal{T}}$ and $\mathbb{Y} \in \boldsymbol{\mathcal{Y}}_{\boldsymbol{\mathcal{T}}}$,

$$\begin{gathered}\mathrm{H}(\mathbf{A}) = \langle \boldsymbol{\mathcal{D}}(\mathbf{A}^\flat, \mathbf{D}); g, \sqsubseteq, \mathcal{T}\rangle, \text{ and}\\ \mathrm{K}(\mathbb{Y}) = \langle \boldsymbol{\mathcal{P}}(\mathbb{Y}^\flat, \mathbb{D}); \vee, \wedge, \neg, \underline{0}, d, \underline{1}\rangle.\end{gathered}$$

That the functors are well defined on morphisms follows from our next lemma.

Lemma 4.4. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be ternary algebras, let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be ternary Priestley spaces, let* $u\colon \mathbf{A}^\flat \to \mathbf{B}^\flat$ *be a homomorphism and let* $\varphi\colon \mathbb{X}^\flat \to \mathbb{Y}^\flat$ *be a Priestley-space morphism.*

(1) *u preserves $\neg$ (and therefore preserves d) if and only if* $\mathrm{H}(u)\colon \mathrm{H}(\mathbf{B}) \to \mathrm{H}(\mathbf{A})$ *preserves g.*

(2) *φ preserves g if and only if* $\mathrm{K}(\varphi)\colon \mathrm{K}(\mathbb{Y}) \to \mathrm{K}(\mathbb{X})$ *preserves $\neg$ (and therefore preserves d).*

Proof. We will prove (1). The proof of (2) is very similar. Assume that u preserves $\neg$. Then, $u \circ \neg = \neg \circ u$, and hence

$$\mathrm{H}(u)(g(y)) = g(y) \circ u = c \circ y \circ \neg \circ u = c \circ y \circ u \circ \neg = g(y \circ u) = g(\mathrm{H}(u)(y)),$$

for all $y \in \boldsymbol{\mathfrak{D}}(\mathbf{B}, \mathbf{D})$. Thus, $\mathrm{H}(u)$ preserves g.

Conversely, assume that $\mathrm{H}(u)$ preserves g. Then, for all $y \in \boldsymbol{\mathfrak{D}}(\mathbf{B}, \mathbf{D})$ we have $\mathrm{H}(u)(g(y)) = g(\mathrm{H}(u)(y))$, whence

$$\begin{aligned} y \circ (u \circ \neg) &= c \circ c \circ (y \circ u) \circ \neg = c \circ g(\mathrm{H}(u)(y)) \\ &= c \circ \mathrm{H}(u)(g(y)) = c \circ (c \circ y \circ \neg) \circ u = y \circ (\neg \circ u). \end{aligned}$$

Since $\boldsymbol{\mathfrak{D}}(\mathbf{B}, \mathbf{D})$ separates the points of B, we conclude that $u \circ \neg = \neg \circ u$. □

Theorem 4.5 (Restricted Priestley duality for $\boldsymbol{\mathfrak{T}}$)**.** *The two functors* $\mathrm{H}\colon \boldsymbol{\mathfrak{T}} \to \boldsymbol{\mathfrak{Y}}_{\boldsymbol{\mathfrak{T}}}$ *and* $\mathrm{K}\colon \boldsymbol{\mathfrak{Y}}_{\boldsymbol{\mathfrak{T}}} \to \boldsymbol{\mathfrak{T}}$ *give a dual category equivalence between the category $\boldsymbol{\mathfrak{T}}$ of ternary algebras and the category $\boldsymbol{\mathfrak{Y}}_{\boldsymbol{\mathfrak{T}}}$ of ternary Priestley spaces. In particular, $\mathbf{A} \cong \mathrm{KH}(\mathbf{A})$ and $\mathbb{Y} \cong \mathrm{HK}(\mathbb{Y})$ for every ternary algebra $\mathbf{A}$ and every ternary Priestley space $\mathbb{Y}$.*

Proof. Since the functors $\mathrm{H}\colon \boldsymbol{\mathfrak{D}} \to \boldsymbol{\mathfrak{P}}$ and $\mathrm{K}\colon \boldsymbol{\mathfrak{P}} \to \boldsymbol{\mathfrak{D}}$ give a dual category equivalence between the category $\boldsymbol{\mathfrak{D}}$ of bounded distributive lattices and the category $\boldsymbol{\mathfrak{P}}$ of Priestley spaces, it remains only to prove that, for all $\mathbf{A} \in \boldsymbol{\mathfrak{T}}$ and all $\mathbb{Y} \in \boldsymbol{\mathfrak{Y}}_{\boldsymbol{\mathfrak{T}}}$,

(1) the isomorphism $e_{\mathbf{A}^\flat}\colon \mathbf{A}^\flat \to \mathrm{KH}(\mathbf{A})^\flat$ in $\boldsymbol{\mathfrak{D}}$ is an isomorphism in $\boldsymbol{\mathfrak{T}}$, and

(2) the isomorphism $\varepsilon_{\mathbb{Y}^\flat}\colon \mathbb{Y}^\flat \to \mathrm{DE}(\mathbb{Y})^\flat$ in $\boldsymbol{\mathfrak{P}}$ is an isomorphism in $\boldsymbol{\mathfrak{Y}}_{\boldsymbol{\mathfrak{T}}}$.

Again, we will prove only (1). Let $\mathbf{A} \in \boldsymbol{\mathfrak{T}}$, let $a \in A$ and let $y \in \boldsymbol{\mathfrak{D}}(\mathbf{A}^\flat, \mathbf{D})$. Since $\neg e_{\mathbf{A}^\flat}(a) = c \circ e_{\mathbf{A}^\flat}(a) \circ g$ in $\mathrm{KH}(\mathbf{A}^\flat)$ and $g(y) = c \circ y \circ \neg$ in $\boldsymbol{\mathfrak{D}}(\mathbf{A}^\flat, \mathbf{D})$, we have

$$\begin{aligned} \big(\neg e_{\mathbf{A}^\flat}(a)\big)(y) = c(e_{\mathbf{A}^\flat}(a)(g(y))) &= c(g(y)(a)) \\ &= c(c(y(\neg a))) = y(\neg a) = e_{\mathbf{A}^\flat}(\neg a)(y). \end{aligned}$$

Hence $\neg e_{\mathbf{A}^\flat}(a) = e_{\mathbf{A}^\flat}(\neg a)$, as required. □

4.2 The Restricted Priestley duality for $\mathfrak{D}$k Very little work is now required to obtain the restricted Priestley dual for don't know algebras. Since the algebras in $\mathfrak{D}$k are not necessarily bounded, we must use the corresponding version of Priestley duality. Let

$$\overline{\mathbf{D}} = \langle \{0,1\}; \vee, \wedge \rangle \quad \text{and} \quad \mathbb{D}_{01} = \langle \{0,1\}; 0, 1, \sqsubseteq, \mathcal{T} \rangle$$

be the two-element lattice and the two-element bounded Priestley space. A straightforward application of the Special NU Strong Duality Theorem 3.1(2) with $C = \{0,1\}$ shows that $\mathbb{D}_{01}$ yields a strong duality between the category $\overline{\mathfrak{D}} = \mathsf{ISP}(\overline{\mathbf{D}})$ of distributive lattices and the category $\mathfrak{P}_{01} = \mathsf{IS_cP^+}(\mathbb{D}_{01})$ of bounded Priestley spaces (with continuous order- and bound-preserving maps as morphisms).

Definition 4.6. A topological structure $\mathbb{Y} = \langle Y; g, 0, 1, \sqsubseteq, \mathcal{T} \rangle$ is a *bounded ternary Priestley space* if

(BPT$_1$) $\mathbb{Y}^\flat := \langle Y; 0, 1, \sqsubseteq, \mathcal{T} \rangle$ is a bounded Priestley space,

(BPT$_2$) $g \colon Y \to Y$ is continuous and order-reversing,

(BPT$_3$) $g \circ g = \mathrm{id}_Y$,

(BPT$_4$) $y \sqsubseteq g(y)$ or $y \sqsupseteq g(y)$, for all $y \in Y$, and

(BPT$_5$) $g(y) \neq y$, for all $y \in Y$.

The category of bounded ternary Priestley spaces with continuous, order-, bound- and g-preserving maps as morphisms is denoted by $\mathfrak{Y}_{\mathfrak{D}\mathrm{k}}$.

