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Duality theory of p-adic Hopf algebras

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Abstract. We show the monoidal functoriality of Schikhof duality, and cultivate new duality theory of p-adic Hopf algebras. Through the duality, we introduce two sorts of p-adic Pontryagin dualities. One is a duality between discrete Abelian groups and affine formal group schemes of specific type, and the other one is a duality between profinite Abelian groups and analytic groups of specific type. We extend Amice transform to a p-adic Fourier transform compatible with the second p-adic Pontryagin duality. As applications, we give explicit presentations of a universal family of irreducible p-adic unitary Banach representations of the open unit disc of the general linear group and its q-deformation in the case of dimension 2.

1 Introduction

Let k be a non-Archimedean local field. We denote by O_k its valuation ring, and by p the characteristic of the residue field of O_k. As a p-adic analogue of the reflexivity of Hilbert spaces, Schikhof duality (cf. [6] Theorem 4.6) gives a contravariant categorical equivalence between Banach k-vector spaces and localisations of compact Hausdorff flat linear topological O_k-modules. We

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show the monoidal functoriality of Schikhof duality in Theorem 3.2, and study a new duality between Hopf monoid objects in Theorem 3.6.

Through the duality restricted to commutative cocommutative Hopf monoid objects, we introduce two sorts of p-adic Pontryagin dualities. One is a contravariant categorical equivalence between discrete Abelian groups (discrete side) and affine formal group schemes over O_k of specific type (compact side), and the other one is a contravariant categorical equivalence between profinite Abelian groups (compact side) and analytic groups over k of specific type (discrete side). Therefore they are variants of the duality between compactness and discreteness.

We explain the relation with classical results. Iwasawa isomorphism is a well-known isomorphism between the Iwasawa algebra $O_k[[\mathbb{Z}_p]]$ and the formal power series algebra $O_k[[T]]$, which represents the affine formal group scheme U_{1/O_k} given as the completion of the multiplicative group scheme $\mathbb{G}_{\mathrm{m}/O_k}$. Through the identification of $O_k[[\mathbb{Z}_p]]$ and the continuous dual of the continuous function algebra $\mathrm{C}(\mathbb{Z}_p, k)$, it gives an isomorphism called Amice transform from the continuous dual of $\mathrm{C}(\mathbb{Z}_p, k)$ to the algebra of global sections on U_{1/O_k}. Therefore Amice transform gives a non-trivial connection between functions on the profinite Abelian group $\mathbb{Z}_p$ and the affine formal group scheme U_{1/O_k}. In addition, the second p-adic Pontryagin duality sends $\mathbb{Z}_p$ to U_{1/O_k}, and hence Amice transform can be regarded as a p-adic analogue of Fourier transform for the specific dual pair $(\mathbb{Z}_p, U_{1/O_k})$. The duality between Hopf monoid objects yields a Hopf monoid isomorphism in a wider case in Theorem 4.23, and hence can be regarded as a p-adic analogue of Fourier transform extending Amice transform.

The notion of a representation of an affine formal group scheme $\mathscr{G}$ of specific type can be formulated in terms of a comodule object. Through the duality between Hopf monoid objects, it can be naturally identified with the notion of a module object over the dual Hopf monoid A of the coordinate ring of $\mathscr{G}$. By the functoriality of $\mathscr{G}$, every A-module object can be regarded as a unitary Banach k-linear representation of the discrete group $\mathscr{G}(O_k)$. Then the regular A-module can be regarded as a universal family of irreducible unitary Banach k-linear representations of $\mathscr{G}(O_k)$ which can be obtained in this way.

As applications of the duality between Hopf monoid objects, we give explicit presentations of the universal family of irreducible unitary Banach k-

linear representations of the open unit disc, that is, the discrete group of the O_k-valued points of the affine formal group scheme given as the completion, of the general linear group and a q-deformation of the universal family in the case of dimension 2.

