Categories and
General Algebraic Structures
with Applications

Volume 14, Number 1, January 2021, 119-165.
https://doi.org/10.29252/cgasa.14.1.119

Schneider-Teitelbaum duality for locally profinite groups

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Abstract. We define monoidal structures on several categories of linear topological modules over the valuation ring of a local field, and study module theory with respect to the monoidal structures. We extend the notion of the Iwasawa algebra to a locally profinite group as a monoid with respect to one of the monoidal structure, which does not necessarily form a topological algebra. This is one of the main reasons why we need monoidal structures. We extend Schneider–Teitelbaum duality to duality applicable to a locally profinite group through the module theory over the generalised Iwasawa algebra, and give a criterion of the irreducibility of a unitary Banach representation.

1 Introduction

Let k denote a non-Archimedean local field, and $O_k \subset k$ the valuation ring of k. The paper is devoted to two topics. One topic is to give monoidal structures on several categories of linear topological O_k-modules. We are interested mainly in the closed symmetric monoidal category $\mathscr{C}_\ell^{\mathrm{cg}}$ of CG

Keywords: Iwasawa theory, p-adic, locally profinite group.
Mathematics Subject Classification [2010]: 11S80, 28C15, 16T05.
Received: 26 July 2019, Accepted: 29 September 2019.
ISSN: Print 2345-5853, Online 2345-5861.

linear topological O_k-modules. A CG linear topological O_k-module is a linear topological O_k-module given as the colimit of totally bounded O_k-submodules. By the definition, it is a module theoretic analogue of a compactly generated topological space. We show that every Banach k-vector space and every compact linear topological O_k-module are CG. Therefore $\mathscr{C}_\ell^{\mathrm{cg}}$ contains both of the categories of Banach k-vector spaces and compact Hausdorff flat linear topological O_k-modules, which play the roles of the foundation in Schneider–Teitelbaum duality (cf. [13] Theorem 2.3).

The other topic is to define a generalised Iwasawa algebra $O_k[[G]]$ associated to a locally profinite group G, and to extend Schneider–Teitelbaum duality, which is applicable to a profinite group, to duality applicable to G by using module theory over $O_k[[G]]$. We note that $O_k[[G]]$ is defined as a monoid in $\mathscr{C}_\ell^{\mathrm{cg}}$, and does not necessarily form a topological O_k-algebra. This is one of the main reasons why we need monoidal structures. As the classical Iwasawa algebra associated to a profinite group is naturally identified with the O_k-algebra of O_k-valued measures, $O_k[[G]]$ is naturally identified with the O_k-algebra of O_k-valued measures on G satisfying a certain property called the normality. As the original Schneider–Teitelbaum duality is given by a module theoretic interpretation of a Banach k-linear representations through the integration of the action along measures (cf. [13] Corollary 2.2), the generalised Schneider–Teitelbaum duality is give by a module theoretic interpretation through the integration of the action of G by normal measures.

As applications, we establish a criterion of the irreducibility of a unitary Banach k-linear representation of G, and give a description of the continuous induction of a unitary Banach k-linear representation of a closed subgroup $P \subset G$ such that the homogeneous space $P \backslash G$ is compact. In particular, we give an explicit description of the continuous parabolic induction for the case G is an algebraic group over a local field so that the representation space of the continuous parabolic induction is independent of the choice of the action of P.

We explain the contents of this paper. In §2.1, we study several categories of linear topological O_k-modules. In §2.2, we introduce a notion of the normality of an O_k-valued measure on a topological space. In §3.1, we define monoidal structures on several categories of linear topological O_k-modules. In §3.2, we define a notion of a CGLT O_k-algebra as a monoid in

$\mathscr{C}_{\ell}^{\mathrm{cg}}$, which is a counterpart of a topological O_k-algebra, and define $O_k[[G]]$ as a CGLT O_k-algebra. In §3.3, we define a notion of a CGLT module over a CGLT O_k-algebra, which is a counterpart of a topological left module over a topological O_k-algebra. In §4.1, we recall a unitary Banach k-linear representation of G and interpret it in terms of a CGLT $O_k[[G]]$-module. In §4.2, we interpret a continuous action of G on a compact Hausdorff flat linear topological O_k-module in terms of a CGLT $O_k[[G]]$-module. In §4.3, we define a notion of the dual of a unitary Banach k-linear representation of G, and extend Schneider–Teitelbaum duality to duality applicable to G. In §5.1, we study the dual of several operations on Banach k-linear representations such as the continuous induction. In §5.2, we give an explicit description of the continuous parabolic induction in the case where G is an algebraic group.

2 Preliminaries

Let k denote a local field, that is, a complete discrete valuation field with finite residue field, $O_k \subset k$ the valuation ring of k, and G a locally profinite group. We denote by ω the set of natural numbers. For a set X, we denote by $\mathscr{P}_{<\omega}(X)$ the set of finite subsets of X. Since we deal with many pairs, we abbreviate $(\bullet_i)_{i=0}^{1}$ to $(\bullet_i)$, $\sum_{i=0}^{1} \bullet_i$ to $\sum \bullet_i$, and $\prod_{i=0}^{1} \bullet_i$ to $\prod \bullet_i$.

Let Θ be a category. We say that Θ is *ω-cocomplete* (respectively, *cocomplete*, *complete*) if it admits all small filtered colimits (respectively, colimits, limits), and is *bicomplete* if it is cocomplete and complete. Let F be a functor. We say that F is *ω-cocontinuous* (respectively, *cocontinuous*, *continuous*) if it commutes with all small filtered colimits (respectively, colimits, limits), and is *bicontinuous* if it is cocontinuous and continuous. We denote by Set the bicomplete category of sets and maps, and by Top the bicomplete category of topological spaces and continuous maps. We abbreviate $\mathrm{Hom}_{\mathrm{Top}}$ to C.

2.1 Linear topological modules

Let M be a topological O_k-module, and $C \subset M$ a subset. We say that C is *pre-compact* (respectively, *complete*) if C is totally bounded (respectively, complete) with respect to the restriction of the uniform structure on M associated to the structure as a topological Abelian group to C. By the definition of the uniformity on M, C is totally bounded if and only if for any open neighbourhood $U \subset M$ of $0 \in M$, there exists a finite subset $C_0 \subset C$ such that $C \subset \{m_0 + m_1 \mid (m_0, m_1) \in U \times C_0\}$. The following are well-known facts (cf. [3] 8.3.2 Theorem, [4], and [3] 8.3.16 Theorem, respectively) on the pre-compactness:

Proposition 2.1. (i) *A $C \subset M$ is pre-compact if and only if every subset of the closure of C in M is pre-compact.*

(ii) *A $C \subset M$ is compact, that is, every open covering admits a finite subcovering, if and only if C is pre-compact and every Cauchy net in C is a convergent net in C.*

(iii) *A $C \subset M$ is compact and Hausdorff if and only if C is pre-compact and complete.*

We denote by $\mathscr{O}(M)$ the set of open O_k-submodules of M, and by $\mathscr{K}(M)$ the set of pre-compact O_k-submodules of M. We say that M is *linear* if $\mathscr{O}(M)$ forms a fundamental system of neighbourhoods of $0 \in M$. We have two examples of linear topological O_k-modules.

Example 2.2. (i) We denote by $\overline{M}$ the underlying O_k-module of M equipped with the topology generated by $\{m + L \mid (m, L) \in M \times \mathscr{O}(M), \#(M/L) < \infty\}$. Then $\overline{M}$ forms a pre-compact linear topological O_k-module, and the identity map $\pi^{\mathrm{c}}_M \colon M \to \overline{M}$ is continuous.

(ii) Let S be a set. A map $f \colon S \to M$ is said to *vanish at infinity* if for any $L \in \mathscr{O}(M)$, there is an $S_0 \in \mathscr{P}_{<\omega}(S)$ such that $f(s) \in L$ for any $s \in S \setminus S_0$. We denote by $\mathrm{C}_0(S, M)$ the O_k-module of maps $f \colon S \to M$ vanishing at infinity equipped with the topology generated by $\{f + \mathrm{C}_0(S, L) \mid (f, L) \in \mathrm{C}_0(S, M) \times \mathscr{O}(M)\}$. Then $\mathrm{C}_0(S, M)$ forms a linear topological O_k-module.

We denote by $\mathscr{C}_\ell$ the O_k-linear category of linear topological O_k-modules and continuous O_k-linear homomorphisms. We abbreviate $\mathrm{Hom}_{\mathscr{C}_\ell}$ to $\mathscr{L}$. Since the pre-image of an open O_k-submodule by a continuous O_k-linear homomorphism is an open O_k-submodule, the correspondence $M \rightsquigarrow \mathscr{O}(M)$ gives a functor $\mathscr{O} \colon \mathscr{C}_\ell^{\mathrm{op}} \to \mathrm{Set}$. On the other hand, the correspondence $M \rightsquigarrow \mathscr{K}(M)$ gives a functor $\mathscr{K} \colon \mathscr{C}_\ell \to \mathrm{Set}$ by the following:

Proposition 2.3. *Let $(M_i) \in \mathrm{ob}(\mathscr{C}_\ell^2)$ and $f \in \mathscr{L}((M_i))$. For any pre-compact subset $C_0 \subset M_0$, $f(C_0) \subset M_1$ is pre-compact.*

Proof. The assertion follows from [3] p. 445 by the uniform continuity of f. □

We will use $\mathscr{O}(M)$ and $\mathscr{K}(M)$ as index sets of limits and colimits. They are filtered and cofiltered with respect to inclusions by Proposition 2.1 (i) and the following:

Proposition 2.4. *The sets $\mathscr{O}(M)$ and $\mathscr{K}(M)$ are closed under finite sum.*

Proof. The assertion for $\mathscr{O}(M)$ immediately follows from [3] p. 433. The assertion for $\mathscr{K}(M)$ immediately follows from Proposition 2.3 and [3] 8.3.3 Theorem, because $\sum M_j$ is the image of the addition $\prod M_i \to M$ for any $(M_i) \in \mathscr{K}(M)^2$. □

As a consequence, we obtain the following variant of [13] Lemma 1.5 i:

Corollary 2.5. *For any pre-compact subset $C \subset M$, $\sum_{m \in C} O_k m$ is pre-compact.*

Proof. Let $L \in \mathscr{O}(M)$. Take a $C_0 \in \mathscr{P}_{<\omega}(C)$ satisfying $C \subset \bigcup_{m \in C_0}(m+L)$. We have $O_k m \in \mathscr{K}(M)$ for any $m \in M$ by Proposition 2.3, and hence $\sum_{m \in C_0} O_k m \in \mathscr{K}(M)$ by Proposition 2.4. Take a $K_0 \in \mathscr{P}_{<\omega}(\sum_{m \in C_0} O_k m)$ satisfying $\sum_{m \in C_0} O_k m \subset \bigcup_{m \in K_0}(m+L)$. We obtain

$$\sum_{m \in C} O_k m \subset \bigcup_{m \in C_0} O_k(m+L) = \bigcup_{m \in C_0} (O_k m + L) \subset \bigcup_{m \in K_0} (m+L).$$

It implies $\sum_{m \in C} O_k m \in \mathscr{K}(M)$. □

We denote by $\mathscr{C}$ the category of O_k-modules and O_k-linear homomorphisms. We denote by $\mathscr{U} : \mathscr{C}_\ell \to \mathrm{Top}$ and $\mathscr{F} : \mathscr{C}_\ell \to \mathscr{C}$ the forgetful functors.

Proposition 2.6. *The category $\mathscr{C}_\ell$ is bicomplete, and $\mathscr{U}$ (respectively, $\mathscr{F}$) is ω-cocontinuous and continuous (respectively, bicontinuous).*

Proof. The completeness of $\mathscr{C}_\ell$ and the continuity of $\mathscr{U}$ and $\mathscr{F}$ follow from the definition of the limits in Top and $\mathscr{C}$. The ω-cocomleteness of $\mathscr{C}_\ell$ and the ω-cocontinuity of $\mathscr{U}$ and $\mathscr{F}$ follow from [6] Proposition 1.3. For any small family $(M_s)_{s \in S}$ in $\mathscr{C}_\ell$, $\bigoplus_{s \in S} \mathscr{F}(M_s)$ forms a linear topological O_k-module with respect to the topology generated by $\{m + \bigoplus_{s \in S} \mathscr{F}(L_s) \mid (m, (L_s)_{s \in S}) \in (\bigoplus_{s \in S} \mathscr{F}(M_s)) \times \prod_{s \in S} \mathscr{O}(M_s)\}$, and satisfies the universality of the direct sum of $(M_s)_{s \in S}$ in $\mathscr{C}_\ell$. Thus $\mathscr{C}_\ell$ is cocomplete, and $\mathscr{F}$ is cocontinuous. □

Since we will introduce several full subcategories of $\mathscr{C}_\ell$, we prepare a convention for colimits (respectively, limits) in order to avoid the ambiguity of categories in which we consider the universality. Let $(M_s)_{s \in S}$ be a small diagram in a full subcategory $\Theta \subset \mathscr{C}_\ell$. We always denote by $\varinjlim_{s \in S} M_s$ (respectively, $\varprojlim_{s \in S} M_s$) the colimit (respectively, limit) of $(M_s)_{s \in S}$ in $\mathscr{C}_\ell$ but not in Θ. As an immediate consequence of Proposition 2.6, we obtain the following:

Corollary 2.7. *Let $(M_s)_{s \in S}$ be a small diagram in $\mathscr{C}_\ell$. For any subset $U \subset \varinjlim_{s \in S} M_s$ (respectively, $U \subset \varprojlim_{s \in S} M_i$), U is open if and only if the preimage of U in M_s is open for any $s \in S$ (respectively, if and only if for any $m \in U$, there is an $(L_s)_{s \in S} \in \prod_{s \in S} \mathscr{O}(M_s)$ satisfying $\{s \in S \mid L_s \neq M_s\} \in \mathscr{P}_{<\omega}(S)$ and $m + \prod_{s \in S} \mathscr{F}(L_s) \subset U$.*

We denote by $\mathscr{C}_\ell^{\mathrm{c}} \subset \mathscr{C}_\ell$ the full subcategory of pre-compact linear topological O_k-modules and by $\mathscr{I}^{\mathrm{c}}$ the inclusion $\mathscr{C}_\ell^{\mathrm{c}} \hookrightarrow \mathscr{C}_\ell$. We put $\mathscr{U}^{\mathrm{c}} := \mathscr{U} \circ \mathscr{I}^{\mathrm{c}}$ and $\mathscr{F}^{\mathrm{c}} := \mathscr{F} \circ \mathscr{I}^{\mathrm{c}}$.

Proposition 2.8. (i) *The correspondence $M \rightsquigarrow \overline{M}$ gives a functor $\overline{(\bullet)} \colon \mathscr{C}_\ell \to \mathscr{C}_\ell^{\mathrm{c}}$ left adjoint to $\mathscr{I}^{\mathrm{c}}$ such that the counit is given as a natural equivalence.*

(ii) *The topological O_k-module M is linear and pre-compact if and only if π_M^{c} is an open map.*

(iii) *The category $\mathscr{C}_\ell^{\mathrm{c}}$ is bicomplete, and the colimit of a small diagram $(M_s)_{s \in S}$ in $\mathscr{C}_\ell^{\mathrm{c}}$ is given by $\overline{\varinjlim_{s \in S} \mathscr{I}^{\mathrm{c}}(M_s)}$.*

Proof. The functoriality of $\overline{(\bullet)}$ and the assertion (ii) immediately follow from the definition. The assertion (iii) immediately follows from the assertion (i) and Proposition 2.6. We show the assertion (i). We consider two functors $F, G \colon \mathscr{C}_\ell^{\mathrm{op}} \times \mathscr{C}_\ell^{\mathrm{c}} \to \mathrm{Set}$ given as $F := \mathscr{L}(\overline{\bullet_0}, \bullet_1)$ and $G := \mathscr{L}(\bullet_0, \mathscr{I}^{\mathrm{c}}(\bullet_1))$. The correspondence $M \rightsquigarrow \pi_M^{\mathrm{c}}$ gives a unit $\pi^{\mathrm{c}} \colon \mathrm{id}_{\mathscr{C}_\ell} \Rightarrow \mathscr{I}^{\mathrm{c}} \circ \overline{(\bullet)}$. We have a counit $(\pi_{\mathscr{I}^{\mathrm{c}}}^{\mathrm{c}})^{-1} \colon \overline{(\bullet)} \circ \mathscr{I}^{\mathrm{c}} \Rightarrow \mathrm{id}_{\mathscr{C}_\ell^{\mathrm{c}}}$, which is a natural equivalence by the assertion (ii). For a $K \in \mathrm{ob}(\mathscr{C}_\ell^{\mathrm{c}})$, we consider maps $T_{M,K} \colon F(M,K) \to G(M,K)$, $f \mapsto f \circ \pi_M^{\mathrm{c}}$ and $T'_{M,K} \colon G(M,K) \to F(M,K)$, $f \mapsto (\pi_{\mathscr{I}^{\mathrm{c}}(K)}^{\mathrm{c}})^{-1} \circ \overline{f}$. The correspondences $(M,K) \rightsquigarrow T_{M,K}, T'_{M,K}$ give natural transformations $T \colon F \Rightarrow G$ and $T' \colon G \Rightarrow F$ satisfying $T \circ T' = \mathrm{id}_G$ and $T' \circ T = \mathrm{id}_F$ by the bijectivity of of values of π^{c}. We obtain adjunction data $(\overline{(\bullet)}, \mathscr{I}^{\mathrm{c}}, T, \pi^{\mathrm{c}}, (\pi_{\mathscr{I}^{\mathrm{c}}}^{\mathrm{c}})^{-1})$ between $\mathscr{C}_\ell^{\mathrm{c}}$ and $\mathscr{C}_\ell$. It implies that $\overline{(\bullet)}$ is left adjoint to $\mathscr{I}^{\mathrm{c}}$. □

Suppose that M is linear in the following in this subsection. Then $\mathscr{K}(M)$ forms a small filtered diagram in $\mathscr{C}_\ell$ by Proposition 2.4. We put $M_{\mathscr{K}} := \varinjlim_{K \in \mathscr{K}(M)} K$. By the universality of the colimit, the system of inclusions induces a continuous injective O_k-linear homomorphism $\iota_M^{\mathrm{cg}} \colon M_{\mathscr{K}} \to M$. By Corollary 2.5, ι_M^{cg} is bijective. We show that ι_M^{cg} preserves the pre-compactness of O_k-submodules.

Proposition 2.9. *Let $K \subset M$ be an O_k-submodule of M. Put $K' := (\iota_M^{\mathrm{cg}})^{-1}(K)$.*

(i) *If K is pre-compact, then $\iota_M^{\mathrm{cg}}|_{K'}$ is a homeomorphism onto K.*

(ii) *The pre-compactness of K is equivalent to that of K'.*

Proof. The assertion (ii) follows from Proposition 2.3 and the assertion (i). We show the assertion (i). By $K \in \mathscr{K}(M)$, we have $\iota_M^{\mathrm{cg}}(K') = K$. Let $L \in \mathscr{O}(M_{\mathscr{K}})$. By $\iota_M^{\mathrm{cg}}(K') = K$ and the injectivity of ι_M^{cg}, we have $\iota_M^{\mathrm{cg}}(L \cap K') = \iota_M^{\mathrm{cg}}(L) \cap K$, and hence $\iota_M^{\mathrm{cg}}(L \cap K') \in \mathscr{O}(K)$. It implies that $\iota_M^{\mathrm{cg}}|_{K'}$ is an open map onto K. □

We say that M is *CG* if ι_M^{cg} is an isomorphism in $\mathscr{C}_\ell$. We denote by $\mathscr{C}_\ell^{\mathrm{cg}} \subset \mathscr{C}_\ell$ the full subcategory of CG linear topological O_k-modules and by $\mathscr{I}^{\mathrm{cg}}$ the inclusion $\mathscr{C}_\ell^{\mathrm{cg}} \hookrightarrow \mathscr{C}_\ell$. We put $\mathscr{U}^{\mathrm{cg}} := \mathscr{U} \circ \mathscr{I}^{\mathrm{cg}}$ and $\mathscr{F}^{\mathrm{cg}} := \mathscr{F} \circ \mathscr{I}^{\mathrm{cg}}$. We study properties of $\mathscr{C}_\ell^{\mathrm{cg}}$ analogous to those of the category of compactly generated topological spaces.

Corollary 2.10. (i) *The correspondence $M \rightsquigarrow M_{\mathscr{K}}$ gives a functor $(\bullet)_{\mathscr{K}} \colon \mathscr{C}_\ell \to \mathscr{C}_\ell^{\mathrm{cg}}$ right adjoint to $\mathscr{I}^{\mathrm{cg}}$ such that the counit is given as a natural equivalence.*

(ii) *The category $\mathscr{C}_\ell^{\mathrm{cg}}$ is bicomplete, and the colimit of a small diagram $(M_s)_{s \in S}$ in $\mathscr{C}_\ell^{\mathrm{cg}}$ is given by $(\varinjlim_{s \in S} \mathscr{I}^{\mathrm{cg}}(M_s))_{\mathscr{K}}$.*

Proof. To begin with, we show that $\mathscr{C}_\ell^{\mathrm{cg}}$ is closed under small colimits in $\mathscr{C}_\ell$. Let $(M_s)_{s \in S}$ be a small diagram in $\mathscr{C}_\ell^{\mathrm{cg}}$. Put $M := \varinjlim_{s \in S} \mathscr{I}^{\mathrm{cg}}(M_s)$. In order to verify that M is pre-compactly generated, it suffices to show $\iota_M^{\mathrm{cg}}(L) \in \mathscr{O}(M)$ for any $L \in \mathscr{O}(M_{\mathscr{K}})$. Let $s \in S$. We denote by L_s the preimage of $\iota_M^{\mathrm{cg}}(L)$ in M_s. Let $K_0 \in \mathscr{K}(M_s)$. We denote by $K \subset M$ the image of K_0. By Proposition 2.3 and Proposition 2.9 (ii), we have $(\iota_M^{\mathrm{cg}})^{-1}(K) \in \mathscr{K}(M_{\mathscr{K}})$. It ensures $L \cap (\iota_M^{\mathrm{cg}})^{-1}(K) \in \mathscr{O}((\iota_M^{\mathrm{cg}})^{-1}(K))$. By Proposition 2.9 (i), we obtain

$\iota_M^{\mathrm{cg}}(L) \cap K \in \mathscr{O}(K)$ and hence $L_s \cap K_0 \in \mathscr{O}(K_0)$. It ensures $L_s \in \mathscr{O}(M_s)$ because M_s is CG. It implies $\iota_M^{\mathrm{cg}}(L) \in \mathscr{O}(M)$ by Corollary 2.7.