We denote the natural hom-functors between $\overline{\mathfrak{D}}$ and $\mathfrak{P}_{01}$ by $\overline{\mathrm{H}}$ and K_{01}. For each $\mathbf{A} \in \mathfrak{D}\mathrm{k}$ and each $\mathbb{Y} \in \mathfrak{Y}_{\mathfrak{D}\mathrm{k}}$, we now let $\mathbf{A}^\flat$ and $\mathbb{Y}^\flat$ be respectively the underlying distributive lattice and bounded Priestley space. Restricting to $\mathfrak{D}$k and $\mathfrak{Y}_{\mathfrak{D}\mathrm{k}}$, we now have functors $\overline{\mathrm{H}} \colon \mathfrak{D}\mathrm{k} \to \mathfrak{Y}_{\mathfrak{D}\mathrm{k}}$ and $\mathrm{K}_{01} \colon \mathfrak{Y}_{\mathfrak{D}\mathrm{k}} \to \mathfrak{D}\mathrm{k}$ given on objects by

$$\begin{aligned} \overline{\mathrm{H}}(\mathbf{A}) &= \langle \overline{\mathfrak{D}}(\mathbf{A}^\flat, \overline{\mathbf{D}}); g, \underline{0}, \underline{1}, \sqsubseteq, \mathcal{T} \rangle, \text{ and} \\ \mathrm{K}_{01}(\mathbb{Y}) &= \langle \mathfrak{P}_{01}(\mathbb{Y}^\flat, \mathbb{D}_{01}); \vee, \wedge, \neg, d \rangle, \end{aligned}$$

for all $\mathbf{A} \in \mathfrak{D}\mathrm{k}$ and $\mathbb{Y} \in \mathfrak{Y}_{\mathfrak{D}\mathrm{k}}$, where $\underline{0}, \underline{1} \colon \mathbf{A}^\flat \to \overline{\mathbf{D}}$ are the constant maps and, as before, $d \colon \mathbb{Y}^\flat \to \mathbb{D}_{01}$ is defined by

$$d(y) = 1 \iff y \sqsupseteq g(y).$$

Theorem 4.7 (Restricted Priestley duality for $\mathfrak{D}\mathrm{k}$)**.** *The two functors* $\overline{\mathrm{H}}\colon \mathfrak{D}\mathrm{k} \to \mathfrak{Y}_{\mathfrak{D}\mathrm{k}}$ *and* $\mathrm{K}_{01}\colon \mathfrak{Y}_{\mathfrak{D}\mathrm{k}} \to \mathfrak{D}\mathrm{k}$ *give a dual category equivalence between the category* $\mathfrak{D}\mathrm{k}$ *of don't know algebras and the category* $\mathfrak{Y}_{\mathfrak{D}\mathrm{k}}$ *of bounded ternary Priestley spaces. In particular,* $\mathbf{A} \cong \mathrm{K}_{01}\overline{\mathrm{H}}(\mathbf{A})$ *and* $\mathbb{Y} \cong \overline{\mathrm{H}}\mathrm{K}_{01}(\mathbb{Y})$ *for every don't know algebra* $\mathbf{A}$ *and every bounded ternary Priestley space* $\mathbb{Y}$*.*

Proof. That $\overline{\mathrm{H}}\colon \mathfrak{D}\mathrm{k} \to \mathfrak{Y}_{\mathfrak{D}\mathrm{k}}$ and $\mathrm{K}_{01}\colon \mathfrak{Y}_{\mathfrak{D}\mathrm{k}} \to \mathfrak{D}\mathrm{k}$ are well-defined functors is a trivial modification of the proof above for $\mathrm{H}\colon \mathfrak{T} \to \mathfrak{Y}_{\mathfrak{T}}$ and $\mathrm{K}\colon \mathfrak{Y}_{\mathfrak{T}} \to \mathfrak{T}$. It is clear that a proof of the theorem can be obtained by making simple changes to the proof of Theorem 4.5. Instead, we will derive the results directly from Theorem 4.5 itself.

Let $\widehat{\mathfrak{D}\mathrm{k}}$ be the full subcategory of $\mathfrak{D}\mathrm{k}$ consisting of bounded don't know algebras in which the top is join-irreducible and the bottom is meet-irreducible. Let $\mathrm{U}\colon \mathfrak{D}\mathrm{k} \to \widehat{\mathfrak{D}\mathrm{k}}$ be the functor that adds new bounds and extends $\neg$ in the obvious way, and let $\mathrm{V}\colon \widehat{\mathfrak{D}\mathrm{k}} \to \mathfrak{D}\mathrm{k}$ be the functor that removes the bounds. Then, $\mathrm{VU}(\mathbf{A}) = \mathbf{A}$ and $\mathrm{UV}(\mathbf{A}) \cong \mathbf{A}$. We regard $\mathfrak{Y}_{\mathfrak{D}\mathrm{k}}$ as a subcategory of $\mathfrak{Y}_{\mathfrak{T}}$, so that both $\mathrm{K}_{01}(\mathbb{Y})$ and $\mathrm{K}(\mathbb{Y})$ are defined. Then, for all $\mathbf{A} \in \mathfrak{D}\mathrm{k}$ and all $\mathbb{Y} \in \mathfrak{Y}_{\mathfrak{D}\mathrm{k}}$, we have

$$\overline{\mathrm{H}}(\mathbf{A}) \cong \mathrm{HU}(\mathbf{A}) \quad \text{and} \quad \mathrm{K}_{01}(\mathbb{Y}) \cong \mathrm{VK}(\mathbb{Y}).$$

Hence, by Theorem 4.5,

$$\begin{aligned} &\mathrm{K}_{01}\overline{\mathrm{H}}(\mathbf{A}) \cong \mathrm{VKHU}(\mathbf{A}) \cong \mathrm{VU}(\mathbf{A}) = \mathbf{A}, \text{ and} \\ &\overline{\mathrm{H}}\mathrm{K}_{01}(\mathbb{Y}) \cong \mathrm{HUVK}(\mathbb{Y}) \cong \mathrm{HK}(\mathbb{Y}) \cong \mathbb{Y}. \end{aligned}$$ □

5 The Restricted Priestley dualities for $\mathcal{K}$ and $\mathcal{L}$

5.1 The Restricted Priestley duality for $\mathcal{K}$ The restricted Priestley duality for Kleene algebras dates back to W. H. Cornish and P. R. Fowler [14, 15] in the 1970s. It was generalised to varieties of Ockham algebras generated by a finite subdirectly irreducible algebra by B. A. Davey and H. A. Priestley [19] in 1987 (see also [13, 7.4.5]). For completeness and for comparison with the results for ternary algebras and don't know algebras, we record the result here.

Definition 5.1. A topological structure $\mathbb{Y} = \langle Y; g, \sqsubseteq, \mathcal{T}\rangle$ is a *Kleene space* if

(PK$_1$) $\mathbb{Y}^\flat := \langle Y; \sqsubseteq, \mathcal{T}\rangle$ is a Priestley space,

(PK$_2$) $g\colon Y \to Y$ is continuous and order-reversing,

(PK$_3$) $g \circ g = \mathrm{id}_Y$, and

(PK$_4$) $y \sqsubseteq g(y)$ or $y \sqsupseteq g(y)$, for all $y \in Y$.

The category of Kleene spaces with continuous, order-preserving and g-preserving maps as morphisms is denoted by $\boldsymbol{\mathcal{Y}}_{\boldsymbol{\mathcal{K}}}$.

As with $\boldsymbol{\mathcal{T}}$, we restrict the domains and codomains of the Priestley-duality functors, to yield functors $\mathrm{H}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{Y}}_{\boldsymbol{\mathcal{K}}}$ and $\mathrm{K}\colon \boldsymbol{\mathcal{Y}}_{\boldsymbol{\mathcal{K}}} \to \boldsymbol{\mathcal{K}}$ given on objects by

$$\mathrm{H}(\mathbf{A}) = \langle \boldsymbol{\mathcal{D}}(\mathbf{A}^\flat, \mathbf{D}); g, \sqsubseteq, \mathcal{T}\rangle, \text{ and}$$
$$\mathrm{K}(\mathbb{Y}) = \langle \boldsymbol{\mathcal{P}}(\mathbb{Y}^\flat, \mathbb{D}); \vee, \wedge, \neg, \underline{0}, \underline{1}\rangle,$$

for all $\mathbf{A} \in \boldsymbol{\mathcal{K}}$ and $\mathbb{Y} \in \boldsymbol{\mathcal{Y}}_{\boldsymbol{\mathcal{K}}}$.

Theorem 5.2 (Restricted Priestley duality for $\boldsymbol{\mathcal{K}}$)**.** *The functors* $\mathrm{H}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{Y}}_{\boldsymbol{\mathcal{K}}}$ *and* $\mathrm{K}\colon \boldsymbol{\mathcal{Y}}_{\boldsymbol{\mathcal{K}}} \to \boldsymbol{\mathcal{K}}$ *give a dual category equivalence between the category* $\boldsymbol{\mathcal{K}}$ *of Kleene algebras and the category* $\boldsymbol{\mathcal{Y}}_{\boldsymbol{\mathcal{K}}}$ *of Kleene spaces. In particular,* $\mathbf{A} \cong \mathrm{KH}(\mathbf{A})$ *and* $\mathbb{Y} \cong \mathrm{HK}(\mathbb{Y})$ *for every Kleene algebra* $\mathbf{A}$ *and every Kleene space* $\mathbb{Y}$.

5.2 The Restricted Priestley duality for $\boldsymbol{\mathcal{L}}$

The restricted Priestley duality for don't know algebras was obtained from the corresponding duality for ternary algebras by simply adding the requirement that the Priestley spaces be bounded. The restricted Priestley duality for Kleene lattices arises from the restricted Priestley duality for Kleene algebras in exactly the same way.

Definition 5.3. A topological structure $\mathbb{Y} = \langle Y; g, 0, 1, \sqsubseteq, \mathcal{T}\rangle$ is a *bounded Kleene space* if

(PL$_1$) $\mathbb{Y}^\flat := \langle Y; 0, 1, \sqsubseteq, \mathcal{T}\rangle$ is a bounded Priestley space,

(PL$_2$) $g\colon Y \to Y$ is continuous and order-reversing,

(PL$_3$) $g \circ g = \mathrm{id}_Y$, and

(PL$_4$) $y \sqsubseteq g(y)$ or $y \sqsupseteq g(y)$, for all $y \in Y$.

The category of bounded Kleene spaces with continuous, order-, bound- and g-preserving maps as morphisms is denoted by $\mathfrak{Y}_{\mathcal{L}}$.