We explain contents of this paper. First, §2 consists of two subsections. In §2.1, we recall compact Hausdorff flat linear topological O_k-modules. In §2.2, we recall Banach k-vector spaces. Secondly, §3 consists of two subsections. In §3.1, we show the monoidal functoriality of Schikhof duality. In §3.2, we study the associated duality between Hopf monoid objects. Thirdly, §4, consists of three subsections. In §4.1, we investigate the p-adic Pontryagin duality between discrete Abelian groups and affine formal groups schemes over O_k of specific type. In §4.2, we investigate the p-adic Pontryagin duality between profinite Abelian groups and analytic groups over k of specific type. In §4.3, we study the p-adic Fourier transform. Finally, §5 consists of two subsections. In 5.1, we give an explicit description of the universal family of irreducible unitary Banach k-linear representations of the open unit disc of the general linear group. In 5.2, we give a similar description of a q-deformation of the universal family.

2 Preliminaries

We fix a Grothendieck universe $\mathscr{U}$ in order to avoid set-theoretic problems. A set x is said to be *$\mathscr{U}$-small* if $x \in \mathscr{U}$. Throughout this paper, let k denote a fixed $\mathscr{U}$-small local field, that is, a complete discrete valuation field with finite residue field, $O_k \subset k$ the valuation ring of k, and p the characteristic of the residue field of O_k.

We introduce the convention and the terminology for categories, which is always assumed to be small but is not assumed to be $\mathscr{U}$-small. We denote by $\mathscr{C}$ (respectively, $\mathscr{C}_k$) the Abelian category of $\mathscr{U}$-small O_k-modules (respectively, k-vector spaces) and O_k-linear homomorphisms. Then the triads $(\mathscr{C}, \otimes_{O_k}, O_k)$ and $(\mathscr{C}_k, \otimes_k, k)$ form symmetric monoidal categories. The correspondence $M \rightsquigarrow k \otimes_{O_k} M$ restricted to $\mathscr{U}$ gives a symmetric monoidal functor $(\mathscr{C}, \otimes_{O_k}, O_k) \to (\mathscr{C}_k, \otimes_k, k)$. We abbreviate "$(\mathscr{C}, \otimes_{O_k}, O_k)$-enriched" (respectively, "$(\mathscr{C}_k, \otimes_k, k)$-enriched") to "$O_k$-linear" (respectively, "k-linear"). For an O_k-linear category Θ, we denote by $k\Theta$ the k-linear category obtained as the localisation of Θ by the symmetric monoidal functor $(\mathscr{C}, \otimes_{O_k}, O_k) \to$

$(\mathscr{C}_k, \otimes_k, k)$. For an O_k-linear functor Φ, we denote by $k\Phi$ the k-linear extension of Φ.

We denote by Set the category of $\mathscr{U}$-small sets and maps, by Top the category of $\mathscr{U}$-small topological spaces and continuous maps, by PTop $\subset$ Top the full subcategory of $\mathscr{U}$-small totally disconnected compact Hausdorff topological spaces, by Grp the category of $\mathscr{U}$-small discrete groups and group homomorphisms, by Ab $\subset$ Grp the full subcategory of $\mathscr{U}$-small discrete Abelian groups, by PGrp the category of $\mathscr{U}$-small profinite groups and continuous group homomorphisms, and by PAb $\subset$ PGrp the full subcategory of $\mathscr{U}$-small profinite Abelian groups. For topological spaces X and Y, we denote by $\mathrm{C}(X, Y)$ the set of continuous maps $X \to Y$.

2.1 Topological module A *topological O_k-module* is an O_k-module M equipped with a topology for which the addition $M \times M \to M$ and the scalar multiplication $O_k \times M \to M$ are continuous. We denote by $\mathscr{O}(M)$ the set of open O_k-submodules of M. We say that M is *linear* if $\mathscr{O}(M)$ forms a fundamental system of neighbourhoods of $0 \in M$, is *flat* if its underlying O_k-module is flat, and is *compact Hausdorff* if its underlying topological space is compact and Hausdorff.

Example 2.1. Let I be a set, and M a topological O_k-module, e.g. O_k. We denote by M^I the direct product of I-copies of the underlying O_k-module of M equipped with the direct product topology. Then M^I forms a topological O_k-module. If M is compact (respectively, Hausdorff, flat, linear), then so is M^I.