We show the assertion (i). Since $\mathscr{C}_\ell^{\mathrm{cg}}$ is closed under small colimits in $\mathscr{C}_\ell$, the correspondence $M \rightsquigarrow M_{\mathscr{K}}$ gives a functor $(\bullet)_{\mathscr{K}} \colon \mathscr{C}_\ell \to \mathscr{C}_\ell^{\mathrm{cg}}$ by Proposition 2.3 and Proposition 2.9 (i). We consider two functors $F, G \colon (\mathscr{C}_\ell^{\mathrm{cg}})^{\mathrm{op}} \times \mathscr{C}_\ell \to \mathrm{Set}$ given as $F := \mathscr{L}(\mathscr{I}^{\mathrm{cg}}(\bullet_0), \bullet_1)$ and $G := \mathscr{L}(\bullet_0, (\bullet_1)_{\mathscr{K}})$. The correspondence $M \rightsquigarrow \iota_M^{\mathrm{cg}}$ gives a unit $\iota^{\mathrm{cg}} \colon \mathscr{I}^{\mathrm{cg}} \circ (\bullet)_{\mathscr{K}} \Rightarrow \mathrm{id}_{\mathscr{C}_\ell}$, and we also have a counit $(\iota_{\mathscr{I}^{\mathrm{cg}}}^{\mathrm{cg}})^{-1} \colon \mathrm{id}_{\mathscr{C}_\ell^{\mathrm{cg}}} \Rightarrow (\bullet)_{\mathscr{K}} \circ \mathscr{I}^{\mathrm{cg}}$, which is a natural equivalence by definition. For an $(M_i) \in \mathrm{ob}(\mathscr{C}_\ell^{\mathrm{cg}} \times \mathscr{C}_\ell)$, we consider maps $T_{(M_i)} \colon F((M_i)) \to G((M_i))$, $f \mapsto f_{\mathscr{K}} \circ (\iota_{\mathscr{I}^{\mathrm{cg}}(M_0)}^{\mathrm{cg}})^{-1}$ and $T'_{(M_i)} \colon G((M_i)) \to F((M_i))$, $f \mapsto \iota_{M_1}^{\mathrm{cg}} \circ f$. The correspondences $(M_i) \rightsquigarrow T_{(M_i)}, T'_{(M_i)}$ give natural transformations $T \colon F \Rightarrow G$ and $T' \colon G \Rightarrow F$ satisfying $T \circ T' = \mathrm{id}_G$ and $T' \circ T = \mathrm{id}_F$ by the bijectivity of values of ι. We obtain adjunction data $(\mathscr{I}^{\mathrm{cg}}, (\bullet)_{\mathscr{K}}, T, \iota^{\mathrm{cg}}, (\iota_{\mathscr{I}^{\mathrm{cg}}}^{\mathrm{cg}})^{-1})$ between $\mathscr{C}_\ell$ and $\mathscr{C}_\ell^{\mathrm{cg}}$. It implies that $(\bullet)_{\mathscr{K}}$ is right adjoint to $\mathscr{I}^{\mathrm{cg}}$.

We show the assertion (ii). By the assertion (i), $(\bullet)_{\mathscr{K}}$ is continuous and $\mathscr{I}^{\mathrm{cg}}$ is cocontinuous. Since the counit $(\iota_{\mathscr{I}^{\mathrm{cg}}}^{\mathrm{cg}})^{-1}$ is a natural equivalence, $\mathscr{C}_\ell^{\mathrm{cg}}$ is complete by Proposition 2.6. Since we have already verified that $\mathscr{C}_\ell^{\mathrm{cg}}$ is closed under small colimits in $\mathscr{C}_\ell$, it implies the assertion (ii) by Proposition 2.6 □

We have three criteria of CG linear topological O_k-modules.

Proposition 2.11. (i) *If M is CG, then so is every closed O_k-submodule of M.*

(ii) *If M is locally compact, then M is CG.*

(iii) *If M is first countable, then M is CG.*

Proof. The assertion (ii) follows from Proposition 2.1 (ii) and Proposition 2.9 (i), because M is locally compact if and only if M admits a compact clopen O_k-submodule. We verify the assertion (i). Let $M_0 \subset M$ be a closed O_k-submodule. Since ι_M^{cg} is an isomorphism in $\mathscr{C}_\ell$, $(\iota_M^{\mathrm{cg}})^{-1}(M_0)$ is closed in $M_{\mathscr{K}}$. Therefore ι_M^{cg} induces a homeomorphism $\varinjlim_{K \in \mathscr{K}(M)}(\mathscr{U}^{\mathrm{c}}(K) \cap \mathscr{U}(M_0)) \to \mathscr{U}(M_0)$ by [6] Lemma 2.23. By Corollary 2.7, we obtain an isomorphism $\varinjlim_{K \in \mathscr{K}(M)}(K \cap M_0) \to M_0$. By Proposition 2.1 (i), $K \cap M_0$ lies in $\mathscr{K}(M_0)$ for any $K \in \mathscr{K}(M)$. It implies that M_0 is CG by Corollary 2.10 (i).

We verify the assertion (iii). Let $L \in \mathscr{O}(M_{\mathscr{K}})$. We show $\iota_M^{\mathrm{cg}}(L) \in \mathscr{O}(M)$. Assume $\iota_M^{\mathrm{cg}}(L) \notin \mathscr{O}(M)$. Take an decreasing sequence $(L_r)_{r\in\omega} \in \mathscr{O}(M)^\omega$ such that $\{L_r \mid r \in \omega\}$ forms a fundamental system of neighbourhoods of $0 \in M$. By the assumption, we have $L_r \setminus \iota_M^{\mathrm{cg}}(L) \neq \emptyset$ for any $r \in \omega$. Take an $(m_r)_{r\in\omega} \in \prod_{r\in\omega}(L_r \setminus \iota_M^{\mathrm{cg}}(L))$. Put $C := \{m_r \mid r \in \omega\}$. We have $C = \bigcup_{h=0}^{r}(m_h + L_r)$ for any $r \in \omega$, and hence C is pre-compact. Put $K := \sum_{m\in C} O_k m \subset M$. By Corollary 2.5, we have $K \in \mathscr{K}(M)$. It ensures $\iota_M^{\mathrm{cg}}(L) \cap K \in \mathscr{O}(K)$. By $0 \in \iota_M^{\mathrm{cg}}(L) \cap K$, there is an $r \in \omega$ such that $L_r \cap K \subset \iota_M^{\mathrm{cg}}(L) \cap K$. We obtain $m_r \in L_r \cap K \subset \iota_M^{\mathrm{cg}}(L) \cap K$, which contradicts $m_r \notin \iota_M^{\mathrm{cg}}(L)$. It implies $\iota_M^{\mathrm{cg}}(L) \in \mathscr{O}(M)$. Thus M is CG. □

We survey Schikhof duality (cf. [10] Theorem 4.6, [13] Theorem 1.2, and [7] Theorem 2.2). We follow the convention of Banach k-vector space in [7] §1.2. We denote by $\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}} \subset \mathscr{C}_\ell$ the full subcategory of compact Hausdorff flat linear topological O_k-modules, by $\mathrm{Ban}(k)$ the k-linear category of Banach k-vector spaces and bounded k-linear homomorphisms, by $\mathrm{Ban}_{\leq}(k) \subset \mathrm{Ban}(k)$ the O_k-linear subcategory of submetric k-linear homomorphisms, and by $\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k) \subset \mathrm{Ban}_{\leq}(k)$ the full subcategory of unramified Banach k-vector spaces. By Proposition 2.1 (ii), $\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}}$ is a full subcategory of $\mathscr{C}_\ell^{\mathrm{c}}$. For a $(V_i) \in \mathrm{ob}(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k))^2$, we denote by $\mathscr{S}((V_i))$ the O_k-module $\mathrm{Hom}_{\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)}((V_i))$ equipped with the topology of pointwise convergence. For a $V \in \mathrm{ob}(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k))$, we put $V^{\mathrm{D_d}} := \mathscr{S}(V, k)$. For a $K \in \mathrm{ob}(\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}})$, we denote by $K^{\mathrm{D_c}}$ the k-vector space $\mathscr{L}(K, k)$ equipped with the supremum norm. The correspondence $V \rightsquigarrow V^{\mathrm{D_d}}$ gives a functor $\mathrm{D_d} \colon \mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)^{\mathrm{op}} \to \mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}}$, and the correspondence $K \rightsquigarrow K^{\mathrm{D_c}}$ gives a functor $\mathrm{D_c} \colon \mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}} \to \mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)^{\mathrm{op}}$.

Theorem 2.12 (Schikhof duality)**.** *The pair* $(\mathrm{D_d}, \mathrm{D_c})$ *is an* O_k*-linear equivalence between* $\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)^{\mathrm{op}}$ *and* $\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}}$.

2.2 Normal Measures

We study a non-Archimedean analogue of the normality of a measure. For this purpose, we introduce a convention of infinite sums. Let S be a set. For an $f \in k^S$, we denote by $\sum_{s\in S} f(s)$ the limit of the net $(\sum_{s\in S_0} f(s))_{S_0 \in \mathscr{P}_{<\omega}(S)}$, where $\mathscr{P}_{<\omega}(S)$ is directed by inclusions. It is elementary to show the following:

Proposition 2.13. *Let S be a set. For any $f \in k^S$ (respectively, O_k^S), $\sum_{s\in S} f(s)$ converges in k (respectively, O_k) if and only if $f \in \mathrm{C}_0(S, k)$ (respectively, $\mathrm{C}_0(S, O_k)$).*

Let X be a topological space. We denote by $\mathrm{CO}(X)$ the set of clopen subsets of X, and by $\mathbb{P}(X)$ the set of subsets $P \subset \mathrm{CO}(X)$ satisfying $X = \bigsqcup_{U \in P} U$. An *$O_k$-valued measure on X* is a map $\mu\colon \mathrm{CO}(X) \to O_k$ such that $\mu(U_0 \cup U_1) = \sum \mu(U_i)$ for any $(U_i) \in \mathrm{CO}(X)^2$ satisfying $U_0 \cap U_1 = \emptyset$. An O_k-valued measure μ on X is said to be *normal* if $\sum_{U' \in P} \mu(U')$ converges to $\mu(U)$ for any $U \in \mathrm{CO}(X)$ and $P \in \mathbb{P}(U)$.

Let $P \in \mathbb{P}(X)$. For a subset $U \subset X$, we put $P|_U \coloneqq \{U' \in P \mid U' \subset U\}$. We define a partial order $P_0 \leq P_1$ on $(P_i) \in \mathbb{P}(X)^2$ as $(P_0|_U)_{U \in P_1} \in \prod_{U \in P_1} \mathbb{P}(U)$. Let $(P_i) \in \mathbb{P}(X)^2$. Then $\{U_0 \cap U_1 \mid (U_i) \in \prod P_i\} \in \mathbb{P}(X)$ forms the least upper bound of $\{P_0, P_1\}$ with respect to $\leq$. In particular, $\mathbb{P}(X)$ is directed with respect to $\leq$. Suppose $P_0 \leq P_1$. Let $f \in \mathrm{C}_0(P_0, O_k)$ and $U \in P_1$. By $P_0|_U \subset P_0$ and Proposition 2.13, $\widetilde{f}(U) \coloneqq \sum_{U' \in P_0|_U} f(U')$ is a converging sum. For any $\epsilon \in (0, \infty)$, there is a $P'_0 \in \mathscr{P}_{<\omega}(P_0)$ such that $|f(U')| < \epsilon$ for any $U' \in P_0 \setminus P'_0$, and hence $P'_1 \coloneqq \{U \in P_1 \mid P'_0 \cap (P_0|_U) \neq \emptyset\}$ is a finite set satisfying $|\widetilde{f}(U)| < \epsilon$ for any $U \in P_1 \setminus P'_1$. It implies that the map $\widetilde{f}\colon P_1 \to X$, $U \mapsto \widetilde{f}(U)$ lies in $\mathrm{C}_0(P_1, O_k)$. We obtain a continuous O_k-linear homomorphism $\mathrm{C}_0(P_0, O_k) \to \mathrm{C}_0(P_1, O_k)$, $f \mapsto \widetilde{f}$ for each $(P_i) \in \mathbb{P}(X)^2$ satisfying $P_0 \leq P_1$, for which $(\mathrm{C}_0(P, O_k))_{P \in \mathbb{P}(X)}$ forms a cofiltered diagram in $\mathscr{C}_\ell$.

We put $\mathbb{M}(X) \coloneqq \varprojlim_{P \in \mathbb{P}(X)} \mathrm{C}_0(P, O_k)$ and $O_k[[X]] \coloneqq \mathbb{M}(X)_{\mathscr{K}}$. The abuse of the notation with the classical Iwasawa algebra is harmless, because we will show in Proposition 2.21 that $O_k[[X]]$ is its generalisation. For a $(\mu, U) \in \mathbb{M}(X) \times \mathrm{CO}(X)$, we denote by $\mu(U)$ the image of μ by the composite of the $\{U, X \setminus U\}$-th projection $\mathbb{M}(X) \twoheadrightarrow \mathrm{C}_0(\{U, X \setminus U\}, O_k)$ and the evaluation $\mathrm{C}_0(\{U, X \setminus U\}, O_k) \twoheadrightarrow O_k$ at U. For a $(P, \epsilon) \in \mathbb{P}(X) \times (0, 1]$, we set $\mathbb{M}(X; P, \epsilon) \coloneqq \{\mu \in \mathbb{M}(X) \mid \forall U \in P, |\mu(U)| < \epsilon\}$. By Corollary 2.7 and the continuity of $\iota^{\mathrm{cg}}_{\mathbb{M}(X)}$, we obtain the following:

Proposition 2.14. *The linear topological O_k-modules $\mathbb{M}(X)$ and $O_k[[X]]$ are Hausdorff, and the set $\{\mathbb{M}(X; P, \epsilon) \mid (P, \epsilon) \in \mathbb{P}(X) \times (0, 1]\}$ forms a fundamental system of neighbourhoods of $0 \in \mathbb{M}(X)$.*

The evaluation map $\mathbb{M}(X) \to O_k^{\mathrm{CO}(X)}$, $\mu \mapsto (\mu(U))_{U \in \mathrm{CO}(X)}$ is injective. We identify $\mathscr{F}(\mathbb{M}(X))$ with the O_k-module of normal O_k-valued measures on X through the evaluation map. For a $U \in \mathrm{CO}(X)$, we denote by $1_U\colon X \to k$ the characteristic function of U.

Proposition 2.15. *If X is compact, then $\mathbb{M}(X)$ is a compact Hausdorff flat linear topological O_k-module, and the map $\mathrm{C}(X,k)^{\mathrm{D_d}} \to O_k^{\mathrm{CO}(X)}$, $\mu \mapsto (\mu(1_U))_{U \in \mathrm{CO}(X)}$ (cf. [7] Example 1.4) induces an isomorphism $\mathrm{C}(X,k)^{\mathrm{D_d}} \to \mathbb{M}(X)$ in $\mathscr{C}_{\mathfrak{fl}}^{\mathrm{ch}}$.*

Proof. By the compactness of X, every O_k-valued measure on X is normal, and hence the map in the assertion gives an O_k-linear homomorphism $\mathrm{C}(X,k)^{\mathrm{D_d}} \to \mathbb{M}(X)$, which is continuous by the finiteness of pairwise disjoint clopen coverings of X. On the other hand, again by the compactness of X, every continuous k-valued function is uniformly approximated by a finite k-linear combination of characteristic functions of clopen subsets. Therefore we obtain the inverse $\mathbb{M}(X) \to \mathrm{C}(X,k)^{\mathrm{D_d}}$, which is continuous because $\mathrm{C}(X,k)^{\mathrm{D_d}}$ is compact and $\mathbb{M}(X)$ is Hausdorff. □

We denote by $\delta_{X,x} \in \mathbb{M}(X)$ the normal O_k-valued measure which assigns 1 if $x \in U$ and 0 otherwise to each $U \in \mathrm{CO}(X)$ for an $x \in X$, by $\delta_X \colon X \to \mathbb{M}(X)$ the map given by setting $\delta_X(x) \coloneqq \delta_{X,x}$ for an $x \in X$, and by $O_k^{\oplus \delta_X} \colon O_k^{\oplus X} \to \mathbb{M}(X)$ the O_k-linear extension of δ_X.

Proposition 2.16. (i) *The map δ_X is continuous.*

(ii) *If X is zero-dimensional, that is, $\mathrm{CO}(X)$ generates the topology of X, and Hausdorff, then $O_k^{\oplus \delta_X}$ is injective.*

(iii) *The image of $O_k^{\oplus \delta_X}$ is dense.*

Proof. We show the assertion (i). Let $U_1 \subset \mathbb{M}(X)$ be an open subset. For any $x \in X$ satisfying $\delta_{X,x} \in U_1$, there is a $(P,\epsilon) \in \mathbb{P}(X) \times (0,1]$ such that $\delta_{X,x} + \mathbb{M}(X;P,\epsilon) \subset U_1$, and hence for any $U_0 \in P$, $x \in U_0$ implies $U_0 \subset \delta_X^{-1}(U_1)$. Therefore δ_X is continuous. We show the assertion (ii). Suppose that X is zero-dimensional and Hausdorff. Let $m \in O_k^{\oplus X} \setminus \{0\}$. Let $X_0 \subset X$ denote a unique non-empty finite subset for which m is presented as $\sum_{x \in X_0} c_x x$ for a $(c_x)_{x \in X_0} \in (O_k \setminus \{0\})^{X_0}$. By the assumption, there is a $P \in \mathbb{P}(X)$ such that $\#(U \cap X_0) \leq 1$ for any $U \in P$. Then $O_k^{\oplus \delta_X}(m)(U) = c_x \neq 0$ for any $(U,x) \in P \times X$ satisfying $x \in U$. It implies $\ker(O_k^{\oplus \delta_X}) = \{0\}$.

We show the assertion (iii). Let $U \subset \mathbb{M}(X)$ be an open neighbourhood of a $\mu \in U$. By Corollary 2.7, there is a $(P,\epsilon) \in \mathbb{P}(X) \times (0,1]$ such that $\mu + \mathbb{M}(X;P,\epsilon) \subset U$. Put $P_0 \coloneqq \{U' \in P \mid |\mu(U')| \geq \epsilon\} \in \mathscr{P}_{<\omega}(P) \setminus \{\emptyset\}$. For each $U' \in P_0$, take an $x_{U'} \in U'$. Then $\mu' \coloneqq O_k^{\oplus \delta_X}(\sum_{U' \in P_0} \mu(U') x_{U'})$

satisfies $|\mu'(U') - \mu(U')| < \epsilon$ for any $U' \in P$. It ensures $\mu' \in U$. Therefore the image of $O_k^{\oplus \delta_X}$ is dense. □

We put $d_X := (\iota^{\mathrm{cg}}_{\mathbb{M}(X)})^{-1} \circ \delta_X$ and $O_k^{\oplus d_X} := (\iota^{\mathrm{cg}}_{\mathbb{M}(X)})^{-1} \circ O_k^{\oplus \delta_X}$. We consider d_G and $O_k^{\oplus d_G}$.

Proposition 2.17. (i) *The map δ_G is a homeomorphism onto the image.*
(ii) *The map d_G is a homeomorphism onto the image.*
(iii) *The image of $O_k^{\oplus d_G}$ is dense.*

In order to verify Proposition 2.17, we study pre-compact subsets of $\mathbb{M}(G)$.

Lemma 2.18. *Let $C \subset \mathbb{M}(G)$ be a pre-compact subset. For any $\epsilon \in (0, 1]$, there is a compact clopen subset $G_0 \subset G$ such that $|\mu(U)| < \epsilon$ for any $(\mu, U) \in C \times \mathrm{CO}(G \setminus G_0)$.*

Proof. Take an open profinite subgroup $K \subset G$. Assume that there is an $\epsilon \in (0, 1]$ such that for any compact clopen subset $G_0 \subset G$, some $(\mu, U) \in C \times \mathrm{CO}(G \setminus G_0)$ satisfies $|\mu(U)| \geq \epsilon$. In particular, G is not compact, because $G_0 = G$ satisfies $\mathrm{CO}(G \setminus G_0) = 1$ and $\mu(\emptyset) = 0$ for any $\mu \in C$. Therefore G/K is an infinite set. We construct $(\mu_r, U_r, C_r) \in C \times \mathrm{CO}(G) \times G/K$ inductively on $r \in \omega$ so that $C_r \neq K$ for any $r \in \omega$, $|\mu_r(U_r)| \geq \epsilon$ for any $r \in \omega$, $U_r \subset C_r$ for any $r \in \omega$, and $C_{r_0} \neq C_{r_1}$ for any $(r_i) \in \omega^2$ satisfying $r_0 \neq r_1$.

By the assumption, there is a $(\mu_0, U_0) \in C \times \mathrm{CO}(G \setminus K)$ such that $|\mu_0(U_0)| \geq \epsilon$. By the normality of μ_0, we have $\mu_0(U_0) = \sum_{C \in G/K} \mu_0(U_0 \cap C)$, and hence $|\mu_0(U_0 \cap C_0)| \geq \epsilon$ for some $C_0 \in G/K$ satisfying $C_0 \neq K$. Replacing U_0 by $U_0 \cap C_0$, we may assume $U_0 \subset C_0$. Let $r \in \omega \setminus \{0\}$. Suppose that we have constructed $(\mu_h, U_h, C_h)_{h=0}^{n-1} \in (C \times \mathrm{CO}(G) \times G/K)^n$ such that $C_h \neq K$, $|\mu_h(U_h)| \geq \epsilon$, and $U_h \subset C_h$ for any $h \in \omega$ satisfying $h < n$, and $C_{h_0} \neq C_{h_1}$ for any $(h_i) \in \omega^2$ satisfying $h_0 \neq h_1$, $h_0 < r$, and $h_1 < r$. By the assumption, there is a $(\mu_r, U_r) \in C \times \mathrm{CO}(G \setminus (K \sqcup \bigsqcup_{h=0}^{r-1} C_h)$ such that $|\mu_r(U_r)| \geq \epsilon$. By the normality of μ_r, we may assume that U_r is contained in a $C_r \in G/K$ satisfying $C_r \neq K$. By induction on $r \in \omega$, we obtain a desired family $(\mu_r, U_r, C_r)_{r \in \omega}$.

Since $(C_r)_{r \in \omega}$ is a system of pairwise disjoint subsets of G, $U_\omega := G \setminus \bigsqcup_{r \in \omega} U_r$ is a clopen subset of G. Put $P := \{U_r \mid r \in \omega \sqcup \{\omega\}\} \in \mathbb{P}(G_0)$.

Since C is pre-compact, so is its image C_P in $\mathrm{C}_0(P, O_k)$ by Proposition 2.3. Therefore there is a $C_{P,0} \in \mathscr{P}_{<\omega}(C_P)$ satisfying $C_P \subset \{\mu \in \mathrm{C}_0(P, O_k) \mid \exists \mu' \in C_{P,0}, \forall U \in P, |\mu(U) - \mu'(U)| < \epsilon\}$. By $C_{P,0} \in \mathscr{P}_{<\omega}(C_P)$, there is a $P_0 \in \mathscr{P}_{<\omega}(P)$ satisfying $\mu(U) < \epsilon$ for any $(\mu, U) \in C_{P,0} \times (P \setminus P_0)$. It ensures $\mu(U) < \epsilon$ for any $(\mu, U) \in C_P \times (P \setminus P_0)$ by the choice of $C_{P,0}$. It contradicts that the inequality $|\mu_r(U_r)| \geq \epsilon$ holds for any $r \in \omega$. This completes the proof of the assertion. □

For an increasing sequence $(X_r)_{r\in\omega}$ of compact clopen subsets of X and a decreasing sequence $(\epsilon_r)_{r\in\omega} \in (0,1)^\omega$, we put $\mathbb{M}(X; (X_r)_{r\in\omega}, (\epsilon_r)_{r\in\omega}) \coloneqq \{\mu \in \mathbb{M}(X) \mid \forall r \in \omega, \forall U \in \mathrm{CO}(X \setminus X_r), |\mu(U)| < \epsilon_r\}$.

Lemma 2.19. *Let $\epsilon \in (0,1)$. A subset of $\mathbb{M}(G)$ is pre-compact if and only if it is contained in $\mathbb{M}(G; (G_r)_{r\in\omega}, (\epsilon^r)_{r\in\omega})$ for an increasing sequence $(G_r)_{r\in\omega}$ of compact clopen subsets of G.*

Proof. Let $C \subset \mathbb{M}(G)$ be a subset. Suppose that C is pre-compact. For each $r \in \omega$, there is a compact clopen subset $G_{r,0} \subset G$ such that $C \subset \{\mu \in \mathbb{M}(G) \mid \forall U \in \mathrm{CO}(G \setminus G_{r,0}), |\mu(U)| < \epsilon^r\}$ by Lemma 2.18. For an $r \in \omega$, put $G_r \coloneqq \bigcup_{s=0}^{r} G_{s,0} \in \mathrm{CO}(G)$. Then $(G_r)_{r\in\omega}$ forms an increasing sequence of compact clopen subsets of G satisfying $C \subset \mathbb{M}(G; (G_r)_{r\in\omega}, (\epsilon^r)_{r\in\omega})$.