As with the restricted Priestley duality for $\mathfrak{D}$k, we restrict the domains and codomains of the Priestley-duality functors (for not-necessarily bounded distributive lattices), to yield functors $\overline{\mathrm{H}}\colon \mathcal{L} \to \mathfrak{Y}_{\mathcal{L}}$ and $\mathrm{K}_{01}\colon \mathfrak{Y}_{\mathcal{L}} \to \mathcal{L}$ given on objects by

$$\overline{\mathrm{H}}(\mathbf{A}) = \langle \overline{\mathfrak{D}}(\mathbf{A}^\flat, \overline{\mathbf{D}}); g, \underline{0}, \underline{1}, \sqsubseteq, \mathcal{T}\rangle, \text{ and}$$
$$\mathrm{K}_{01}(\mathbb{Y}) = \langle \mathfrak{P}_{01}(\mathbb{Y}^\flat, \mathbb{D}_{01}); \vee, \wedge, \neg\rangle,$$

for all $\mathbf{A} \in \mathcal{L}$ and $\mathbb{Y} \in \mathfrak{Y}_{\mathcal{L}}$, where $\underline{0}, \underline{1}\colon \mathbf{A}^\flat \to \overline{\mathbf{D}}$ are the constant maps. The following theorem was first stated by H. Gaitán [24].

Theorem 5.4 (Restricted Priestley duality for $\mathcal{L}$)**.** *The functors* $\mathrm{H}\colon \mathcal{L} \to \mathfrak{Y}_{\mathcal{L}}$ *and* $\mathrm{K}\colon \mathfrak{Y}_{\mathcal{L}} \to \mathcal{L}$ *give a dual category equivalence between the category* $\mathcal{L}$ *of Kleene lattices and the category* $\mathfrak{Y}_{\mathcal{L}}$ *of bounded Kleene spaces. In particular,* $\mathbf{A} \cong \mathrm{KH}(\mathbf{A})$ *and* $\mathbb{Y} \cong \mathrm{HK}(\mathbb{Y})$ *for every Kleene lattice* $\mathbf{A}$ *and every bounded Kleene space* $\mathbb{Y}$*.*

6 Translating between the natural and Priestley duals for $\mathfrak{T}$

We now return our attention to the dualities for the variety $\mathfrak{T}$ of ternary algebras. To increase the utility of both the natural dual $\mathfrak{X}_{\mathfrak{T}}$ and the restricted Priestley dual $\mathfrak{Y}_{\mathfrak{T}}$, by allowing them to work in tandem, we now give explicit descriptions (up to a natural isomorphism) of the translation functors between the two dual categories:

$$\mathrm{P} := \mathrm{H} \circ \mathrm{E}\colon \mathfrak{X}_{\mathfrak{T}} \to \mathfrak{Y}_{\mathfrak{T}} \quad \text{and} \quad \mathrm{N} := \mathrm{D} \circ \mathrm{K}\colon \mathfrak{Y}_{\mathfrak{T}} \to \mathfrak{X}_{\mathfrak{T}}.$$

The corresponding functors for don't know algebras, Kleene algebras and Kleene lattices are described briefly at the end of the section.

Let $\mathbb{X} = \langle X; \preccurlyeq, \sim, \mathcal{T}\rangle$ belong to $\mathfrak{X}_{\mathfrak{T}}$. Informally, we define $\mathrm{P}(\mathbb{X})$ to be the Priestley space obtained by placing the order-theoretic dual $\mathbb{X}^\partial$ 'below' $\mathbb{X}$ and defining the order between $\mathbb{X}^\partial$ and $\mathbb{X}$ via $\sim$, then defining g to be the map that flips the top and bottom. Formally, we define

$$\mathrm{P}(\mathbb{X}) := \langle \widehat{X} \mathbin{\dot\cup} X; g, \sqsubseteq, \mathcal{T}\rangle,$$

where $\widehat{X} := \{\widehat{x} \mid x \in X\}$. The relation $\sqsubseteq$ is defined on $\widehat{X} \,\dot{\cup}\, X$ by

$$x \sqsubseteq y, \text{ if } x \preccurlyeq y, \quad \widehat{x} \sqsubseteq \widehat{y}, \text{ if } x \succcurlyeq y, \quad \text{and} \quad \widehat{x} \sqsubseteq y, \text{ if } x \sim y,$$

for all $x, y \in X$, (note that $x \sqsubseteq \widehat{y}$ never holds), the map g is defined by

$$g(x) = \widehat{x} \text{ and } g(\widehat{x}) = x, \quad \text{for all } x \in X,$$

and $\mathcal{T}$ is the disjoint union topology. We extend P to a functor from $\mathfrak{X}_{\mathfrak{T}}$ to $\mathfrak{Y}_{\mathfrak{T}}$ in the obvious way: if $\varphi\colon \mathbb{X} \to \mathbb{Y}$ is an $\mathfrak{X}_{\mathfrak{T}}$-morphism, then

$$\mathrm{P}(\varphi)(x) = \varphi(x) \quad \text{and} \quad \mathrm{P}(\varphi)(\widehat{x}) = \widehat{\varphi(x)}, \quad \text{for all } x \in X.$$

We will give an indirect proof that $\mathrm{P}(\mathbb{X})$ is a ternary Priestley space in Theorem 6.5.

Our initial step will be to prove that $\mathrm{P} \circ \mathrm{D} \cong \mathrm{H}$, but first we require a couple of background lemmas that are based in the theory of piggyback dualities. No knowledge of piggyback dualities is required here, but the interested reader may like to consult the Clark–Davey text [13, Section 7.2] and the references given there.

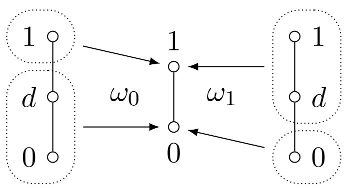

Figure 4 The maps ω_0 and ω_1

Let $\omega_0, \omega_1 \in \mathfrak{D}(\mathbf{T}^\flat, \mathbf{D})$ be the homomorphisms shown in Figure 4. Let $\mathbf{A}$ be a ternary algebra and define $\mu_{\mathbf{A}}\colon \mathrm{PD}(\mathbf{A}) \to \mathrm{H}(\mathbf{A})$ by

$$\mu_{\mathbf{A}}(x) = \omega_1 \circ x \quad \text{and} \quad \mu_{\mathbf{A}}(\widehat{x}) = \omega_0 \circ x \quad \text{for all } x \in \mathfrak{T}(\mathbf{A}, \mathbf{T}).$$

Some general theory tells us that $\mu_{\mathbf{A}}$ is surjective.

Lemma 6.1. $\mu_{\mathbf{A}}\colon \mathrm{PD}(\mathbf{A}) \to \mathrm{H}(\mathbf{A})$ *is surjective for every ternary algebra* $\mathbf{A}$.

Proof. Since $\mathbf{T}$ is the only subdirectly irreducible algebra in $\mathfrak{T}$, the homomorphisms from $\mathbf{A}$ to $\mathbf{T}$ separate the points of A. Hence, for all $a \neq b$ in A, there exists $x \in \mathfrak{T}(\mathbf{A}, \mathbf{T})$ and $i \in \{0, 1\}$ such that $\omega_i(x(a)) \neq \omega_i(x(b))$. It follows at once from [13, Lemma 7.2.2] that $\mu_{\mathbf{A}}$ is surjective. $\square$

To prove that $\mu_{\mathbf{A}}$ is an isomorphism in $\mathcal{Y}_{\mathfrak{T}}$, we require a description of the order on $\mathrm{H}(\mathbf{A})$ in terms of the structure on $\mathrm{D}(\mathbf{A})$. This is a special case of a very general result by Cabrer and Priestley [12, Theorem 2.3]. As the arguments are very easy in our special case, we present the details.

Let $\mathbf{A}$ be a ternary algebra and let R be a $\{0,1\}$-sublattice of $\mathbf{A}^\flat$. Then, provided $d \in R$, there is a unique maximal ternary-algebra subuniverse R° of $\mathbf{A}$ contained in R. This is a consequence of the fact that $\neg$ is a dual endomorphism of $\mathbf{A}^\flat$ that fixes d—see Davey and Priestley [19, Lemma 3.5] and Clark and Davey [13, Exercise 7.5]. We apply this to subsets of T^2 of the form

$$(\omega_i, \omega_j)^{-1}(\sqsubseteq) := \{\, (c,d) \in T^2 \mid \omega_i(c) \sqsubseteq \omega_j(d) \,\}.$$

Indeed, if $(d,d) \in (\omega_i,\omega_j)^{-1}(\sqsubseteq)$, then we define R_{ij} to be $(\omega_i,\omega_j)^{-1}(\sqsubseteq)^\circ$. As usual, $R_{ij}^{\mathrm{D}(\mathbf{A})}$ denotes the pointwise extension of R_{ij} to $\mathfrak{T}(\mathbf{A},\mathbf{T})$.

Lemma 6.2. *Let $\mathbf{A}$ be a ternary algebra, let $x, y \in \mathfrak{T}(\mathbf{A},\mathbf{T})$ and let $i, j \in \{0,1\}$. The following are equivalent.*

(1) $\omega_i \circ x \sqsubseteq \omega_j \circ y$ *in* $\mathrm{H}(\mathbf{A})$,

(2) $\omega_i(d) \sqsubseteq \omega_j(d)$ *and* $(x,y) \in R_{ij}^{\mathrm{D}(\mathbf{A})}$.

