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Distributive lattices and some related topologies in comparison with zero-divisor graphs

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Abstract. In this paper, for a distributive lattice $\mathcal{L}$, we study and compare some lattice theoretic features of $\mathcal{L}$ and topological properties of the Stone spaces $\mathrm{Spec}(\mathcal{L})$ and $\mathrm{Max}(\mathcal{L})$ with the corresponding graph theoretical aspects of the zero-divisor graph $\Gamma(\mathcal{L})$. Among other things, we show that the Goldie dimension of $\mathcal{L}$ is equal to the cellularity of the topological space $\mathrm{Spec}(\mathcal{L})$ which is also equal to the clique number of the zero-divisor graph $\Gamma(\mathcal{L})$. Moreover, the domination number of $\Gamma(\mathcal{L})$ will be compared with the density and the weight of the topological space $\mathrm{Spec}(\mathcal{L})$.

For a 0-distributive lattice $\mathcal{L}$, we investigate the compressed subgraph $\Gamma_E(\mathcal{L})$ of the zero-divisor graph $\Gamma(\mathcal{L})$ and determine some properties of this subgraph in terms of some lattice theoretic objects such as associated prime ideals of $\mathcal{L}$.

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1 Introduction

In 1988, Beck [6] introduced the zero-divisor graph $\Gamma_0(R)$ of a commutative ring R whose vertices are elements of R and two distinct vertices x and y are adjacent if and only if $xy = 0$. A subgraph $\Gamma(R)$ of $\Gamma_0(R)$ has been considered and investigated by Anderson and Livingston [1] in which they restricted the vertex set of $\Gamma(R)$ to all nonzero zero-divisors of R (see also [2]). In the recent decades, many authors have studied zero-divisor graph and similar graphs associated to rings, semigroups or other algebraic structures (see for example [1, 2, 4, 6, 7, 23]).

The study of equivalence classes in a zero-divisor graph of a commutative ring was first appeared in [19, Sec. 3] and later, it has been investigated and extended by S. Spiroff and C. Wickham in [24]. The main purpose of this process was to obtain a subgraph of the zero-divisor graph of a ring which preserves many properties of the original graph, but it is easier to deal with because it has smaller vertex and edge sets. In [5], it has been also generalized to the reduction graph of an arbitrary graph.

Azarpanah and Motamedi [3] studied and investigated the zero-divisor graph of the ring $C(X)$ containing all continuous real-valued functions on a topological space X and compared some topological properties of X and corresponding ring theoretic features of the ring $C(X)$ with some graph theoretical invariants of the assigned zero-divisor graph $\Gamma(C(X))$.

As a generalization, Samei [23] investigated the relation among the ring theoretic properties of an arbitrary commutative ring R, the topological fearturs of Spec(R) and the graph theoretic aspects of the zero-divisor graph $\Gamma(R)$.

There are many papers which interlink graph theory with poset and lattice theory. For instance, Halaš and Jukl [14] introduced the zero-divisor graph of posets. These investigations has been continued by Xue and Liu [28] and Tamizh Chelvam and Nithya [25]. In [8, 10, 12, 15, 17, 25], the authors discuss the properties of graphs derived from lattices. The *annihilator graph* of a 0-distributive lattice has been investigated by authors as an extension of the zero-divisor graph of $\mathcal{L}$ (see [4]). Reduced zero-divisor graph of a poset has been studied and investigated in [20]. In [18], the graph $\Gamma_E(\mathcal{L})$ of equivalance classes of zero divisors of a meet semilattice $\mathcal{L}$ with 0 has been introduced and some of its properties have been studied.

For a distributive lattice $\mathcal{L}$, the *Stone space* Spec$(\mathcal{L})$ and the subspace

$\mathrm{Max}(\mathcal{L})$ are well known (for example see [13, Section II-5] and [21]). For a pm-lattice $\mathcal{L}$, Joshi and Khiste [16] investigated some properties of the zero-divisor graph $\Gamma(\mathcal{L})$ such as the diameter and the eccentricity. They also studied some algebraic and topological conditions under which the zero-divisor graph $\Gamma(\mathcal{L})$ is triangulated or hypertriangulated.

In this article, at first for a distributive lattice $\mathcal{L}$, we study and investigate some relations among lattice theoretic featurs of $\mathcal{L}$ such as Goldie dimension and minimal prime ideals of $\mathcal{L}$ and some topological properties of the Stone spaces $\mathrm{Spec}(\mathcal{L})$ and $\mathrm{Max}(\mathcal{L})$ like cellularity, density and weight of them and compare them with some graph theoretic aspects of the zero-divisor graph $\Gamma(\mathcal{L})$ such as its clique number and domination number. After that, for a 0-distributive lattice $\mathcal{L}$, we investigate the compressed subgraph $\Gamma_E(\mathcal{L})$ of the zero-divisor graph $\Gamma(\mathcal{L})$ and determine some properties of this subgraph in terms of the corresponding lattice theoretic properties such as associated prime ideals of $\mathcal{L}$.

In Section 2, we recall some preliminary concepts from graph theory, lattice theory, and topology. Section 3 is devoted to an investigation of the zero-divisor graph $\Gamma(\mathcal{L})$ in comparison with some topological aspects of $\mathrm{Spec}(\mathcal{L})$ and $\mathrm{Max}(\mathcal{L})$. In Theorem 3.1, we establish equalities among the Goldie dimension of a distributive lattice $\mathcal{L}$, the cellularity of its Stone space $\mathrm{Spec}(\mathcal{L})$ and the clique number of the corresponding zero-divisor graph. We will see that in case $\mathcal{L}$ is semiprimitive, they are also equal to the cellularity of the subspace $\mathrm{Max}(\mathcal{L})$ (see Corollary 3.4). Among other things, we show that if $\mathcal{L}$ is a semiprimitive pm-lattice with $|\mathrm{Min}(\mathcal{L})| \geq 3$, then the domination number $\mathrm{dt}(\Gamma(\mathcal{L}))$ lies between the density $\mathrm{dens}(\mathrm{Spec}(\mathcal{L}))$ and the weight $\mathrm{w}(\mathrm{Spec}(\mathcal{L}))$ (see Theorem 3.10). Moreover, under an additional mild condition on $\mathrm{Spec}(\mathcal{L})$, we obtain the equality $\mathrm{dens}(\mathrm{Spec}(\mathcal{L})) = \mathrm{dt}(\Gamma(\mathcal{L}))$ (see Theorem 3.11). Also, we give a necessary and sufficient condition under which the set of isolated points of $\mathrm{Spec}(\mathcal{L})$ is a dense subset (see Theorem 3.19). In Section 4, we study the the compressed zero-divisor graph $\Gamma_E(\mathcal{L})$ of a bounded 0-distributive lattice $\mathcal{L}$. We show that for nontrivial cases, this graph can not be a complete, complete bipartite graph or a cycle (see Theorems 4.9, 4.10 and 4.11). Furthermore, we will see that this graph is closely related to the associated prime ideals of $\mathcal{L}$ (see Corollaries 4.15 and 4.16 and Theorems 4.19 and 4.21).

2 Preliminaries

In this section, for convenience, we recall some preliminary definitions and notations. For more details about the standard notations and terminologies in graph theory, lattice theory, and topology the reader is referred to [26, 27], [13], and [9, 21], respectively.

2.1 Some notions from graph and lattice theory In a graph G, the *distance* between two distinct vertices a and b, denoted by $\mathrm{d}(a,b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, we set $\mathrm{d}(a,b)=\infty$. The *diameter* of a graph G is defined as $\mathrm{diam}(G)=\sup\{\mathrm{d}(x,y) : x,y\in V(G)\}$. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K^n to denote the complete graph with n vertices. For a positive integer r, an *r-partite graph* is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite* is one in which each vertex is joined to every vertex that is not in the same subset. The *complete bipartite graph* with parts of sizes m and n is denoted by $K^{m,n}$. A nonempty subset S of $V(G)$ is called a *dominating set* if every vertex in $V(G)\setminus S$ is adjacent to at least one vertex in S. The *domination number* dt(G) of G is the minimum cardinality of the dominating sets in G. A *clique* of a graph G is a complete subgraph of G and the maximum size of cliques in G is called the *clique number* of G and is denoted by $\omega(G)$.