On the other hand, suppose that C is contained in $\mathbb{M}(G; (G_r)_{r\in\omega}, (\epsilon^r)_{r\in\omega})$ for an increasing sequence $(G_r)_{r\in\omega}$ of compact clopen subsets of G. Let $L \in \mathscr{O}(\mathbb{M}(G))$. By Corollary 2.7, there is a $(P, \epsilon') \in \mathbb{P}(X) \times (0,1]$ such that $\mathbb{M}(G; P, \epsilon') \subset L$. By $\epsilon \in (0,1)$, there is an $r \in \omega$ such that $\epsilon^r \leq \epsilon'$. By the compactness of G_r, there is a $P_0 \in \mathscr{P}_{<\omega}(P)$ such that $G_r \subset \bigsqcup_{U \in P_0} U$. Since O_k is compact, there is an $S \in \mathscr{P}_{<\omega}(O_k)$ such that $O_k = \bigcup_{c \in S}\{c' \in O_k \mid |c' - c| < \epsilon^r\}$. By $\# S^{P_0} = (\# S)^{\# P_0} < \infty$, there is a $C_0 \in \mathscr{P}_{<\omega}(C)$ such that $C = \bigcup_{\mu \in C_0}\{\mu \in C \mid \forall U \in P_0, |\mu'(U) - \mu(U)| < \epsilon^r\}$. It implies $C \subset \bigcup_{\mu \in C_0} \mu + L$. Thus C is pre-compact. □

Lemma 2.20. *Let $M \in \mathrm{ob}(\mathscr{C}_\ell)$. Then a map $f \colon G \to M$ is continuous if and only if $(\iota_M^{\mathrm{cg}})^{-1} \circ f$ is continuous.*

Proof. Take an open profinite subgroup $K \subset G$. The direct implication follows from the continuity of ι_M^{cg}. Suppose that f is continuous. Let $U \subset M_{\mathscr{K}}$ be an open subset. Let $g \in G$. Suppose $(\iota_M^{\mathrm{cg}})^{-1}(f(g)) \in U$. Since $f(gK) \subset M$ is compact, $\iota_M^{\mathrm{cg}}(U) \cap f(gK)$ is an open subset of $f(gK)$ by

Proposition 2.1 (ii) and Corollary 2.5. By the continuity of f, $f^{-1}(\iota_M^{\mathrm{cg}}(U) \cap f(gK))$ is an open subset of $f^{-1}(f(gK))$. It ensures that $f^{-1}(\iota_M^{\mathrm{cg}}(U)) \cap gK$ is an open subset of gK. Since gK is an open subset of G, $f^{-1}(\iota_M^{\mathrm{cg}}(U)) = ((\iota_M^{\mathrm{cg}})^{-1} \circ f)^{-1}(U)$ is an open neighbourhood of g in G. It implies that $(\iota_M^{\mathrm{cg}})^{-1} \circ f$ is continuous. □

Proof of Proposition 2.17. Take an open profinite subgroup $K \subset G$. Then G/K gives an element $\{gK \mid g \in G\}$ of $\mathbb{P}(G)$. For any $g \in G$, $\delta_G|_{gK}$ is a closed continuous map by Proposition 2.16 (i) because gK is compact and $\mathbb{M}(G)$ is Hausdorff, and its image is contained in $\delta_{G,g} + \mathbb{M}(G; G/K, 1)$. Therefore δ_G is an injective local homeomorphism onto the image by Proposition 2.16 (ii), because $\{\delta_{G,g} + \mathbb{M}(G; G/K, 1) \mid g \in G\}$ forms a covering of the image of δ_G consisting of pairwise disjoint clopen subsets of $\mathbb{M}(G)$. It implies that δ_G is a homeomorphism onto the image, and so is d_G by Lemma 2.20.

Let $U \subset O_k[[G]]$ be a non-empty open subset. Take a $\mu \in U$. By Lemma 2.19, the pre-compact subset $\{\iota_{\mathbb{M}(G)}^{\mathrm{cg}}(\mu)\} \subset \mathbb{M}(G)$ is contained in $K \coloneqq \mathbb{M}(G; (G_r)_{r \in \omega}, (\epsilon^r)_{r \in \omega})$ for an increasing sequence $(G_r)_{r \in \omega} \in \mathrm{CO}(G)^\omega$ and an $\epsilon \in (0,1)$, and K itself is a pre-compact O_k-submodule of $\mathbb{M}(G)$. By Corollary 2.7, there is a $(P, \epsilon') \in \mathbb{P}(G) \times (0,1]$ such that $\{\mu' \in K \mid \forall U' \in P, |\mu'(U') - \iota_{\mathbb{M}(G)}^{\mathrm{cg}}(\mu)(U')| < \epsilon'\} \subset \iota_{\mathbb{M}(G)}^{\mathrm{cg}}(U)$. By $\epsilon \in (0,1)$, there is an $r \in \omega$ such that $\epsilon^r \leq \epsilon'$. By the compactness of G_r, there is a $P_0 \in \mathscr{P}_{<\omega}(P) \setminus \{\emptyset\}$ such that $G_r \subset \bigsqcup_{U' \in P_0} U'$. For each $U' \in P_0$, take an $x_{U'} \in U'$. Then $\mu' \coloneqq O_k^{\oplus \delta_G}(\sum_{U' \in P_0} \iota_{\mathbb{M}(G)}^{\mathrm{cg}}(\mu)(U') x_{U'})$ satisfies $|\mu'(U') - \iota_{\mathbb{M}(G)}^{\mathrm{cg}}(\mu)(U')| < \epsilon^r$ for any $U' \in P$. It ensures $(\iota_{\mathbb{M}(G)}^{\mathrm{cg}})^{-1}(\mu') \in U$. Therefore the image of $O_k^{\oplus d_G}$ is dense. □

We show the relation between $O_k[[G]]$ and the classical Iwasawa algebra. We denote by $\mathscr{O}(G)$ the set of open normal subgroups of G, which is filtered and cofiltered by inclusions. For a $(\wp, K) \in \mathscr{O}(O_k) \times \mathscr{O}(G)$, we equip $(O_k/\wp)[G/K]$ with the discrete topology so that it forms a linear topological O_k-module.

Proposition 2.21. *Suppose that G is a profinite group. Then the system of the canonical projections $O_k[G] \twoheadrightarrow (O_k/\wp)[G/K]$ indexed by $(\wp, K) \in$*

$\mathscr{O}(O_k) \times \mathscr{O}(G)$ *induces a unique isomorphism*

$$O_k[[G]] \to \varprojlim_{(\wp,K)\in\mathscr{O}(O_k)\times\mathscr{O}(G)} (O/\wp)[G/K]$$

in $\mathscr{C}_\ell$. *In particular,* $O_k[[G]]$ *forms a compact Hausdorff flat linear topological* O_k*-module.*

Proof. The assertion follows from Proposition 2.1 (ii), Proposition 2.8 (ii), Proposition 2.15, and the fact that the classical Iwasawa algebra over O_k associated to G has an interpretation as an O_k-module of O_k-valued measures on G. □

3 Monoidal structures

We define symmetric monoidal structures on the categories introduced in §2.1, and an O_k-algebra structure on $O_k[[G]]$ in terms of a monoid in one of them. We note that $O_k[[G]]$ does not necessarily form a topological O_k-algebra, that is, a monoid object in the Cartesian monoidal category of topological O_k-modules and continuous O_k-linear homomorphisms. This is one of the main reasons why we need monoidal structures.

3.1 Topological tensor products We define symmetric monoidal structures on $\mathscr{C}_\ell$, $\mathscr{C}_\ell^{\mathrm{c}}$, and $\mathscr{C}_\ell^{\mathrm{cg}}$. First, we study $\mathscr{C}_\ell$. Let $(M_i) \in \mathrm{ob}(\mathscr{C}_\ell^2)$. We denote by $(L_i)_{(M_i)} \subset \mathscr{F}(M_0) \otimes_{O_k} \mathscr{F}(M_1)$ the kernel of the natural projection $\mathscr{F}(M_0) \otimes_{O_k} \mathscr{F}(M_1) \twoheadrightarrow \mathscr{F}(M_0/L_0) \otimes_{O_k} \mathscr{F}(M_1/L_1)$ for O_k-submodules $L_0 \subset M_0$ and $L_1 \subset M_1$, and by $M_0 \otimes^\ell M_1$ the O_k-module $\mathscr{F}(M_0) \otimes_{O_k} \mathscr{F}(M_1)$ equipped with the topology generated by the set $\{m + (L_i)_{(M_i)} \mid (m,(L_i)) \in (\mathscr{F}(M_0) \otimes_{O_k} \mathscr{F}(M_1)) \times \prod \mathscr{O}(M_i)\}$. Then $M_0 \otimes^\ell M_1$ forms a linear topological O_k-module. By the definition of the topology of $M_0 \otimes^\ell M_1$, the O_k-bilinear homomorphism $\nabla_{(M_i)} \colon \prod \mathscr{U}(M_i) \to \mathscr{U}(M_0 \otimes^\ell M_1)$, $(m_i) \mapsto m_0 \otimes m_1$ is continuous. The correspondence $(M_i) \rightsquigarrow M_0 \otimes^\ell M_1$ gives a functor $\otimes^\ell \colon \mathscr{C}_\ell^2 \to \mathscr{C}_\ell$, and the correspondence $(M_i) \rightsquigarrow \nabla_{(M_i)}$ gives a natural transformation $\nabla \colon \prod \mathscr{U}(\bullet_i) \Rightarrow \mathscr{U}(\bullet_0 \otimes^\ell \bullet_1)$. Let $(M_s)_{s\in S}$ be a small diagram in $\mathscr{C}_\ell$. By the functoriality of $\otimes^\ell$ and the universality of the colimit, the system of canonical morphisms $M_{s_0} \to \varinjlim_{s \in S} M_s$ indexed by $s_0 \in S$ induces a morphism $\mathrm{S}_{(M_s)_{s \in S}, M} \colon \varinjlim_{s \in S}(M_s \otimes^{\ell} M) \to (\varinjlim_{s \in S} M_s) \otimes^{\ell} M$ for an $M \in \mathrm{ob}(\mathscr{C}_{\ell})$. We note that $\otimes^{\ell}$ seems not to be cocontinuous.

Proposition 3.1. *The triad $(\mathscr{C}_{\ell}, \otimes, O_k)$ forms a symmetric monoidal category.*

Proof. We denote by $(\mathrm{A}, \mathrm{L}, \mathrm{R}, \mathrm{B})$ the data of the associator, the left unitor, the right unitor, and the braiding of $(\mathscr{C}, \otimes_{O_k}, O_k)$. We have

$$\mathscr{F}(\bullet_0 \otimes^{\ell} \bullet_1) = \mathscr{F}(\bullet_0) \otimes_{O_K} \mathscr{F}(\bullet_1)$$

by definition. Since $\mathscr{F} : \mathscr{C}_{\ell} \to \mathscr{C}$ is faithful, it suffices to verify that every value of $\Phi \circ \mathscr{F}$ lies in the image of $\mathscr{F}$ for any $\Phi \in \{\mathrm{A}, \mathrm{L}, \mathrm{R}, \mathrm{B}\}$. By $O_k \in \mathscr{O}(O_k)$, every value of $\mathrm{L} \circ \mathscr{F}$ (respectively, $\mathrm{R} \circ \mathscr{F}$) lies in the image of $\mathscr{F}$. By the symmetry of the sub-base of the topology of every value of $\otimes^{\ell}$, every value of $\mathrm{B} \circ \mathscr{F}$ lies in the image of $\mathscr{F}$. Let $(M_i)_{i=0}^{2} \in \mathrm{ob}(\mathscr{C}_{\ell}^{3})$. We show that the O_k-linear homomorphism $\mathrm{A}_{M_0, M_1, M_2} \colon (M_0 \otimes^{\ell} M_1) \otimes^{\ell} M_2 \to M_0 \otimes^{\ell} (M_1 \otimes^{\ell} M_2)$, $m \mapsto \mathrm{A}_{\mathscr{F}(M_0), \mathscr{F}(M_1), \mathscr{F}(M_2)}(\mathscr{F}(m))$ is continuous. Let $(L_0, L_{1,2}) \in \mathscr{O}(M_0) \times \mathscr{O}(M_1 \otimes^{\ell} M_2)$. Take an $(L_{i+1}) \in \prod \mathscr{O}(M_{i+1})$ satisfying $(L_{i+1})_{(M_{i+1})} \subset L_{1,2}$. We have

$$\begin{aligned} &((L_i)_{(M_i)}, L_2)_{M_0 \otimes^{\ell} M_1, M_2} \\ &\qquad = (L_0, (L_{i+1})_{(M_{i+1})})_{M_0, M_1 \otimes^{\ell} M_2} \subset \mathrm{A}_{M_0, M_1, M_2}^{-1}((L_0, L_{1,2})_{M_0, M_1 \otimes^{\ell} M_2}) \end{aligned}$$

by the right exactness of $\otimes_{O_k}$. Therefore A_{M_0, M_1, M_2} is a continuous map satisfying $\mathscr{F}(\mathrm{A}_{M_0, M_1, M_2}) = \mathrm{A}_{\mathscr{F}(M_0), \mathscr{F}(M_1), \mathscr{F}(M_2)}$. $\square$

Next, we study $\mathscr{C}_{\ell}^{\mathrm{c}}$. Let $(K_i) \in \mathrm{ob}((\mathscr{C}_{\ell}^{\mathrm{c}})^2)$. Then $K_0 \otimes^{\ell} K_1$ is pre-compact by $\#((K_0 \otimes^{\ell} K_1)/(L_i)_{(K_i)}) = \#(K_0/L_0 \otimes^{\ell} K_1/L_1) \leq \prod \#(K_i/L_i) < \infty$ for any $(L_i) \in \prod \mathscr{O}(K_i)$. Therefore the correspondence $(K_i) \rightsquigarrow K_0 \otimes^{\ell} K_1$ gives a functor $\otimes^{\mathrm{c}} \colon (\mathscr{C}_{\ell}^{\mathrm{c}})^2 \to \mathscr{C}_{\ell}^{\mathrm{c}}$, and the correspondence $(K_i) \rightsquigarrow \nabla_{(K_i)}$ gives a natural transformation $\nabla^{\mathrm{c}} \colon \prod \mathscr{U}^{\mathrm{c}}(\bullet_i) \Rightarrow \mathscr{U}^{\mathrm{c}}(\bullet_0 \otimes^{\mathrm{c}} \bullet_1)$. Since $\mathscr{C}_{\ell}^{\mathrm{c}}$ is a full subcategory of $\mathscr{C}_{\ell}$, we obtain the following by Proposition 3.1:

Proposition 3.2. *The triad $(\mathscr{C}_{\ell}^{\mathrm{c}}, \otimes^{\mathrm{c}}, O_k)$ forms a symmetric monoidal category.*

We put $\mathscr{L}((K_0, M_1), L) \coloneqq \{f \in \mathscr{L}(K_0, M_1) \mid f(K_0) \subset L\}$ for an $L \in \mathscr{O}(M_1)$, and denote by $\mathscr{H}\mathrm{om}^{\mathrm{c}}(K_0, M_1)$ the O_k-module $\mathscr{L}(K_0, M_1)$ equipped with the topology generated by the set $\{f + \mathscr{L}((K_0, M_1), L) \mid (f, L) \in K_0 \times \mathscr{O}(M_1)\}$. Then $\mathscr{H}\mathrm{om}^{\mathrm{c}}(K_0, M_1)$ forms a linear topological O_k-module. By Proposition 2.3 and Corollary 2.10 (i), the correspondence $(K_0, M_1) \rightsquigarrow \mathscr{H}\mathrm{om}^{\mathrm{c}}(K_0, M_1)$ gives a functor $\mathscr{H}\mathrm{om}^{\mathrm{c}} \colon (\mathscr{C}_\ell^{\mathrm{c}})^{\mathrm{op}} \times \mathscr{C}_\ell \to \mathscr{C}_\ell$. By Theorem 2.12, the transpose map ${}^{\mathrm{T}}(\bullet)_{(K_i)} \colon \mathscr{H}\mathrm{om}^{\mathrm{c}}((K_i)) \to \mathscr{S}((K_{1-i}^{\mathrm{D_c}}))$ is bijective. We have a comparison of the endomorphism algebras, which corresponds to [13] Lemma 1.6 in the case $\mathrm{ch}(k) = 0$.

Proposition 3.3. *The map ${}^{\mathrm{T}}(\bullet)_{(K_i)}$ is an isomorphism in $\mathscr{C}_\ell$.*

Proof. Let $(v, \epsilon) \in K_1^{\mathrm{D_c}} \times (0, \infty)$. Put $L \coloneqq \{f \in \mathscr{S}((K_{1-i}^{\mathrm{D_c}})) \mid |f(v)| < \epsilon\}$. We show ${}^{\mathrm{T}}(\bullet)_{(K_i)}^{-1}(L) \in \mathscr{O}(\mathscr{H}\mathrm{om}^{\mathrm{c}}((K_i)))$. Put $L_1 \coloneqq \{m \in K_1 \mid |v(m)| < 2^{-1}\epsilon\} \in \mathscr{O}(K_1)$. Let $f \in \mathscr{L}((K_i), L_1)$. We have $\|{}^{\mathrm{T}}f_{(K_i)}(v)\| = \sup_{m \in K_1} |v(f(m))| \leq \sup_{m \in L_1} |v(m)| \leq 2^{-1}\epsilon < \epsilon$. It ensures ${}^{\mathrm{T}}f_{(K_i)} \in L$. It implies $\mathscr{L}((K_i), L_1) \subset {}^{\mathrm{T}}(\bullet)_{(K_i)}^{-1}(L)$. We obtain ${}^{\mathrm{T}}(\bullet)_{(K_i)}^{-1}(L) \in \mathscr{O}(\mathscr{H}\mathrm{om}^{\mathrm{c}}((K_i)))$. Therefore ${}^{\mathrm{T}}(\bullet)_{(K_i)}$ is continuous.

Let $L_1 \in \mathscr{O}(K_1)$. We show ${}^{\mathrm{T}}(\bullet)_{(K_i)}(\mathscr{L}((K_i), L_1)) \in \mathscr{O}(\mathscr{S}((K_{1-i}^{\mathrm{D_c}})))$. By Theorem 2.12, there is an $(S, \epsilon) \in \mathscr{P}_{<\omega}(K_0^{\mathrm{D_c}}) \times (0, \infty)$ such that $\{m \in K_1 \mid \forall v \in S, |v(m)| < \epsilon\} \subset L_1$. Put $L \coloneqq \{f \in \mathscr{S}((K_{1-i}^{\mathrm{D_c}})) \mid \forall v \in S, \|f(v)\| < \epsilon\} \in \mathscr{O}(\mathscr{S}((K_{1-i}^{\mathrm{D_c}})))$. Let $f \in L$. We show ${}^{\mathrm{T}}(\bullet)_{(K_i)}^{-1}(f) \in \mathscr{L}((K_i), L_1)$. Let $m \in K_0$. We have $|v({}^{\mathrm{T}}(\bullet)_{(K_i)}^{-1}(f)(m))| = |f(v)(m)| \leq \|f(v)\| < \epsilon$ for any $v \in S$, and hence ${}^{\mathrm{T}}(\bullet)_{(K_i)}^{-1}(f)(m) \in L_1$. It ensures ${}^{\mathrm{T}}(\bullet)_{(K_i)}^{-1}(f) \in \mathscr{L}((K_i), L_1)$. It implies $L \subset {}^{\mathrm{T}}(\bullet)_{(K_i)}(\mathscr{L}((K_i), L_1))$. Therefore ${}^{\mathrm{T}}(\bullet)_{(K_i)}$ is an open map. □

We denote by $\mathrm{C}_{\mathrm{L}}^{\mathrm{c}}, \mathrm{C}_{\mathrm{R}}^{\mathrm{c}} \colon ((\mathscr{C}_\ell^{\mathrm{c}})^{\mathrm{op}})^2 \times \mathscr{C}_\ell \to \mathrm{Set}$ the functors given as $\mathrm{C}_{\mathrm{L}}^{\mathrm{c}} \coloneqq \mathscr{L}(\mathscr{I}^{\mathrm{c}}(\bullet_0 \otimes^{\mathrm{c}} \bullet_1), \bullet_2)$ and $\mathrm{C}_{\mathrm{R}}^{\mathrm{c}} \coloneqq \mathscr{L}(\mathscr{I}^{\mathrm{c}}(\bullet_0), \mathscr{H}\mathrm{om}^{\mathrm{c}}((\bullet_{i+1})))$. We construct an adjunction $\mathrm{T}^{\mathrm{c}} \colon \mathrm{C}_{\mathrm{L}}^{\mathrm{c}} \Rightarrow \mathrm{C}_{\mathrm{R}}^{\mathrm{c}}$. Let f be an O_k-linear homomorphism $K_0 \otimes^{\mathrm{c}} K_1 \to M_2$ for a $((K_i), M_2) \in \mathrm{ob}((\mathscr{C}_\ell^{\mathrm{c}})^2 \times \mathscr{C}_\ell)$. We characterise the continuity of f.

Proposition 3.4. *The map f is continuous if and only if $f \circ \nabla_{(K_i)}^{\mathrm{c}}$ is continuous.*

Proof. The inverse implication follows from the continuity of $\nabla_{(K_i)}^{\mathrm{c}}$. Suppose that $f \circ \nabla_{(K_i)}^{\mathrm{c}}$ is continuous. Let $L_2 \in \mathscr{O}(M_2)$. We show $f^{-1}(L_2) \in$

$\mathscr{O}(K_0 \otimes^{\mathrm{c}} K_1)$. By the continuity of $f \circ \nabla^{\mathrm{c}}_{(K_i)}$, there is an $(L_i) \in \prod \mathscr{O}(K_i)$ such that $\prod L_i \subset (f \circ \nabla^{\mathrm{c}}_{(K_i)})^{-1}(L_2)$. Put $i_0 := 0$ (respectively, $i_0 := 1$). Take a $K_{i_0,0} \in \mathscr{P}_{<\omega}(K_{i_0})$ satisfying $K_{i_0} \subset \bigcup_{m \in K_{i_0,0}}(m + L_{i_0})$. For each $m \in K_{i_0,0}$, there is an $L_{i_0,0,m} \in \mathscr{O}(K_{i_0})$ such that $L_{i_0,0,m} \times \{m\}$ (respectively, $\{m\} \times L_{i_0,0,m}$) is contained in $(f \circ \nabla^{\mathrm{c}}_{(K_i)})^{-1}(L_2)$ by the continuity of $f \circ \nabla^{\mathrm{c}}_{(K_i)}$. Put $L_{i_0,0} := L_{i_0} \cap \bigcap_{m \in K_{i_0,0}} L_{i_0,0,m} \in \mathscr{O}(K_{i_0})$. By $L_2 + L_2 = L_2$, we obtain $(L_{i,0})_{(K_i)} \subset f^{-1}(L_2)$. Therefore f is continuous. □

Suppose that f is continuous. Let $m_0 \in K_0$. We denote by $f(m_0 \otimes^{\mathrm{c}} \bullet)$ the O_k-linear homomorphism $K_1 \to M_2,\ m_1 \mapsto f(m_0 \otimes m_1)$. Then $f(m_0 \otimes^{\mathrm{c}} \bullet)$ is the composite of f, $\nabla^{\mathrm{c}}_{(K_i)}$, and the map $\mathscr{U}^{\mathrm{c}}(K_1) \hookrightarrow \prod \mathscr{U}^{\mathrm{c}}(K_i),\ m_1 \mapsto (m_i)$, and hence is continuous. We obtain an O_k-linear homomorphism $\mathrm{T}^{\mathrm{c}}_{(K_i),M_2}(f) \colon K_0 \to \mathscr{H}\mathrm{om}^{\mathrm{c}}(K_1, M_2),\ m_0 \mapsto f(m_0 \otimes^{\mathrm{c}} \bullet)$.