Proof. Assume that $\omega_i \circ x \sqsubseteq \omega_j \circ y$ in $\mathrm{H}(\mathbf{A})$. Then $\omega_i(x(a)) \sqsubseteq \omega_j(y(a))$, for all $a \in A$. In particular, $\omega_i(d) = \omega_i(x(d)) \sqsubseteq \omega_j(x(d)) = \omega_j(d)$. It follows that $(d,d) \in (\omega_i,\omega_j)^{-1}(\sqsubseteq)$ and hence $R_{ij} := (\omega_i,\omega_j)^{-1}(\sqsubseteq)^\circ$ is a non-empty subuniverse of $\mathbf{T}^2$. Thus we can argue as follows.

Define $x \sqcap y \colon \mathbf{A} \to \mathbf{T}^2$ by $(x \sqcap y)(a) = (x(a), y(a))$. Then, since $(x \sqcap y)(A)$ is a non-empty subuniverse of $\mathbf{T}^2$,

$$\begin{aligned}
& \omega_i \circ x \sqsubseteq \omega_j \circ y \quad \text{in } \mathrm{H}(\mathbf{A}) \\
\iff & (\forall a \in A)\ \omega_i(x(a)) \sqsubseteq \omega_j(y(a)) \\
\iff & (\forall a \in A)\ (x(a), y(a)) \in (\omega_i,\omega_j)^{-1}(\sqsubseteq) \\
\iff & (x \sqcap y)(A) \subseteq (\omega_i,\omega_j)^{-1}(\sqsubseteq) \\
\iff & (d,d) \in (\omega_i,\omega_j)^{-1}(\sqsubseteq) \quad \text{and} \quad (x \sqcap y)(A) \subseteq (\omega_i,\omega_j)^{-1}(\sqsubseteq)^\circ = R_{ij} \\
\iff & \omega_i(d) \sqsubseteq \omega_j(d) \quad \text{and} \quad (\forall a \in A)\ (x(a), y(a)) \in R_{ij} \\
\iff & \omega_i(d) \sqsubseteq \omega_j(d) \quad \text{and} \quad (x,y) \in R_{ij}^{\mathrm{D}(\mathbf{A})}.
\end{aligned}$$

□

The relations $R_{ij} = (\omega_i, \omega_j)^{-1}(\sqsubseteq)^\circ$, for $i, j \in \{0,1\}$, are very easy to calculate.

Lemma 6.3. *For $i, j \in \{0,1\}$, we have $\omega_i(d) \sqsubseteq \omega_j(d)$ if and only if $ij \neq 10$. Moreover, $R_{00} = \succcurlyeq$, $R_{11} = \preccurlyeq$ and $R_{01} = \sim$.*

Proof. A quick look at Figure 4 reveals that $\omega_i(d) \sqsubseteq \omega_j(d)$ if and only if $ij \neq 10$. We leave the reader to draw three copies of $\mathbf{T}^2$ and thereby verify the claim concerning R_{00}, R_{11} and R_{01}. $\square$

We require one further calculation.

Lemma 6.4. *Let $\mathbf{A}$ be a ternary algebra. Then, for all $x \in \mathfrak{T}(\mathbf{A}, \mathbf{T})$, we have $g(\omega_0 \circ x) = \omega_1 \circ x$ and $g(\omega_1 \circ x) = \omega_0 \circ x$ in $\mathrm{H}(\mathbf{A})$.*

Proof. Since $g(\omega_0 \circ x) = c \circ \omega_0 \circ x \circ \neg$, we have, for all $a \in A$,

$$\begin{aligned} g(\omega_0 \circ x)(a) = 1 &\Leftrightarrow c(\omega_0(x(\neg a))) = 1 \Leftrightarrow \omega_0(x(\neg a)) = 0 \\ &\Leftrightarrow \neg x(a) = x(\neg a) \in \{0, d\} \Leftrightarrow x(a) \in \{1, d\} \Leftrightarrow (\omega_1 \circ x)(a) = 1, \end{aligned}$$

whence $g(\omega_0 \circ x) = \omega_1 \circ x$. The proof that $g(\omega_1 \circ x) = \omega_0 \circ x$ is similar. $\square$

Theorem 6.5. $\mathrm{P}(\mathbb{X})$ *is a ternary Priestley space, for all $\mathbb{X} \in \mathfrak{X}_{\mathfrak{T}}$, and $\mathrm{P}\colon \mathfrak{X}_{\mathfrak{T}} \to \mathfrak{Y}_{\mathfrak{T}}$ is a functor satisfying* $\mathrm{PD}(\mathbf{A}) \cong \mathrm{H}(\mathbf{A})$*, for all $\mathbf{A} \in \mathfrak{T}$. Indeed, $\mu\colon \mathrm{PD} \to H$ is a natural isomorphism.*

Proof. Once we know that P is a functor, a very simple calculation shows that $\mu\colon \mathrm{PD} \to \mathrm{H}$ is a natural transformation, that is, that $\mathrm{H}(u) \circ \mu_{\mathbf{B}} = \mu_{\mathbf{A}} \circ \mathrm{PD}(u)$, for all $\mathbf{A}, \mathbf{B} \in \mathfrak{T}$ and all $u \in \mathfrak{T}(\mathbf{A}, \mathbf{B})$. It is clear that P is a functor provided it is well defined, that is, provided $\mathrm{P}(\mathbb{X})$ is a ternary Priestley space, for all $\mathbb{X} \in \mathfrak{X}_{\mathfrak{T}}$. Let $\mathbb{X} \in \mathfrak{X}_{\mathfrak{T}}$. Up to isomorphism, $\mathbb{X}$ is of the form $\mathrm{D}(\mathbf{A})$, for some $\mathbf{A} \in \mathfrak{T}$. Since $\mathrm{H}(\mathbf{A})$ is a ternary Priestley space, for all $\mathbf{A} \in \mathfrak{T}$, to show that $\mathrm{P}(\mathbb{X})$ is a ternary Priestley space, and to complete the proof of the theorem, it suffices to prove that $\mu_{\mathbf{A}}\colon \mathrm{PD}(\mathbf{A}) \to \mathrm{H}(\mathbf{A})$ is an isomorphism of topological structures, for all $\mathbf{A} \in \mathfrak{T}$.

Let $\mathbf{A} \in \mathfrak{T}$. By Lemma 6.1, $\mu_{\mathbf{A}}$ is surjective and by the definition of $\sqsubseteq$ on $\widehat{X} \,\dot\cup\, X$ along with Lemmas 6.2 and 6.3, we have $x \sqsubseteq y$ in $\mathrm{PD}(\mathbf{A})$ if and only if $\mu_{\mathbf{A}}(x) \sqsubseteq \mu_{\mathbf{A}}(y)$. Let $x \in \mathfrak{T}(\mathbf{A}, \mathbf{T})$. By Lemma 6.4, we have

$$\begin{aligned} g(\mu_{\mathbf{A}}(x))(a) = 1 &\Leftrightarrow c(\omega_1(x(\neg a))) = 1 \Leftrightarrow \omega_1(x(\neg a)) = 0 \\ &\Leftrightarrow \neg x(a) = x(\neg a) = 0 \Leftrightarrow x(a) = 1 \Leftrightarrow \omega_0(x(a)) = 1 \\ &\Leftrightarrow \mu_{\mathbf{A}}(\widehat{x})(a) = 1 \Leftrightarrow \mu_{\mathbf{A}}(g(x))(a) = 1, \end{aligned}$$

whence $g(\mu_{\mathbf{A}}(x)) = \mu_{\mathbf{A}}(g(x))$. Similarly,

$$\begin{aligned} g(\mu_{\mathbf{A}}(\widehat{x}))(a) = 1 &\Leftrightarrow c(\omega_0(x(\neg a))) = 1 \Leftrightarrow \omega_1(x(\neg a)) = 1 \\ &\Leftrightarrow \neg x(a) = x(\neg a) \in \{0, d\} \Leftrightarrow x(a) \in \{1, d\} \Leftrightarrow \omega_1(x(a)) = 1 \\ &\Leftrightarrow \mu_{\mathbf{A}}(x)(a) = 1 \Leftrightarrow \mu_{\mathbf{A}}(g(\widehat{x}))(a) = 1, \end{aligned}$$

whence $g(\mu_{\mathbf{A}}(\widehat{x})) = \mu_{\mathbf{A}}(g(\widehat{x}))$. Thus $\mu_{\mathbf{A}}$ preserves g. The continuity of $\mu_{\mathbf{A}}$ follows from the fact that, for all $a \in A$ and $i \in \{0, 1\}$,

$$\begin{aligned} \mu_{\mathbf{A}}^{-1}(U_{a,i}) &= \{\, x \mid x \in \mathfrak{T}(\mathbf{A}, \mathbf{T}) \ \&\ \omega_1(x(a)) = i \,\} \\ &\qquad \cup \{\, \widehat{x} \mid x \in \mathfrak{T}(\mathbf{A}, \mathbf{T}) \ \&\ \omega_0(x(a)) = i \,\} \\ &= \{\, x \mid x \in \mathfrak{T}(\mathbf{A}, \mathbf{T}) \ \&\ x(a) \in \omega_1^{-1}(i) \,\} \\ &\qquad \cup \{\, \widehat{x} \mid x \in \mathfrak{T}(\mathbf{A}, \mathbf{T}) \ \&\ x(a) \in \omega_0^{-1}(i) \,\}, \end{aligned}$$

which is open in $\mathrm{PD}(\mathbf{A})$. □

We turn now to the functor $\mathrm{N}\colon \mathcal{Y}_{\mathfrak{T}} \to \mathcal{X}_{\mathfrak{T}}$. Let $\mathbb{Y} = \langle Y; g, \sqsubseteq, \mathcal{T} \rangle$ belong to $\mathcal{Y}_{\mathfrak{T}}$. We define $\mathrm{N}(\mathbb{Y}) = \langle \mathbb{Y}^+; \preccurlyeq, \sim, \mathcal{T} \rangle$ where

- $\mathbb{Y}^+ = \{\, x \in Y \mid \mathbb{Y} \models g(x) \sqsubseteq x \,\}$,
- $\preccurlyeq\ =\ \sqsubseteq\restriction_{\mathbb{Y}^+}$,
- $(\forall x, y \in \mathbb{Y}^+)\ x \sim y \iff \mathbb{Y} \models x \sqsupseteq g(y)$, and
- $\mathcal{T}$ is the restriction of the topology on $\mathbb{Y}$.