Let $\mathcal{L}$ be a modular lattice with 0 and 1. Following [11] a finite subset $\{a_i|i\in I\}$ of $\mathcal{L}\setminus\{0\}$ is said to be *join-independent* if $a_i\wedge(\bigvee_{j\neq i}a_j)=0$ for every $i\in I$. An arbitrary subset of $\mathcal{L}\setminus\{0\}$ is said to be join-independent, if all its finite subsets are join-independent. If a bounded modular lattice $\mathcal{L}$ does not contain any infinite join-independent subset, then there exists a smallest positive integer n such that any join-independent subset of $\mathcal{L}$ has cardinality $\leq n$. In this case n is said to be the *Goldie dimension* of $\mathcal{L}$. If $\mathcal{L}$ contains an infinite join-independent subset, then $\mathcal{L}$ is said to have *infinite Goldie dimension.* A subset I of a lattice $\mathcal{L}$ is called an *ideal* if it is a sublattice of $\mathcal{L}$ and for $x\in I$ and $a\in\mathcal{L}$ imply that $x\wedge a\in I$. An ideal I of $\mathcal{L}$ is *proper* if $I\neq\mathcal{L}$. A proper ideal I of $\mathcal{L}$ is called a *prime ideal* if $a,b\in\mathcal{L}$ and $a\wedge b\in I$ imply that $a\in I$ or $b\in I$. For every $a\in\mathcal{L}$, we set $(a] = \{x \in \mathcal{L} : x \leq a\}$. In fact, $(a]$ is an ideal of $\mathcal{L}$ which is called the *principal ideal* generated by a. A bounded distributive lattice $\mathcal{L}$ is called a *pm-lattice* if every prime ideal of $\mathcal{L}$ is contained in a unique maximal ideal and a lattice $\mathcal{L}$ with 0 is said to be *semiprimitive* if the intersection of all maximal ideals of $\mathcal{L}$ is 0. Also a lattice $\mathcal{L}$ with the bottom element 0 is said to be *0-distributive* if the equalities $a \wedge b = 0 = a \wedge c$ imply that $a \wedge (b \vee c) = 0$.

2.2 Some notions from Ston spaces $\mathrm{Spec}(\mathcal{L})$ and $\mathrm{Max}(\mathcal{L})$ Let X be a topological space. The *cellularity* of X, denoted by $c(X)$, is the smallest cardinal number m such that every set of pairwise disjoint nonempty open subsets of X has cardinality at most m. The *weight* of a topological space X, denoted by w(X), is the smallest element in the set of all cardinal numbers of the form $|B|$, where B is a base for the open subsets of X. The *density* of a topological space X, denoted by $\mathrm{dens}(X)$, is defined as the smallest cardinal number of the form $|Y|$, where Y is a dense subset of X. For a distributive lattice $\mathcal{L}$, we denote by $\mathrm{Max}(\mathcal{L})$, $\mathrm{Spec}(\mathcal{L})$, and $\mathrm{Min}(\mathcal{L})$ the set of all maximal ideals, the set of all prime ideals and the set of all minimal prime ideals of $\mathcal{L}$ respectively. Let I be an ideal of $\mathcal{L}$ and $a \in \mathcal{L}$. We set

$$V(a) = \{P \in \mathrm{Spec}(\mathcal{L}) : a \in P\} \quad , \qquad D(a) = \mathrm{Spec}(\mathcal{L}) \setminus V(a).$$

Then the sets $V(I) = \bigcap_{a \in I} V(a) = \{P \in \mathrm{Spec}(\mathcal{L}) : I \subseteq P\}$ satisfy the axioms for the closed sets of a topology on $\mathrm{Spec}(\mathcal{L})$, called the *Stone topology*. Also the sets $D(I) = \bigcup_{a \in I} D(a)$ satisfy the axioms for the open sets of this topology. We set

$$V'(a) = V(a) \cap \mathrm{Min}(\mathcal{L}) \quad , \quad D'(a) = D(a) \cap \mathrm{Min}(\mathcal{L}),$$

$$M(a) = V(a) \cap \mathrm{Max}(\mathcal{L}) \quad , \quad D_m(a) = \mathrm{Max}(\mathcal{L}) \setminus M(a),$$

and for an ideal I of $\mathcal{L}$, $D_m(I) = \mathrm{Max}(\mathcal{L}) \setminus \{M \in \mathrm{Max}(\mathcal{L}) : I \subseteq M\}$. Given a subset μ of a topolgical space, the closure of μ is defined as the intersection of all closed sets containing μ, and the interior of μ is defined as the union of all open sets contained in μ.

2.3 The zero-divisor graph $\Gamma(\mathcal{L})$ and the compressed zero–divisor graph $\Gamma_E(\mathcal{L})$ of a lattice $\mathcal{L}$ For a lattice $\mathcal{L}$ with the bottom element 0, the set

$$Z(\mathcal{L}) = \{x \in \mathcal{L} : \exists a \in \mathcal{L} \setminus \{0\}; x \wedge a = 0\}$$

of zero-divisors is considered and the *zero-divisor graph* of $\mathcal{L}$ is defined as the (undirected) graph with the vertex set $Z(\mathcal{L})^\star = Z(\mathcal{L}) \setminus \{0\}$ and two distinct vertices x and y are assumed to be adjacent if and only if $x \wedge y = 0$.

An equivalence relation $\sim$ on a 0-distributive lattice $\mathcal{L}$, is defined by $x \sim y$ if and only if $\mathrm{ann}_{\mathcal{L}}(x) = \mathrm{ann}_{\mathcal{L}}(y)$. The *compressed zero-divisor graph* $\Gamma_E(\mathcal{L})$ is the graph whose vertices are the equivalence classes induced by $\sim$ other than $[0]$ and $[1]$, such that distinct vertices $[x]$ and $[y]$ are adjacent in $\Gamma_E(\mathcal{L})$ if and only if $x \wedge y = 0$.

3 Some relations between the zero-divisor graph $\Gamma(\mathcal{L})$ and the topological space $\mathrm{Spec}(\mathcal{L})$

It has been shown in [23] that for a commutative reduced ring R, the clique number of the zero-divisor graph $\Gamma(R)$ is the same as the cellularity of the topological space $rmSpec(R)$ and this is also equal to the Goldie dimension of the ring R. For a distributive lattice $\mathcal{L}$, the Goldie dimension of $\mathcal{L}$ has been defined and investigated in the literature (see for example [11]). Also the Stone spaces $\mathrm{Spec}(\mathcal{L})$ and $\mathrm{Max}(\mathcal{L})$ are well known (see for example [13, Section II-5]). In [16] V. Joshi and A. Khiste stated some results about the zero-divisor graph of a distributive lattice.

In this section, we state more results on the zero-divisor graph $\Gamma(\mathcal{L})$ of a distributive lattice $\mathcal{L}$ relating it with topological properties of $\mathrm{Spec}(\mathcal{L})$ and $\mathrm{Max}(\mathcal{L})$ and some lattice theoretic aspects of $\mathcal{L}$.

Theorem 3.1. *For a distributive lattice $\mathcal{L}$ with 0, we have*

$$\omega(\Gamma(\mathcal{L})) = c(\mathrm{Spec}(\mathcal{L})) = \mathrm{Gdim}(\mathcal{L}).$$

Proof. Let H be a clique in $\Gamma(\mathcal{L})$. Then for all distinct vertices a and b in H, $a \wedge b = 0$ implies that $D(a) \cap D(b) = \emptyset$. In this way, we obtain a family $\mathcal{C} = \{D(a) : a \in H\}$ of pairwise disjoint nonempty open subsets of $\mathrm{Spec}(\mathcal{L})$. This yields that $\omega(\Gamma(\mathcal{L})) \leq c(\mathrm{Spec}(\mathcal{L}))$. Now consider the collection $\mathfrak{B} = \{A_\lambda : \lambda \in \Lambda\}$ of pairwise disjoint nonempty open subsets of $\mathrm{Spec}(\mathcal{L})$. For every $A_\lambda \in \mathfrak{B}$ there exists an element $a_\lambda \in \mathcal{L} \setminus \{0\}$ such that $D(a_\lambda) \subseteq A_\lambda$. Clearly, for every two distinct indices λ and λ' in Λ, we have $a_\lambda \wedge a_{\lambda'} = 0$. Hence the set $\mathcal{G} = \{a_\lambda : \lambda \in \Lambda\}$ forms a clique in $\Gamma(\mathcal{L})$. This means that $\omega(\Gamma(\mathcal{L})) \geq c(\mathrm{Spec}(\mathcal{L}))$ and therefore the first equality holds.