Proposition 3.5. *The O_k-linear homomorphism $\mathrm{T}^{\mathrm{c}}_{(K_i),M_2}(f)$ is continuous.*

Proof. Let $L_2 \in \mathscr{O}(M_2)$. By the continuity of f, there is an $(L_i) \in \prod \mathscr{O}(K_i)$ such that $(L_i)_{(K_i)} \subset f^{-1}(L_2)$. Take a $K_{1,0} \in \mathscr{P}_{<\omega}(K_1)$ satisfying $K_1 \subset \bigcup_{m_1 \in K_{1,0}}(m_1 + L_1)$. For each $m_1 \in K_{1,0}$, there is an $L_{0,0,m_1} \in \mathscr{O}(K_0)$ such that $f(m_0 \otimes m_1) \in L_2$ for any $m_0 \in L_{0,0,m_1}$ by the continuity of f, $\nabla^{\mathrm{c}}_{(K_i)}$, and the map $K_0 \hookrightarrow \prod K_i,\ m_0 \mapsto (m_i)$. By $0 \in K_1$, we have $K_{1,0} \neq \emptyset$. Put $L_{0,0} := L_0 \cap \bigcap_{m_1 \in K_{1,0}} L_{0,0,m_1} \in \mathscr{O}(K_0)$. By $L_2 + L_2 = L_2$, we obtain $f(m_0 \otimes m_1) \in L_2$ for any $(m_i) \in L_{0,0} \times K_1$. It ensures $L_{0,0} \subset \mathrm{T}^{\mathrm{c}}_{(K_i),M_2}(f)^{-1}(\mathscr{L}((K_1, M_2), L_2))$. Thus $\mathrm{T}^{\mathrm{c}}_{(K_i),M_2}(f)$ is continuous. □

By Proposition 3.5, the correspondence $((K_i), M_2) \rightsquigarrow \mathrm{T}^{\mathrm{c}}_{(K_i),M_2}$ gives a natural transformation $\mathrm{T}^{\mathrm{c}} \colon \mathrm{C}^{\mathrm{c}}_{\mathrm{L}} \Rightarrow \mathrm{C}^{\mathrm{c}}_{\mathrm{R}}$.

Proposition 3.6. *The natural transformation T^{c} is a natural equivalence.*

Proof. We have $\mathscr{F}^{\mathrm{c}}(\bullet_0 \otimes^{\mathrm{c}} \bullet_1) = \mathscr{F}^{\mathrm{c}}(\bullet_0) \otimes_{O_k} \mathscr{F}^{\mathrm{c}}(\bullet_1)$ and $\mathscr{F}^{\mathrm{c}} \circ \mathrm{T}^{\mathrm{c}}$ coincides with the restriction of the adjunction between $\otimes_{O_k}$ and the internal-hom functor on $\mathscr{C}$. Since $\mathscr{F}^{\mathrm{c}}$ is faithful, $\mathrm{T}^{\mathrm{c}}_{(K_i),M_2}$ is injective. Let $f \in \mathrm{C}^{\mathrm{c}}_{\mathrm{R}}((K_i), M_2)$. We show that the O_k-linear homomorphism $\tilde{f} \colon K_0 \otimes^{\mathrm{c}} K_1 \to M_2,\ (m_i) \mapsto f(m_0)(m_1)$ is continuous. Let $L_2 \in \mathscr{O}(M_2)$. By the continuity of f, there is an $L_0 \in \mathscr{O}(K_0)$ such that $L_0 \subset f^{-1}(\mathscr{L}((K_1, M_2), L_2))$. It ensures $(L_0, K_1)_{(K_i)} \subset \tilde{f}^{-1}(L_2)$. Therefore $\tilde{f}$ is continuous. We have

$\mathrm{T}^{\mathrm{c}}_{(K_i),M_2}(\tilde{f}) = f$. It implies that $\mathrm{T}^{\mathrm{c}}_{(K_i),M_2}$ is surjective. Thus T^{c} is a natural equivalence. □

By Proposition 3.6, we obtain an adjoint property between $\otimes^{\mathrm{c}}$ and $\mathscr{H}\mathrm{om}^{\mathrm{c}}$. It does not ensure that $\otimes^{\mathrm{c}}$ is cocontinuous, because we used $\mathscr{I}^{\mathrm{c}}$ in the description of the adjoint property. On the other hand, we have a commutativity between $\otimes^{\mathrm{c}}$ and colimits in $\mathscr{C}^{\mathrm{c}}_{\ell}$ in a special case. Let $(K_s)_{s \in S}$ be a small diagram in $\mathscr{C}^{\mathrm{c}}_{\ell}$. We put $M \coloneqq \varinjlim_{s \in S} \mathscr{I}^{\mathrm{c}}(K_s)$. We recall that the colimit of $(K_s)_{s \in S}$ in $\mathscr{C}^{\mathrm{c}}_{\ell}$ is given as $\overline{M}$ by Proposition 2.8 (i). Therefore if M is pre-compact, then $\mathrm{S}_{(\mathscr{I}^{\mathrm{c}}(K_s))_{s \in S}, \mathscr{I}^{\mathrm{c}}(K)}$ gives a morphism $\mathrm{S}^{\mathrm{c}}_{(K_s)_{s \in S},K} \colon \varinjlim_{s \in S} \mathscr{I}^{\mathrm{c}}(K_s \otimes^{\mathrm{c}} K) \to \mathscr{I}^{\mathrm{c}}(M \otimes^{\mathrm{c}} K)$ in $\mathscr{C}_{\ell}$ for any $K \in \mathrm{ob}(\mathscr{C}^{\mathrm{c}}_{\ell})$.

Proposition 3.7. *If M is pre-compact, then $\mathrm{S}_{(K_s)_{s \in S},K}$ is an isomorphism in $\mathscr{C}_{\ell}$ for any $K \in \mathrm{ob}(\mathscr{C}^{\mathrm{c}}_{\ell})$.*

Proof. For any $M' \in \mathrm{ob}(\mathscr{C}_{\ell})$, $\mathscr{L}(\mathrm{S}_{(K_s)_{s \in S},K}, M')$ is given as the composite of $\mathrm{T}^{\mathrm{c}}_{M,K,M'}$, the natural map

$$\mathrm{C}^{\mathrm{cg}}_{\mathrm{R}}(M,K,M') \to \varprojlim_{s \in S} \mathrm{C}^{\mathrm{cg}}_{\mathrm{R}}(K_s,K,M'), \ \varprojlim_{s \in S} (\mathrm{T}^{\mathrm{c}}_{K_s,K,M'})^{-1},$$

and the natural map $\varprojlim_{s \in S} \mathrm{C}^{\mathrm{cg}}_{\mathrm{L}}(K_s,K,M') \to \mathscr{L}(\varinjlim_{s \in S} \mathscr{I}^{\mathrm{c}}(K_s \otimes^{\mathrm{c}} K), M')$, which are bijective by Proposition 3.6 and the universality of colimits. Therefore $\mathrm{S}_{(K_s)_{s \in S},K}$ is an isomorphism in $\mathscr{C}_{\ell}$. □

Finally, we study $\mathscr{C}^{\mathrm{cg}}_{\ell}$. We put $M_0 \otimes^{\mathrm{cg}} M_1 \coloneqq \varinjlim_{(K_i) \in \prod \mathscr{K}(M_i)} K_0 \otimes^{\ell} K_1$ and $M_0 \times^{\mathrm{cg}} M_1 \coloneqq \varinjlim_{(K_i) \in \prod \mathscr{K}(M_i)} \prod \mathscr{U}^{\mathrm{c}}(K_i)$. By Proposition 2.9 (i) and Corollary 2.10, $M_0 \otimes^{\mathrm{cg}} M_1$ forms a CG linear topological O_k-module. By Corollary 2.7 and the naturality of ∇^{c}, the system $(\nabla^{\mathrm{c}}_{(K_i)})_{(K_i) \in \prod \mathscr{K}(M_i)}$ induces a continuous O_k-bilinear homomorphism

$$\nabla^{\mathrm{cg}\otimes}_{(M_i)} \colon M_0 \times^{\mathrm{cg}} M_1 \to \mathscr{U}^{\mathrm{cg}}(M_0 \otimes^{\mathrm{cg}} M_1).$$

Suppose $(M_i) \in \mathrm{ob}((\mathscr{C}^{\mathrm{cg}}_{\ell})^2)$ in the following in this subsection. By the universality of the colimit, the system of the inclusions $\prod \mathscr{U}^{\mathrm{c}}(K_i) \hookrightarrow \prod \mathscr{U}^{\mathrm{cg}}(M_i)$ indexed by $(K_i) \in \prod \mathscr{K}(M_i)$ induces a bijective continuous map $\nabla^{\mathrm{cg}\times}_{(M_i)} \colon M_0 \times^{\mathrm{cg}} M_1 \to \prod \mathscr{U}^{\mathrm{cg}}(M_i)$. By Proposition 2.3, the correspondences $(M_i) \rightsquigarrow M_0 \otimes^{\mathrm{cg}} M_1, M_0 \times^{\mathrm{cg}} M_1$ give functors $\otimes^{\mathrm{cg}} \colon (\mathscr{C}^{\mathrm{cg}}_{\ell})^2 \to \mathscr{C}^{\mathrm{cg}}_{\ell}$

and $\bullet_0 \times^{\mathrm{cg}} \bullet_1 \colon (\mathscr{C}_\ell^{\mathrm{cg}})^2 \to \mathrm{Top}$, respectively, and the correspondences $(M_i) \rightsquigarrow \nabla^{\mathrm{cg}\otimes}_{(M_i)}, \nabla^{\mathrm{cg}\times}_{(M_i)}$ give natural transformations $\nabla^{\mathrm{cg}\otimes} \colon \bullet_0 \times^{\mathrm{cg}} \bullet_1 \Rightarrow \mathscr{U}^{\mathrm{cg}}(\bullet_0 \otimes^{\mathrm{cg}} \bullet_1)$ and $\nabla^{\mathrm{cg}\times} \colon \bullet_0 \times^{\mathrm{cg}} \bullet_1 \Rightarrow \prod \mathscr{U}^{\mathrm{cg}}(\bullet_i)$, respectively.

Theorem 3.8. *The triad $(\mathscr{C}_\ell^{\mathrm{cg}}, \otimes^{\mathrm{cg}}, O_k)$ forms a closed symmetric monoidal category.*

We construct an exponential functor on $\mathscr{C}_\ell^{\mathrm{cg}}$. We put $\mathscr{L}((M_i), K, L) \coloneqq \{f \in \mathscr{L}((M_i)) \mid f(K) \subset L\}$ for a $(K, L) \in \mathscr{K}(M_0) \times \mathscr{O}(M_1)$, and denote by $\mathscr{H}\mathrm{om}^{\mathrm{cg}}((M_i))$ the O_k-module $\mathscr{L}((M_i))$ equipped with the topology generated by the set $\{f + \mathscr{L}((M_i), K, L) \mid (f, K, L) \in M_0 \times \mathscr{K}(M_0) \times \mathscr{O}(M_1)\}$. Then $\mathscr{H}\mathrm{om}^{\mathrm{cg}}((M_i))$ forms a linear topological O_k-module. We put $M_1^{M_0} \coloneqq \mathscr{H}\mathrm{om}^{\mathrm{cg}}((M_i))_{\mathscr{K}}$. By Proposition 2.3 and Corollary 2.10 (i), the correspondence $((M_i)) \rightsquigarrow M_1^{M_0}$ gives a functor $(\bullet_1)^{\bullet_0} \colon (\mathscr{C}_\ell^{\mathrm{cg}})^{\mathrm{op}} \times \mathscr{C}_\ell^{\mathrm{cg}} \to \mathscr{C}_\ell^{\mathrm{cg}}$. We denote by $\mathrm{C}_{\mathrm{L}}^{\mathrm{cg}}, \mathrm{C}_{\mathrm{R}}^{\mathrm{cg}} \colon ((\mathscr{C}_\ell^{\mathrm{cg}})^{\mathrm{op}})^2 \times \mathscr{C}_\ell^{\mathrm{cg}} \to \mathrm{Set}$ the functors given as $\mathrm{C}_{\mathrm{L}}^{\mathrm{cg}} \coloneqq \mathscr{L}(\bullet_0 \otimes^{\mathrm{cg}} \bullet_1, \bullet_2)$ and $\mathrm{C}_{\mathrm{R}}^{\mathrm{cg}} \coloneqq \mathscr{L}(\bullet_0, \bullet_2^{\bullet_1})$. We construct an adjunction $\mathrm{T}^{\mathrm{cg}} \colon \mathrm{C}_{\mathrm{L}}^{\mathrm{cg}} \Rightarrow \mathrm{C}_{\mathrm{R}}^{\mathrm{cg}}$. Let $m_0 \in M_0$.

Lemma 3.9. *The map $(m_0, \bullet) \colon \mathscr{U}^{\mathrm{cg}}(M_1) \hookrightarrow M_0 \times^{\mathrm{cg}} M_1, m_1 \mapsto (m_i)$ is continuous.*

Proof. Let $U \subset M_0 \times^{\mathrm{cg}} M_1$ be an open subset. By Corollary 2.5, we have $O_k m_0 \in \mathscr{K}(M_0)$. For any $K \in \mathscr{K}(M_1)$, the map $(m_0, \bullet)_K \colon K \hookrightarrow O_k m_0 \times K,\ m_1 \mapsto (m_0, M_1)$ is continuous, and hence $(m, \bullet)^{-1}(U) \cap K = (m, \bullet)_K^{-1}(U \cap (O_k m_0 \times K))$ is open in K. It implies $(m, \bullet)^{-1}(U)$ is open in M_1 by Corollary 2.7. Thus $(m_0, \bullet)$ is continuous. □

Let f be an O_k-linear homomorphism $M_0 \otimes^{\mathrm{cg}} M_1 \to M_2$ for a $M_2 \in \mathrm{ob}(\mathscr{C}_\ell^{\mathrm{cg}})$. By Corollary 2.7 and Proposition 3.4, we have the following characterisation of the continuity of f:

Proposition 3.10. *The map f is continuous if and only if $f \circ \nabla^{\mathrm{cg}\otimes}_{(M_i)}$ is continuous.*

Suppose that f is continuous. By Lemma 3.9, $f \circ \nabla^{\mathrm{cg}\otimes}_{(M_i)} \circ (m_0, \bullet)$ is continuous. We obtain an O_k-linear homomorphism

$$f_{\mathrm{R}} \colon M_0 \to \mathscr{H}\mathrm{om}^{\mathrm{cg}}((M_{i+1})),\ m_0 \mapsto f \circ \nabla^{\mathrm{cg}\otimes}_{(M_i)} \circ (m_0 \otimes \bullet).$$

Lemma 3.11. *The O_k-linear homomorphism f_{R} is continuous.*

Proof. Let $(K_1, L_2) \in \mathscr{K}(M_1) \times \mathscr{O}(M_2)$. Put $L := f_{\mathrm{R}}^{-1}(\mathscr{L}((M_i), K_1, L_2))$. We show $L \in \mathscr{O}(M_0)$. Let $K_0 \in \mathscr{K}(M_0)$. We denote by $f_{(K_i)} \colon K_0 \otimes^{\mathrm{c}} K_1 \to M_2$ the composite of f and the canonical morphism $K_0 \otimes^{\mathrm{c}} K_1 \to M_0 \otimes^{\mathrm{cg}} M_1$. By the continuity of f, $f_{(K_i)}$ is continuous. By Proposition 3.5, we have $L \cap K_0 \in \mathscr{O}(K_0)$. By Corollary 2.7, we obtain $L \in \mathscr{O}(M_0)$. Thus f_{R} is continuous. $\square$

The O_k-linear homomorphism $\mathrm{T}^{\mathrm{cg}}_{M_0,M_1,M_2}(f) \colon M_0 \to M_2^{M_1}$ given as the composite $(\iota^{\mathrm{cg}}_{\mathscr{H}\mathrm{om}^{\mathrm{cg}}((M_{i+1}))})^{-1} \circ f_{\mathrm{R}}$ is continuous by Corollary 2.10 (i) and Lemma 3.11. We obtain a map

$$\mathrm{T}^{\mathrm{cg}}_{M_0,M_1,M_2} \colon \mathrm{C}^{\mathrm{cg}}_{\mathrm{L}}(M_0, M_1, M_2) \to \mathrm{C}^{\mathrm{cg}}_{\mathrm{R}}(M_0, M_1, M_2),\ f \mapsto \mathrm{T}^{\mathrm{cg}}_{M_0,M_1,M_2}(f).$$

The correspondence $(M_i)_{i=0}^2 \rightsquigarrow \mathrm{T}^{\mathrm{cg}}_{M_0,M_1,M_2}$ gives a natural transformation $\mathrm{T}^{\mathrm{cg}} \colon \mathrm{C}^{\mathrm{cg}}_{\mathrm{L}} \Rightarrow \mathrm{C}^{\mathrm{cg}}_{\mathrm{R}}$.

Proof of Theorem 3.8. We denote by $(\mathrm{A}, \mathrm{L}, \mathrm{R}, \mathrm{B})$ the data of the associator, the left unitor, the right unitor, and the braiding of $(\mathscr{C}_\ell, \otimes^\ell, O_k)$. Let $M \in \mathrm{ob}(\mathscr{C}^{\mathrm{cg}}_\ell)$. The system $(\mathrm{L}_K|_{(\wp,K)_{(O_k,K)}})_{(\wp,K) \in \mathscr{K}(O_k) \times \mathscr{K}(M)}$ induces a morphism $\tilde{\mathrm{L}}_M \colon O_k \otimes^{\mathrm{cg}} M \to M$ in $\mathscr{C}^{\mathrm{cg}}_\ell$ by the functoriality of L and the universality of the colimit. We show that $\tilde{\mathrm{L}}_M$ is an isomorphism in $\mathscr{C}^{\mathrm{cg}}_\ell$. Let $L \in \mathscr{O}(O_k \otimes^{\mathrm{cg}} M)$. Since the preimage of L in $O_k \otimes^{\mathrm{c}} K$ is open and L_K is a homeomorphism, we have $\tilde{\mathrm{L}}_M(L) \cap K \in \mathscr{O}(K)$ for any $K \in \mathscr{K}(M)$. It ensures $\tilde{\mathrm{L}}_M(L) \in \mathscr{O}(M)$ by Corollary 2.7. Therefore $\tilde{\mathrm{L}}_M$ is an isomorphism in $\mathscr{C}^{\mathrm{cg}}_\ell$. The correspondence $M \rightsquigarrow \tilde{\mathrm{L}}_M$ gives a natural equivalence $\tilde{\mathrm{L}} \colon O_k \otimes^{\mathrm{cg}} \bullet \Rightarrow \mathrm{id}_{\mathscr{C}^{\mathrm{cg}}_\ell}$. Similarly, we also have a natural equivalence $\tilde{\mathrm{R}} \colon \bullet \otimes^{\mathrm{cg}} O_k \Rightarrow \mathrm{id}_{\mathscr{C}^{\mathrm{cg}}_\ell}$. Let $(M_i) \in \mathrm{ob}((\mathscr{C}^{\mathrm{cg}}_\ell)^2)$. The system $(\mathrm{B}_{(K_i)})_{(K_i) \in \prod \mathscr{K}(M_i)}$ induces a morphism $\tilde{\mathrm{B}}_{(M_i)} \colon M_0 \otimes^{\mathrm{cg}} M_1 \to M_1 \otimes^{\mathrm{cg}} M_0$ in $\mathscr{C}^{\mathrm{cg}}_\ell$ by the functoriality of B and the filtered colimit. The correspondence $(M_i) \rightsquigarrow \tilde{\mathrm{B}}_{(M_i)}$ gives a natural transformation $\tilde{\mathrm{B}} \colon \bullet_0 \otimes^{\mathrm{cg}} \bullet_1 \Rightarrow \bullet_1 \otimes^{\mathrm{cg}} \bullet_0$. By $\mathrm{B}^2 = \mathrm{id}_{\mathscr{C}_\ell \times \mathscr{C}_\ell}$, we obtain $\tilde{\mathrm{B}}^2 = \mathrm{id}_{\mathscr{C}^{\mathrm{cg}}_\ell \times \mathscr{C}^{\mathrm{cg}}_\ell}$.

Let $(M_i)_{i=0}^2 \in \mathrm{ob}((\mathscr{C}^{\mathrm{cg}}_\ell)^3)$. We define a morphism $\tilde{\mathrm{A}}_{M_0,M_1,M_2} \colon (M_0 \otimes^{\mathrm{cg}} M_1) \otimes^{\mathrm{cg}} M_2 \to M_0 \otimes^{\mathrm{cg}} (M_1 \otimes^{\mathrm{cg}} M_2)$ in $\mathscr{C}^{\mathrm{cg}}_\ell$. Let $(K_{0,1,0}, K_2) \in \mathscr{K}(M_0 \otimes^{\mathrm{cg}} M_1) \times \mathscr{K}(M_2)$. We denote by $K_{0,1} \subset M_0 \otimes^{\mathrm{cg}} M_1$ the closure of $K_{0,1,0}$, which is pre-compact by Proposition 2.1 (i). Let $(K_i) \in \prod \mathscr{K}(M_i)$. We denote by

$(K_i)_{K_{0,1}} \subset K_0 \otimes^{\mathrm{c}} K_1$ the preimage of $K_{0,1}$, and by

$$\mathrm{A}_{K_{0,1,0},K_0,K_1,K_2} \colon ((K_i)_{K_{0,1}}, K_2)_{K_0 \otimes^{\mathrm{c}} K_1, K_2} \to M_0 \otimes^{\mathrm{cg}} (M_1 \otimes^{\mathrm{cg}} M_2)$$

the composite of the inclusion $((K_i)_{K_{0,1}}, K_2)_{K_0 \otimes^{\mathrm{c}} K_1, K_2} \hookrightarrow (K_0 \otimes^{\mathrm{c}} K_1) \otimes^{\mathrm{c}} K_2$, A_{K_0,K_1,K_2}, and the natural morphism $K_0 \otimes^{\mathrm{c}} (K_1 \otimes^{\mathrm{c}} K_2) \to M_0 \otimes^{\mathrm{cg}} (M_1 \otimes^{\mathrm{cg}} M_2)$ in $\mathscr{C}_\ell$. By Proposition 2.6 and [6] Lemma 2.23, the system of inclusions $(K_i)_{K_{0,1}} \hookrightarrow K_{0,1}$ indexed by $(K_i) \in \prod \mathscr{K}(M_i)$ induces an isomorphism $\varinjlim_{(K_i) \in \prod \mathscr{K}(M_i)} (K_i)_{K_{0,1}} \to K_{0,1}$ in $\mathscr{C}_\ell^{\mathrm{c}}$. Therefore the system $(\mathrm{A}_{K_{0,1,0},K_0,K_1,K_2})_{(K_i) \in \prod \mathscr{K}(M_i)}$ induces a morphism $\mathrm{A}_{K_{0,1,0},K_2} \colon K_{0,1} \otimes^{\mathrm{c}} K_2 \to M_0 \otimes^{\mathrm{cg}} (M_1 \otimes^{\mathrm{cg}} M_2)$ in $\mathscr{C}_\ell$ by Proposition 3.7. We denote by $\tilde{\mathrm{A}}_{K_{0,1,0},K_2} \colon K_{0,1,0} \otimes^{\mathrm{c}} K_2 \to M_0 \otimes^{\mathrm{cg}} (M_1 \otimes^{\mathrm{cg}} M_2)$ the composite of the natural morphism $K_{0,1,0} \otimes^{\mathrm{c}} K_2 \to K_{0,1} \otimes^{\mathrm{c}} K_2$ in $\mathscr{C}_\ell$ and $\mathrm{A}_{K_{0,1,0},K_2}$. By the universality of the colimit, the system $(\tilde{\mathrm{A}}_{K_{0,1,0},K_2})_{(K_{0,1,0},K_2) \in \mathscr{K}(M_0 \otimes^{\mathrm{cg}} M_1) \times \mathscr{K}(M_2)}$ induces a morphism $\tilde{\mathrm{A}}_{M_0,M_1,M_2} \colon (M_0 \otimes^{\mathrm{cg}} M_1) \otimes^{\mathrm{cg}} M_2 \to M_0 \otimes^{\mathrm{cg}} (M_1 \otimes^{\mathrm{cg}} M_2)$ in $\mathscr{C}_\ell^{\mathrm{cg}}$. The correspondence $(M_i)_{i=0}^2 \rightsquigarrow \tilde{\mathrm{A}}_{M_0,M_1,M_2}$ induces a natural transformation $\tilde{\mathrm{A}} \colon (\bullet_0 \otimes^{\mathrm{cg}} \bullet_1) \otimes^{\mathrm{cg}} \bullet_2 \Rightarrow \bullet_0 \otimes^{\mathrm{cg}} (\bullet_1 \otimes^{\mathrm{cg}} \bullet_2)$. Similarly, we obtain a natural formation of the opposite direction, which is the inverse of $\tilde{\mathrm{A}}$.