Let M_0 and M_1 be compact Hausdorff linear topological O_k-modules. For each $(L_0, L_1) \in \mathscr{O}(M_0) \times \mathscr{O}(M_1)$, we equip $M_0/L_0 \otimes_{O_k} M_1/L_1$ with the discrete topology so that it again forms a compact Hausdorff linear topological O_k-module by the finiteness of its underlying set. We denote by $M_0 \hat{\otimes}_{O_k} M_1$ the inverse limit of $(M_0/L_0 \otimes_{O_k} M_1/L_1)_{(L_0,L_1) \in \mathscr{O}(M_0) \times \mathscr{O}(M_1)}$ with respect to canonical projections equipped with the inverse limit topology. We give an explicit example of the completed tensor product.

Proposition 2.2. *Let I_0 and I_1 be sets. Then there is a natural homeomorphic O_k-linear isomorphism $O_k^{I_0 \times I_1} \to O_k^{I_0} \hat{\otimes}_{O_k} O_k^{I_1}$.*

Proof. Let $(L_0, L_1) \in \mathscr{O}(O_k^{I_0}) \times \mathscr{O}(O_k^{I_1})$. Put $M_{L_0,L_1} \coloneqq O_k^{I_0}/L_0 \otimes_{O_k} O_k^{I_1}/L_1$. Let $h \in \{0, 1\}$. For each $i \in I_h$, we denote by $\delta_{I_h,i} \in O_k^{I_h}$ the characteristic function of $\{i\} \subset I_h$. By the definition of the direct product topology, there is a finite subset $J_h \subset I_h$ such that $\{(m_i)_{i \in I_h} \in O_k^{I_h} \mid \forall i \in J_h, m_i = 0\} \subset L_h$. We denote by f_{L_0,L_1} the composite of the canonical projection $O_k^{I_0 \times I_1} \twoheadrightarrow O_k^{J_0 \times J_1}$ and the following O_k-linear homomorphism, which is continuous by the continuity of the O_k-module structure of M_{L_0,L_1}:

$$\begin{array}{rcl} O_k^{J_0 \times J_1} & \to & M_{L_0,L_1} \\ (c_{i_0,i_1})_{(i_0,i_1) \in J_0 \times J_1} & \mapsto & \displaystyle\sum_{(i_0,i_1) \in J_0 \times J_1} c_{I_0,i_1}(\delta_{I_0,i_0} + L_0) \otimes (\delta_{I_1,i_1} + L_1) \end{array}$$

Then f_{L_0,L_1} is independent of the choice of (J_0, J_1) by $(\delta_{I_0,i_0} + L_0) \otimes (\delta_{I_1,i_1} + L_1) = 0 \in M_{L_0,L_1}$ for any $(i_0, i_1) \in (I_0 \setminus J_0) \times (I_1 \setminus J_1)$, and is surjective because $\{(\delta_{I_0,i_0} + L_0) \otimes (\delta_{I_1,i_1} + L_1) \mid (i_0, i_1) \in I_0 \times I_1\}$ generates M_{L_0,L_1}.

By the definition, $(f_{L_0,L_1})_{(L_0,L_1)}$ forms a compatible system of surjective continuous O_k-linear homomorphisms, and hence induces a continuous O_k-linear homomorphism $f \colon O_k^{I_0 \times I_1} \to O_k^{I_0} \hat{\otimes}_{O_k} O_k^{I_1}$, whose image is dense by the surjectivity of the composite with any canonical projection. Let $c = (c_{i_0,i_1})_{(i_0,i_1) \in I_0 \times I_1} \in \ker(f)$. For any $(i_0, i_1) \in I_0 \times I_1$, we have $c_{i_0,i_1} = 0$ by $(c_{i_0,i_1} + \wp)(\delta_{I_0,i_0} + L_0) \otimes (\delta_{I_1,i_1} + L_1) = f_{L_0,L_1}(c) = 0 \in M_{L_0,L_1} \cong (O_k/\wp)^{\{i_0\}} \otimes_{O_k} (O_k/\wp)^{\{i_1\}} \cong O_k/\wp$ for any open ideal $\wp \subset O_k$, where L_0 and L_1 denote $O_k^{I_0 \setminus \{i_0\}} \times \wp O_k^{\{i_0\}} \in \mathscr{O}(O_k^{I_0})$ and $O_k^{I_1 \setminus \{i_1\}} \times \wp O_k^{\{i_1\}} \in \mathscr{O}(O_k^{I_1})$ respectively. It implies $c = 0$. Therefore f is injective. Since $O_k^{I_0 \times I_1}$ is compact and $O_k^{I_0} \hat{\otimes}_{O_k} O_k^{I_1}$ is Hausdorff, f is a homeomorphism. □

By the definition, the completed tensor product of compact Hausdorff linear topological O_k-modules is again a compact Hausdorff linear topological O_k-module. Moreover, we show that it also preserves the flatness.