In order to prove the second equality, let $\mathrm{Gdim}(\mathcal{L}) = c$ and $\{a_\lambda : \lambda \in \Lambda\}$ be a join-independent subset of $\mathcal{L}$ such that $|\Lambda| \leq c$. Since for distinct elements $\lambda, \lambda^{'} \in \Lambda$, $a_\lambda \wedge a_{\lambda'} = 0$, then $D(a_\lambda) \cap D(a_{\lambda'}) = \emptyset$ and this implies that $\mathfrak{C} = \{D(a_\lambda) : \lambda \in \Lambda\}$ is a collection of mutually disjoint nonempty open subsets in $\mathrm{Spec}(\mathcal{L})$. Therefore, $c(\mathrm{Spec}(\mathcal{L})) \geq c$. Conversely, let $c(\mathrm{Spec}(\mathcal{L})) = |\Lambda|$ and $\{G_\lambda : \lambda \in \Lambda\}$ be any collection of mutually disjoint nonempty open subsets in $\mathrm{Spec}(\mathcal{L})$. By axiom of choice, for every $\lambda \in \Lambda$, we can select a unique a_λ such that $D(a_\lambda) \subseteq G_\lambda$. We claim that the set $\{a_\lambda : \lambda \in \Lambda\}$ is a join-independent subset of $\mathcal{L}$. Let $x = a_\lambda \wedge (\bigvee_{\lambda \neq \lambda' \in \Lambda'} a_{\lambda'})$, where $\Lambda^{'} \subseteq \Lambda$ and $|\Lambda^{'}|$ is finite. Now since $x \leq a_\lambda$ and $x \leq \bigvee_{\lambda \neq \lambda' \in \Lambda'} a_{\lambda'}$, then $D(x) \subseteq D(a_\lambda)$ and

$$D(x) \subseteq D(\bigvee_{\lambda \neq \lambda' \in \Lambda'} a_{\lambda'}) = \bigcup_{\lambda \neq \lambda' \in \Lambda'} D(a_{\lambda'}).$$

Hence

$$D(x) \subseteq D(a_\lambda) \cap (\bigcup_{\lambda \neq \lambda' \in \Lambda'} D(a_{\lambda'})) \subseteq \bigcup_{\lambda \neq \lambda' \in \Lambda'} (G_\lambda \cap G_{\lambda'}) = \emptyset.$$

Therefore, $x = 0$ and this means that $c(\mathrm{Spec}(\mathcal{L})) \leq c = \mathrm{Gdim}(\mathcal{L})$. □

Corollary 3.2. *Let $\mathcal{L}$ be a bounded distibutive lattice and every maximal ideal of $\mathcal{L}$ is generated by a nonzero zero-divisor. Then* $\mathrm{Spec}(\mathcal{L}) = \mathrm{Max}(\mathcal{L})$ *is a discrete space and*

$$\omega(\Gamma(\mathcal{L})) = c(\mathrm{Spec}(\mathcal{L})) = \mathrm{Gdim}(\mathcal{L}) = |\mathrm{Spec}(\mathcal{L})| = |\mathrm{Max}(\mathcal{L})|.$$

Proof. Suppose that $y \in Z(\mathcal{L})^\star$ and $M = (y]$ is a maximal ideal. Then there exists a nonzero element $x \in \mathcal{L}$ such that $y \wedge x = 0$ and $\{M\} = D(x)$. Since every prime ideal P is contained in a maximal ideal M which is generated by a nonzero zero-divisor y, it can be concluded that $\mathrm{Spec}(\mathcal{L}) = \mathrm{Max}(\mathcal{L})$ is a discrete space. Now the last statement is an immediate consequence of Theorem 3.1. □

Lemma 3.3. *[16, Corollary 1.4] Let $\mathcal{L}$ be a semiprimitive distributive lattice. Then* $\mathrm{Max}(\mathcal{L})$ *is a dense subspace of* $\mathrm{Spec}(\mathcal{L})$.

Corollary 3.4. *Let $\mathcal{L}$ be a semiprimitive distibutive lattice. Then*

$$\omega(\Gamma(\mathcal{L})) = c(\mathrm{Spec}(\mathcal{L})) = \mathrm{Gdim}(\mathcal{L}) = c(\mathrm{Max}(\mathcal{L})).$$

Proof. Let $\mathrm{Gdim}(\mathcal{L}) = c$ and $\{a_i : i \in I\}$ be a join-independent subset of $\mathcal{L}$ with $|I| = c$. Now, $a_i \wedge a_j = 0$, for all distinct elements $i, j \in I$. Hence $D_m(a_i) \cap D_m(a_j) = \emptyset$ implies that $F = \{D_m(a_i) : i \in I\}$ is a collection of pairwise disjoint nonempty open subsets in $\mathrm{Max}(\mathcal{L})$. This means that $c(\mathrm{Max}(\mathcal{L})) \geq \mathrm{Gdim}(\mathcal{L})$. On the other hand, if $\{G_i \neq \emptyset : i \in I\}$ is a collection of pairwise disjoint nonempty open subsets in $\mathrm{Max}(\mathcal{L})$, then it is in correspondence with a collection of mutually disjoint nonempty open subsets in $\mathrm{Spec}(\mathcal{L})$ by Lemma 3.3. Therefore, $c(\mathrm{Max}(\mathcal{L})) \leq c(\mathrm{Spec}(\mathcal{L}))$ and the proof is complete by Theorem 3.1. □

Lemma 3.5. [16, Lemma 1.9] *For a distributive lattice $\mathcal{L}$ with 0, if $a \in \mathcal{L}$, then $V(\mathrm{ann}_{\mathcal{L}}(a)) = \overline{D(a)}$, where $\mathrm{ann}_{\mathcal{L}}(a) = \{x \in \mathcal{L} : x \wedge a = 0\}$ and $\overline{D(a)}$ is the closure of $D(a)$ in $\mathrm{Spec}(\mathcal{L})$.*

Corollary 3.6. *Let $\mathcal{L}$ be a distributive lattice with 0. For every element $a \in \mathcal{L}$, $D(\mathrm{ann}_{\mathcal{L}}(a)) = \mathrm{int}(V(a))$.*

Proof. By Lemma 3.5 and [9, Theorem 1.1.5], we have

$$V(\mathrm{ann}_{\mathcal{L}}(a)) = \overline{D(a)} = \mathrm{Spec}(\mathcal{L}) \setminus \mathrm{int}(V(a)).$$

Therefore, $D(\mathrm{ann}_{\mathcal{L}}(a)) = \mathrm{int}(V(a))$. □

Lemma 3.7. *[16, Lemma 2.15] For a pm-lattice $\mathcal{L}$ let U be an open subset in $\mathrm{Spec}(\mathcal{L})$ and $P \in \mathrm{Spec}(\mathcal{L})$ with $V(P) \subseteq U$. Then there exists an element $a \in \mathcal{L}$ such that $P \in \mathrm{int}(V(a)) \subseteq U$.*

Remark 3.8. Let $\mathcal{L}$ be a pm-lattice and U_m be a nonempty open subset in $\mathrm{Max}(\mathcal{L})$ such that $M_1 \in U_m$. Then there exists a nonempty open subset U_s in $\mathrm{Spec}(\mathcal{L})$ such that $U_m = U_s \cap \mathrm{Max}(\mathcal{L})$ and $M_1 \in \{M_1\} = V(M_1) \subseteq U_s$. By Lemma 3.7, there exists an element $a \in \mathcal{L}$ such that $M_1 \in \mathrm{int}(V(a)) \subseteq U_s$. By Corollary 3.6, $M_1 \in D(\mathrm{ann}_{\mathcal{L}}(a)) \cap \mathrm{Max}(\mathcal{L}) \subseteq U_s \cap \mathrm{Max}(\mathcal{L}) = U_m$. Therefore, $M_1 \in D_m(\mathrm{ann}_{\mathcal{L}}(a)) \subseteq U_m$. This means that $\{D_m(\mathrm{ann}_{\mathcal{L}}(a)) : a \in \mathcal{L}\}$ is a base for the open subsets in $\mathrm{Max}(\mathcal{L})$.

Theorem 3.9. [16, Theorem 2.18] *Let $\mathcal{L}$ be a distributive lattice with the bottom element 0 and there exists a vertex a in $\Gamma(\mathcal{L})$ which is adjacent to every other vertex in $\Gamma(\mathcal{L})$. Then $|\mathrm{Min}(\mathcal{L})| = 2$.*

Theorem 3.10. *For a semiprimitive pm-lattice* $\mathcal{L}$ *if* $|\mathrm{Min}(\mathcal{L})| \geq 3$*, then*

$$\mathrm{dens}(\mathrm{Spec}(\mathcal{L})) \leq \mathrm{dt}(\Gamma(\mathcal{L})) \leq \mathrm{w}(\mathrm{Spec}(\mathcal{L})).$$

Proof. Suppose that D is a dominating set with minimum cardinality. By Theorem 3.9, $|D| \geq 2$. For every $a \in D$ there exists an element $b \in Z(\mathcal{L})^{\star}$ such that $a \wedge b = 0$ and since $\bigcap_{M \in \mathrm{Max}(\mathcal{L})} M = \{0\}$, there is a maximal ideal $M \in \mathrm{Max}(\mathcal{L})$ such that $b \notin M$. Therefore, $\mathrm{ann}_{\mathcal{L}}(a) \nsubseteq M$, i.e. $M \in D_m(\mathrm{ann}_{\mathcal{L}}(a))$. Also there exists a maximal ideal $M^{'} \in D_m(a)$. For every $a \in D$, using axiom of choice, we can choose one M_a in $D_m(\mathrm{ann}_{\mathcal{L}}(a))$ and one $M_a^{'}$ in $D_m(a)$. We show that the set

$$\mathfrak{B} = \{M_a : a \in D\} \cup \{M_a^{'} : a \in D\}$$

is dense in $\mathrm{Max}(\mathcal{L})$ and thus in $\mathrm{Spec}(\mathcal{L})$, by Lemma 3.3.