Given $\mathbb{X}, \mathbb{Y} \in \mathcal{Y}_{\mathfrak{T}}$ and a ternary-space morphism $\varphi\colon \mathbb{X} \to \mathbb{Y}$, it is easy to see that $\varphi(\mathbb{X}^+) \subseteq \mathbb{Y}^+$, so we may define $N(\varphi) := \varphi\restriction_{\mathbb{X}^+}$.

Theorem 6.6. $\mathrm{N}\colon \mathcal{Y}_{\mathfrak{T}} \to \mathcal{X}_{\mathfrak{T}}$ *is a well-defined functor with* $\mathrm{NP}(\mathbb{X}) = \mathbb{X}$*, for all* $\mathbb{X} \in \mathcal{X}_{\mathfrak{T}}$*.*

Proof. It is clear that N is a functor provided $\mathrm{N}(\mathbb{Y}) \in \boldsymbol{\mathcal{X}}_{\mathfrak{T}}$, for all $\mathbb{Y} \in \boldsymbol{\mathcal{Y}}_{\mathfrak{T}}$, that is, provided $\mathrm{N}(\mathbb{Y})$ satisfies axioms (NT_1)–(NT_3) of Theorem 3.9.

(NT_1): By Lemma 4.2, $\mathbb{Y}^+$ is a closed subset of $\mathbb{Y}$ and hence $\langle \mathbb{Y}^+; \preccurlyeq, \mathcal{T}\rangle$ is a Priestley space.

(NT_2): Since $g\colon Y \to Y$ is continuous and $\sqsubseteq$ is a closed subset of $\mathbb{Y} \times \mathbb{Y}$, by Lemma 3.13(4), the set

$$\sim\, = \{\, (x, y) \in \mathbb{Y}^+ \times \mathbb{Y}^+ \mid g(x) \sqsubseteq y \,\} = (g \times \mathrm{id}_{\mathbb{Y}})^{-1}(\sqsubseteq) \cap (\mathbb{Y}^+ \times \mathbb{Y}^+)$$

is closed in $\mathbb{Y}^+ \times \mathbb{Y}^+$.

(NT_3)(a): Let $x \in \mathbb{Y}^+$. Then $g(x) \sqsubseteq x$ and hence $x \sim x$.

(NT_3)(b): Let $x, y, z \in \mathbb{Y}^+$ and assume that $x \sim y$ and $y \preccurlyeq z$. Then, since g is order-reversing, in $\mathbb{Y}$ we have

$$x \sqsupseteq g(y) \ \ \&\ \ y \sqsubseteq z \quad\Longrightarrow\quad x \sqsupseteq g(y) \ \ \&\ \ g(y) \sqsupseteq g(z) \quad\Longrightarrow\quad x \sqsupseteq g(z)$$

and hence $x \sim z$.

We now prove that $\mathrm{NP}(\mathbb{X}) = \mathbb{X}$. It follows immediately from the definition of $\mathrm{P}(\mathbb{X})$ that $\mathrm{P}(\mathbb{X})^+ = X$, so $\mathrm{NP}(\mathbb{X})$ and $\mathbb{X}$ have the same underlying set, namely X. It is immediate from the construction that $\preccurlyeq^{\mathrm{NP}(\mathbb{X})} = \preccurlyeq^{\mathbb{X}}$. For all $x, y \in X$, we have

$$\begin{aligned} x \sim y \text{ in } \mathbb{X} &\iff \widehat{x} \sqsubseteq y \text{ in } \mathrm{P}(\mathbb{X}) \\ &\iff x, y \in X \text{ and } g(x) \sqsubseteq y \text{ in } \mathrm{P}(\mathbb{X}) \\ &\iff x, y \in \mathrm{P}(\mathbb{X})^+ \text{ and } x \sqsupseteq g(y) \text{ in } \mathrm{P}(\mathbb{X})^+ \\ &\iff x \sim y \text{ in } \mathrm{NP}(\mathbb{X}). \end{aligned}$$

Hence $\mathrm{NP}(\mathbb{X}) = \mathbb{X}$. □

Combining these results gives the category equivalence that we seek.

Theorem 6.7. *The functors* $\mathrm{P}\colon \boldsymbol{\mathcal{X}}_{\mathfrak{T}} \to \boldsymbol{\mathcal{Y}}_{\mathfrak{T}}$ *and* $\mathrm{N}\colon \boldsymbol{\mathcal{Y}}_{\mathfrak{T}} \to \boldsymbol{\mathcal{X}}_{\mathfrak{T}}$ *satisfy* $\mathrm{NP} = \mathrm{id}_{\boldsymbol{\mathcal{X}}_{\mathfrak{T}}}$ *and* $\mathrm{PN} \cong \mathrm{id}_{\boldsymbol{\mathcal{Y}}_{\mathfrak{T}}}$*, and so give a category equivalence between* $\boldsymbol{\mathcal{X}}_{\mathfrak{T}}$ *and* $\boldsymbol{\mathcal{Y}}_{\mathfrak{T}}$*. Moreover,*

$$\mathrm{PD} \cong \mathrm{H} \quad \textit{and} \quad \mathrm{NH} \cong \mathrm{D}.$$

$$
\begin{tikzcd}
 & \mathscr{X}_{\mathfrak{T}} \arrow[dl, shift left, "E"] \arrow[dd, shift right, "P"'] \\
\mathfrak{T} \arrow[ur, shift left, "D"] \arrow[dr, shift right, "H"'] & \\
 & \mathscr{Y}_{\mathfrak{T}} \arrow[ul, shift right, "K"'] \arrow[uu, shift right, "N"']
\end{tikzcd}
$$

Figure 5 The six functors

Proof. By Theorems 6.5 and 6.6, we have $\mathrm{PD} \cong \mathrm{H}$ and $\mathrm{NP} = \mathrm{id}_{\mathscr{X}_{\mathfrak{T}}}$. Hence

$$\mathrm{NH} \cong \mathrm{NPD} \cong \mathrm{id}_{\mathscr{X}_{\mathfrak{T}}}\mathrm{D} \cong \mathrm{D}$$

and

$$\mathrm{PN} \cong \mathrm{PNid}_{\mathscr{Y}_{\mathfrak{T}}} \cong \mathrm{PNHK} \cong \mathrm{PDK} \cong \mathrm{HK} \cong \mathrm{id}_{\mathscr{Y}_{\mathfrak{T}}}. \qquad \square$$

We close this section with descriptions of the translation functors for the other three varieties.

6.1 Don't know algebras The translation functors for $\mathfrak{D}\mathrm{k}$ between $\mathscr{X}_{\mathfrak{D}\mathrm{k}}$ and $\mathscr{Y}_{\mathfrak{D}\mathrm{k}}$ are defined exactly as they were for $\mathfrak{T}$: the bounds of $\mathrm{P}(\mathbb{X})$ are given by $d \in X$ and $\widehat{d} \in \widehat{X}$, and the element d of $\mathrm{N}(\mathbb{Y})$ is given by the top of $\mathbb{Y}$.

6.2 Kleene algebras The translation functors for $\mathfrak{K}$ between $\mathscr{X}_{\mathfrak{K}}$ and $\mathscr{Y}_{\mathfrak{K}}$ were described and generalised by Davey and Priestley [19] (see also [13, Section 7.5]). As [19] uses a multi-sorted natural duality for $\mathfrak{K}$, the description of the translation functors in our single-sorted situation is implicit rather than explicit and requires extraction.

Given $\mathbb{X} = \langle X; \preccurlyeq, \sim, B, \mathcal{T}\rangle$ in $\mathscr{X}_{\mathfrak{K}}$, the structure $\langle X; \preccurlyeq, \sim, \mathcal{T}\rangle$ belongs to $\mathscr{X}_{\mathfrak{T}}$. Thus we may define $\mathrm{P}_{\mathfrak{K}}(\mathbb{X}) \in \mathscr{Y}_{\mathfrak{K}}$ by

$$\mathrm{P}_{\mathfrak{K}}(\mathbb{X}) = \mathrm{P}(\mathbb{X})/\theta_B,$$

where θ_B is the equivalence relation on $\mathrm{P}(X)$ that identifies x and $\widehat{x}$, for all $x \in B$.