Proposition 2.3. *For any compact Hausdorff flat linear topological O_k-modules M_0 and M_1, $M_0 \hat{\otimes}_{O_k} M_1$ is also a compact Hausdorff flat linear topological O_k-module.*

Proof. It is reduced to the case where M_0 and M_1 are direct products of copies of O_k by [8] Expose VII_B 0.3.8. Corollaire. When M_0 and M_1 are direct products of copies of O_k, then the assertion immediately follows from Proposition 2.2. □

We denote by $\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}$ the O_k-linear category of $\mathscr{U}$-small compact Hausdorff flat linear topological O_k-modules and continuous O_k-linear homomorphisms. By Proposition 2.3, the correspondence $(M_0, M_1) \rightsquigarrow M_0 \hat{\otimes}_{O_k} M_1$ restricted to $\mathscr{U}$ gives an O_k-linear bifunctor $\hat{\otimes}_{O_k} \colon \mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}} \times \mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}} \to \mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}$ and a k-linear bifunctor $\hat{\otimes}_{O_k} \colon k\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}} \times k\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}} \to k\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}$, for which $\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}$ and $k\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}$ naturally form symmetric monoidal categories.

A topological O_k-algebra is said to be *compact Hausdorff flat linear* if its underlying topological O_k-module is a compact Hausdorff flat linear topological O_k-module. It is elementary to show that the notion of a $\mathscr{U}$-small compact Hausdorff flat linear topological O_k-algebra is equivalent to that of a monoid object in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$.

Let I be a set. We equip O_k^I with the pointwise multiplication $O_k^I \times O_k^I \to O_k^I$, for which it forms a commutative compact Hausdorff flat linear topological O_k-algebra and hence a commutative monoid object in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$ as long as I is $\mathscr{U}$-small.

Let G be a profinite group. We denote by $O_k[[G]]$ the compact Hausdorff flat linear topological O_k-algebra given as the inverse limit of $((O_k/\wp)[G/K])_{(\wp,K)}$ with respect to the canonical projections, where $\wp$ runs through all open ideals of O_k and K runs though all open normal subgroups of G, and hence forms a monoid object in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$ as long as G is $\mathscr{U}$-small. The canonical embedding $\iota_G^{\mathrm{c}} \colon O_k[G] \hookrightarrow O_k[[G]]$ is an O_k-algebra homomorphism whose image generates a dense O_k-subalgebra, and hence $O_k[[G]]$ is an analogue of $O_k[G]$.

We denote by $\mathrm{Alg}(O_k)$ the category of monoid objects in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$ and monoid homomorphisms, and by $\mathrm{CAlg}(O_k) \subset \mathrm{Alg}(O_k)$ the full subcategory of commutative monoid objects in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}(k), \hat{\otimes}_{O_k}, O_k)$. The correspondence $G \rightsquigarrow O_k[[G]]$ restricted to $\mathscr{U}$ gives a functor $O_k[[\bullet]] \colon \mathrm{PGrp} \to \mathrm{Alg}(O_k)$, which induces a functor $\mathrm{PAb} \to \mathrm{CAlg}(O_k)$.

2.2 Banach space A *Banach k-vector space* is a k-vector space V equipped with a map $\| \bullet \| \colon V \to [0, \infty)$ called *the norm* satisfying the following:

(i) The map $V \times V \to [0, \infty)$, $(v_0, v_1) \mapsto \|v_0 - v_1\|$ is a complete ultrametric.

(ii) For any $(c, v) \in k \times V$, $\|cv\| = |c| \, \|v\|$ holds.