By Remark 3.8, $\{D_m(\mathrm{ann}_{\mathcal{L}}(a)) : a \in Z(\mathcal{L})^{\star}\}$ is a base for the nonempty open subsets of $\mathrm{Max}(\mathcal{L})$. Therefore, it is sufficient to observe that for every $a \in Z(\mathcal{L})^{\star}$, $\mathfrak{B} \cap D_m(\mathrm{ann}_{\mathcal{L}}(a)) \neq \emptyset$. If $a \in D$, then $M_a \in \mathfrak{B} \cap D_m(\mathrm{ann}_{\mathcal{L}}(a))$. Otherwise, if $a \notin D$, then there exists $b \in D$ such that $a \wedge b = 0$. Thus $M_b^{'} \in \mathfrak{B} \cap D_m(\mathrm{ann}_{\mathcal{L}}(a))$. According to the description given, $\mathfrak{B}$ is dense in $\mathrm{Spec}(\mathcal{L})$. This shows that

$$\mathrm{dens}(\mathrm{Spec}(\mathcal{L})) \leq \mathrm{dens}(\mathrm{Max}(\mathcal{L})) \leq |\mathfrak{B}|.$$

Now we claim that $\mathrm{dens}(\mathrm{Spec}(\mathcal{L})) \leq |D|$. If D is an infinite set, then $\mathrm{dens}(\mathrm{Spec}(\mathcal{L})) \leq |\mathfrak{B}| \leq 2|D| = |D|$. Therefore, we assume that $D = \{a_1, \ldots, a_n\}$. We claim that, in this case we can construct $\mathfrak{B}$ in such a way that $|\mathfrak{B}| \leq |D|$; by deleting the redundant maximal ideals in $\mathfrak{B}$. If $n = 2k$, then we partition the set

$$D = \{a_1, a_2, \ldots, a_{2k-1}, a_{2k}\}$$

into k disjoint two-element subsets

$$\{a_1, a_2\}, \ldots, \{a_{2k-1}, a_{2k}\}.$$

As $\mathrm{d}(a_1, a_2) \leq 3$, we have three cases:

Case(I): If $a_1 \wedge a_2 = 0$, then we can take $M_{a_1} = M_{a_2}^{'}$ and $M_{a_2} = M_{a_1}^{'}$.

Case(II): If $\mathrm{d}(a_1, a_2) = 2$, then $a_1 \wedge a_2 \neq 0$ and there is a vertex x in $\Gamma(\mathcal{L})$ such that $x \wedge a_1 = 0 = x \wedge a_2$. Therefore, we can take $M'_{a_1 \wedge a_2} = M'_{a_1} = M'_{a_2}$ and $M'_x = M_{a_1} = M_{a_2}$.

Case(III): In case $\mathrm{d}(a_1, a_2) = 3$, there are distinct vertices x_1 and x_2 in $\Gamma(\mathcal{L})$ such that $x_1 \wedge a_1 = 0, x_1 \wedge a_2 \neq 0$ and $x_2 \wedge a_2 = 0, x_2 \wedge a_1 \neq 0$. Therefore, we can take $M'_{x_1 \wedge a_2} = M'_{a_2} = M_{a_1}$ and $M'_{x_2 \wedge a_1} = M'_{a_1} = M_{a_2}$.

Now if $n = 2k+1$, then we divide D into $k-1$ mutually disjoint two-element subsets and one subset $\{a_{n-2}, a_{n-1}, a_n\}$ with three elements. If there is at least one edge in the subset with three elements, then considering all possible cases, it is straightforward to see that
$|\{M_{a_n}, M_{a_{n-1}}, M_{a_{n-2}}, M'_{a_n}, M'_{a_{n-1}}, M'_{a_{n-2}}\}| \leq 3$. Otherwise, if there is no pair of adjacent vertices in $\{a_{n-2}, a_{n-1}, a_n\}$, we have two different cases:

Case(1): Let $a_n \wedge a_{n-1} \wedge a_{n-2} \neq 0$. This case in turn can be divided into the following subcases:

Subcase(1-1): If there exists a vertex $x_1 \in Z(\mathcal{L})^\star$ such that $x_1 \wedge a_{n-1} = x_1 \wedge a_{n-2} = x_1 \wedge a_n = 0$, then we can take $M'_{a_n \wedge a_{n-1} \wedge a_{n-2}} = M'_{a_{n-1}} = M'_{a_{n-2}} = M'_{a_n}$ and $M'_{x_1} = M_{a_{n-1}} = M_{a_{n-2}} = M_{a_n}$.

Subcase(1-2): If there exist distinct $i, j \in \{n-2, n-1, n\}$ such that $\mathrm{d}(a_i, a_j) = 2$, then there is a vertex $x \in Z(\mathcal{L})^\star$ such that $x \wedge a_i = x \wedge a_j = 0$ and if x is not adjacent to the third vertex $y \in \{a_{n-2}, a_{n-1}, a_n\}$, then we can take $M_{a_i} = M_{a_j} = M'_y = M'_{x \wedge y}$ and $M'_{a_i} = M'_{a_j} = M'_{a_i \wedge a_j}$.

Subcase(1-3): Let for every distinct $i, j \in \{n-2, n-1, n\}$, $\mathrm{d}(a_i, a_j) = 3$ and $x_1 \wedge a_{n-2} = 0$, $x_1 \wedge a_{n-1} \neq 0$, $x_1 \wedge a_n \neq 0$, $x_3 \wedge a_{n-2} \neq 0$, $x_3 \wedge a_n = 0$. If $x_1 \wedge a_{n-1} \wedge a_n \neq 0$, then we can take $M'_{x_1 \wedge a_{n-1} \wedge a_n} = M'_{a_{n-1}} = M'_{a_n} = M_{a_{n-2}}$ and $M'_{x_3 \wedge a_{n-2}} = M'_{a_{n-2}} = M_{a_n}$. Otherwise, if $x_1 \wedge a_{n-1} \wedge a_n = 0$, we can take $M'_{x_1 \wedge a_{n-1}} = M'_{a_{n-1}} = M_{a_n} = M_{a_{n-2}}$ and $M'_{x_1 \wedge a_n} = M'_{a_n} = M_{a_{n-1}}$.

Case(2): Now let $a_n \wedge a_{n-1} \wedge a_{n-2} = 0$, $a_{n-1} \wedge a_{n-2} \neq 0$, $a_n \wedge a_{n-1} \neq 0$ and $a_n \wedge a_{n-2} \neq 0$. Then we can take $M'_{a_n} = M'_{a_{n-1}} = M_{a_{n-2}}$ and $M'_{a_{n-2}} = M_{a_n}$.

Hence $\mathrm{dens}(\mathrm{Spec}(\mathcal{L})) \leq |\mathfrak{B}| \leq |D| \leq \mathrm{dt}(\Gamma(\mathcal{L}))$.

In order to show that $\mathrm{dt}(\Gamma(\mathcal{L})) \leq \mathrm{w}(\mathrm{Spec}(\mathcal{L}))$, suppose that $B = \{B_\lambda : \lambda \in \Lambda\}$ is a base for the open subsets of $\mathrm{Spec}(\mathcal{L})$. Since $\{D(a) : a \in \mathcal{L}\}$ is also a base for open subsets of $\mathrm{Spec}(\mathcal{L})$, then for every $B_\lambda \in B$ there exists an a_λ such that $D(a_\lambda) \subseteq B_\lambda$. We claims that $D = \{a_\lambda : \lambda \in \Lambda\}$ is a dominating set. To see this, let $b \in \Gamma(\mathcal{L}) \setminus D$; then there exists $B_\lambda \in B$ such that $B_\lambda \subseteq \mathrm{int}(V(b))$. Therefore, $D(a_\lambda) \subseteq \mathrm{int}(V(b))$, that is, $a_\lambda \wedge b = 0$ and consequently D is a dominating set. Now $\mathrm{dt}(\Gamma(\mathcal{L})) \leq |D| \leq |B|$ for every base of open subsets of $\mathrm{Spec}(\mathcal{L})$ such as B. This means that $\mathrm{dt}(\Gamma(\mathcal{L})) \leq \mathrm{w}(\mathrm{Spec}(\mathcal{L}))$. □

A family $B(x)$ of neighbourhoods of x is called a *base* for a topological space X at the point x if for any neighbourhood V of x there exists an element $U \in B(x)$ such that $x \in U \subseteq V$. The *character* of a point x in a topological space X is defined as the smallest cardinal number of the form $|B(x)|$, where $B(x)$ is a base for X at the point x; this cardinal number is denoted by $\chi(x, X)$. The *character of a topological space* X is defined as the supremum of all numbers $\chi(x, X)$, where $x \in X$. This cardinal number is denoted by $\chi(X)$.