By the construction, the data $(\tilde{\mathrm{A}}, \tilde{\mathrm{L}}, \tilde{\mathrm{R}}, \tilde{\mathrm{B}}, \mathrm{T}^{\mathrm{cg}})$ is sent to the data of the associator, the left unitor, the right unitor, the braiding, and the Currying of $(\mathscr{C}, \otimes_{O_k}, O_k)$ through $\mathscr{F}^{\mathrm{cg}}$ and ι^{cg}. Since $\mathscr{F}^{\mathrm{cg}}$ is faithful, it ensures the coherence so that $(\tilde{\mathrm{A}}, \tilde{\mathrm{L}}, \tilde{\mathrm{R}}, \tilde{\mathrm{B}})$ forms data of an associator, a left unitor, a right unitor, a braiding, and an injective Currying of $(\mathscr{C}_\ell^{\mathrm{cg}}, \otimes^{\mathrm{cg}}, O_k)$. We have only to verify that $\mathrm{T}^{\mathrm{cg}}_{M_0,M_1,M_2}$ is surjective. Let $f \in \mathrm{C}^{\mathrm{cg}}_{\mathrm{R}}(M_0, M_1, M_2)$. Put

$$f' \coloneqq \iota^{\mathrm{cg}}_{\mathscr{H}\mathrm{om}^{\mathrm{cg}}((M_{i+1}))} \circ f \colon M_0 \to \mathscr{H}\mathrm{om}^{\mathrm{cg}}((M_{i+1})).$$

Let $(K_i) \in \prod \mathscr{K}(M_i)$. We show that the O_k-linear homomorphism $f'_{(K_i)} \colon K_0 \otimes^{\mathrm{c}} K_1 \to M_2$, $(m_i) \mapsto f'(m_0)(m_1)$ is continuous. Let $L_2 \in \mathscr{O}(M_2)$. Put $L_0 \coloneqq (f')^{-1}(\mathscr{L}((K_1, M_2), L_2) \cap K_0$. By the continuity of f', we have $L_0 \in \mathscr{O}(K_0)$. It implies $(f'_{(K_i)})^{-1}(L_2) \in \mathscr{O}(K_0 \otimes^{\mathrm{c}} K_1)$ by $(L_0, K_1)_{(K_i)} \subset (f'_{(K_i)})^{-1}(L_2)$. Therefore $f'_{(K_i)}$ is continuous. By the universality of the colimit, the system $(f'_{(K_i)})_{(K_i) \in \prod \mathscr{K}(M_i)}$ gives a morphism $f_{\mathrm{L}} \colon M_0 \otimes^{\mathrm{cg}} M_1 \to M_2$ in $\mathscr{C}_\ell$. By the construction, we have $\mathrm{T}^{\mathrm{cg}}_{M_0,M_1,M_2}(f_{\mathrm{L}}) = f$. Thus $\mathrm{T}^{\mathrm{cg}}_{M_0,M_1,M_2}$ is surjective. □

As a consequence of Theorem 3.8, we obtain the following:

Corollary 3.12. *The functor* $\otimes^{\mathrm{cg}}$ *is cocontinuous.*

3.2 CGLT algebras A *CGLT* O_k*-algebra* is a monoid in $(\mathscr{C}_\ell^{\mathrm{cg}}, \otimes^{\mathrm{cg}}, O_k)$. We will verify that $O_k[[G]]$ forms a CGLT O_k-algebra. Before that, we give examples of CGLT O_k-algebras. For this purpose, we compare $\otimes^\ell$, the tensor product $\hat{\otimes}_k$ of Banach k-vector spaces (cf. [1] p. 12), and the tensor product of compact Hausdorff flat linear topological O_k-modules given as the inverse limit of the algebraic tensor product of finite quotients. For this purpose, we recall an elementary property of $\hat{\otimes}_k$.

Proposition 3.13. *For any* $(X, V) \in \mathrm{ob}(\mathrm{Top} \times \mathrm{Ban}_{\leq}^{\mathrm{ur}}(k))$*, the multiplication* $\mathrm{C}(X, k) \times V \to \mathrm{C}(X, V)$ *extends to a unique isomorphism* $\mathrm{C}(X, k) \hat{\otimes}_k V \to \mathrm{C}(X, V)$ *in* $\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)$*.*

Proof. The assertion immediately follows from the orthonormalisability of an unramified Banach k-vector space (cf. [8] IV 3 Corollaire 1, [2] 2.5.2 Lemma 2, and the proof of [11] Proposition 10.1). □

The underlying linear topological O_k-module of any Banach k-vector space is CG by Proposition 2.11 (iii). We denote by $\mathscr{I}_k \colon \mathrm{Ban}(k) \to \mathscr{C}_\ell^{\mathrm{cg}}$ the forgetful functor. Let $(V_i) \in \mathrm{ob}(\mathrm{Ban}(k)^2)$. By the definition of $\otimes^\ell$, $\mathscr{I}^{\mathrm{cg}}(\mathscr{I}_k(V_0)) \otimes^\ell \mathscr{I}^{\mathrm{cg}}(\mathscr{I}_k(V_1))$ is first countable. The natural embedding $\mathscr{I}^{\mathrm{cg}}(\mathscr{I}_k(V_0)) \otimes^\ell \mathscr{I}^{\mathrm{cg}}(\mathscr{I}_k(V_1)) \hookrightarrow \mathscr{I}^{\mathrm{cg}}(\mathscr{I}_k(V_0 \hat{\otimes}_k V_1))$ is a homeomorphism onto the dense image by the definition of $\otimes^\ell$ and $\hat{\otimes}_k$, and hence induces a homeomorphism $T_{(V_i)}^{\hat{\otimes}_k, \otimes^{\mathrm{cg}}} \colon \mathscr{I}_k(V_0) \otimes^{\mathrm{cg}} \mathscr{I}_k(V_1) \hookrightarrow \mathscr{I}_k(V_0 \hat{\otimes}_k V_1)$ onto the dense image by Proposition 2.11 (iii). The correspondence $(V_i) \rightsquigarrow T_{(V_i)}^{\hat{\otimes}_k, \otimes^{\mathrm{cg}}}$ gives a natural transformation $T^{\hat{\otimes}_k, \otimes^{\mathrm{cg}}} \colon \mathscr{I}_k(\bullet_0) \otimes^{\mathrm{cg}} \mathscr{I}_k(\bullet_1) \to \mathscr{I}_k(\bullet_0 \hat{\otimes}_k \bullet_1)$. As a consequence, we obtain the following:

Proposition 3.14. *Every Banach* k*-algebra, that is, monoid in* $(\mathrm{Ban}(k), \hat{\otimes}_k, k)$*, forms a CGLT* O_k*-algebra through* $\mathscr{I}_k$ *and* $T^{\hat{\otimes}_k, \otimes^{\mathrm{cg}}}$*.*

By [7] Corollary 2.8 (i), if G is a profinite group, then $\mathrm{C}(G, k)$ admits a unique Hopf monoid structure in $(\mathrm{Ban}(O_k), \hat{\otimes}_k, k)$ extending the pointwise k-algebra structure. Therefore by Proposition 3.14, we obtain the following:

Corollary 3.15. *If* G *is a profinite group, then* $\mathrm{C}(G, k)$ *admits a unique structure of a commutative CGLT* O_k*-algebra such that the multiplication is a continuous* O_k*-linear extension of the pointwise multiplication.*

Every compact topological O_k-module is CG by Proposition 2.1 (ii) and Proposition 2.11 (ii). We denote by $\mathscr{I}_{O_k} \colon \mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}} \hookrightarrow \mathscr{C}_{\ell}^{\mathrm{cg}}$ the inclusion. The natural O_k-linear homomorphism $\mathscr{I}^{\mathrm{c}}(K_0) \otimes^{\mathrm{c}} \mathscr{I}^{\mathrm{c}}(K_1) \to \mathscr{I}_{O_k}(K_0 \hat{\otimes}_{O_k} K_1)$ is a homeomorphism onto the dense image by the definition of $\otimes^{\mathrm{c}}$ and $\hat{\otimes}_{O_k}$, and it induces a homeomorphism $T_{(K_i)}^{\hat{\otimes}_{O_k}, \otimes^{\mathrm{c}}} \colon \mathscr{I}_{O_k}(K_0) \otimes^{\mathrm{cg}} \mathscr{I}_{O_k}(K_1) \hookrightarrow \mathscr{I}_{O_k}(K_0 \hat{\otimes}_{O_k} K_1)$ onto the dense image. The correspondence $(K_i) \rightsquigarrow T_{(K_i)}^{\hat{\otimes}_{O_k}, \otimes^{\mathrm{cg}}}$ gives a natural transformation

$$T^{\hat{\otimes}_{O_k}, \otimes^{\mathrm{cg}}} \colon \mathscr{I}_{O_k}(\bullet_0) \otimes^{\mathrm{cg}} \mathscr{I}_{O_k}(\bullet_1) \to \mathscr{I}_{O_k}(\bullet_0 \hat{\otimes}_{O_k} \bullet_1).$$

As a consequence, we obtain the following:

Proposition 3.16. *Every monoid in $(\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$ forms a CGLT O_k-algebra through $\mathscr{I}_{O_k}$ and $T^{\hat{\otimes}_{O_k}, \otimes^{\mathrm{cg}}}$.*

By Proposition 2.21 and [7] Proposition 2.7, if G is a profinite group, then $O_k[[G]]$ admits a unique Hopf monoid structure in $(\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}}. \hat{\otimes}_{O_k}, O_k)$ extending the Hopf O_k-algebra structure of $O_k[G]$. Therefore by Proposition 3.16, we obtain the following:

Corollary 3.17. *If G is a profinite group, then $O_k[[G]]$ admits a unique structure of a CGLT O_k-algebra extending the Hopf O_k-algebra structure of $O_k[G]$.*

We note that Corollary 3.17 will be extended to the case where G is not necessarily a profinite group, as we mentioned in the beginning of this subsection. Another simple example of a CGLT O_k-algebra is given by a topological O_k-algebra.

Proposition 3.18. *Let A be a topological O_k-algebra. If the underlying topological O_k-module M of A is linear and CG, then M admits a unique structure of a CGLT O_k-algebra whose multiplication is an O_k-linear extension of the multiplication of A through $\nabla_{M,M}^{\mathrm{cg}\otimes} \circ (\nabla_{M,M}^{\mathrm{cg}\times})^{-1}$.*

Proof. We denote by $f_{(K_i)} \colon K_0 \otimes^{\mathrm{c}} K_1 \to M$ the O_k-linear extension of the multiplication of A restricted to $\prod \mathscr{U}^{\mathrm{c}}(K_i) \subset \mathscr{U}^{\mathrm{cg}}(M)^2$, which is continuous for any $(K_i) \in \mathscr{K}(M)^2$ by Proposition 3.4. The system $(f_{K_0,k_1})_{(K_0,k_1) \in \mathscr{K}(M)^2}$ induces a continuous O_k-linear homomorphism $f \colon M \otimes^{\mathrm{cg}} M \to M$ by the universality of the colimit. We denote by ϵ the map $O_k \to M,\ c \mapsto c1$. Since the identity map $A \to (M, f, \epsilon)$ preserves the multiplication and the unit, (M, f, ϵ) satisfies the axiom of a monoid in $\mathscr{C}_\ell^{\mathrm{cg}}$. □

The rest of this subsection is devoted to the following extension of Corollary 3.17:

Theorem 3.19. *The CG linear topological O_k-module $O_k[[G]]$ admits a unique structure of a CGLT O_k-algebra such that $O_k^{\oplus d_G}$ is an O_k-algebra homomorphism.*

In order to verify Theorem 3.19, we define a convolution product on $\mathbb{M}(G)$. Let $(\mu_i) \in \mathbb{M}(G)^2$. We define elements $\prod \mu_i \in \mathbb{M}(G^2)$ and $\mu_0 * \mu_1 \in \mathbb{M}(G)$. Let $U' \in \mathrm{CO}(G^2)$. To begin with, suppose that U' is compact. Take an $S \in \mathscr{P}_{<\omega}(\mathrm{CO}(G)^2)$ satisfying $U' = \bigsqcup_{(U_i) \in S} \prod U_i$. We put $(\prod \mu_i)(U') \coloneqq \sum_{(U_i) \in S} \prod \mu_i(U_i)$. By the finite additivity of μ_0 and μ_1, $(\prod \mu_i)(U')$ depends only on U'. In particular, the equality $(\prod \mu_i)(\prod U_i) = \prod \mu_i(U_i)$ holds for any compact clopen subsets U_0 and U_1 of G.

Next, we consider the case where U' is not necessarily compact. Take a compact clopen subgroup $K \subset G$. Then $(G/K)^2 = \{(g_i K) \mid (g_i) \in G^2\}$ gives an element of $\mathbb{P}(G^2)$ consisting of compact clopen subsets. Put $(\prod \mu_i)(U') \coloneqq \sum_{(C_i) \in (G/K)^2} (\prod \mu_i)(U' \cap \prod C_i)$. By the normality of μ_0 and μ_1, the infinite sum in the right hand side actually converges, and $(\prod \mu_i)(U')$ is independent of the choice of K. We obtain a normal O_k-valued measure $\prod \mu_i$ on G^2.

For a $U \in \mathrm{CO}(G)$, we denote by $\widetilde{U} \subset G^2$ the preimage of U by the multiplication $G^2 \to G$. Set $(\mu_0 * \mu_1)(U) \coloneqq (\prod \mu_i)(\widetilde{U})$. Since $\prod \mu_i$ is a normal O_k-valued measure on G^2, so is $\mu_0 * \mu_1$ on G. We have constructed an element $\mu_0 * \mu_1 \in \mathbb{M}(G)$. By the construction, the convolution product $*_G \colon \mathscr{F}(\mathbb{M}(G))^2 \to \mathscr{F}(\mathbb{M}(G)) \colon (\mu_i) \mapsto \mu_0 * \mu_1$ is compatible with $O_k^{\oplus \delta_G}$ and the multiplication $O_k[G]^2 \to O_k[G]$. We note that $*_G$ is not necessarily continuous.

Lemma 3.20. *For any $(K_i) \in \mathscr{K}(\mathbb{M}(G))^2$, $\{\mu_0 * \mu_1 \mid (\mu_i) \in \prod K_i\} \subset \mathbb{M}(G)$ is pre-compact.*

Proof. Put $K \coloneqq \{\mu_0 * \mu_1 \mid (\mu_i) \in \prod K_i\}$. For each $i \in \{0, 1\}$, there is an increasing sequence $(G_{i,r})_{r \in \omega}$ of compact clopen subsets such that

$K_i \subset \mathbb{M}(G, (G_{i,r})_{r\in\omega}, (2^{-r})_{r\in\omega})$ by Lemma 2.19. For an $r \in \omega$, put $G_r \coloneqq \bigcup_{h=0}^{r} \{g_0 g_1 \mid (g_i) \in G_{r-h,0} \times G_{h,1}\}$. Then $(G_r)_{r\in\omega}$ forms an increasing sequence of compact clopen subsets of G satisfying $K \subset \mathbb{M}(G, (G_r)_{r\in\omega}, (2^{-r})_{r\in\omega})$ by the definition of $*_G$. Therefore K is pre-compact by Lemma 2.19. □

By the bijectivity of $\iota^{\mathrm{cg}}_{\mathbb{M}(G)}$ and $\nabla^{\mathrm{cg}\times}_{\mathbb{M}(G),\mathbb{M}(G)}$, $*_G$ induces an O_k-bilinear homomorphism $*^{\mathrm{cg}}_G \colon O_k[[G]] \times^{\mathrm{cg}} O_k[[G]] \to \mathscr{U}^{\mathrm{cg}}(O_k[[G]])$, $(\mu_i) \mapsto \mu_0 * \mu_1$ compatible with $O_k^{\oplus d_G}$ and the multiplication $O_k[G]^2 \to O_k[G]$.

Lemma 3.21. *The convolution product* $*^{\mathrm{cg}}_G$ *is continuous.*

Proof. Let $U \subset O_k[[G]]$ be an open neighbourhood of $\mu_0 * \mu_1$ for a $(\mu_i) \in \mathbb{M}(G)^2$. It suffices to show that for any $(K_i) \in \mathscr{K}(\mathbb{M}(G))^2$ satisfying $(\mu_i) \in \prod K_i$, the preimage of $(*^{\mathrm{cg}}_G)^{-1}(U)$ in $\prod K_i$ is open. Put $K \coloneqq \{\mu'_0 * \mu'_1 \mid (\mu'_i) \in \prod K_i\}$. By Lemma 3.20, K lies in $\mathscr{K}(\mathbb{M}(G))$. By Corollary 2.7, $\iota^{\mathrm{cg}}_{\mathbb{M}(G)}(U) \cap K$ is an open subset of K, and there is a $(P, \epsilon) \in \mathbb{P}(G) \times (0, 1]$ such that $(\mu_0 * \mu_1 + \mathbb{M}(G; P, \epsilon)) \cap K \subset \iota^{\mathrm{cg}}_{\mathbb{M}(G)}(U) \cap K$. Let $i \in \{0, 1\}$. By Lemma 2.18, there is a compact clopen subset $G_i \subset G$ such that $|\mu(U)| < \epsilon$ for any $(\mu, U) \in K_i \times \mathrm{CO}(G \setminus G_i)$. We obtain $\mu'_0 * \mu'_1 \in \iota^{\mathrm{cg}}_{\mathbb{M}(G)}(U) \cap K$ for any $(\mu'_i) \in \prod((\mu_i + \mathbb{M}(G; \{G_i, G \setminus G_i\}, \epsilon)) \cap K_i)$ by the definition of $*_G$. It implies that the preimage of $(*^{\mathrm{cg}}_G)^{-1}(U)$ in $\prod K_i$ is open. □

Proof of Theorem 3.19. The uniqueness follows from Proposition 2.14 and Proposition 2.17 (iii). By Corollary 2.7 and the cocontinuity of the forgetful functor Top → Set, $*_G$ induces an O_k-linear homomorphism $\otimes^{\mathrm{cg}}_G \colon O_k[[G]] \otimes^{\mathrm{cg}} O_k[[G]] \to O_k[[G]]$. The composite $\otimes^{\mathrm{cg}} \circ \nabla^{\mathrm{cg}\otimes}_{\mathbb{M}(G),\mathbb{M}(G)}$ coincides with $*^{\mathrm{cg}}_G$ by the construction, and hence is continuous by Lemma 3.21. The embedding $O_k^{\oplus d_G}$ sends the multiplication of $O_k[G]$ to $\otimes^{\mathrm{cg}}_G$ and the identity to $d_{G,1}$ by the construction. Since $O_k[G]$ satisfies the axiom of a monoid in $\mathscr{C}$, $O_k[[G]]$ forms a CGLT O_k-algebra with respect to the convolution product $\otimes^{\mathrm{cg}}_G$ and the unit $O_k \to O_k[[G]]$, $c \mapsto c d_{G,1}$ by Proposition 2.14, Proposition 2.17 (iii), and the continuity of $\otimes^{\mathrm{cg}}_G$. □

We have examples of the computation of the generalised Iwasawa algebra $O_k[[G]]$.

Example 3.22. (i) If G is discrete, then $O_k[[G]]$ is identified with $\mathrm{C}_0(G, O_k)$ equipped with the unique continuous extension of the O_k-algebra structure of the group algebra $O_k[G]$ through the correspondence in Proposition 3.14.

(ii) If G is a profinite group, then the algebra structure of $O_k[[G]]$ coincides with the one induced by the homeomorphic O_k-linear isomorphism in Proposition 2.21, and $O_k[[G]]$ is identified with the classical Iwasawa algebra associated to G through the correspondence in Proposition 3.16.

(iii) If G admits a closed subgroup $H \subset G$ and a compact subset $C \subset G$ such that the multiplication $H \times C \to G$ is bijective, then the multiplication is actually a homeomorphism by [6] Lemma 2.13, and hence $O_k[[G]]$ admits a natural homeomorphic O_k-linear isomorphism to $O_k[[H \times C]]$.

(iv) For any open subgroup $H \subset G$ and a discrete subset $D \subset G$ such that the multiplication $H \times D \to G$ is bijective, the multiplication is a homeomorphism by [3] p. 433, and hence $O_k[[G]]$ admits a natural homeomorphic O_k-linear isomorphism to $O_k[[H \times D]]$. In particular, $\mathscr{F}^{\mathrm{cg}}(O_k[[G]])$ admits a natural O_k-linear isomorphism to the ideal-adic completion of $\mathscr{F}^{\mathrm{cg}}(O_k[[H]])^{\oplus D}$ by Lemma 2.19.

(v) If G admits an increasing sequence $(G_r)_{r \in \omega}$ of open subgroups satisfying $\bigcup_{r \in \omega} G_r = G$, then $\mathscr{F}^{\mathrm{cg}}(O_k[[G]])$ admits a natural O_k-algebra isomorphism to the ideal-adic completion of $\varinjlim_{r \in \omega} \mathscr{F}^{\mathrm{cg}}(O_k[[G_r]])$ by Lemma 2.19.

As an application of Example 3.22 (ii) and (vi), we immediately obtain the following:

Proposition 3.23. *Let p denote the residual characteristic of k, and $\varpi \in O_k$ a uniformiser. Then $\mathscr{F}^{\mathrm{cg}}(O_k[[\mathbb{Q}_p]])$ admits a natural O_k-algebra isomorphism to the ϖ-adic completion of the filtered colimit of $\mathscr{F}^{\mathrm{c}}(O_k[[T]])$ with respect to the continuous O_k-algebra homomorphism $O_k[[T]] \to O_k[[T]]$, $T \mapsto (T+1)^p - 1$.*

3.3 CGLT modules

Let A be a CGLT O_k-algebra. A *CGLT A-module* is a left A-module in $(\mathscr{C}_\ell^{\mathrm{cg}}, \otimes^{\mathrm{cg}}, O_k)$. We give three examples of CGLT modules as immediate consequences of Proposition 3.14, Proposition 3.16, and Proposition 3.18, respectively:

Proposition 3.24. *Let $\mathscr{A}$ be a Banach k-algebra. Then every Banach left $\mathscr{A}$-module, that is, left $\mathscr{A}$-module in $(\mathrm{Ban}(k), \hat{\otimes}_k, k)$, forms a CGLT $\mathscr{I}_k(\mathscr{A})$-module through $\mathscr{I}_k$ and $T^{\hat{\otimes}_k, \otimes^{\mathrm{cg}}}$.*

Proposition 3.25. *Let $\mathscr{A}$ be a monoid in $(\mathscr{C}_{\mathfrak{f}\ell}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$. Then every left $\mathscr{A}$-module in $(\mathscr{C}_{\mathfrak{f}\ell}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$ forms a CGLT $\mathscr{I}_{O_k}(\mathscr{A})$-module through $\mathscr{I}_{O_k}$ and $T^{\hat{\otimes}_{O_k}, \otimes^{\mathrm{cg}}}$.*

Proposition 3.26. *Let $\mathscr{A}$ be a topological O_k-algebra whose underlying topological O_k-module is linear and CG. Then every topological left $\mathscr{A}$-module whose underlying topological O_k-module is linear and CG forms a CGLT $\mathscr{A}$-module through $\nabla^{\mathrm{cg}\otimes} \circ (\nabla^{\mathrm{cg}\times})^{-1}$.*

A *BT A-module* is a CGLT A-module V whose underlying O_k-module structure extends to a k-vector space structure equipped with a complete non-Archimedean norm on the underlying k-vector space of V giving its original topology. Let V be a BT A-module. Then V forms a topological k-vector space because it forms a Banach k-vector space. We say that V is *bounded* if there is an $R \in (0, \infty)$ such that $\|fv\| \leq R\|v\|$ for any $(f, v) \in A \times V$, is *submetric* if $\|fv\| \leq \|v\|$ for any $(f, v) \in A \times V$, and is *unitary* if it is submetric and the underlying Banach k-vector space V is unramified. We denote by $\mathrm{BT}(A)$ the k-linear category of bounded BT A-modules and bounded A-linear homomorphisms, by $\mathrm{BT}_{\leq}(A) \subset \mathrm{BT}(A)$ the O_k-linear subcategory of submetric BT A-modules and submetric A-linear homomorphisms, and by $\mathrm{BT}_{\leq}^{\mathrm{ur}}(A) \subset \mathrm{BT}_{\leq}(A)$ the full subcategory of unitary BT A-modules.