We say that V is *unramified* if $\|v\| \in \{|c| \mid c \in k\}$ for any $v \in V$. We put $V^{\circ} := \{v \in V \mid \|v\| \leq 1\}$.

Example 2.4. Let X be a topological space, and V a Banach k-vector space, e.g. k. We denote by $\mathrm{C}_{\mathrm{bd}}(X, V) \subset \mathrm{C}(X, V)$ the k-vector subspace of bounded continuous maps. Then $\mathrm{C}_{\mathrm{bd}}(X, V)$ forms a Banach k-vector space with respect to the supremum norm $\| \bullet \| \colon \mathrm{C}_{\mathrm{bd}}(X, V) \to [0, \infty)$, $f \mapsto \sup_{x \in X} \|f(x)\|$. If V is unramified, then so is $\mathrm{C}_{\mathrm{bd}}(X, V)$, because $\{|c| \mid c \in k\}$ is closed in $[0, \infty)$ by the discreteness of the valuation. We denote by $\mathrm{C}_0(X, V) \subset \mathrm{C}_{\mathrm{bd}}(X, V)$ the k-vector subspace of functions $f \colon X \to V$ such that for any $\epsilon \in (0, \infty)$, there is a compact subset $C \subset X$ such that $\|f(x)\| < \epsilon$ for any $x \in X \setminus C$. Then $\mathrm{C}_0(X, V)$ is closed in $\mathrm{C}_{\mathrm{bd}}(X, V)$, and hence forms a Banach k-vector space with respect to the restriction of the supremum norm. Suppose that X is compact. Then we have $\mathrm{C}_0(X, V) = \mathrm{C}_{\mathrm{bd}}(X, V) = \mathrm{C}(X, V)$ by the maximal modulus principle, and hence regard $\mathrm{C}(X, V)$ as a Banach k-vector space.

Example 2.5. Let M be a compact Hausdorff flat linear topological O_k-module. We denote by $M^{\mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}}} \subset \mathrm{C}(M, k)$ the k-vector subspace of continuous O_k-linear homomorphisms $M \to k$. Then $M^{\mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}}}$ is closed in $\mathrm{C}_{\mathrm{bd}}(M, k)$, and hence forms a Banach k-vector space with respect to the restriction of the supremum norm.

Let V_0 and V_1 be Banach k-vector spaces, and $f \colon V_0 \to V_1$ a k-linear homomorphism. We say that f is *bounded* if there is a $C \in (0, \infty)$ such that $\|f(v)\| \leq C\|v\|$ for any $v \in V_0$, and is *submetric* (respectively, *isometric*) if $\|f(v)\| \leq \|v\|$ (respectively, $\|f(v)\| = \|v\|$) for any $v \in V_0$. Since the valuation of k is not trivial, a k-linear homomorphism between Banach k-vector spaces is bounded if and only if it is continuous by [2] 2.1.8 Corollary 3.

Let V_0 and V_1 be Banach k-vector spaces. We denote by $V_1 \hat{\otimes}_k V_2$ the completed non-Archimedean tensor product of V_1 and V_2 (cf. [1] p. 12). We give an explicit example of the completed tensor product analogous to the one in Proposition 2.2.

Proposition 2.6. *Let I_0 and I_1 be discrete sets. Then there is a natural isometric k-linear isomorphism* $\mathrm{C}_0(I_0, k) \hat{\otimes}_k \mathrm{C}_0(I_1, k) \to \mathrm{C}_0(I_0 \times I_1, k)$.

Proof. Take a uniformiser $\varpi \in O_k$. Put $V_0 := \mathrm{C}_0(I_0, k)\hat{\otimes}_k \mathrm{C}_0(I_1, k)$ and $V_1 := \mathrm{C}_0(I_0 \times I_1, k)$. Since V_0° and $V_1^\circ$are naturally identified with the ϖ-adic completion of $O_k^{\oplus I_0} \otimes_{O_k} O_k^{\oplus I_1}$ and $O_k^{\oplus(I_0 \times I_1)}$ respectively, the O_k-linear isomorphism $O_k^{\oplus I_0} \otimes_{O_k} O_k^{\oplus I_1} \to O_k^{\oplus(I_0 \times I_1)}$ associated to $\mathrm{id}_{I_0 \times I_1}$ extends to a unique continuous O_k-linear isomorphism $V_0^\circ \to V_1^\circ$, which extends to a unique isometric k-linear isomorphism $V_0 \to V_1$. □