In the following theorem, we see that under a mild condition on the space $\mathrm{Spec}(\mathcal{L})$, the first inequality in Theorem 3.10, can be replaced by an equality.

Theorem 3.11. *Let $\mathcal{L}$ be a semiprimitive pm-lattice with $|\mathrm{Min}(\mathcal{L})| \geq 3$ and $\chi(\mathrm{Spec}(\mathcal{L})) \leq \mathrm{dens}(\mathrm{Spec}(\mathcal{L}))$. Then $\mathrm{dt}(\Gamma(\mathcal{L})) = \mathrm{dens}(\mathrm{Spec}(\mathcal{L}))$.*

Proof. Theorem 3.10, shows that $\mathrm{dens}(\mathrm{Spec}(\mathcal{L})) \leq \mathrm{dt}(\Gamma(\mathcal{L}))$. Now we must prove the other inequality. Suppose that $\mathfrak{C} = \{P_\lambda : \lambda \in \Lambda\}$ is a dense subset of $\mathrm{Spec}(\mathcal{L})$. For every $\lambda \in \Lambda$, there exists a base $\mathcal{B}_\lambda$ of the topological space $\mathrm{Spec}(\mathcal{L})$ at P_λ such that $|\mathcal{B}_\lambda| \leq \chi(\mathrm{Spec}(\mathcal{L})) \leq \mathrm{dens}(\mathrm{Spec}(\mathcal{L}))$. Considering $\{\mathcal{B}_{\lambda_i} \in \mathcal{B}_\lambda : \lambda_i \in I_\lambda\}$ there exists $x_{\lambda_i} \in \mathcal{L}$, such that $P_\lambda \in D(x_{\lambda_i}) \subseteq \mathcal{B}_{\lambda_i}$. Now if $\chi(\mathrm{Spec}(\mathcal{L}))$ is finite. We set $\bigwedge_{\lambda_i \in I_\lambda} x_{\lambda_i} = a_\lambda \neq 0$ and show that $\mathfrak{D} = \{a_\lambda : \lambda \in \Lambda\}$ is a dominating set. If $b \in \Gamma(\mathcal{L}) \setminus \mathfrak{D}$, then $\mathrm{int}(V(b)) \neq \emptyset$ and since $\mathfrak{C}$ is a dense subset, therefore, there exists $\lambda_0 \in \Lambda$ such that $P_{\lambda_0} \in \mathrm{int}(V(b)) \cap \mathfrak{C}$ and, according to the above descriptions, there exists $\mathcal{B}_{\lambda_{0i}} \in \mathcal{B}_{\lambda_0}$ in which $P_{\lambda_0} \in D(a_{\lambda_0}) \subseteq \mathcal{B}_{\lambda_{0i}} \subseteq \mathrm{int}(V(b))$; then $a_{\lambda_0} \wedge b = 0$. Inasmuch as $|\mathfrak{D}| \leq |\mathfrak{C}|$, we have $\mathrm{dt}(\Gamma(\mathcal{L})) \leq \mathrm{dens}(\mathrm{Spec}(\mathcal{L}))$ and hence $\mathrm{dt}(\Gamma(\mathcal{L})) = \mathrm{dens}(\mathrm{Spec}(\mathcal{L}))$. Otherwise, if $\chi(\mathrm{Spec}(\mathcal{L}))$ is infinite, we set

$\mathfrak{D}_1 = \{x_{\lambda_i} : \lambda_i \in I_\lambda, \lambda \in \Lambda\}$, and with a similar argument as above, we see that $\mathfrak{D}_1$ is a dominating set and also $|\mathfrak{D}_1| \leq \sum_{P_\lambda \in \mathfrak{C}} |\mathcal{B}_\lambda| \leq |\mathfrak{C}||\mathfrak{C}| = |\mathfrak{C}|$. □

Lemma 3.12. *Suppose that $\mathcal{L}$ is a semiprimitive bounded distributive lattice. Then for $y \in Z(\mathcal{L})^\star$, $M = (y]$ is a maximal ideal in $\mathcal{L}$ if and only if M is an isolated point in* $\mathrm{Max}(\mathcal{L})$.

Proof. Let $M = (y]$ be a maximal ideal, where $y \in Z(\mathcal{L})^\star$ with $x \wedge y = 0 (x \neq 0)$. Then $\{M\} = D_m(x)$. Conversely, suppose that M is an isolated point in $\mathrm{Max}(\mathcal{L})$. Then there exists a nonzero element $x \in \mathcal{L}$ such that $\{M\} = D_m(x)$. For every $m \in M$, $m \wedge x \in \bigcap_{M' \in Max(\mathcal{L})} M' = \{0\}$ and $M \vee (x] = \mathcal{L}$. Hence there is $y \in M$ such that $y \vee x = 1$. Now, we claim that $M = (y]$. To see this, we note that an arbitrary element $m \in M$ can be written as $m = m \wedge (y \vee x)$ which implies that $m = m \wedge y \in (y]$. Therefore, $M = (y]$. □

Lemma 3.13. *Let $\mathcal{L}$ be a semiprimitive bounded distributive lattice. Then the set of isolated points in the space* $\mathrm{Spec}(\mathcal{L})$ *is the same as the set of isolated points in the space* $\mathrm{Max}(\mathcal{L})$.

Proof. Let $P_0(\mathcal{L})$ and $M_0(\mathcal{L})$ be the sets of isolated points of the spaces $\mathrm{Spec}(\mathcal{L})$ and $\mathrm{Max}(\mathcal{L})$ respectively, suppose that $M \in M_0(\mathcal{L})$. By Lemma 3.12, there exist $y, x \in Z(\mathcal{L})^\star$, such that $x \wedge y = 0$ and $M = (y]$. Hence for each $P \in \mathrm{Spec}(\mathcal{L})$ with $P \neq M$, P contains x. This means that $\bigcap_{P \in \mathrm{Spec}(\mathcal{L}) \setminus \{\mathrm{M}\}} P \nsubseteq M$, that is, $D(x) = \{M\}$. Conversely, suppose that $P \in P_0(\mathcal{L})$. Since $\mathcal{L}$ is a semiprimitive lattice, P is a maximal ideal. □

We recall that the *eccentricity* $e(a)$ of a vertex a of a graph G is defined to be $e(a) = \max\{\mathrm{d}(a, b) : b \in G, b \neq a\}$. A *centeral vertex* in a graph G is a vertex a_0 in G with minimum eccentricity. The set of all central vertices in a graph G is called the *center* of G.

Theorem 3.14. *[16, Theorem 2.20] For a semiprimitive pm-lattice $\mathcal{L}$, let a be a vertex in $\Gamma(\mathcal{L})$ with $e(a) \neq 1$. Then $e(a) = 2$ if and only if $|\overline{D(a)}| = 1$.*

Lemma 3.15. *[16, Lemma 1.2] For a distributive lattice $\mathcal{L}$ and every subset $\mu \subseteq \mathrm{Spec}(\mathcal{L})$, the closure of μ is $\overline{\mu} = \{P' \in \mathrm{Spec}(\mathcal{L}) : \bigcap_{P \in \mu} P \subseteq P'\}$.*

With a similar argument as in [22, Proposition 4.1], it can be seen that:

Theorem 3.16. *Let $\mathcal{L}$ be a semiprimitive pm-lattice and $P_0(\mathcal{L})$, $M_0(\mathcal{L})$ and $I_0(\mathcal{L})$ be the sets of isolated points of the spaces* $\mathrm{Spec}(\mathcal{L})$, $\mathrm{Max}(\mathcal{L})$, *and* $\mathrm{Min}(\mathcal{L})$, *respectively. Then $P_0(\mathcal{L}) = M_0(\mathcal{L}) = I_0(\mathcal{L})$.*

A point P of $\mathrm{Spec}(\mathcal{L})$ is said to be a *quasi-isolated point* if P is a minimal prime ideal and P is not contained in the union of all minimal prime ideals of $\mathcal{L}$ different from P. In this case, there exists an element $x \in \mathcal{L}$ such that $\{P\} = V^{'}(x) = D^{'}(\mathrm{ann}_{\mathcal{L}}(x))$.

Remark 3.17. It is not hard to prove that for a semiprimitive pm-lattice $\mathcal{L}$ a point in $\mathrm{Min}(\mathcal{L})$ is quasi-isolated if and only if it is isolated.

A graph G is called *triangulated* if each vertex of G is a vertex of a triangle.