Let M be a CGLT A-module. We say that M is a *CHFLT A-module* if the underlying linear topological O_k-module of M is a compact Hausdorff flat linear topological O_k-module. We give a characterisation of a CHFLT A-module.

Proposition 3.27. *Let $K \in \mathrm{ob}(\mathscr{C}_{\ell}^{\mathrm{c}})$. For any a map $\rho \colon A \times K \to K$, K forms a CGLT A-module with respect to the O_k-linear extension of ρ if and only if $\mathscr{F}^{\mathrm{cg}}(K)$ forms a left $\mathscr{F}^{\mathrm{cg}}(A)$-module and ρ is continuous.*

Proof. We denote by $\tilde{\rho} \colon A \otimes^{\mathrm{cg}} K \to K$ the O_k-linear extension of ρ. The direct implication follows from Proposition 3.10 and the continuity of $\nabla_{A,K}^{\mathrm{cg}\otimes}$. Suppose that K forms a CGLT A-module with respect to $\tilde{\rho}$. By the naturality of $\nabla^{\mathrm{cg}\times}$, $\mathscr{F}^{\mathrm{cg}}(K)$ forms a left $\mathscr{F}^{\mathrm{cg}}(A)$-module with respect to ρ. Let $U \subset K$ be an open subset. Let $(f, m) \in A \times K$. We show that if $\rho(f, m) \in U$, then $\rho^{-1}(U)$ is an open neighbourhood of (f, m). Put $L \coloneqq \{f' \in A \mid \forall m' \in K, \rho(f + f', m') \in U\}$. Let $K_0 \in \mathscr{K}(A)$. By Proposition 2.1 (ii) and the continuity of the $\tilde{\rho}$, there is an $(L_i) \in \mathscr{O}(K_0) \times \mathscr{O}(K)$

such that $\rho(f', m') \in U$ for any $(f', m') \in ((f + L_0) \times K) \cup (K_0 \times (m + L_1))$. In particular, we have $L_0 \subset L \cap K_0$ and hence $L \cap K_0 \in \mathscr{O}(K_0)$. It implies $L \in \mathscr{O}(A)$ by Corollary 2.7. By $L \times K \subset \rho^{-1}(U)$, $\rho^{-1}(U)$ is an open neighbourhood of (f, m). Thus ρ is continuous. $\square$

A left A-submodule $K \subset M$ is said to be a *core of* M if K is compact, the inclusion $K \hookrightarrow M$ induces an isomorphism $k \otimes_{O_k} \mathscr{F}^{\mathrm{c}}(K) \to \mathscr{F}^{\mathrm{cg}}(M)$ in $\mathscr{C}$, and every O_k-submodule $L \subset M$ satisfying $cL \cap K \in \mathscr{O}(K)$ for any $c \in O_k \backslash \{0\}$ is open. We say that M is a *CGHLT A-module* if M is Hausdorff and admits a core. If M is a CGHLT A-module, then M forms a topological k-vector space because $\mathscr{O}(M)$ is stable under the action of $k^{\times}$. We denote by $\mathrm{Mod}_{\mathrm{f}\ell}^{\mathrm{ch}}(A)$ the O_k-linear category of CHFLT A-modules and continuous A-linear homomorphisms, and by $\mathrm{Mod}_{\ell}^{\mathrm{cgh}}(A)$ the k-linear category of CGHLT A-modules and continuous A-linear homomorphisms.

We give an example of a CGHLT A-module. Let $K \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{f}\ell}^{\mathrm{ch}}(A))$. We denote by K_k the left $\mathscr{F}^{\mathrm{cg}}(A)$-module $k \otimes_{O_k} \mathscr{F}^{\mathrm{c}}(K)$ equipped with the strongest topology for which K_k forms a topological k-vector space and the natural embedding $\iota_K^{\mathrm{c}} \colon K \hookrightarrow K_k$ is continuous. We identify $\mathscr{F}^{\mathrm{c}}(K)$ with its image in $k \otimes_{O_k} \mathscr{F}^{\mathrm{c}}(K)$. The following is an analogue of [13] Lemma 1.4:

Proposition 3.28. *The linear topological O_k-module K_k forms a CGHLT A-module, and ι_K^{c} is a homeomorphism onto a core.*

In order to verify Proposition 3.28, we characterise the topology of K_k.

Lemma 3.29. *A subset $U \subset K_k$ is open if and only if $(\iota_K^{\mathrm{c}})^{-1}(cU) \subset K$ is open for any $c \in O_k \setminus \{0\}$.*

Proof. We denote by $\mathscr{O}$ the set of subsets $U \subset K_k$ such that $(\iota_K^{\mathrm{c}})^{-1}(cU) \subset K$ is open for any $c \in O_k \setminus \{0\}$. Then $\mathscr{O}$ satisfies the open set axiom of the underlying set of K_k, for which ι_K^{c} is continuous and K_k forms a topological k-vector space because $\mathscr{O}$ is stable under the action of $k^{\times}$. Therefore by the universality of the strongest topology, $\mathscr{O}$ coincides with the set of open subsets of K_k. $\square$

Proof of Proposition 3.28. Take a uniformiser $\varpi \in O_k$. Put $K_r \coloneqq K$ for an $r \in \omega$, and denote by K_ω the colimit in $\mathscr{C}_\ell$ of $(K_r)_{r \in \omega}$ with respect to the transition maps $K_r \to K_{r+1}$, $m \mapsto cm$ indexed by $r \in \omega$. Then K_ω forms a CG linear topological O_k-module by Proposition 2.1 (ii), Proposition 2.9 (i), and Corollary 2.10. It is Hausdorff by the same computation as that in the proof of [6] Proposition 1.27 using Corollary 2.7 and a well-known property of T_1 normal topological spaces. By Corollary 3.12 and the functoriality of the colimit, the scalar multiplication $A \otimes^{\mathrm{cg}} K \to K$ induces a continuous O_k-linear homomorphism $A \otimes^{\mathrm{cg}} K_\omega \to K_\omega$, for which K_ω forms a CGLT A-module.

By the universality of the colimit and the flatness of K, ι_K^{c} induces a continuous bijective O_k-linear homomorphism $k\iota_K^{\mathrm{c}} \colon K_\omega \to K_k$. By Corollary 2.7, the map $K_\omega \to K_\omega$, $m \mapsto \varpi m$ is an isomorphism in $\mathscr{C}_\ell$, and hence K_ω forms a topological k-vector space. We show that $k\iota_K^{\mathrm{c}}$ is an open map. Let $L \in \mathscr{O}(K_\omega)$. For any $c \in O_k \setminus \{0\}$, $(\iota_K^{\mathrm{c}})^{-1}(c(k\iota_K^{\mathrm{c}})(L))$ coincides with the preimage of cL in K_0, and hence is open by the continuity of the canonical embedding $K_0 \hookrightarrow K_\omega$. It ensures $(k\iota_K^{\mathrm{c}})(L) \in \mathscr{O}(K_k)$ by Lemma 3.29. Therefore $k\iota_K^{\mathrm{c}}$ is an isomorphism in $\mathscr{C}_\ell$, and K_k forms a Hausdorff CGLT A-module. Since K is compact and K_k is Hausdorff, ι_K^{c} is a homeomorphism onto the image, which is a core of K_k. □

We obtain a characterisation of a CGHLT A-module.

Proposition 3.30. *If M is a CGHLT A-module with a core $K \subset M$, then the bijective O_k-linear homomorphism $K_k \to M$ induced by the inclusion $K \hookrightarrow M$ is an isomorphism in* $\mathrm{Mod}_\ell^{\mathrm{cgh}}(A)$.

Proof. We denote by $\varphi \colon K_k \to M$ the map in the assertion, and by $i \colon K \hookrightarrow K_k$ the canonical embedding. By the universality of the strongest topology, φ is continuous. Let $L \in K_k$. For any $c \in O_k \setminus \{0\}$, we have $c\varphi(L) \cap K = \varphi(cL) \cap K = i^{-1}(cL) \in \mathscr{O}(K)$. It implies $\varphi(L) \in \mathscr{O}(M)$. Therefore φ is an open map. □

The correspondence $K \rightsquigarrow K_k$ gives an O_k-linear functor $\Phi_A \colon \mathrm{Mod}_{\mathrm{f}\ell}^{\mathrm{ch}}(A) \to \mathrm{Mod}_\ell^{\mathrm{cgh}}(A)$ by Proposition 3.28. We denote by $\Phi_{A,k} \colon k\mathrm{Mod}_{\mathrm{f}\ell}^{\mathrm{ch}}(A) \to \mathrm{Mod}_\ell^{\mathrm{cgh}}(A)$ its k-linear extension.

Proposition 3.31. *The k-linear functor $\Phi_{A,k}$ is fully faithful and essentially surjective.*

Proof. The faithfulness of $\Phi_{A,k}$ follows from the faithfulness of Φ_A and the flatness of hom objects. The fullness follows from the same computation as that in the proof of [13] Lemma 1.5 ii and iii using Baire category theorem. The essential surjectivity follows from Proposition 3.30. □

By Proposition 2.1 (ii), Proposition 3.28, and Proposition 3.29, O_k forms a commutative CGLT O_k-algebra. By Proposition 3.27, $\mathscr{I}_{O_k}$ induces an equivalence $\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}} \to \mathrm{Mod}_{\mathrm{f}\ell}^{\mathrm{ch}}(O_k)$ of categories. By Proposition 3.28 and Proposition 3.29, we obtain an O_k-linear functor $\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}} \to \mathrm{Mod}_{\ell}^{\mathrm{cgh}}(O_k)$, which extends to a fully faithful essentially surjective k-linear functor $k\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}} \to \mathrm{Mod}_{\ell}^{\mathrm{cgh}}(O_k)$.

4 Modules over Iwasawa algebras

We study relation between module theory over $O_k[[G]]$ and representation theory of G. As a main result, we generalise Schneider–Teitelbaum duality to duality applicable to G, and give a criterion of the irreducibility of unitary Banach k-linear representations of G.

4.1 Unitary Banach representations A *Banach k-linear representation of G* is a pair (V, ρ) of a $V \in \mathrm{ob}(\mathrm{Ban}(k))$ and a continuous map $\rho \colon G \times V \to V$ giving a k-linear action of G on V. Let (V, ρ) be a Banach k-linear representation of G. We say that (V, ρ) is *unitarisable* if there is an $R \in (0, \infty)$ such that $\|\rho(g, v)\| \leq R\|v\|$ for any $(g, v) \in G \times V$, is *isometric* if $\|\rho(g, v)\| = \|v\|$ for any $(g, v) \in G \times V$, and is said to be *unitary* if V is unramified and (V, ρ) is isometric. A map between Banach k-linear representations is said to be a *$k[G]$-linear homomorphism* if it is a G-equivariant k-linear homomorphism. We denote by $\mathrm{Rep}_G(\mathrm{Ban}(k))$ the k-linear category of unitarisable Banach k-linear representations of G and bounded $k[G]$-linear homomorphisms, by $\mathrm{Rep}_G(\mathrm{Ban}_{\leq}(k)) \subset \mathrm{Rep}_G(\mathrm{Ban}(k))$ the O_k-linear subcategory of isometric Banach k-linear representations of G and submetric $k[G]$-linear homomorphisms, and by $\mathrm{Rep}_G(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)) \subset \mathrm{Rep}_G(\mathrm{Ban}_{\leq}(k))$ the full subcategory of unitary Banach k-linear representations of G.

We compare the notion of a BT $O_k[[G]]$-module and the notion of a Banach k-linear representation of G. For this purpose, we consider a partial generalisation of Banach–Steinhaus theorem (cf. [11] Corollary 6.16). Let $(X_0, (V_i)) \in \mathrm{ob}(\mathrm{Top} \times \mathrm{Ban}(k)^2)$.

Proposition 4.1. *A map $\varphi \colon X_0 \to \mathscr{S}((V_i))$ is continuous if and only if the map $X_0 \times V_1 \to V_2 \colon (x, v) \mapsto \varphi(x)(v)$ is continuous.*

Proof. We denote by $\rho\colon X_0 \times V_1 \to V_2$ the induced map. The direct implication follows from the continuity of the map $X_0 \to X_0 \times V_1,\ x \mapsto (x, v)$ for any $v \in V_1$. Suppose that φ is continuous. Let $U_2 \subset V_2$ be an open subset. Let $(x, v) \in X_0 \times V_1$. Suppose $\rho(x, v) \in U_2$. Take an $\epsilon \in (0, \infty)$ satisfying $\{v' \in V_2 \mid \|v' - \rho(x, v)\| < \epsilon\} \subset U_2$. Put $U_1 \coloneqq \{v' \in V_1 \mid \|v' - v\| < \epsilon\}$. By the continuity of φ, there is an open neighbourhood $U_0 \subset X_0$ of x such that $\|\rho(x', v) - \rho(x, v)\| < \epsilon$ for any $x' \in U_0$. We obtain $\|\rho(x', v') - \rho(x, v)\| \leq \max\{\|\rho(x', v' - v)\|, \|\rho(x', v) - \rho(x, v)\|\} < \epsilon$ for any $(x', v') \in \prod U_i$, and hence $\prod U_i \subset \rho^{-1}(U_2)$. It implies that ρ is continuous. $\square$

Let $(X, (V, \rho)) \in \mathrm{ob}(\mathrm{Top} \times \mathrm{Rep}_G(\mathrm{Ban}_{\leq}(k)))$. By Proposition 4.1, the monoid homomorphism $\varphi_\rho\colon G \to \mathscr{S}(V)^\times$ induced by ρ is continuous. In order to obtain a submetric BT $O_k[[G]]$-module structure on V associated to ρ, we prepare a partial generalisation of [13] Lemma 2.1 for the Banach space side.

Proposition 4.2. *The map* $\mathscr{L}(\mathbb{M}(X), \mathscr{S}(V)) \to \mathrm{Hom}_{\mathrm{Top}}(X, \mathscr{U}(\mathscr{S}(V)))$, $F \mapsto F \circ \delta_X$ *is bijective.*

Proof. Denote by δ_X^* the map in the assertion. By Proposition 2.16 (iii), δ_X^* is injective. Let $\varphi \in \mathrm{Hom}_{\mathrm{Top}}(X, \mathscr{U}(\mathscr{S}(V)))$. Denote by $O_k^{\oplus \varphi}\colon O_k^{\oplus X} \to \mathscr{S}(V)$ the O_k-linear extension of φ. Let $(v, \epsilon) \in V \times (0, 1]$. Put $L \coloneqq \{f \in \mathscr{S}(V) \mid |f(v)| < \epsilon\}$. By the continuity of φ, the set of the preimages of open balls in V of radius ϵ by the map $X \to V,\ x \mapsto \varphi(x)(v)$ gives a $P \in \mathbb{P}(X)$. We have $(O_k^{\oplus \delta_X})^{-1}(\mathbb{M}(X; P, \epsilon)) \subset (O_k^{\oplus \varphi})^{-1}(L)$. Therefore $O_k^{\oplus \varphi}$ extends to a unique continuous O_k-linear homomorphism $\tilde{\varphi}\colon \mathbb{M}(X) \to \mathscr{S}(V)$ by Proposition 2.16 (iii). We have $\delta_X^*(\tilde{\varphi}) = \varphi$. Thus δ_X^* is surjective. $\square$

As a consequence of Theorem 3.19 and Proposition 4.2, we obtain the following:

Corollary 4.3. *For any continuous monoid homomorphism* $\varphi\colon G \to \mathscr{S}(V)^\times$ *there is a unique continuous* O_k*-linear homomorphism* $F\colon \mathbb{M}(G) \to \mathscr{S}(V)$ *satisfying* $F \circ \delta_G = \varphi$*, and* $F \circ \iota^{\mathrm{cg}}_{\mathbb{M}(G)}$ *preserves the multiplication and the unit.*

By Corollary 4.3, φ_ρ induces a continuous O_k-linear homomorphism $\tilde{\Pi}_\rho\colon \mathbb{M}(G) \to \mathscr{S}(V)$such that $\tilde{\Pi} \circ \iota^{\mathrm{cg}}_{\mathbb{M}(G)}$ preserves the multiplication and the unit. We have a comparison between the closed unit balls of $\mathscr{H}\mathrm{om}^{\mathrm{cg}}$ and $\mathscr{B}$.

Proposition 4.4. *The identity map*

$$\mathscr{H}\mathrm{om}^{\mathrm{cg}}(\mathscr{I}_k(V), \mathscr{I}_k(V)) \cap \mathrm{End}_{\mathrm{Ban}_{\leq}(k)}(V) \to \mathscr{S}(V)$$

is an isomorphism in $\mathscr{C}_\ell$.

Proof. Denote by i the map in the assertion. By Corollary 2.5, i is continuous. Let $L \in \mathscr{O}(\mathscr{H}\mathrm{om}^{\mathrm{cg}}(\mathscr{I}_k(V), \mathscr{I}_k(V)) \cap \mathrm{End}_{\mathrm{Ban}_{\leq}(k)}(V))$. Take a $(K, \epsilon) \in \mathscr{K}(\mathscr{I}_k(V)) \times (0, \infty)$ satisfying $\{f \in \mathrm{End}_{\mathrm{Ban}_{\leq}(k)}(V) \mid \forall v \in K, |f(v)| < \epsilon\} \subset L$ and a $K_0 \in \mathscr{P}_{<\omega}(K)$ satisfying $K \subset \bigcup_{v \in K_0} \{v' \in V \mid |v' - v| < \epsilon\}$. We have $\{f \in \mathscr{S}(V) \mid \forall v \in K_0, |f(v)| < \epsilon\} \subset i(L)$, and hence $i(L) \in \mathscr{O}(\mathscr{S}(V))$. Therefore i is an isomorphism in $\mathscr{C}_\ell$. □

By Corollary 2.10 (i) and Proposition 4.4, $\tilde{\Pi}_\rho$ induces a continuous O_k-linear homomorphism $\Pi_\rho \colon O_k[[G]] \to \mathscr{I}_k(V)^{\mathscr{I}_k(V)}$ preserving the multiplication and the unit. By Theorem 3.8, Π_ρ gives a CGLT $O_k[[G]]$-module structure on $\mathscr{I}_k(V)$, for which $\mathscr{I}_k(V)$ forms a submetric BT $O_k[[G]]$-module $\int_G(V, \rho)$. By the construction, the correspondence $(V, \rho) \rightsquigarrow \int_G(V, \rho)$ gives O_k-linear functors $\int^{\mathrm{d}}_{G,\leq} \colon \mathrm{Rep}_G(\mathrm{Ban}_{\leq}(k)) \to \mathrm{BT}_{\leq}(O_k[[G]])$ and

$$\int^{\mathrm{d}}_{G,\mathrm{ur}} \colon \mathrm{Rep}_G(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k)) \to \mathrm{BT}^{\mathrm{ur}}_{\leq}(O_k[[G]]).$$

Each step of the construction of $\int_G(V, \rho)$ is obviously invertible, and hence we obtain a comparison between the notion of a submetric BT $O_k[[G]]$-module and the notion of an isometric Banach k-linear representation of G.

Theorem 4.5. *The functors* $\int^{\mathrm{d}}_{G,\leq}$ *and* $\int^{\mathrm{d}}_{G,\mathrm{ur}}$ *are equivalences of* O_k*-linear categories.*

We also consider a similar comparison without the assumption of the submetric condition. Let $(V, \rho) \in \mathrm{ob}(\mathrm{Rep}_G(\mathrm{Ban}(k)))$. Take a $c \in k^\times$ satisfying $\|\rho(g, v)\| \leq |c| \, \|v\|$ for any $(g, v) \in G \times V$. By Proposition 4.1, the map $G \to \mathscr{S}(V)$ induced by the continuous map $G \times V \to V,\ (g, v) \mapsto c^{-1}\rho(g, v)$ is continuous. By Proposition 4.2, it induces a continuous O_k-linear homomorphism $\mathbb{M}(G) \to \mathscr{S}(V)$, which does not necessarily preserve the multiplication. By Corollary 2.10 (i) and Proposition 4.4, it induces a continuous O_k-linear homomorphism $O_k[[G]] \to \mathscr{I}_k(V)^{\mathscr{I}_k(V)}$. Multiplying c, we obtain a continuous O_k-linear homomorphism $\Pi_\rho \colon O_k[[G]] \to \mathscr{I}_k(V)^{\mathscr{I}_k(V)}$ independent of the choice of c preserving the multiplication and the unit. By Theorem 3.8, Π_ρ gives a CGLT $O_k[[G]]$-module structure on $\mathscr{I}_k(V)$, for which $\mathscr{I}_k(V)$ forms a bounded BT $O_k[[G]]$-module $\int_G(V,\rho)$. By the construction, the correspondence $(V,\rho) \rightsquigarrow \int_G(V,\rho)$ gives a k-linear functor $\int_G^{\mathrm{d}} \colon \mathrm{Rep}_G(\mathrm{Ban}(k)) \to \mathrm{BT}(O_k[[G]])$. Each step of the construction of $\int_G(V,\rho)$ is obviously invertible, and hence we obtain a comparison between the notion of a bounded BT $O_k[[G]]$-module and the notion of a unitarisable Banach k-linear representation of G.

Theorem 4.6. *The functor $\int_G^{\mathrm{d}}$ is a k-linear equivalence of categories.*

4.2 CHFLT modules A *CGLT O_k-linear representation of G* is a pair (M,ρ) of an $M \in \mathrm{ob}(\mathscr{C}_\ell^{\mathrm{cg}})$ and a continuous map $\rho \colon G \times M \to M$ giving an O_k-linear action of G on M. A map between CGLT O_k-linear representations is said to be an *$O_k[G]$-linear homomorphism* if it is a G-equivariant O_k-linear homomorphism. Let (M,ρ) be a CGLT O_k-linear representation of G. We say that (M,ρ) is a *CHFLT O_k-linear representation of* G if $M \in \mathrm{ob}(\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}})$. We denote by $\mathrm{Rep}_G(\mathscr{C}_\ell^{\mathrm{cg}})$ the O_k-linear category of CGLT O_k-linear representations of G and continuous $O_k[G]$-linear homomorphisms, and by $\mathrm{Rep}_G(\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}}) \subset \mathrm{Rep}_G(\mathscr{C}_\ell^{\mathrm{cg}})$ the full subcategory of CHFLT O_k-linear representations of G.

We compare the notion of a CHFLT $O_k[[G]]$-module and the notion of a CHFLT O_k-linear representation of G. For this purpose, we consider a compact analogue of Banach–Steinhaus theorem (cf. [11] Corollary 6.16). We denote by Unf the category of compact uniform spaces and uniformly continuous maps. Let $(X_0,(C_{i+1})) \in \mathrm{ob}(\mathrm{Top} \times \mathrm{Unf}^2)$. We equip $\mathrm{Hom}_{\mathrm{Unf}}((C_{i+1}))$ the topology of uniform convergence.

Proposition 4.7. *A map $\varphi \colon X_0 \to \mathrm{Hom}_{\mathrm{Unf}}((C_{i+1}))$ is continuous if and only if the induced map $X_0 \times C_1 \to C_2 \colon (x,m) \mapsto \varphi(x)(m)$ is continuous.*

Proof. If $C_1 = \emptyset$, then the assertion is obvious. We assume $C_1 \neq \emptyset$. We denote by $\rho \colon X_0 \times C_1 \to C_2$ the induced map. Suppose that φ is continuous. Let $U_2 \subset C_2$ be an open neighbourhood of $\rho(x_0,m_0)$ for a $(x_0,m_0) \in X_0 \times C_1$. Take entourages $E_0, E_1 \subset C_2^2$ satisfying $\{m_1 \in C_2 \mid (\rho(x_0,m_0),m_1) \in E_0\} \subset U_2$ and that for any $(m_i)_{i=0}^2 \in C_2^3$, $((m_i),(m_{i+1})) \in E_1^2$ implies $(m_{2i}) \in E_0$.