Given a Kleene space $\mathbb{Y} = \langle Y; g, \sqsubseteq, \mathcal{T}\rangle$, we define $\mathrm{N}_{\mathfrak{K}}(\mathbb{Y})$ in terms of $\mathrm{N}(\mathbb{Y})$ via $\mathrm{N}_{\mathfrak{K}}(\mathbb{Y}) = \langle \mathbb{Y}^{+}; \preccurlyeq, \sim, B, \mathcal{T}\rangle$ where

- $\mathrm{N}(\mathbb{Y}) = \langle \mathbb{Y}^+; \preccurlyeq, \sim, \mathcal{T} \rangle$, and
- $B = \{\, y \in Y \mid g(y) = y \,\}$.

6.3 Kleene lattices As in the case of don't know algebras, the translation functors for $\boldsymbol{\mathcal{L}}$ are defined as they were for Kleene algebras: the bounds of $\mathrm{P}_{\boldsymbol{\mathcal{L}}}(\mathbb{X})$ are given by $d \in X$ and $\widehat{d} \in \widehat{X}$, and the element d of $\mathrm{N}_{\boldsymbol{\mathcal{L}}}(\mathbb{Y})$ is given by the top of $\mathbb{Y}$.

7 Applications of the dualities and the translation functors for $\boldsymbol{\mathcal{T}}$

The following simple observation will be used twice below: the relational product of $\succcurlyeq$ and $\preccurlyeq$ equals the relation $\sim$, more formally,

$$\succcurlyeq \cdot \preccurlyeq := \{\, (a, b) \in T^2 \mid (\exists c \in T)\ a \succcurlyeq c \ \ \&\ \ c \preccurlyeq b \,\} = \sim. \tag{\$}$$

This extends to powers of T: let S be a non-empty set and let $\succcurlyeq^{T^S}$, $\preccurlyeq^{T^S}$ and $\sim^{T^S}$ be the pointwise extensions of $\succcurlyeq$, $\preccurlyeq$ and $\sim$ to T^S; then $\succcurlyeq^{T^S} \cdot \preccurlyeq^{T^S} = \sim^{T^S}$. (We saw in Subsection 3.6 that this does not work on subsets of powers.)

7.1 Term functions By the property known as (CLO), which holds for every natural duality (see [13, Prop. 2.2.3]), for every non-empty set S, a map $\varphi \colon T^S \to T$ is an S-ary term function of $\mathbf{T}$ if and only if φ is a morphism from $\mathbb{T}^S$ to $\mathbb{T}$. By (\$), if $\varphi \colon T^S \to T$ preserves $\preccurlyeq$ then it must also preserve $\sim$. This gives us the following extension of Mukaidono's [33] description of the finitary term functions on $\mathbf{T}$.

Theorem 7.1. *Let S be a non-empty set and endow T^S with the product topology coming from the discrete topology on T. A map $\varphi \colon T^S \to T$ is an S-ary term function of $\mathbf{T}$ if and only if it is continuous and preserves the uncertainty order $\preccurlyeq$. In particular, for $n \in \mathbb{N}$, a map $\varphi \colon T^n \to T$ is an n-ary term function of $\mathbf{T}$ if and only if it preserves the uncertainty order.*

7.2 Free algebras It follows at once from Theorem 7.1 that the free S-generated ternary algebra $\mathbf{F}_{\boldsymbol{\mathcal{T}}}(S)$ can be represented as the subalgebra of $\mathbf{T}^{T^S}$ consisting of the continuous, $\preccurlyeq$-preserving maps with the projections as the free generators. As an alternative approach, we now apply the translation functor $\mathrm{P}\colon \mathfrak{X}_{\mathfrak{T}} \to \mathfrak{Y}_{\mathfrak{T}}$ to study the ordered set of join-irreducible elements of $\mathbf{F}_{\mathfrak{T}}(n)$, for $n \in \mathbb{N}$.

For a finite distributive lattice $\mathbf{L}$, denote the ordered set of join-irreducible elements of $\mathbf{L}$ by $\mathfrak{J}\mathbf{i}(\mathbf{L})$. The Priestley dual $\mathrm{H}(\mathbf{L})$ is isomorphic to the order-theoretic dual $\mathfrak{J}\mathbf{i}(\mathbf{L})^{\partial}$ of $\mathfrak{J}\mathbf{i}(\mathbf{L})$. Since every ternary algebra is order-theoretically self-dual, for a finite ternary algebra $\mathbf{A}$ we may ignore the $^{\partial}$ and hence we have $\mathfrak{J}\mathbf{i}(\mathbf{A}) \cong \mathrm{H}(\mathbf{A})$. We will apply this in the case that $\mathbf{A} = \mathbf{F}_{\mathfrak{T}}(n)$, but first we require one further order-theoretic construct.

Let $\mathbf{P} = \langle P; \leqslant \rangle$ be a finite ordered set. By analogy with the definition of $\mathrm{P}(\mathbb{X})$, we define

$$\Diamond(\mathbf{P}) := \langle \widehat{P} \mathbin{\dot\cup} P; \sqsubseteq \rangle,$$

where $\widehat{P} := \{\widehat{x} \mid x \in P\}$. The relation $\sqsubseteq$ is defined on $\widehat{P} \mathbin{\dot\cup} P$ by

$$x \sqsubseteq y, \text{ if } x \leqslant y, \quad \widehat{x} \sqsubseteq \widehat{y}, \text{ if } x \geqslant y,$$
$$\text{and} \quad \widehat{x} \sqsubseteq y, \text{ if } (\exists z \in P)\ x \geqslant z \ \ \&\ \ z \leqslant y,$$

for all $x, y \in P$. The relation $\sqsubseteq$ is easily seen to be an order: $\sqsubseteq$ is reflexive as $\leqslant$ is, $\sqsubseteq$ is antisymmetric as $\leqslant$ is and since $\widehat{x} \sqsupseteq y$ never holds, and a very simple calculation with four cases shows that $\sqsubseteq$ is transitive. It is very easy to convert a diagram for $\mathbf{P}$ into a diagram for $\Diamond(\mathbf{P})$: place $\mathbf{P}^{\partial}$ immediately below $\mathbf{P}$ and add a line from $\widehat{m}$ to m, for each minimal element m of $\mathbf{P}$ (see Figures 6 and 7). Our next result is an alternative to the descriptions of $\mathfrak{J}\mathbf{i}(\mathbf{F}_{\mathfrak{T}}(n))$ given by Berman and Mukaidono [2, pp. 30–33] and by Balbes [1, Thm. 5.3, Cor. 5.2] and gives additional information on the structure of the ordered set.

Theorem 7.2. *Let $\mathbf{U} = \langle \{0, d, 1\}; \preccurlyeq \rangle$ be the underlying ordered set of $\mathbb{T}$. For all $n \in \mathbb{N}$, we have*

$$\mathfrak{J}\mathbf{i}(\mathbf{F}_{\mathfrak{T}}(n)) \cong \Diamond(\mathbf{U}^n).$$

In particular, $|\mathfrak{J}\mathbf{i}(\mathbf{F}_{\mathfrak{T}}(n))| = 2 \cdot 3^n$.

Proof. A very basic result from the theory of natural dualities [13, 2.2.4] tells us that $\mathrm{D}(\mathbf{F}_{\mathfrak{T}}(n)) \cong \mathbb{T}^n$. Since $\mathrm{P}(\mathbb{T}^n) = \Diamond(\mathbf{U}^n)$, by Theorem 6.7, we have

$$\mathfrak{J}\mathbf{i}(\mathbf{F}_{\mathfrak{T}}(n)) \cong \mathrm{H}(\mathbf{F}_{\mathfrak{T}}(n)) \cong \mathrm{PD}(\mathbf{F}_{\mathfrak{T}}(n)) \cong \mathrm{P}(\mathbb{T}^n) \cong \Diamond(\mathbf{U}^n).$$

It follows immediately that $|\mathfrak{J}\mathbf{i}(\mathbf{F}_{\mathfrak{T}}(n))| = 2 \cdot 3^n$. $\square$

The diagrams of $\Diamond(\mathbf{U})$ and of $\mathbf{F}_{\mathfrak{T}}(1)$ are given in Figure 6, and the diagram of $\Diamond(\mathbf{U}^2)$ is given in Figure 7, where we abbreviate (a, b) to ab. Diagrams of $\mathbf{F}_{\mathfrak{T}}(1)$ and $\mathfrak{Ji}(\mathbf{F}_{\mathfrak{T}}(2))$ also appear in [1] and [2]. Their diagrams of $\mathfrak{Ji}(\mathbf{F}_{\mathfrak{T}}(2))$ are slightly different from one another but, unlike our diagram in Figure 7, both have the disadvantage that they do not expose the role of $\mathbf{U}^2$ nor of the $\Diamond$ construction.