Theorem 3.18. [16, Theorem 3.6] *Let $\mathcal{L}$ be a distributive lattice with* 0. *Then* $\Gamma(\mathcal{L})$ *is a triangulated graph if and only if* $\mathrm{Spec}(\mathcal{L})$ *has no quasi-isolated points.*

Theorem 3.19. *Let $\mathcal{L}$ be a semiprimitive pm-lattice and* $|\mathrm{Min}(\mathcal{L})| \geq 3$. *Then the following assertions are equivalent.*

(1) $\Gamma(\mathcal{L})$ *is not triangulated and the center of* $\Gamma(\mathcal{L})$ *is a dominating set.*

(2) *The set of all isolated points of* $\mathrm{Spec}(\mathcal{L})$ *is a dense subset of* $\mathrm{Spec}(\mathcal{L})$.

Proof. (1) $\Rightarrow$ (2) As $\Gamma(\mathcal{L})$ is not a triangulated graph, by Theorem 3.18, there exists at least one quasi-isolated point P_0 in $\mathrm{Spec}(\mathcal{L})$. Now, by Remark 3.17 and Theorem 3.16, $D(a_0) = \{P_0\}$, and by Lemma 3.15, we have $|\overline{D(a_0)}| = 1$. Also, by Theorems 3.14 and 3.9, $e(a_0) = 2$. Thus if we denote the center of $\Gamma(\mathcal{L})$ by $\Gamma^{'}(\mathcal{L})$, we have $\Gamma^{'}(\mathcal{L}) = \{a \in \Gamma(\mathcal{L}) : |\overline{D(a)}| = 1\}$. We show that $Y = \{P^{'}_a : a \in \Gamma^{'}(\mathcal{L}), D(a) = \{P^{'}_a\}\}$ is dense in $\mathrm{Spec}(\mathcal{L})$. On contrary, suppose that U is a nonempty open subset of $\mathrm{Spec}(\mathcal{L})$ such that $U \cap Y = \emptyset$. By density of $\mathrm{Max}(\mathcal{L})$ in $\mathrm{Spec}(\mathcal{L})$, it can be shown that $U \cap \mathrm{Max}(\mathcal{L}) \neq \emptyset$ and $|U \cap \mathrm{Max}(\mathcal{L})| \geq 2$. Thus there are distinct maximal ideals $M, M^{'} \in U$ and elements x, y, b and $b^{'}$ such that $x \in P_0 \setminus M$, $y \in M^{'} \setminus M \cup P_0$, $b = x \wedge y \in P_0 \cap M^{'} \setminus M$ and $M \in D(b^{'}) \subseteq U$. Therefore, $M \in D(b) \cap D(b^{'}) = D(b \wedge b^{'}) \subseteq U$ and $P_0, M^{'} \notin D(b \wedge b^{'})$), by Lemma 3.7, there exists $c \in \Gamma(\mathcal{L})$ such that $M \in \mathrm{int}(V(c)) \subseteq D(b \wedge b^{'}) \subseteq U$. Consequently, $P_0, M^{'} \in \overline{D(c)}$, that is, $c \notin \Gamma^{'}(\mathcal{L})$. Since $\Gamma^{'}(\mathcal{L})$ is a dominating set, there exists an element $a \in \Gamma^{'}(\mathcal{L})$ with $a \wedge c = 0$. Hence

$P'_a \in D(a) \subseteq \text{int}(V(c)) \subseteq U$, that is, U contains an isolated point of $\text{Spec}(\mathcal{L})$, which is a contradiction. Therefore, $U \cap Y \neq \emptyset$, and then Y is dense in $\text{Spec}(\mathcal{L})$.

(2) $\Rightarrow$ (1) If $Y = \{P_\lambda : \lambda \in \Lambda\}$ is the set of isolated points of $\text{Spec}(\mathcal{L})$, then, by Theorem 3.18, $\Gamma(\mathcal{L})$ is not triangulated. Consider $D = \{a_\lambda \in \Gamma(\mathcal{L}) : D(a_\lambda) = \{P_\lambda\}\}$. By Theorem 3.9, $e(a) \geq 2$, for every $a \in \Gamma(\mathcal{L})$ and since P_λ is an isolated point in $\text{Spec}(\mathcal{L})$, then $|\overline{D(a_\lambda)}| = 1$, for every $\lambda \in \Lambda$. By Theorem 3.14, D is the center of $\Gamma(\mathcal{L})$. Now for every $b \in \Gamma(\mathcal{L}) \setminus D$, $\text{int}(V(b)) \neq \emptyset$ and by density of Y in $\text{Spec}(\mathcal{L})$, we can take a prime ideal $P_\lambda \in \text{int}(V(b)) \cap Y$. Therefore, $D(a_\lambda) \subseteq \text{int}(V(b))$, which implies that $a_\lambda \wedge b = 0$, that is, D is a dominating set. □

4 Some properties of the compressed zero-divisor graph $\Gamma_E(\mathcal{L})$

In this section we study and investigate some properties of the compressed zero-divisor graph of a 0-distributive lattice.

Let $\mathcal{L}$ be a $0-$distributive lattice. Then for every element $x \in \mathcal{L}$, $\text{ann}_{\mathcal{L}}(x) = \{y \in \mathcal{L} : x \wedge y = 0\}$ is an ideal of $\mathcal{L}$(see [4, Lemma 2.3]).

With a similar argument as in [14, Lemmas 2.2 and 2.4], it can be shown that:

Lemma 4.1. *Let $\mathcal{L}$ be a $0-$distributive lattice and $\omega(\Gamma(\mathcal{L})) < \infty$. Then every nonempty subset of $\Im = \{\text{ann}_{\mathcal{L}}(x) : x \in \mathcal{L}, x \neq 0\}$ has a maximal element (with respect to inclusion). Moreover, every maximal element of $\Im$ is a prime ideal of $\mathcal{L}$.*

Definition 4.2. Let $\mathcal{L}$ be a 0-distributive lattice. A prime ideal P of $\mathcal{L}$ is called an *associated prime ideal* of $\mathcal{L}$ if there exists an element $x \in Z(\mathcal{L})^\star$ such that $P = \text{ann}_{\mathcal{L}}(x)$.

The set of all associated prime ideals of $\mathcal{L}$ is denoted by $Ass(\mathcal{L})$. As we saw in Lemma 4.1, any maximal element of $\Im = \{\text{ann}_{\mathcal{L}}(x) : 0 \neq x \in \mathcal{L}\}$ is an associated prime ideal.

For two elements x and y in a bounded 0-distributive lattice $\mathcal{L}$, we say that $x \sim y$ if and only if $\text{ann}_{\mathcal{L}}(x) = \text{ann}_{\mathcal{L}}(y)$. In this way, $\sim$ defines an equivalance relation on $\mathcal{L}$. Furthermore, for elements $x_1, x_2, y \in \mathcal{L}$, if

$x_1 \sim x_2$ and $x_1 \wedge y = 0$, then $x_2 \wedge y = 0$. If $[x]$ denotes the equivalance class of x, then the meet $[x] \sqcap [y] = [x \wedge y]$ and the join $[x] \sqcup [y] = [x \vee y]$ make sense. This means that the operations meet and join are well-defined on the set $\mathcal{L}_E = \{[x] : x \in \mathcal{L}\}$ of all equivalence classes of $\sim$. Note that $[0] = \{0\}$ and $[1] = \mathcal{L} \setminus Z(\mathcal{L})$. If $\mathcal{L}$ is also a $0-$distributive lattice, then the equivalence relation $\sim$ is a congruence. Therefore, $\mathcal{L}_E$ is a $0-$distributive lattice.

Definition 4.3. The *compressed zero-divisor graph* of a bounded 0-distributive lattice $\mathcal{L}$, denoted by $\Gamma_E(\mathcal{L}) = \Gamma(\mathcal{L}_E)$, is a graph whose vertices are the elements of $Z(\mathcal{L}_E)^\star = \mathcal{L}_E \setminus \{[0], [1]\}$, and each pair of distinct classes $[x]$ and $[y]$ are joined by an edge if and only if $[x] \sqcap [y] = [0]$.

There is a natural injective map from $\mathrm{Ass}(\mathcal{L})$ to the vertex set of $\Gamma_E(\mathcal{L})$ given by $P \mapsto [y]$, where $P = \mathrm{ann}_{\mathcal{L}}(y)$.

In order to proceed further, we need the following two lemmas, which both of them are essentially similar to [24, Lemma 1.2].