By the uniform continuity of $\varphi(x_0)$, there is an entourage $E_2 \subset C_1^2$ such that every $(m_i) \in E_2$ satisfies $(\varphi(x_0)(m_i)) \in E_1$. By the continuity of φ, there exists an open neighbourhood $U_0 \subset X$ of x_0 such that $(\varphi(x_{1-i})(m_1)) \in E_1$ for any $(x_1, m_1) \in U_0 \times C_1$. Put $U_1 \coloneqq \{m_1 \in C_1 \mid (m_i) \in E_2\}$. Then for any $(x_1, m_1) \in \prod U_i$, we have $((\rho(x_0, m_i)), (\rho((x_i, m_1))) \in E_1^2$, and hence $(\rho(x_i, m_i)) \in E_0$. It implies that $\prod U_i \subset \rho^{-1}(U_2)$. Therefore ρ is continuous.

Suppose that ρ is continuous. Let $U \subset \mathrm{Hom}_{\mathrm{Unf}}((C_{i+1}))$ be an open neighbourhood of $\varphi(x_0)$ for a $x_0 \in X_0$. For an entourage $E \subset C_2^2$, set $U_E \coloneqq \{f \in \mathrm{Hom}_{\mathrm{Unf}}((C_{i+1})) \mid \forall m \in C_1, (\varphi(x_0)(m), f(m)) \in E\}$. Then the collection of subsets of the form U_E forms a fundamental system of neighbourhoods of $\varphi(x_0)$. Take entourages $E_0, E_1 \subset C_2^2$ satisfying $U_{E_0} \subset U$ and that for any $(m_i)_{i=0}^2 \in C_2^3$, $((m_i), (m_{2i})) \in E_1^2$ implies $(m_{i+1}) \in E_0$. For each $m_0 \in C_1$, there are open neighbourhoods $U_0 \subset X$ and $U_1 \subset C_1$ of x_0 and m_0, respectively, such that $(\rho(x_i, m_i)) \in E_1$ for any $(x_1, m_1) \in \prod U_i$ by the continuity of ρ. We denote by S the set of such an $(m_0, (U_i))$ satisfying $m_0 \in C_1$. Since C_1 is compact and non-empty, there is an $S_0 \in \mathscr{P}_{<\omega}(S) \backslash \{\emptyset\}$ such that $C_1 = \bigcup_{(m_0,(U_i)) \in S_0} U_1$. Put $V_0 \coloneqq \bigcap_{(m_0,(U_i)) \in S_0} U_0$. Let $x_1 \in V_0$. We show $\varphi(x_1) \in U_{E_0}$. Let $m_0 \in C_1$. Take an $(m_1, (U_i)) \in S_0$ satisfying $m_0 \in U_1$. We have $((\rho(x_0, m_i)), (\rho(x_i, m_i))) \in E_1^2$ by the choice of m_1 and U_1. Therefore we obtain $(\varphi(x_i)(m_1)) = (\rho(x_i, m_1)) \in E_0$ by the choice of E_1. It ensures $\varphi(x_1) \in U_{E_0}$. It implies that $V_0 \subset \varphi^{-1}(U_{E_0})$. Thus φ is continuous. $\square$

Let $(K, \rho) \in \mathrm{ob}(\mathrm{Rep}_G(\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}}))$. The monoid homomorphism $\varphi_\rho \colon G \to \mathscr{H}\mathrm{om}^{\mathrm{c}}(K, K)^\times$ induced by ρ is continuous by Proposition 4.7. In order to obtain a CHFLT $O_k[[G]]$-module structure on K associated to ρ, we prepare a partial generalisation of [13] Lemma 2.1 for the compact side. By Proposition 3.3 and Proposition 4.2, we obtain the following:

Proposition 4.8. *The map*

$$\mathscr{L}(\mathbb{M}(X), \mathscr{H}\mathrm{om}^{\mathrm{c}}(K, K)) \to \mathrm{C}(X, \mathscr{H}\mathrm{om}^{\mathrm{c}}(K, K)),\ F \mapsto F \circ \delta_X$$

is bijective.

By Theorem 3.19 and Proposition 4.8, we obtain a locally profinite counterpart of [13] Corollary 2.2 for the compact side.

Corollary 4.9. *For any continuous monoid homomorphism*

$$\varphi\colon G \to \mathscr{H}\mathrm{om}^{\mathrm{c}}(K,K)^{\times},$$

there is a unique continuous O_k-linear homomorphism

$$F\colon \mathbb{M}(G) \to \mathscr{H}\mathrm{om}^{\mathrm{c}}(K,K)$$

such that $\mathbb{F} \circ \delta_G = \varphi$, and $F \circ \iota^{\mathrm{cg}}_{\mathbb{M}(G)}$ preserves the multiplication and the unit.

By Corollary 2.10 (i) and Corollary 4.9, φ_ρ induces a continuous O_k-linear homomorphism $\Pi_\rho\colon O_k[[G]] \to \mathscr{I}_{O_k}(K)^{\mathscr{I}_{O_k}(K)}$ preserving the multiplication and the unit. By Theorem 3.8, Π_ρ gives a CGLT $O_k[[G]]$-module structure on $\mathscr{I}_{O_k}(K)$, for which $\mathscr{I}_{O_k}(K)$ forms a CHFLT $O_k[[G]]$-module $\int_G(K,\rho)$. By the construction, the correspondence $(K,\rho) \rightsquigarrow \int_G(K,\rho)$ gives an O_k-linear functor $\int^{\mathrm{c}}_G\colon \mathrm{Rep}_G(\mathscr{C}^{\mathrm{ch}}_{\mathrm{f}\ell}) \to \mathrm{Mod}^{\mathrm{ch}}_{\mathrm{f}\ell}(O_k[[G]])$. Each step of the construction of $\int_G(K,\rho)$ is obviously invertible, and hence we obtain a comparison between the notion of a CHFLT $O_k[[G]]$-module and the notion of a CHFLT O_k-linear representation of G.

Theorem 4.10. *The functor $\int^{\mathrm{c}}_G$ is an O_k-linear equivalence of categories.*

Let $(M,\rho) \in \mathrm{ob}(\mathrm{Rep}_G(\mathscr{C}^{\mathrm{cg}}_\ell))$. A G-stable O_k-submodule $K \subset M$ is said to be a *core of* (M,ρ) if K is compact, the inclusion $K \hookrightarrow M$ induces an isomorphism $k \otimes_{O_k} \mathscr{F}^{\mathrm{c}}(K) \to \mathscr{F}^{\mathrm{cg}}(M)$ in $\mathscr{C}$, and every O_k-submodule $L \subset M$ satisfying $cL \cap K \in \mathscr{O}(K)$ for any $c \in O_k \setminus \{0\}$ is open. We say that (M,ρ) is a *CGHLT k-linear representation of G* if M is Hausdorff and (M,ρ) admits a core. If (M,ρ) is a CGHLT k-linear representation of G, then M forms a topological k-vector space because $\mathscr{O}(M)$ is closed under the action of $k^{\times}$. We denote by $\mathrm{Rep}_G(k\mathscr{C}^{\mathrm{ch}}_{\mathrm{f}\ell}) \subset \mathrm{Rep}_G(\mathscr{C}^{\mathrm{cg}}_\ell)$ the full subcategory of CGHLT k-linear representations of G. We give an example of a CGHLT k-linear representation of G. We denote by $(K,\rho)_k$ the pair of $K_k \in \mathrm{ob}(\mathrm{Mod}^{\mathrm{cgh}}_\ell(O_k))$ and the k-linear extension of ρ.

Proposition 4.11. *The pair $(K,\rho)_k$ forms a CGHLT k-linear representation of G, and ι^{c}_K is a homeomorphic $O_k[G]$-linear isomorphism onto a core.*

Proof. By Proposition 3.28 applied to $A = O_k$, K_k is a CGHLT O_k-module, and ι_K^{c} is a homeomorphism onto a core of K_k. By Corollary 2.7, the k-linear extension of ρ gives a continuous map $G \times K_k \to K_k$. Therefore $(K,\rho)_k$ forms a CGLT O_k-linear representation of G. Since ι_K^{c} is $O_k[G]$-linear, $\iota_K^{\mathrm{c}}(K)$ forms a core of $(K,\rho)_k$. $\square$

By Proposition 4.11, the correspondence $(K,\rho) \rightsquigarrow (K,\rho)_k$ gives an O_k-linear functor $\Psi\colon \mathrm{Rep}_G(\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}}) \to \mathrm{Rep}_G(k\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}})$. We denote by

$$\Psi_k\colon k\mathrm{Rep}_G(\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}}) \to \mathrm{Rep}_G(k\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}})$$

its k-linear extension. By a similar argument to that in the proof of Proposition 3.31, we obtain a characterisation of a CGHLT k-linear representation of G.

Proposition 4.12. *The k-linear functor Ψ_k is fully faithful and essentially surjective.*

We compare the notion of a CGHLT $O_k[[G]]$-module and the notion of a CGHLT k-linear representation of G. Let $(M,\rho) \in \mathrm{ob}(\mathrm{Rep}_G(k\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}}))$. Take a core $K \subset (M,\rho)$. We abbreviate the pair of K and the restriction $G \times K \to K$ of ρ to (K,ρ). The scalar multiplication $O_k[[G]] \otimes^{\mathrm{cg}} \int_G (K,\rho) \to \int_G (K,\rho)$ induces a continuous O_k-linear homomorphism $O_k[[G]] \otimes^{\mathrm{cg}} K_k \to K_k$ by Corollary 3.12 and the functoriality of the colimit. Through the isomorphism $(K,\rho)_k \to (M,\rho)$ in $\mathrm{Rep}_G(k\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}})$ induced by the inclusion $K \hookrightarrow M$, we obtain a continuous O_k-linear homomorphism $O_k[[G]] \otimes^{\mathrm{cg}} M \to M$, for which M forms a CGHLT O_k-module $\int_G (M,\rho)$ with a core K. By the construction, the correspondence $(M,\rho) \rightsquigarrow \int_G (M,\rho)$ gives a k-linear functor $\int_{G,k}^{\mathrm{c}}\colon \mathrm{Rep}_G(k\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}}) \to \mathrm{Mod}_\ell^{\mathrm{cgh}}(O_k[[G]])$. We obtain a comparison between the notion of a CGHLT $O_k[[G]]$-module and the notion of a CGHLT k-linear representation of G.

Theorem 4.13. *The functor $\int_{G,k}^{\mathrm{c}}$ is a k-linear equivalence of categories.*

Proof. We construct an inverse. Let $M \in \mathrm{ob}(\mathrm{Mod}_\ell^{\mathrm{cgh}}(O_k[[G]]))$. We denote by M_0 the underlying CGHLT O_k-module of M. We show that the map $\rho_M\colon G \times M_0 \to M_0,\ (g,m) \mapsto d_{G,g} m$ is continuous. Take a core $K_1 \subset M$. We consider the composite $O_k[[G]] \otimes^{\mathrm{cg}} K_1 \to M$ of the O_k-linear homomorphism $O_k[[G]] \otimes^{\mathrm{cg}} K_1 \to O_k[[G]] \otimes^{\mathrm{cg}} M$ induced by the inclusion $K_1 \hookrightarrow M$, which is continuous by the functoriality of $\otimes^{\mathrm{cg}}$, and the scalar multiplication $O_k[[G]] \otimes^{\mathrm{cg}} M \to M$. Since K_1 is a left $O_k[[G]]$-submodule, it factors through $K_1 \subset M$. We obtain a continuous O_k-linear homomorphism $O_k[[G]] \otimes^{\mathrm{cg}} K_1 \to K_1$, for which K_1 forms a CHFLT $O_k[[G]]$-module.

Let $U_2 \subset M_0$ be an open subset. Take an open profinite subgroup $H \subset G$. For a $g \in G$, put $U_{g,1} \coloneqq gH$ and $U_{g,2} \coloneqq \{m \in M_0 \mid \forall g' \in U_{g,1}, \rho_M(g', m) \in U_2\}$. Then we have $\rho_M^{-1}(U_2) = \bigcup_{g \in G} \prod U_{g,i}$. Therefore in order to show that $\rho_M^{-1}(U_2)$ is open, it suffices to show $U_{g,2}$ is open for any $g \in G$. Let $g \in G$. We show that $cU_{g,2} \cap K_1$ is open in K_1 for any $c \in O_k \setminus \{0\}$. Let $c \in O_k \setminus \{0\}$. Let $m \in cU_{g,2} \cap K_1$. By Proposition 2.1 (ii), Corollary 2.5, and Proposition 2.17 (ii), we have $K_0 \coloneqq \sum_{g' \in H} O_k d_{G,gg'} \in \mathscr{K}(O_k[[G]])$. By Proposition 2.1 (ii) and the continuity of the scalar multiplication $O_k[[G]] \otimes^{\mathrm{cg}} K_1 \to K_1$, there is an $(L_i) \in \prod \mathscr{O}(K_i)$ such that $(d_{G,g} \otimes m) + (L_i)_{(K_i)}$ (cf. §3.1) is contained in the preimage of $\rho_M^{-1}(cU_2)$ in $K_0 \otimes^{\mathrm{c}} K_1$. In particular, we have $m + L_1 \subset cU_{g,2} \cap K_1$. It ensures that $cU_{g,2} \cap K_1$ is open in K_1. By Lemma 3.29 and Proposition 3.30, $U_{g,2}$ is open. It implies that ρ_M is continuous. We obtain a CGHLT k-linear representation (M_0, ρ_M) with a core K_1. The correspondence $M \rightsquigarrow (M_0, \rho_M)$ gives a functor $\mathrm{Mod}_\ell^{\mathrm{cgh}}(O_k[[G]]) \to \mathrm{Rep}_G(k\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}})$ which is a strict inverse of $\int_{G,k}^{\mathrm{c}}$. □

4.3 Generalised Schneider-Teitelbaum duality

Imitating the method of [13] Theorem 2.3, we extend $(\mathrm{D_d}, \mathrm{D_c})$ to an O_k-linear equivalence $(\mathscr{D}_{\mathrm{d}}, \mathscr{D}_{\mathrm{c}})$ of $\mathrm{Rep}_G(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k))^{\mathrm{op}}$ and $\mathrm{Mod}_{\mathrm{f}\ell}^{\mathrm{ch}}(O_k[[G]])$. Let $(V, \rho) \in \mathrm{ob}(\mathrm{Rep}_G(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)))$. For a $(g, m) \in G \times V^{\mathrm{D_d}}$, we denote by $\rho^{\mathrm{D_d}}(g, m)$ the submetric k-linear homomorphism $V \to k,\ v \mapsto m(\rho(g^{-1}, v))$. We obtain a map $\rho^{\mathrm{D_d}} \colon G \times V^{\mathrm{D_d}} \to V^{\mathrm{D_d}} \colon (g, m) \mapsto \rho^{\mathrm{D_d}}(g, m)$.

Proposition 4.14. *The map $\rho^{\mathrm{D_d}}$ is continuous.*

Proof. By Proposition 4.1, ρ induces a continuous monoid homomorphism $\varphi \colon G \to \mathscr{S}(V)^{\times}$. The map $G \to \mathscr{H}\mathrm{om}^{\mathrm{c}}(V^{\mathrm{D_d}}, V^{\mathrm{D_d}})^{\times},\ g \mapsto {}^{\mathrm{T}}(\bullet)_{V^{\mathrm{D_d}}, V^{\mathrm{D_d}}}^{-1}(\varphi(g^{-1}))$ is a continuous by Proposition 3.3. Therefore $\rho^{\mathrm{D_d}}$ is continuous by Proposition 4.7. □

By Proposition 4.14, the correspondence $(V, \rho) \rightsquigarrow (V^{\mathrm{D_d}}, \rho^{\mathrm{D_d}})$ gives an

O_k-linear functor $d\mathscr{D}_{\mathrm{d}} \colon \mathrm{Rep}_G(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k))^{\mathrm{op}} \to \mathrm{Rep}_G(\mathscr{C}^{\mathrm{ch}}_{\mathrm{f}\ell})$. We denote by $\mathscr{D}_{\mathrm{d}} \colon \mathrm{Rep}_G(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k))^{\mathrm{op}} \to \mathrm{Mod}^{\mathrm{ch}}_{\mathrm{f}\ell}(O_k[[G]])$ the composite of $\int^{\mathrm{c}}_G$ and $d\mathscr{D}_{\mathrm{d}}$.

Let $K \in \mathrm{ob}(\mathrm{Mod}^{\mathrm{ch}}_{\mathrm{f}\ell}(O_k[[G]]))$. We denote by $K_0 \in \mathrm{ob}(\mathscr{C}^{\mathrm{ch}}_{\mathrm{f}\ell})$ the underlying topological O_k-module of K. For a $(g, v) \in G \times K_0^{\mathrm{D_c}}$, we denote by $\rho_K(g, v)$ the continuous O_k-linear homomorphism $K_0 \to k$, $m \mapsto v(d_{G,g^{-1}} m)$. We obtain a map $\rho_K \colon G \times K_0^{\mathrm{D_c}} \to K_0^{\mathrm{D_c}} \colon (g, v) \mapsto \rho_K(g, v)$.

Proposition 4.15. *The map ρ_K is continuous.*

Proof. By Proposition 2.17 (ii) and Proposition 3.27, the map $G \times K \to K$, $(g, m) \mapsto d_{G,g} m$ is continuous. By Proposition 4.7, it induces a continuous monoid homomorphism $\varphi \colon G \to \mathscr{H}\mathrm{om}^{\mathrm{c}}(K_0, K_0)^{\times}$. The map $G \to \mathscr{H}\mathrm{om}^{\mathrm{c}}(K_0^{\mathrm{D_c}}, K_0^{\mathrm{D_c}})^{\times}$, $g \mapsto {}^{\mathrm{T}}\varphi(g^{-1})_{K_0,K_0}$ is continuous by Proposition 3.3. Therefore ρ_K is continuous by Proposition 4.1. □

We put $K^{\mathscr{D}_{\mathrm{c}}} := (K_0^{\mathrm{D_c}}, \rho_K)$. By Proposition 4.15, the correspondence $K \rightsquigarrow K^{\mathscr{D}_{\mathrm{c}}}$ gives an O_k-linear functor $\mathscr{D}_{\mathrm{c}} \colon \mathrm{Mod}^{\mathrm{ch}}_{\mathrm{f}\ell}(O_k[[G]]) \to \mathrm{Rep}_G(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k))^{\mathrm{op}}$. By Proposition 2.12, we obtain the following:

Theorem 4.16. *The pair $(\mathscr{D}_{\mathrm{d}}, \mathscr{D}_{\mathrm{c}})$ is an O_k-linear equivalence between $\mathrm{Rep}_G(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k))^{\mathrm{op}}$ and $\mathrm{Mod}^{\mathrm{ch}}_{\mathrm{f}\ell}(O_k[[G]])$.*

We obtain a generalised Schneider–Teitelbaum duality (cf. [13] Theorem 2.3).

Theorem 4.17. *The composite $k\mathrm{Mod}^{\mathrm{ch}}_{\mathrm{f}\ell}(O_k[[G]]) \to \mathrm{Rep}_G(\mathrm{Ban}(k))$ of $k\mathscr{D}_{\mathrm{c}}$ and the k-linear extension $k\mathrm{Rep}_G(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k)) \to \mathrm{Rep}_G(\mathrm{Ban}(k))$ of the inclusion $\mathrm{Rep}_G(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k)) \hookrightarrow \mathrm{Rep}_G(\mathrm{Ban}(k))$ is fully faithful and essentially surjective.*

Proof. The assertion follows from Theorem 4.5, Theorem 4.6, and Theorem 4.16 because the composite of the k-linear functor $k\mathrm{Rep}_G(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k)) \to \mathrm{Rep}_G(\mathrm{Ban}(k))$ and $\int^{\mathrm{d}}_G$ coincides with the composite of $k \int_{G,\mathrm{ur}}$ and the k-linear extension $k\mathrm{BT}^{\mathrm{ur}}_{\leq}(O_k[[G]]) \to \mathrm{BT}(O_k[[G]])$ of the inclusion

$$\mathrm{BT}^{\mathrm{ur}}_{\leq}(O_k[[G]]) \hookrightarrow \mathrm{BT}(O_k[[G]]).$$

□

Let $(V,\rho) \in \mathrm{ob}(\mathrm{Rep}_G(\mathrm{Ban}(k)))$ (respectively, $M \in \mathrm{ob}(\mathrm{Mod}_\ell^{\mathrm{cgh}}(O_k[[G]]))$). We say that (V,ρ) (respectively, M) is *irreducible* (respectively, *simple*) if it admits exactly two closed G-stable k-vector subspaces (respectively, closed left $O_k[[G]]$-submodules which are k-vector spaces). As an analogue of [13] Corollary 3.6, we obtain a criterion for the irreducibility.

Theorem 4.18. *Suppose that (V,ρ) is unitary. Then (V,ρ) is irreducible if and only if $((V,\rho)^{\mathscr{D}_{\mathrm{d}}})_k$ is simple.*

Proof. Suppose that $((V,\rho)^{\mathscr{D}_{\mathrm{d}}})_k$ is simple. We show that (V,ρ) is irreducible. We have $(V,\rho)^{\mathscr{D}_{\mathrm{d}}} \neq \{0\}$ by $((V,\rho)^{\mathscr{D}_{\mathrm{d}}})_k \neq \{0\}$, and hence $V \neq \{0\}$. Let $V_0 \subset V$ be a proper closed G-stable k-vector subspace. Then (V_0,ρ) forms a unitary Banach k-linear representation of G. By Hahn–Banach theorem (cf. [5] Theorem 3 and [11] Proposition 9.2), the restriction map $\pi\colon (V^{\mathrm{D}_{\mathrm{d}}})_k \to (V_0^{\mathscr{D}_{\mathrm{d}}})_k$ is surjective and $\ker \pi$ is a non-zero closed $O_k[[G]]$-submodule of $((V,\rho)^{\mathscr{D}_{\mathrm{d}}})_k$ which is a k-vector space. Since $((V,\rho)^{\mathscr{D}_{\mathrm{d}}})_k$ is simple, we obtain $\ker \pi = ((V,\rho)^{\mathscr{D}_{\mathrm{d}}})_k$. It ensures $V_0^{\mathscr{D}_{\mathrm{d}}} = \{0\}$, and hence $V_0 = \{0\}$ again by Hahn–Banach theorem. It implies that (V,ρ) is irreducible.

Suppose that (V,ρ) is irreducible. We show that $((V,\rho)^{\mathscr{D}_{\mathrm{d}}})_k$ is simple. Let $M_0 \subset ((V,\rho)^{\mathscr{D}_{\mathrm{d}}})_k$ be a proper closed $O_k[[G]]$-submodule which is a k-vector space. The identity map $\mathscr{F}^{\mathrm{c}}((V,\rho)^{\mathscr{D}_{\mathrm{d}}}) \to \mathrm{Hom}_{\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)}(V,k)$ induces a bijective k-linear homomorphism $\mathscr{F}^{\mathrm{cg}}((V^{\mathrm{D}_{\mathrm{d}}})_k) \to \mathrm{Hom}_{\mathrm{Ban}(k)}(V,k)$, through which we regard $\mathscr{F}(M_0)$ as a k-vector subspace of $\mathrm{Hom}_{\mathrm{Ban}(k)}(V,k)$. We have $((V,\rho)^{\mathscr{D}_{\mathrm{d}}})_k \neq \{0\}$ by $V \neq \{0\}$ and Hahn–Banach theorem. Put $V_0 \coloneqq \bigcap_{m \in M_0} \ker(m) \subset V$. We show $V_0 \neq \{0\}$. Let M denote the quotient $(V^{\mathrm{D}_{\mathrm{d}}})_k/M_0$. Since M_0 is a proper closed $O_k[[G]]$-submodule of $(V^{\mathrm{D}_{\mathrm{d}}})_k$ which is a k-vector space, M is a non-zero Hausdorff linear topological O_k-module which is a topological k-vector space. Therefore there is a non-zero continuous O_k-linear homomorphism $\overline{v}\colon M \to k$ by [6] Theorem 2.1. By the compactness of $V^{\mathrm{D}_{\mathrm{d}}}$ and the continuity of $\iota^{\mathrm{c}}_{V^{\mathrm{D}_{\mathrm{d}}}}$ and the canonical projection $((V,\rho)^{\mathscr{D}_{\mathrm{d}}})_k \twoheadrightarrow M$, we have $\sup_{m \in V^{\mathrm{D}_{\mathrm{d}}}} \overline{v}(\iota^{\mathrm{c}}_{V^{\mathrm{D}_{\mathrm{d}}}}(m) + M_0) < \infty$. Therefore there is a $v \in V \setminus \{0\}$ such that $\overline{v}(\iota^{\mathrm{c}}_{V^{\mathrm{D}_{\mathrm{d}}}}(m) + M_0) = m(v)$ for any $m \in V^{\mathrm{D}_{\mathrm{d}}}$ by Theorem 2.12. It ensures $m(v) = \overline{v}(0) = 0$ for any $m \in M_0$. We obtain $v \in V_0$ and hence $V_0 \neq \{0\}$. Since V_0 is a closed G-stable k-vector subspace of (V,ρ) and (V,ρ) is irreducible, we obtain $V_0 = V$. It ensures $M_0 = \{0\}$. It implies that $((V,\rho)^{\mathscr{D}_{\mathrm{d}}})_k$ is simple. □

5 Applications

As applications of the module theory in the monoidal structure, we give an explicit description of a continuous parabolic induction of unitary Banach k-linear representations.