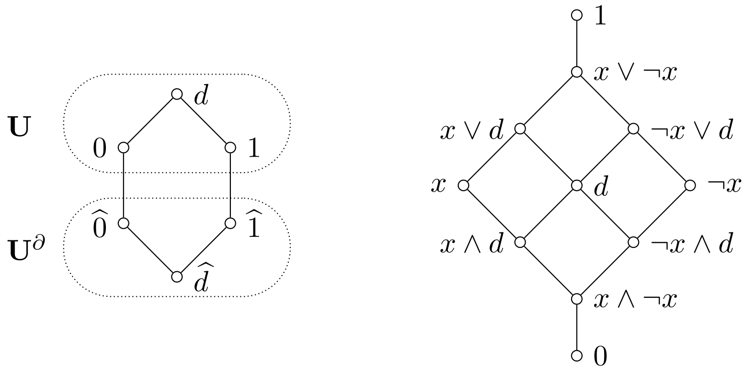

Figure 6 $\mathfrak{J}(\mathbf{F}_{\mathfrak{T}}(1)) = \Diamond(\mathbf{U})$ and $\mathbf{F}_{\mathfrak{T}}(1)$

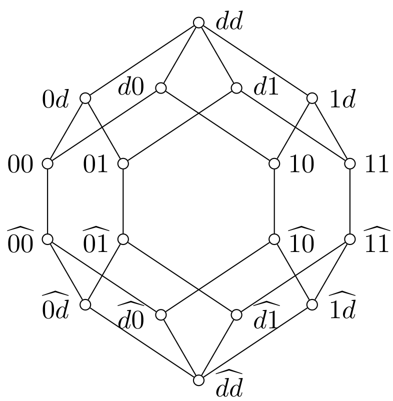

Figure 7 $\mathfrak{J}(\mathbf{F}_{\mathfrak{T}}(2)) = \Diamond(\mathbf{U}^2)$

While we cannot draw $\mathbf{F}_{\mathfrak{T}}(2)$, we can use the structure of $\Diamond(\mathbf{U}^2)$ to calculate the size of $\mathbf{F}_{\mathfrak{T}}(2)$. We will use the fact that, the Priestley dual

$\mathrm{K}(\mathbf{P})$ of a finite ordered set $\mathbf{P}$ is isomorphic to the lattice $\mathcal{U}(\mathbf{P})$ of up-sets of $\mathbf{P}$.

Corollary 7.3. *$|\mathbf{F}_{\mathfrak{J}}(2)| = 197$.*

Proof. Since $\mathfrak{Ji}(\mathbf{F}_{\mathfrak{J}}(2)) \cong \Diamond(\mathbf{U}^2)$, by Theorem 7.2, we have

$$\mathbf{F}_{\mathfrak{J}}(2) \cong \mathrm{KH}(\mathbf{F}_{\mathfrak{J}}(2)) \cong \mathcal{U}(\mathfrak{Ji}(\mathbf{F}_{\mathfrak{J}}(2))) \cong \mathcal{U}(\Diamond(\mathbf{U}^2)).$$

An up-set V of $\Diamond(\mathbf{U}^2)$ is either (A) an up-set of $\mathbf{U}^2$ or (B) has a non-empty intersection with $\widehat{U^2}$.

Case (A): We shall prove that $|\mathcal{U}(\mathbf{U}^2)| = 48$. Priestley duality tells us that $\mathcal{U}(\mathbf{U}^2)$ is the coproduct $\mathcal{U}(\mathbf{U}) \sqcup \mathcal{U}(\mathbf{U})$ in $\mathfrak{D}$. By Davey [17], $\mathcal{U}(\mathbf{U}) \sqcup \mathcal{U}(\mathbf{U})$ is isomorphic to the lattice of order-preserving maps from $\mathbf{U}$ into $\mathcal{U}(\mathbf{U})$. Label $\mathcal{U}(\mathbf{U})$ as in Figure 8.

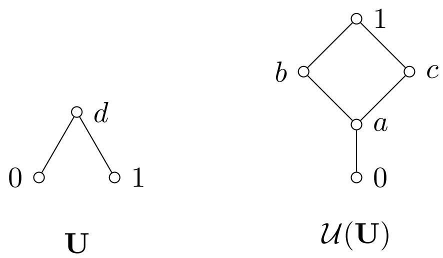

Figure 8 $\mathbf{U}$ and $\mathcal{U}(\mathbf{U})$

An order-preserving map φ from $\mathbf{U}$ to $\mathcal{U}(\mathbf{U})$ corresponds to choosing $\varphi(d)$ in $\mathcal{U}(\mathbf{U})$ and then choosing a pair of elements from ${\downarrow}\varphi(d)$ as values for $\varphi(0)$ and $\varphi(1)$. Choosing $\varphi(d)$, in order, to be 0, a, b, c, 1 gives the following total number of maps:

$$1 + 2^2 + 3^2 + 3^2 + 5^2 = 48.$$

Case(B): Let V be an up-set of $\Diamond(\mathbf{U}^2)$ that intersects $\widehat{U^2}$. The intersection of V with the set $\max(\mathbf{U}^2) = \{\widehat{00}, \widehat{01}, \widehat{10}, \widehat{11}\}$ is one of the 15 non-empty subsets of $\max(\mathbf{U}^2)$. The up-set V must contain $\overline{V} := \bigcup\{\, {\uparrow}a \mid a \in V \cap \max(\widehat{\mathbf{U}^2}) \,\}$. The only possibility for $V \setminus \overline{V}$ is an up-set of $\mathbf{P} \mathbin{\dot\cup} \mathbf{Q}$, where

$$P = U \setminus \overline{V}, \text{ and}$$
$$Q = \widehat{U^2} \setminus \big(\max(\widehat{U^2}) \cup \bigcup\{\, {\downarrow}b \mid b \in \max(\widehat{\mathbf{U}^2}) \setminus V \,\}\big).$$

The number of possibilities for V for a given $V \cap \max(\widehat{\mathbf{U}^2})$ will be

$$|\mathcal{U}(\mathbf{P} \mathbin{\dot{\cup}} \mathbf{Q})| = |\mathcal{U}(\mathbf{P})| \times |\mathcal{U}(\mathbf{Q})|.$$

We will consider five sub-cases.

Case (B1): $|V \cap \max(\widehat{\mathbf{U}^2})| = 1$. In this case $\mathbf{P}$ is a fence of size five and hence $|\mathcal{U}(\mathbf{P})| = 13$. (A fence of size n has f_{n+2} up-sets, where f_k is the kth Fibonacci number—see Davey and Priestley [21, Exercise 1.16].) Since $\mathbf{Q}$ is empty, this case yields $4 \times 13 = 52$ up-sets.

Case (B2): $V \cap \max(\widehat{\mathbf{U}^2}) \in \big\{\{\widehat{00}, \widehat{01}\}, \{\widehat{00}, \widehat{10}\}, \{\widehat{01}, \widehat{11}\}, \{\widehat{10}, \widehat{11}\}\big\}$. In each case $\mathbf{P}$ is a fence of size three, while $\mathbf{Q}$ has size one. Hence $|\mathcal{U}(\mathbf{P})| \times |\mathcal{U}(\mathbf{Q})| = 5 \times 2 = 10$, and thus this case yields $4 \times 10 = 40$ up-sets.

Case (B3): $V \cap \max(\widehat{\mathbf{U}^2}) \in \big\{\{\widehat{00}, \widehat{11}\}, \{\widehat{01}, \widehat{10}\}\big\}$. In both cases, $\mathbf{P}$ is a two-element antichain, whence $|\mathcal{U}(\mathbf{P})| = 4$, and $\mathbf{Q}$ is empty. Thus this case yields $2 \times 4 = 8$ up-sets.

Case (B4): $|V \cap \max(\widehat{\mathbf{U}^2})| = 3$. In this case, $\mathbf{P}$ has one element and $\mathbf{Q}$ is a two-element antichain, so $|\mathcal{U}(\mathbf{P})| \times |\mathcal{U}(\mathbf{Q})| = 2 \times 4 = 8$. Thus this case yields $4 \times 8 = 32$ up-sets.

Case (B5): $\max(\widehat{\mathbb{Y}^2}) \subseteq V$. Now $\mathbf{P}$ is empty and $\mathbf{Q} \cong \mathbf{1} \oplus \overline{\mathbf{4}}$ (the linear sum of a one-element ordered set and a four-element antichain). This case yields a further $|\mathcal{U}(\mathbf{Q})| = |\mathcal{U}(\mathbf{1} \oplus \overline{\mathbf{4}})| = 17$ up-sets.

Summing the numbers from Case (A) and Cases (B1)–(B5) gives,

$$48 + 52 + 40 + 8 + 32 + 17 = 197$$

up-sets. □

That $|\mathbf{F}_{\mathfrak{T}}(2)| = 197$ is stated without proof by Berman and Mukaidono [2]; they also used a computer to calculate the sizes of $\mathbf{F}_{\mathfrak{T}}(3)$ and $\mathbf{F}_{\mathfrak{T}}(4)$.

Remark 7.4. The Priestley dual of a coproduct of ternary algebras is easily studied via the translation functors. For ternary algebras $\mathbf{A}$ and $\mathbf{B}$, denote their coproduct in $\mathfrak{T}$ by $\mathbf{A} \sqcup \mathbf{B}$. One of the advantages of the natural dual is that $\mathrm{D}(\mathbf{A} \sqcup \mathbf{B}) \cong \mathrm{D}(\mathbf{A}) \times \mathrm{D}(\mathbf{B})$, with the product in $\mathfrak{X}_{\mathfrak{T}}$ being the usual pointwise cartesian product. In comparison, products in $\mathfrak{Y}_{\mathfrak{T}}$ are not pointwise. The translation functor gives us

$$\mathrm{H}(\mathbf{A} \sqcup \mathbf{B}) \cong \mathrm{P}(\mathrm{D}(\mathbf{A}) \times \mathrm{D}(\mathbf{B})).$$

Alternatively, given $\mathbb{X}, \mathbb{Y} \in \boldsymbol{\mathcal{Y}}_{\mathfrak{T}}$, their product $\mathbb{X} \sqcap \mathbb{Y}$ in $\boldsymbol{\mathcal{Y}}_{\mathfrak{T}}$ is given by

$$\mathbb{X} \sqcap \mathbb{Y} \cong \mathrm{P}(\mathrm{N}(\mathbb{X}) \times \mathrm{N}(\mathbb{Y})).$$