Lemma 4.4. *Let $\mathcal{L}$ be a $0-$distributive lattice and $x, y \in \mathcal{L}$. If $\mathrm{ann}_{\mathcal{L}}(x)$ and $\mathrm{ann}_{\mathcal{L}}(y)$ are distinct prime ideals of $\mathcal{L}$, then $[x] \sqcap [y] = [0]$. In particular, any pair of distinct vertices in $\Gamma_E(\mathcal{L})$ which are corresponded to associated prime ideals of $\mathcal{L}$ are adjacent.*

The converse of Lemma 4.4 is not true. In fact, every pair of adjacent vertices in $\Gamma_E(\mathcal{L})$ need not be corresponded to associated prime ideals (see Example 4.17 and Remark 4.18). However, the following lemma shows that every vertex in $\Gamma_E(\mathcal{L})$ is some how related to an associated prime ideal of $\mathcal{L}$. Namely, with a similar argument as in [18, Theorem 7] it can be shown that:

Lemma 4.5. *Let $\mathcal{L}$ be a 0-distributive lattice and $\omega(\Gamma(\mathcal{L})) < \infty$. Then for every vertex $[v]$ of $\Gamma_E(\mathcal{L})$ either $\mathrm{ann}_{\mathcal{L}}(v)$ is an associated prime ideal or there exists a vertex $[x] \in V(\Gamma_E(\mathcal{L}))$ which is adjacent to $[v]$ and $\mathrm{ann}_{\mathcal{L}}(x)$ is a maximal element in $\Im$.*

Corollary 4.6. *Let $\mathcal{L}$ be a $0-$distributive lattice and $\omega(\Gamma(\mathcal{L})) < \infty$. Then the set $\{[y] : \mathrm{ann}_{\mathcal{L}}(y) \in Ass(\mathcal{L})\}$ is a dominating set for $\Gamma_E(\mathcal{L})$ and $|Ass(\mathcal{L})| \leq \omega(\Gamma(\mathcal{L})) < \infty$.*

Remark 4.7. Let $\mathcal{L}$ be a 0-distributive lattice and $\omega(\Gamma(\mathcal{L})) < \infty$. Then by Lemma 4.5, the set of vertices corresponded to maximal elements of $\Im$ is also a dominating set for $\Gamma_E(\mathcal{L})$.

In [24] for a commutative Noetherian ring with unit, it has been shown that the compressed zero-divisor graph $\Gamma_E(R)$ is connected and its diameter is at most 3. A similar result has been proven by Joshi, Waphare and Pourali in [18] for a meet semilattice with 0. With a similar argument for a 0-distributive lattice $\mathcal{L}$ we have:

Theorem 4.8. *Let $\mathcal{L}$ be a 0-distributive lattice and $\omega(\Gamma(\mathcal{L})) < \infty$. Then the compressed zero-divisor graph $\Gamma_E(\mathcal{L})$ is connected and* $\operatorname{diam}(\Gamma_E(\mathcal{L})) \leq 3$.

Theorem 4.9. *Let $\mathcal{L}$ be a 0-distributive lattice with $\omega(\Gamma(\mathcal{L})) < \infty$. If $|V(\Gamma_E(\mathcal{L}))| \geq 3$, then $\Gamma_E(\mathcal{L})$ can not be a complete graph.*

Proof. Using Lemma 4.1, $\Im$ contains a maximal element, say $\operatorname{ann}_{\mathcal{L}}(z)$. Now let $\Gamma_E(\mathcal{L})$ be a complete graph, $[x], [y]$, and $[z]$ be 3 distinct vertices in $\Gamma_E(\mathcal{L})$. By maximality, $\operatorname{ann}_{\mathcal{L}}(z)$ is not a subset of $\operatorname{ann}_{\mathcal{L}}(x)$ or $\operatorname{ann}_{\mathcal{L}}(y)$. Thus we can consider elements $a \in \operatorname{ann}_{\mathcal{L}}(z) \setminus \operatorname{ann}_{\mathcal{L}}(x)$ and $b \in \operatorname{ann}_{\mathcal{L}}(z) \setminus \operatorname{ann}_{\mathcal{L}}(y)$. Therefore, $(a \vee b) \wedge z = 0$ and thus $[a \vee b]$ is a vertex of $\Gamma_E(\mathcal{L})$, $[a \vee b] \notin \{[x], [y]\}$, and $[a \vee b] \sqcap [x] \neq [0]$ which contradicts the completeness of $\Gamma_E(\mathcal{L})$. □

For a vertex v of a simple graph G, the set of vertices which are adjacent to v is called the *neighborhood* of v and is denoted by $N_G(v)$ or $N(v)$.

Theorem 4.10. *Let $\mathcal{L}$ be a 0-distributive lattice with $\omega(\Gamma(\mathcal{L})) < \infty$ such that $\Gamma_E(\mathcal{L})$ is a complete r-partite graph. Then $r = 2$ and $\Gamma_E(\mathcal{L}) = K^{1,1}$.*

Proof. Let $\Gamma_E(\mathcal{L}) = K^{n_1,n_2,\ldots,n_r}$. By Theorem 4.9, if $|V(\Gamma_E(\mathcal{L}))| \geq 3$, then $r \neq 1$ and $n_i \geq 2$; for some $i \in \{1, \ldots, r\}$. Without loss of generality, we can assume that $n_1 > 1$ and there are two non-adjacent distinct vertices $[a_1]$ and $[a_2]$. Since a_1 and a_2 are not equivalent, there exists $z \in \operatorname{ann}_{\mathcal{L}}(a_1) \setminus \operatorname{ann}_{\mathcal{L}}(a_2)$. Therefore $[z] \sqcap [a_1] = [0]$ and $[z] \sqcap [a_2] \neq [0]$, whereas $[z] \neq [a_2]$. But since $\Gamma_E(\mathcal{L})$ is a complete r-partite graph, $[a_1]$ and $[a_2]$ are located in the same partition, $[a_1]$ and $[a_2]$ have the same set of neighbors and this is a contradiction. Therefore, $|V(\Gamma_E(\mathcal{L}))| = 2$ and $\Gamma_E(\mathcal{L}) = K^{1,1}$.

□

Theorem 4.11. *Let $\mathcal{L}$ be a 0-distributive lattice with $\omega(\Gamma(\mathcal{L})) < \infty$. Then $\Gamma_E(\mathcal{L})$ is not a cycle.*

Proof. By Theorems 4.10 and 4.9, $\Gamma_E(\mathcal{L})$ is not a cycle of length 3 or 4. It suffices to show that cycles in $\Gamma_E(\mathcal{L})$ with a path of length five are not possible. Suppose that $[x] - [y] - [z] - [\alpha] - [\beta]$ is a path in the graph with distinct vertices of degree 2. Then $y \wedge \alpha \neq 0$ and $z \in \operatorname{ann}_{\mathcal{L}}(y \wedge \alpha) \setminus (\operatorname{ann}_{\mathcal{L}}(x) \cup \operatorname{ann}_{\mathcal{L}}(z) \cup \operatorname{ann}_{\mathcal{L}}(\beta))$. Therefore, $[y \wedge \alpha]$ is a vertex different from $[x], [z]$ and $[\beta]$ and it is annihilated by $[x], [z], [\beta]$ and thus $\deg[y \wedge \alpha] \geq 3$ which is a contradiction. □

Lemma 4.12. *[24, Lemma 1.9] Let G be a simple finite graph with the property that two distinct vertices v and w of G are non-adjacent if and only if $N_G(v) = N_G(w)$. Then G is a complete r-partite graph for some $r \in \mathbb{N}$.*

Theorem 4.13. *Let $\mathcal{L}$ be a 0-distributive lattice with $\omega(\Gamma(\mathcal{L})) < \infty$ and $\Gamma_E(\mathcal{L})$ be a finite regular graph. Then $|V(\Gamma_E(\mathcal{L}))| \leq 2$.*

Proof. It is easy to see that regular graphs with 3 or 4 vertices are isomorphic to K^3, K^4 or $K^{2,2}$ respectively. Also a regular graph with 5 vertices is either a cycle or isomorphic to K^5. According to the above statements there is no regular graph $\Gamma_E(\mathcal{L})$ with 3, 4 or 5 vertices. Therefore, suppose that $\Gamma_E(\mathcal{L})$ is a regular graph of degree $d \geq 3$ with at least 6 vertices. Since $\Gamma_E(\mathcal{L})$ is not a complete graph, there are two non-adjacent distinct vertices $[y_1]$ and $[y_2]$. If $N([y_1]) \neq N([y_2])$, then without loss of generality, we may assume that there is a vertex $[u] \in N([y_1]) \setminus N([y_2])$ whereas $[u] \neq [y_2]$. Thus $u \wedge y_1 = 0, u \wedge y_2 \neq 0$. This implies that $[u] \in N([y_1 \wedge y_2])$ and so $N([y_2]) \neq N([y_1 \wedge y_2])$. But $N([y_2]) \subseteq N([y_1 \wedge y_2])$ and, since each set has cardinality d, we must have equality, which leads to a contradiction. Therefore, any two non-adjacent distinct vertices on the graph have the same neighborhood and clearly, the converse is true. Thus by Lemma 4.12 and Theorem 4.10, $\Gamma_E(\mathcal{L}) = K^{1,1}$, that leads to a contradiction. □

Theorem 4.14. *Let $\mathcal{L}$ be a* 0*-distributive lattice and $\{x, y\} \subseteq Z(\mathcal{L})^\star$. If $\operatorname{ann}_{\mathcal{L}}(x) \subset \operatorname{ann}_{\mathcal{L}}(y)$, then $N_{\Gamma_E(\mathcal{L})}([x]) \subseteq N_{\Gamma_E(\mathcal{L})}([y])$. In particular, when $\Gamma_E(\mathcal{L})$ is finite, we have* $\deg[x] < \deg[y]$.