5.1 Duality of operations Let $P \subset G$ be a closed subgroup. Suppose that $P \backslash G$ is compact. We study relations between the dual functors in §4.3 and operations on representations. Let $(V, \rho) \in \mathrm{ob}(\mathrm{Rep}_G(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)))$. We put $\mathrm{Res}_P^G(V, \rho) := (V, \rho|_{P \times V})$. The correspondence $(V, \rho) \rightsquigarrow \mathrm{Res}_P^G(V, \rho)$ gives an O_k-linear functor $\mathrm{Res}_P^G \colon \mathrm{Rep}_G(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)) \to \mathrm{Rep}_P(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k))$.

Let $K \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{f}\ell}^{\mathrm{ch}}(O_k[[G]]))$. We denote by $\mathrm{Res}_{O_k[[P]]}^{O_k[[G]]}(K)$ the scalar restriction of K by the natural embedding $O_k[[P]] \hookrightarrow O_k[[G]]$. The correspondence $K \rightsquigarrow \mathrm{Res}_{O_k[[P]]}^{O_k[[G]]}(K)$ gives an O_k-linear functor

$$\mathrm{Res}_{O_k[[P]]}^{O_k[[G]]} \colon \mathrm{Mod}_{\mathrm{f}\ell}^{\mathrm{ch}}(O_k[[G]]) \to \mathrm{Mod}_{\mathrm{f}\ell}^{\mathrm{ch}}(O_k[[P]]).$$

We have $\mathscr{D}_{\mathrm{d}} \circ \mathrm{Res}_P^G = \mathrm{Res}_{O_k[[P]]}^{O_k[[G]]} \circ \mathscr{D}_{\mathrm{d}} \colon \mathrm{Rep}_G(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)) \to \mathrm{Mod}_{\mathrm{f}\ell}^{\mathrm{ch}}(O_k[[P]])$ by the construction.

Let $(V_0, \rho_0) \in \mathrm{ob}(\mathrm{Rep}_P(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)))$. We denote by $\rho \colon G \times \mathrm{C}_{\mathrm{bd}}(G, V_0) \to \mathrm{C}_{\mathrm{bd}}(G, V_0)$ the map given by setting $\rho(g, f)(g') := f(g'g)$ for an $(f, g, g') \in \mathrm{C}_{\mathrm{bd}}(G, V_0) \times G^2$, which is not necessarily continuous. We set $\mathrm{Ind}_P^G(V_0) := \{f \in \mathrm{C}_{\mathrm{bd}}(G, V_0) \mid \forall (h, v) \in P \times G, f(hg) = \rho_0(h, f(g))\}$. Then $\mathrm{Ind}_P^G(V_0) \subset \mathrm{C}_{\mathrm{bd}}(G, V_0)$ is a closed G-equivariant k-vector subspace. We denote by $\mathrm{Ind}_P^G(\rho_0) \colon G \times \mathrm{Ind}_P^G(V_0) \to \mathrm{Ind}_P^G(V_0)$ the restriction of ρ. It can be easily verified that $\mathrm{Ind}_P^G(\rho_0)$ is continuous by Banach–Steinhaus theorem (cf. [11] Corollary 6.16), and $\mathrm{Ind}_P^G(V_0, \rho_0) := (\mathrm{Ind}_P^G(V_0), \mathrm{Ind}_P^G(\rho_0))$ forms a unitary Banach k-linear representation of G. The correspondence $(V_0, \rho_0) \rightsquigarrow \mathrm{Ind}_P^G(V_0, \rho_0)$ gives an O_k-linear functor

$$\mathrm{Ind}_P^G \colon \mathrm{Rep}_P(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)) \to \mathrm{Rep}_G(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)).$$

Let $K_0 \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{f}\ell}^{\mathrm{ch}}(O_k[[P]]))$. We describe $\mathrm{Ind}_P^G(K_0^{\mathscr{D}_{\mathrm{c}}})^{\mathscr{D}_{\mathrm{d}}}$ explicitly by G and K_0. Since the underlying topological space of G is a disjoint union of compact clopen subspaces, a map $\varphi \colon G \to K_0^{\mathscr{D}_{\mathrm{c}}}$ is continuous if and only if the induced map $G \times K_0 \to k\colon (g,m) \mapsto \varphi(g)(m)$ is continuous by Proposition 4.7. Therefore we obtain an isometric k-linear homomorphism $\mathrm{C}_{\mathrm{bd}}(G, K_0^{\mathscr{D}_{\mathrm{c}}}) \hookrightarrow \mathrm{C}_{\mathrm{bd}}(G \times K_0, k)$ onto the closed image. We consider the map $\rho\colon G \times \mathrm{C}_{\mathrm{bd}}(G \times K_0, k) \to \mathrm{C}_{\mathrm{bd}}(G \times K_0, k)$ given by setting $\rho(g,f)(g',m) := f(g'g,m)$ for a $(g,f,g',m) \in G \times \mathrm{C}_{\mathrm{bd}}(G \times K_0, k) \times G \times K_0$, which is not necessarily continuous. The inclusion $\mathrm{Ind}_P^G(K_0^{\mathscr{D}_{\mathrm{c}}}) \hookrightarrow \mathrm{C}_{\mathrm{bd}}(G, K_0^{\mathscr{D}_{\mathrm{c}}}) \subset \mathrm{C}_{\mathrm{bd}}(G \times K_0, k)$ is an isometric G-equivariant k-linear homomorphism, and its image is the closed G-stable k-vector subspace consisting of functions $f\colon G \times K_0 \to k$ satisfying the following:

(I) The equality $f(g,cm) = cf(g,m)$ holds for any $(g,c,m) \in G \times O_k \times K_0$.

(II) The equality $f(g, \sum m_i) = \sum f(g,m_i)$ holds for any $(g,(m_i)) \in G \times K_0^2$.

(III) The equality $f(hg,m) = f(g, \delta_{G,h}^{-1} m)$ holds for any $(h,g,m) \in P \times G \times K_0$.

The inclusion $\mathrm{Ind}_P^G(K_0^{\mathscr{D}_{\mathrm{c}}}) \hookrightarrow \mathrm{C}_{\mathrm{bd}}(G \times K_0, k)$ induces a continuous surjective G-equivariant O_k-linear homomorphism $\varphi_{G,P}\colon \mathrm{C}_{\mathrm{bd}}(G \times K_0, k)^{\mathrm{D_d}} \twoheadrightarrow \mathrm{Ind}_P^G(K_0^{\mathscr{D}_{\mathrm{c}}})^{\mathrm{D_d}}$ by Hahn–Banach theorem (cf. [5] Theorem 3 and [11] Proposition 9.2). Since the target and the source of $\varphi_{G,P}$ are compact and Hausdorff, the target is homeomorphic to the coimage. We determine $\ker(\varphi_{G,P})$ in order to describe the target. We denote by $e_{g,m}$ the submetric k-linear homomorphism $\mathrm{C}_{\mathrm{bd}}(G \times K_0, k) \to k$, $f \mapsto f(g,m)$ for a $(g,m) \in G \times K_0$. We put $\mu^{\mathrm{I}}_{g,c,m} := ce_{g,m} - e_{g,cm}$ for a $(g,c,m) \in G \times O_k \times K_0$, $\mu^{\mathrm{II}}_{g,(m_i)} := e_{g,\sum m_i} - \sum e_{g,m_i}$ for a $(g,(m_i)) \in G \times K_0^2$, and $\mu^{\mathrm{III}}_{g,h,m} := e_{hg,m} - e_{g,d_{G,h^{-1}}m}$ for a $(g,h,m) \in G \times P \times K_0$. We denote by $\mu^{\mathrm{I}} + \mu^{\mathrm{II}} + \mu^{\mathrm{III}} \subset \mathrm{C}_{\mathrm{bd}}(G \times K_0, k)^{\mathrm{D_d}}$ the closed O_k-submodule generated by the union of $\{\mu^{\mathrm{I}}_{g,c,m} \mid (g,c,m) \in G \times O_k \times K_0\}$, $\{\mu^{\mathrm{II}}_{g,(m_i)} \mid (g,(m_i)) \in G \times K_0^2\}$, and $\{\mu^{\mathrm{III}}_{g,h,m} \mid (g,h,m) \in G \times P \times K_0\}$.

Proposition 5.1. *The equality* $\ker(\varphi_{G,P}) = \mu^{\mathrm{I}} + \mu^{\mathrm{II}} + \mu^{\mathrm{III}}$ *holds.*

Proof. We have $\mu^{\mathrm{I}} + \mu^{\mathrm{II}} + \mu^{\mathrm{III}} \subset \ker(\varphi_{G,P})$ by the characterisation of the image of $\mathrm{Ind}_P^G(K_0^{\mathscr{D}_{\mathrm{c}}})$ in $\mathrm{C}_{\mathrm{bd}}(G \times K_0, k)$. Let $\mu \in \ker(\varphi_{G,P})$. We show $\mu \in \mu^{\mathrm{I}} + \mu^{\mathrm{II}} + \mu^{\mathrm{III}}$. Let $f \in \mathrm{C}_{\mathrm{bd}}(G \times K_0, k)$ and $\epsilon \in (0,\infty)$. We verify that there is a $\mu' \in \mu^{\mathrm{I}} + \mu^{\mathrm{II}} + \mu^{\mathrm{III}}$ such that $|\mu(f) - \mu'(f)| < \epsilon$. In the case

$f \in \mathrm{Ind}_P^G(K_0^{\mathscr{D}_c})$, we have $\mu(f) = \varphi_{G,P}(\mu)(f) = 0$, and hence $\mu' := 0$ satisfies the desired inequality. Suppose $f \notin \mathrm{Ind}_P^G(V_0)$. Then f does not satisfy at least one of the conditions (I)–(III) in the characterisation of of the image of $\mathrm{Ind}_P^G(K_0^{\mathscr{D}_c})$ in $\mathrm{C}_{\mathrm{bd}}(G \times K_0, k)$. First, suppose that f does not satisfy (I). Take a $(g, c, m) \in G \times O_k \times K_0$ satisfying $f(g, cm) - cf(g, m) \neq 0$. Set $\mu' := (f(g, cm) - cf(g, m))^{-1}\mu(f)\mu^{\mathrm{I}}_{g,c,m}$. Then we have $\mu'(f) = \mu(f)$ by the construction, and hence $|\mu(f) - \mu'(f)| = 0 < \epsilon$. Next, suppose that f does not satisfy (II). Take a $(g, (m_i)) \in G \times K_0^2$ satisfying $f(g, \sum m_i) - \sum f(g, m_i) \neq 0$. Set $\mu' := (f(g, \sum m_i) - \sum f(g, m_i))^{-1}\mu(f)\mu^{\mathrm{II}}_{g,m,m'}$. Then we have $\mu'(f) = \mu(f)$ by the construction, and hence $|\mu(f) - \mu'(f)| = 0 < \epsilon$. Finally, suppose that f does not satisfy (III). Take a $(g, h, m) \in G \times P \times K_0$ satisfying $f(hg, m) - f(g, \delta_{G,h}^{-1} m) \neq 0$. Set $\mu' := (f(hg, m) - f(g, \delta_{G,h}^{-1} m))^{-1}\mu(f)\mu^{\mathrm{III}}_{g,h,m}$. Then we have $\mu'(f) = \mu(f)$, and hence $|\mu(f) - \mu'(f)| = 0 < \epsilon$. It ensures $\mu \in \mu^{\mathrm{I}} + \mu^{\mathrm{II}} + \mu^{\mathrm{III}}$. We obtain $\ker(\varphi_{G,P}) = \mu^{\mathrm{I}} + \mu^{\mathrm{II}} + \mu^{\mathrm{III}}$. □

We set $\mathrm{Ind}_{O_k[[P]]}^{O_k[[G]]}(K_0) := \mathrm{C}_{\mathrm{bd}}(G \times K_0, k)^{\mathrm{D_d}}/(\mu^{\mathrm{I}} + \mu^{\mathrm{II}} + \mu^{\mathrm{III}})$. By Proposition 5.1, we obtain the following:

Theorem 5.2. *The continuous surjective O_k-linear homomorphism $\varphi_{G,P}$ induces a homeomorphic O_k-linear isomorphism*

$$\mathrm{Ind}_{O_k[[P]]}^{O_k[[G]]}(K_0) \to \mathrm{Ind}_P^G(K_0^{\mathscr{D}_c})^{\mathscr{D}_d}.$$

We equip $\mathrm{Ind}_{O_k[[P]]}^{O_k[[G]]}(K_0)$ with a CHFLT $O_k[[G]]$-module structure by pulling back that of $\mathrm{Ind}_P^G(K_0^{\mathscr{D}_c})^{\mathscr{D}_d}$ by the isomorphism in Theorem 5.2. The correspondence $K_0 \rightsquigarrow \mathrm{Ind}_{O_k[[P]]}^{O_k[[G]]}(K_0)$ gives an O_k-linear functor

$$\mathrm{Ind}_{O_k[[P]]}^{O_k[[G]]} \colon \mathrm{Mod}_{\mathrm{f}\ell}^{\mathrm{ch}}(O_k[[P]]) \to \mathrm{Mod}_{\mathrm{f}\ell}^{\mathrm{ch}}(O_k[[G]]).$$

By Theorem 4.16 and Theorem 5.2, we obtain the following:

Corollary 5.3. *There is a natural equivalence* $\mathrm{Ind}_P^G \Rightarrow \mathscr{D}_c \circ \mathrm{Ind}_{O_k[[P]]}^{O_k[[G]]} \circ \mathscr{D}_d$.

5.2 Continuous parabolic inductions As an application of Corollary 5.3, we compute the continuous parabolic induction. For this purpose, we give a more practical description of $\mathrm{Ind}_{O_k[[P]]}^{O_k[[G]]}$. To begin with, we prepare a compact complete representative $C \subset G$ of $P\backslash G$. We denote by Σ the set of open subsets $U \subset P\backslash G$ admitting a continuous section $U \hookrightarrow G$ of the canonical projection $G \twoheadrightarrow P\backslash G$. Take an open profinite subgroup $G_0 \subset G$. Since G is a topological group, the canonical projection $G \twoheadrightarrow P\backslash G$ is an open map. Therefore the image $\overline{G_0 g} \subset P\backslash G$ of $G_0 g$ is an open subset, and the map $G_0 \hookrightarrow G,\ h \mapsto hg$ induces a homeomorphism $(P \cap G_0)\backslash G_0 \to \overline{G_0 g}$ for any $g \in G$. It implies that Σ forms an open covering of $P\backslash G$ by [9] Theorem 2. Take a $\Sigma_0 \in \mathscr{P}_{<\omega}(\Sigma)$ satisfying $P\backslash G = \bigsqcup_{U \in \Sigma_0} U$. Gluing continuous sections on each $U \in \Sigma_0$, we obtain a continuous section $P\backslash G \hookrightarrow G$, whose image forms a compact subset $C \subset G$ such that the multiplication $P \times C \to G$ is a continuous bijective map. Conversely, let $C \subset G$ be an arbitrary compact subset such that the multiplication $P \times C \to G$ is a continuous bijective map. As is mentioned in Example 3.22 (iii), the multiplication $P \times C \to G$ is a homeomorphism, and induces a homeomorphic O_k-linear isomorphism $O_k[[P \times C]] \to O_k[[G]]$. We denote by $\pi_0 \colon G \twoheadrightarrow P$ (respectively, $\pi_1 \colon G \twoheadrightarrow C$) the composite of the inverse $G \to P \times C$ of the multiplication and the canonical projection $P \times C \twoheadrightarrow P$ (respectively, $P \times C \twoheadrightarrow P$). As a result, C is obtained as the image of the continuous section $P \setminus G \hookrightarrow G$ induced by π_1.

Let F be a local field, $\mathbb{G}$ an algebraic group over $\mathrm{Spec}(F)$, and $\mathbb{P} \subset \mathbb{G}$ a parabolic subgroup. Then $\mathbb{G}(F)$ forms a locally profinite group with respect to the topology induced by the valuation of F, and $\mathbb{P}(F)$ is naturally identified with a closed subgroup of $\mathbb{G}(F)$. Since $\mathbb{P}\backslash\mathbb{G}$ forms a proper algebraic variety over $\mathrm{Spec}(F)$, $\mathbb{P}(F)\backslash\mathbb{G}(F)$ forms a totally disconnected compact Hausdorff topological space. Henceforth, we consider the case $G = \mathbb{G}(F)$ and $P = \mathbb{P}(F)$.

Let $(V_0, \rho_0) \in \mathrm{ob}(\mathrm{Rep}_P(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)))$. We consider the composite $r_{C,V_0} \colon \mathrm{Ind}_P^G(V_0) \to \mathrm{C}(C, V_0)$ of the inclusion $\mathrm{Ind}_P^G(V_0) \hookrightarrow \mathrm{C}_{\mathrm{bd}}(G, V_0)$ and the restriction map $\mathrm{C}_{\mathrm{bd}}(C, V_0) \to \mathrm{C}(C, V_0)$. Then r_{C,V_0} is injective by the conditions (III) in §5.1 and $PC = G$. The quotient norm on the source of r_{C,V_0} coincides with the norm restricted to the image of r_{C,V_0} because P acts isometrically on V_0. Therefore r_{C,V_0} is isometric. For any $f \in \mathrm{C}(C, V_0)$, the map $\tilde{f} \colon G \to V_0,\ g \mapsto \rho_0(\pi_0(g), (f \circ \pi_1(g)))$ lies in $\mathrm{Ind}_P^G(K_0)$. We obtain an isometric section $\mathrm{C}(C, V_0) \to \mathrm{Ind}_P^G(V_0),\ f \mapsto \tilde{f}$, and hence r_{C,V_0} is an isomorphism in $\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)$. Pulling back $\mathrm{Ind}_P^G(\rho_0)$ by r_{C,V_0} and the isomorphism

$\mathrm{C}(C,k)\hat{\otimes}_k V_0 \to \mathrm{C}(C,V_0)$ in $\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)$ introduced in Proposition 3.13, we equip $\mathrm{C}(C,k)\hat{\otimes}_k V_0$ with a continuous action $C\hat{\otimes}_k \rho_0$ of G. By Theorem 5.2, we obtain an isomorphism $\mathrm{Ind}_{O_k[[P]]}^{O_k[[G]]}((V_0,\rho_0)^{\mathscr{D}_{\mathrm{d}}}) \to (\mathrm{C}(C,k)\hat{\otimes}_k V_0, C\hat{\otimes}_k \rho_0)^{\mathscr{D}_{\mathrm{d}}}$ in $\mathrm{Mod}_{\mathrm{f}\ell}^{\mathrm{ch}}(O_k[[G]])$. By Proposition 2.15 and [7] Theorem 2.2, we have a natural isomorphism $O_k[[C]]\hat{\otimes}_{O_k} V_0^{\mathrm{D_c}} \to (\mathrm{C}(C,k)\hat{\otimes}_k V_0)^{\mathrm{D_d}}$ in $\mathscr{C}_{\mathrm{f}\ell}^{\mathrm{ch}}$. Pulling back the scalar multiplication of $O_k[[G]]$ on $(\mathrm{C}(C,k)\hat{\otimes}_k V_0, C\hat{\otimes}_k \rho_0)^{\mathscr{D}_{\mathrm{d}}}$, we regard $O_k[[C]]\hat{\otimes}_{O_k} V_0^{\mathrm{D_c}}$ as a CHFLT $O_k[[G]]$-module. By Theorem 4.16, we obtain the following:

Theorem 5.4. *The continuous parabolic induction* $\mathrm{Ind}_P^G(V_0,\rho_0)$ *admits a natural isomorphism to* $(O_k[[C]]\hat{\otimes}_{O_k} V_0^{\mathrm{D_d}})^{\mathscr{D}_{\mathrm{c}}}$ *in* $\mathrm{Rep}_G(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k))$.

The induced action of $O_k[[G]]$ on $(O_k[[C]]\hat{\otimes}_{O_k} V_0^{\mathrm{D_d}})^{\mathscr{D}_{\mathrm{c}}}$ is a little complicated, but this presentation enable us to describe the deformation of $\mathrm{Ind}_P^G(V_0,\rho)$ associated to a deformation of ρ_0 as a deformation of actions of G on a single Banach k-vector space $(O_k[[C]]\hat{\otimes}_{O_k} V_0^{\mathrm{D_d}})^{\mathrm{D_c}}$.

Example 5.5. Let $n \in \omega$. We denote by $\mathrm{B}_n^+(k) \subset \mathrm{GL}_n(k)$ the Borel subgroup consisting of upper triangular invertible matrices, by $\mathrm{C}_n^- \subset \mathrm{GL}_n(k)$ the compact subset consisting of lower triangular invertible matrix whose entries are contained in O_k and whose diagonals are 1, and by $\mathscr{S}_n \subset \mathrm{GL}_n(k)$ the finite subgroup consisting of permutations of the canonical basis. By the LUP-decomposition, $\mathrm{GL}_n(k)$ is expressed as the product $\mathrm{B}_n^+(k)\mathrm{C}_n^-\mathscr{S}_n$, and the multiplication $\mathrm{B}_n^+(k) \times \mathrm{C}_n^-\mathscr{S}_n \to \mathrm{GL}_n(k)$ is bijective. Therefore for a $(V_0,\rho) \in \mathrm{ob}(\mathrm{Rep}_{\mathrm{B}_n^+(k)}(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)))$, we have a natural isomorphism $\mathrm{Ind}_{rB_n^+}^{\mathrm{GL}_n(k)}(V_0,\rho_0) \to (O_k[[\mathrm{C}_n^-\mathscr{S}_n]]\hat{\otimes}_{O_k} V_0^{\mathrm{D_c}})^{\mathscr{D}_{\mathrm{c}}}$ in $\mathrm{Rep}_{\mathrm{GL}_n(k)}(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k))$ by the argument above, and also a natural isomorphism $(O_k[[\mathrm{C}_n^-\mathscr{S}_n]]\hat{\otimes}_{O_k} V_0^{\mathrm{D_c}})^{\mathscr{D}_{\mathrm{c}}} \to (((O_k[[\mathrm{C}_n^-]]\hat{\otimes}_{O_k} K_0))^{\mathrm{D_c}})^{\mathscr{S}_n}$ in $\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)$.

Acknowledgement

I am extremely grateful to Takeshi Tsuji for constructive advices in seminars. I express my deep gratitude to Atsushi Yamashita for instructing me on topological groups. I am profoundly thankful to Takuma Hayashi and Frédéric Paugum for instructing me on the elementary categorical convention. I thank my colleague for daily discussions. I greatly appreciate my family's deep affection. I was a research fellow of Japan Society for the Promotion of Science. I am also thankful to the referee for the careful reading and the helpful comments. This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

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