7.3 Process spaces and subset-pair algebras Process spaces were introduced by R. Negulescu in 1995 as a formalism for modelling interacting systems [35, 36]. We refer to [6, 8, 35, 36] for the background motivation. Given a non-empty set X, a *process over* X is an ordered pair (U, V) of subsets of X such that $U \cup V = X$. The set $\mathcal{P}(X)$ of all processes over X is called the *process space* over X and yields a ternary algebra $\boldsymbol{\mathcal{P}}(X) = \langle \mathcal{P}(X); \vee, \wedge, \neg, 0, d, 1\rangle$ with operations defined by

$$(U,V) \vee (S,T) = (U \cap S, V \cup T), \qquad (U,V) \wedge (S,T) = (U \cup S, V \cap T),$$
$$\neg(U,V) = (V,U), \quad 0 = (X, \varnothing), \quad d = (X,X), \text{ and } \quad 1 = (\varnothing, X).$$

Note that

$$(U,V) \leqslant (S,T) \text{ in } \boldsymbol{\mathcal{P}}(X) \iff U \supseteq S \;\&\; V \subseteq T. \tag{¢}$$

A subalgebra of the process space ternary algebra $\boldsymbol{\mathcal{P}}(X)$ is called a *subset-pair algebra* over X. Brzozowski, Lou and Neglescu [7] proved that every finite non-trivial ternary algebra is isomorphic to a subset-pair algebra and Ésik [23] extended this to arbitrary ternary algebras. In both cases, the proofs were somewhat indirect. We now apply the natural duality for $\mathfrak{T}$ to give a direct proof that every ternary algebra $\mathbf{A}$ embeds into the process space algebra $\boldsymbol{\mathcal{P}}(X)$, where $X = \mathfrak{T}(\mathbf{A}, \mathbf{T})$ and therefore is isomorphic to a subset-pair algebra.

Let $\mathbb{X} = \langle X; \preccurlyeq, \sim, \mathcal{T}\rangle$ be the natural dual space of a non-trivial ternary algebra. We say that (U, V) is a *compatible process over* $\mathbb{X}$ if (U, V) is a process over X (i.e., $U \cup V = X$), both U and V are clopen upsets in $\mathbb{X}$, and $x \not\sim y$, for all $x \in U \setminus V$ and $y \in V \setminus U$. The set $\mathcal{P}_{\mathrm{c}}(\mathbb{X})$ of all compatible processes over $\mathbb{X}$ forms a subalgebra $\boldsymbol{\mathcal{P}}_{\mathrm{c}}(\mathbb{X})$ of $\boldsymbol{\mathcal{P}}(X)$.

The following lemma shows that compatible processes over $\mathbb{X}$ are in a one-to-one correspondence with morphisms $\alpha \colon \mathbb{X} \to \mathbb{T}$. It follows almost immediately from Lemma 3.14.

Lemma 7.5. *Let $\mathbb{X} \in \boldsymbol{\mathfrak{X}}_{\mathfrak{T}}$. A map $\alpha \colon X \to T$ is an $\boldsymbol{\mathfrak{X}}_{\mathfrak{T}}$-morphism if and only if $\big(\alpha^{-1}(\{0, d\}), \alpha^{-1}(\{1, d\})\big)$ is a compatible process over $\mathbb{X}$.*

Theorem 7.6.

(1) *Let* $\mathbb{X} \in \boldsymbol{\mathfrak{X}}_{\mathfrak{T}}$ *be the natural dual of a ternary algebra. The map*

$$u\colon \alpha \mapsto \big(\alpha^{-1}(\{0,d\}), \alpha^{-1}(\{1,d\})\big), \quad \textit{for all } \alpha \in \boldsymbol{\mathfrak{X}}_{\mathfrak{T}}(\mathbb{X}, \mathbb{T})$$

is an isomorphism from $\mathrm{E}(\mathbb{X})$ *to* $\boldsymbol{\mathcal{P}}_{\mathrm{c}}(\mathbb{X})$*.*

(2) *Every ternary algebra is isomorphic to a subset-pair algebra.*

(3) *Every finite ternary algebra* $\mathbf{A}$ *is isomorphic to a subset-pair algebra over a set* X *with* $|X| = \frac{1}{2}|\boldsymbol{\mathfrak{J}}\mathbf{i}(\mathbf{A})|$*.*

Proof. (1) The forward implication of Lemma 7.5 implies that the map u is well defined. Let (U, V) be a compatible process over $\mathbb{X}$ and define $\alpha\colon X \to T$ by

$$\alpha(x) = 1, \text{ if } x \in V \setminus U, \alpha(x) = d, \text{ if } x \in U \cap V, \alpha(x) = 0, \text{ if } x \in U \setminus V.$$

Since $\big(\alpha^{-1}(\{0,d\}), \alpha^{-1}(\{1,d\})\big) = (U, V)$, the backward implication of Lemma 7.5 implies that α is a morphism from $\mathbb{X}$ to $\mathbb{T}$. Hence u is surjective.

Let $\alpha \leqslant \beta$ in $\mathrm{E}(\mathbb{X})$. Since the order on $\mathrm{E}(\mathbb{X})$ is pointwise from the order on $\mathbf{T}$ and the order in $\boldsymbol{\mathcal{P}}_{\mathrm{c}}(\mathbb{X})$ is given by (¢), the following equivalences are straightforward to establish:

$$\begin{aligned} & \alpha \leqslant \beta \text{ in } \mathrm{E}(\mathbb{X}) \\ \iff & \alpha^{-1}(\{0,d\}) \supseteq \beta^{-1}(\{0,d\}) \quad \& \quad \alpha^{-1}(\{1,d\}) \subseteq \beta^{-1}(\{1,d\}) \\ \iff & u(\alpha) \leqslant u(\beta) \text{ in } \boldsymbol{\mathcal{P}}_{\mathrm{c}}(\mathbb{X}). \end{aligned}$$

Thus u is an order-isomorphism. Since $\neg\alpha$ is calculated pointwise in $\mathrm{E}(\mathbb{X})$, we have $(\neg\alpha)^{-1}(0) = \alpha^{-1}(1)$, $(\neg\alpha)^{-1}(1) = \alpha^{-1}(0)$ and $(\neg\alpha)^{-1}(d) = \alpha^{-1}(d)$. Thus

$$u(\neg\alpha) = \big(\alpha^{-1}(\{1,d\}), \alpha^{-1}(\{0,d\})\big) = \neg\big(\alpha^{-1}(\{0,d\}), \alpha^{-1}(\{1,d\})\big) = \neg u(\alpha).$$

The three nullary operations in $\mathrm{E}(\mathbb{X})$ are the constant maps $\underline{0}$, $\underline{d}$ and $\underline{1}$, and

$$u(\underline{0}) = (X, X) \quad u(\underline{d}) = (X, \varnothing), \text{ and } u(\underline{1}) = (\varnothing, X).$$

Thus u preserves the nullary operations. Hence u is an isomorphism.

Let $\mathbf{A}$ be a ternary algebra. Since $\mathbf{A} \cong \mathrm{ED}(\mathbf{A})$ by Theorem 3.8, (2) follows immediately from (1) with $\mathbb{X} = \mathrm{D}(\mathbf{A})$, and (3) follows from (1), again with $\mathbb{X} = \mathrm{D}(\mathbf{A})$, and the fact that, for a finite ternary algebra $\mathbf{A}$, we have $|\boldsymbol{\mathfrak{J}}\mathbf{i}(\mathbf{A})| = 2|\mathrm{D}(\mathbf{A})|$ since $\boldsymbol{\mathfrak{J}}\mathbf{i}(\mathbf{A}) \cong \mathrm{H}(\mathbf{A}) \cong \mathrm{PD}(\mathbf{A})$, by Theorem 6.5. □

7.4 The reflection from $\mathcal{L}$ into $\mathcal{T}$ via the natural dualities

There is a natural forgetful functor $\mathrm{U}\colon \mathcal{T} \to \mathcal{L}$ that forgets the three nullary operations. If we view $\mathcal{T}$ as a subcategory of $\mathcal{L}$, then the reflection $\mathrm{V}\colon \mathcal{L} \to \mathcal{T}$ of $\mathcal{L}$ into $\mathcal{T}$ is simply the left adjoint to U. The enrichment functor $\mathrm{S}\colon \mathfrak{X}_{\mathcal{T}} \to \mathfrak{X}_{\mathcal{L}}$ dual to U was described in the proof of Theorem 3.9 in Subsection 3.7, and a right adjoint $\mathrm{T}\colon \mathfrak{X}_{\mathcal{L}} \to \mathfrak{X}_{\mathcal{T}}$ to S will be dual to V. It is very easy to see that the right adjoint to S is given by the forgetful functor defined on objects by

$$\mathrm{T}\big(\langle X; d, \preccurlyeq, \sim, B, \mathcal{T}\rangle\big) := \langle X; \preccurlyeq, \sim, \mathcal{T}\rangle.$$

Put another way, the free ternary algebra generated by Kleene lattice $\mathbf{A}$ is the ternary algebra dual to object $\langle \mathcal{L}(\mathbf{A}, \mathbf{L}); \preccurlyeq, \sim, \mathcal{T}\rangle$ of $\mathfrak{X}_{\mathcal{T}}$.

The reflections of $\mathcal{L}$ into $\mathcal{K}$ and $\mathcal{D}$k and the reflections of $\mathcal{K}$ and $\mathcal{D}$k into $\mathcal{T}$ can be obtained in the same way from the appropriate forgetful functors between the dual categories.

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