We denote by $\mathrm{Ban}(k)$ (respectively, $\mathrm{Ban}_{\leq}(k)$) the k-linear (respectively, O_k-linear) category of $\mathscr{U}$-small Banach k-vector spaces and bounded (respectively, submetric) k-linear homomorphisms, by $\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k) \subset \mathrm{Ban}_{\leq}(k)$ the full subcategory of $\mathscr{U}$-small unramified Banach k-vector spaces. The correspondence $I \rightsquigarrow \mathrm{C}_0(I, k)$ restricted to $\mathscr{U}$ gives a functor $\mathrm{C}_0(\bullet, k) \colon \mathrm{Set} \to \mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)$. The correspondence $(V_0, V_1) \rightsquigarrow V_0 \hat{\otimes}_k V_1$ restricted to $\mathscr{U}$ gives a k-linear (respectively, an O_k-linear) bifunctor $\hat{\otimes}_k \colon \mathrm{Ban}(k) \times \mathrm{Ban}(k) \to \mathrm{Ban}(k)$ (respectively, $\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k) \times \mathrm{Ban}_{\leq}^{\mathrm{ur}}(k) \to \mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)$), for which $\mathrm{Ban}(k)$ (respectively, $\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k)$) naturally forms a symmetric monoidal category.

Proposition 2.7. *The localisation* $k\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k) \to \mathrm{Ban}(k)$ *of the inclusion* $\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k) \hookrightarrow \mathrm{Ban}(k)$ *is an equivalence of category.*

Proof. The following proof is essentially the same as the argument in the proof of [9] Theorem 1.2, in which k is assumed to be of characteristic 0. Denote by F the functor in the assertion. The fullness of F follows from the definition of a bounded homomorphism and non-triviality of the valuation of k. The faithfulness of F follows from the flatness of hom modules. For a Banach k-vector space V, denote by V^{ur} the underlying k-vector space of V equipped with the norm $V^{\mathrm{ur}} \to [0, \infty),\ v \mapsto \inf\{|c| \mid c \in k \wedge \|v\| \leq |c|\}$. Then V^{ur} forms a Banach k-vector space, and the identity map $I_V \colon V \to V^{\mathrm{ur}}$ is an isomorphism in $\mathrm{Ban}(k)$. The correspondence $V \rightsquigarrow V^{\mathrm{ur}}$ restricted to $\mathscr{U}$ gives an endofunctor $(\bullet)^{\mathrm{ur}}$ of $\mathrm{Ban}(k)$, and the correspondence $V \rightsquigarrow I_V$ restricted to $\mathscr{U}$ gives a natural equivalence $I \colon \mathrm{id}_{\mathrm{Ban}(k)} \Rightarrow (\bullet)^{\mathrm{ur}}$. Moreover, I_V is the identity for any unramified Banach k-vector space V. Therefore $(\bullet)^{\mathrm{ur}}$ induces a quasi-inverse of F, because F is fully faithful. □

A *Banach k-algebra* is a Banach k-vector space A equipped with a k-algebra structure satisfying $\|f_0 f_1\| \leq \|f_0\|\ \|f_1\|$ for any $(f_0, f_1) \in A^2$ and $\|1\| = 1$ as long as A is not a zero ring. It is elementary to show that the notion of a $\mathscr{U}$-small Banach k-algebra is equivalent to that of a monoid object in $(\mathrm{Ban}_{\leq}^{\mathrm{ur}}, \hat{\otimes}_k, k)$. We note that there are several distinct formulations of the notion of a Banach k-algebra, and such an equivalence does not necessarily hold if one applies another formulation.

Let X be a topological space. We equip $\mathrm{C}_{\mathrm{bd}}(X, k)$ with the pointwise multiplication $\mathrm{C}_{\mathrm{bd}}(X, k) \times \mathrm{C}_{\mathrm{bd}}(X, k) \to \mathrm{C}_{\mathrm{bd}}(X, k)$, for which it forms a commutative Banach k-algebra and hence a commutative monoid object in $(\mathrm{Ban}_{\leq}^{\mathrm{ur}}, \hat{\otimes}_k, k)$ as long as X is $\mathscr{U}$-small.