Proof. If $[u] \in N([x])$, then $u \wedge x = 0$ and thus $u \wedge y = 0$. Now since $\mathrm{ann}_{\mathcal{L}}(x) \subset \mathrm{ann}_{\mathcal{L}}(y)$ and $y \neq 0$, we have $x \wedge y \neq 0$ and $0 \neq x \in \mathrm{ann}_{\mathcal{L}}(u) \setminus \mathrm{ann}_{\mathcal{L}}(y)$. Therefore, $[y] \neq [u] \in N([y])$ and thus $N_{\Gamma_E(\mathcal{L})}([x]) \subseteq N_{\Gamma_E(\mathcal{L})}([y])$. For the last part of statement, we note that there exists $z \in \mathrm{ann}_{\mathcal{L}}(y) \setminus \mathrm{ann}_{\mathcal{L}}(x)$, such that $[z] \in Z(\mathcal{L}_E)^{\star}$ and $[z] \in N([y]) \setminus N([x])$. □

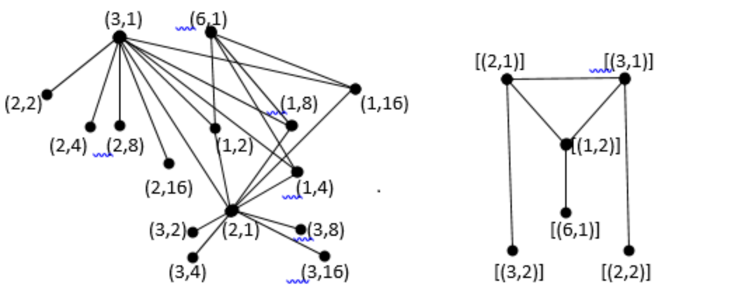
Corollary 4.15. *Let $\mathcal{L}$ be a 0-distributive lattice. If y is an element of $Z(\mathcal{L})^*$ such that $\deg[y] > \deg[x]$ for every $[x]$ in $\Gamma_E(\mathcal{L})$, then $\mathrm{ann}_{\mathcal{L}}(y)$ is maximal in $\Im$ and hence it is an associated prime ideal.*

Proof. Assume to the contrary that $\mathrm{ann}_{\mathcal{L}}(y) \subset \mathrm{ann}_{\mathcal{L}}(x)$ for some $x \in Z(\mathcal{L})^{\star}$. Then by Theorem 4.14, $\deg[x] \geq \deg[y]$, which is a contradiction. □

Corollary 4.16. *Let $\mathcal{L}$ be a 0-distributive lattice with $2 < |V(\Gamma_E(\mathcal{L}))| < \infty$. Then every vertex of maximum degree in $\Gamma_E(\mathcal{L})$ is corresponded to a maximal element in $\Im$.*

Proof. Let the maximum degree of $\Gamma_E(\mathcal{L})$ is $\Delta(\Gamma_E(\mathcal{L})) = d$. Then the connectedness of $\Gamma_E(\mathcal{L})$ implies that $d \geq 2$. Suppose that $[y_1]$ is a vertex of degree d, hence $\mathrm{ann}_{\mathcal{L}}(y_1) \neq \{0\}$. If $\mathrm{ann}_{\mathcal{L}}(y_1)$ is not maximal in $\Im$, then by Lemma 4.1, there exists $y_2 \in Z(\mathcal{L})^{\star}$ such that $\{0\} \neq \mathrm{ann}_{\mathcal{L}}(y_1) \subset \mathrm{ann}_{\mathcal{L}}(y_2)$ and $\mathrm{ann}_{\mathcal{L}}(y_2)$ is maximal in $\Im$. By Theorem 4.14, we have $d = \deg[y_1] < \deg[y_2] \leq d$, which is a contradiction. Therefore, $\mathrm{ann}_{\mathcal{L}}(y_1)$ is maximal in $\Im$ and it is an associated prime ideal. □

Example 4.17. The associated prime ideals of $\mathcal{L} = \mathcal{D}(6) \times \mathcal{D}(16)$ are $\mathrm{ann}_{\mathcal{L}}(1,2)$, $\mathrm{ann}_{\mathcal{L}}(2,1)$, and $\mathrm{ann}_{\mathcal{L}}(3,1)$.



(a) $\Gamma(D(6)\times D(16))$. (b) $\Gamma_E(D(6)\times D(16))$.

Fig. 1.

Remark 4.18. The vertices of a dominating set $\Gamma_E(\mathcal{L})$ are necessarily correspoded to associated prime ideals. As we can see in **Fig.1**, $\{[(6,1)],[(3,2)],[(2,2)]\}$ is a dominating set for $\Gamma_E(\mathcal{L})$ whereas none of the ideals $\mathrm{ann}_{\mathcal{L}}(6,1),\mathrm{ann}_{\mathcal{L}}(3,2)$, and $\mathrm{ann}_{\mathcal{L}}(2,2)$ are not associated prime ideals.

Theorem 4.19. *Let $\mathcal{L}$ be a 0-distributive lattice with $\omega(\Gamma(\mathcal{L}))<\infty$. Then vertex set of $\Gamma_E(\mathcal{L})$ is infinite if and only if there exists at least one maximal element $\mathrm{ann}_{\mathcal{L}}(x)$ in $\Im$ with $\deg[x]=\infty$.*

Proof. Clearly, if some vertex has infinite degree, then the vertex set of $\Gamma_E(\mathcal{L})$ is infinite. Conversely, suppose that the vertex set of $\Gamma_E(\mathcal{L})$ is infinite. If $\Im$ has infinite number of maximal elements, then, by Lemmas 4.1 and 4.4 the assertion holds immediately. Otherwise, let $\{\mathrm{ann}_{\mathcal{L}}(x_1),...,\mathrm{ann}_{\mathcal{L}}(x_r)\}$ be the set of all maximal elements in $\Im$. Thus the assertion holds, by Remark 4.7. □

Definition 4.20. A *leaf* in a graph is a vertex of degree 1.

Theorem 4.21. *If $\mathcal{L}$ is a 0-distributive lattice with $\omega(\Gamma(\mathcal{L}))<\infty$ such that $|V(\Gamma_E(\mathcal{L}))|>3$, then there are no leaf vertices in $\Gamma_E(\mathcal{L})$ corresponded to the associated prime ideals of $\mathcal{L}$.*

Proof. Suppose that $\mathrm{ann}_{\mathcal{L}}(y)$ is an associated prime ideal and $\deg([y])=1$. Then there is one and only one class $[x]$ such that $[x]\neq[y]$ and $[x]\sqcap[y]=[0]$. By Corollary 4.16, $\{\mathrm{ann}_{\mathcal{L}}(y),\mathrm{ann}_{\mathcal{L}}(x)\}\subseteq Ass(\mathcal{L})$. There are at least two other vertices which are connected to $[x]$. If $[z]$ is another vertex and $\deg[z]\geq 2$, then for some $[\omega]$, $z\wedge\omega=0\in\mathrm{ann}_{\mathcal{L}}(y)$; hence $z\in\mathrm{ann}_{\mathcal{L}}(y)$ or $\omega\in\mathrm{ann}_{\mathcal{L}}(y)$. Since this is not possible, it must be that $\deg[z]=\deg[\omega]=1$. Now, since $\mathrm{ann}_{\mathcal{L}}(z)=\mathrm{ann}_{\mathcal{L}}(\omega)=[x]$, and $[z]=[\omega]$ yields a contradiction. □

As we saw in Theorem 3.1, the clique number of the zero-divisor graph $\Gamma(\mathcal{L})$ is equal to the cellularity of the topological space $\mathrm{Spec}(\mathcal{L})$, which is also equal to the Goldie dimension of $\mathcal{L}$. Combining this with [5, Corollary 3.2], we obtain a similar result for the lattice $\mathcal{L}_E$ of equivalance classes of $\mathcal{L}$:

Corollary 4.22. *For a distributive lattice $\mathcal{L}$ with 0, we have*

$$\omega(\Gamma(\mathcal{L}))=\omega(\Gamma_E(\mathcal{L}))=c(\mathrm{Spec}(\mathcal{L}_E))=\mathrm{Gdim}(\mathcal{L}_E).$$

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