In order to introduce another example of a monoid object in $(\mathrm{Ban}_{\leq}^{\mathrm{ur}}, \hat{\otimes}_k, k)$, we recall the notion of a (possibly uncountable) infinite sum. Let I be a discrete set, and $f \colon I \to k$ a map. We denote by $\sum_{i \in I} f(i)$ the limit of the net $(\sum_{i \in J} f(i))_{J \subset I, \# J < \infty}$ indexed by the set of finite subsets $J \subset I$ directed with respect to the inclusion relation. When $I = \{i \in \mathbb{N} \mid i < n\}$ for some $n \in \mathbb{N}$ (respectively, $I = \mathbb{N}$), then $\sum_{i \in I} f(i)$ coincides with the usual finite sum $\sum_{i=0}^{n-1} f(i)$ (respectively, the usual infinite sum $\sum_{i=0}^{\infty} f(i)$). Moreover, it is elementary to show that $\sum_{i \in I} f(i)$ converges in k if and only if f lies in $\mathrm{C}_0(I, k)$.

Let G be a discrete group. We equip $\mathrm{C}_0(G, k)$ with the convolution product

$$\mathrm{C}_0(G, k) \times \mathrm{C}_0(G, k) \to \mathrm{C}_0(G, k), \ ((c_g)_{g \in G}, (d_g)_{g \in G}) \mapsto (\sum_{h \in G} c_h d_{h^{-1}g})_{g \in G},$$

where the sum in the definition makes sense by the argument above, for which it forms a Banach k-algebra and hence a monoid object in $(\mathrm{Ban}_{\leq}^{\mathrm{ur}}, \hat{\otimes}_k, k)$ as long as G is $\mathscr{U}$-small. The canonical embedding $\iota_G^{\mathrm{d}} \colon k[G] \hookrightarrow \mathrm{C}_0(G, k)$ is an k-algebra homomorphism whose image is dense, and hence $\mathrm{C}_0(G, k)$ is an analogue of $k[G]$.

We denote by $\mathrm{Alg}(k)$ the category of monoid objects in $(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k), \hat{\otimes}_k, k)$ and monoid homomorphisms, and by $\mathrm{CAlg}(k) \subset \mathrm{Alg}(k)$ the full subcategory of commutative monoid objects in $(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k), \hat{\otimes}_k, k)$. The correspondence $G \rightsquigarrow \mathrm{C}_0(G, k)$ restricted to $\mathscr{U}$ gives a functor $\mathrm{C}_0(\bullet, k) \colon \mathrm{Grp} \to \mathrm{Alg}(k)$, which induces a functor $\mathrm{Ab} \to \mathrm{CAlg}(k)$. The correspondence $X \rightsquigarrow \mathrm{C}_{\mathrm{bd}}(X, k)$ restricted to $\mathscr{U}$ gives a functor $\mathrm{C}_{\mathrm{bd}}(\bullet, k) \colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{CAlg}(k)$. We recall non-Archimedean Gel'fand–Naimark theorem.

Proposition 2.8. *The functor* $\mathrm{C}_{\mathrm{bd}}(\bullet, k)$ *restricted to* PTop *is fully faithful.*

Proof. The assertion immediately follows from [1] 9.2.7 Corollary. □

3 Monoidal Structure

We introduce symmetric monoidal structures on $\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k)$, $\mathrm{Ban}(k)$, $\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}$, and $k\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}$, and verify the monoidal functoriality of Schikhof duality (cf. [6] Theorem 4.6 and [9] Theorem 1.2). As a corollary, we obtain a duality between Hopf monoid objects.

3.1 Monoidal functoriality Let V be a Banach k-vector space. We denote by $V^{\mathrm{D}_{\mathrm{Ban}}}$ the O_k-module of submetric k-linear homomorphism $V \to k$ equipped with the topology of pointwise convergence. Then $V^{\mathrm{D}_{\mathrm{Ban}}}$ forms a compact Hausdorff flat linear topological O_k-module because $O_k^{V^\circ}$ equipped with the direct product topology is a compact Hausdorff flat linear topological O_k-module and the evaluation map $V^{\mathrm{D}_{\mathrm{Ban}}} \to O_k^{V^\circ}$, $m \mapsto (m(v))_{v \in V^\circ}$ is a homeomorphism onto the closed image. The correspondence $V \rightsquigarrow V^{\mathrm{D}_{\mathrm{Ban}}}$ restricted to $\mathscr{U}$ gives a functor $\mathrm{D}_{\mathrm{Ban}} \colon \mathrm{Ban}_{\leq}(k)^{\mathrm{op}} \to \mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}$. We put $\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}} := \mathrm{D}_{\mathrm{Ban}}|_{\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k)^{\mathrm{op}}}$.

Example 3.1. For any profinite group G, $\mathrm{C}(G,k)^{\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}}$ is naturally identified with $O_k[[G]]$ through a unique continuous O_k-bilinear extension $O_k[[G]] \times \mathrm{C}(G,k) \to k$ of the canonical pairing $O_k[G] \times \mathrm{C}(G,k) \to k$, $(c_g)_{g \in G} \otimes f \mapsto \sum_{g \in G} c_g f(g)$ through $\iota^{\mathrm{c}}_G \times \mathrm{id}_{\mathrm{C}(G,k)}$.

On the other hand, let M be a compact Hausdorff flat linear topological O_k-module. For any continuous O_k-linear homomorphism $v \colon M \to k$, its supremum norm $\|v\| := \sup_{m \in M} |v(m)|$ is finite by the compactness of M. We denote by $M^{\mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}}}$ the k-vector space of continuous O_k-linear homomorphisms $M \to k$ equipped with the supremum norm. Then $M^{\mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}}}$ forms an unramified Banach k-vector space by the discreteness of the valuation of k.

For a Banach k-vector space V, we denote by $\eta_{\mathrm{Ban}}(V) \colon V \to V^{\mathrm{D}_{\mathrm{Ban}} \mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}}}$ the evaluation map. For a compact Hausdorff flat linear topological O_k-module M, we denote by $\eta^{\mathrm{ch}}_{\mathrm{fl}}(M) \colon M \to M^{\mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}} \mathrm{D}_{\mathrm{Ban}}}$ the evaluation map. The correspondences $V \rightsquigarrow \eta_{\mathrm{Ban}}(V)$ and $M \rightsquigarrow \eta^{\mathrm{ch}}_{\mathrm{fl}}(M)$ restricted to $\mathscr{U}$ give natural transformations $\eta_{\mathrm{Ban}} \colon \mathrm{id}_{\mathrm{Ban}_{\leq}(k)^{\mathrm{op}}} \Rightarrow \mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}} \circ \mathrm{D}_{\mathrm{Ban}}$ and $\eta^{\mathrm{ch}}_{\mathrm{fl}} \colon \mathrm{id}_{\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}} \Rightarrow \mathrm{D}_{\mathrm{Ban}} \circ \mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}}$. By Schikhof duality, $\eta_{\mathrm{Ban}}(V)$ and $\eta^{\mathrm{ch}}_{\mathrm{fl}}(M)$ (respectively, $k\eta^{\mathrm{ch}}_{\mathrm{fl}}(M)$) are isomorphisms in $\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k)$ and $\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}$ (respectively, $\mathrm{Ban}(k)$ and $k\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}$) respectively for any $\mathscr{U}$-small unramified Banach k-vector space V (respectively, $\mathscr{U}$-small Banach k-vector space V) and any $\mathscr{U}$-small compact Hausdorff flat linear topological O_k-module M, and $(\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}, \mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}})$ (respectively, $(\mathrm{D}_{\mathrm{Ban}}, k\mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}})$) gives an equivalence of categories between $\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k)$ and $\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}$ (respectively, $\mathrm{Ban}(k)$ identified with $k\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k)$ by Proposition 3.13 and $k\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}$). Further, we show the compatibility with the other structures.

Theorem 3.2. *The pair* $(\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}, \mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}})$ *(respectively,* $(\mathrm{D}_{\mathrm{Ban}}, k\mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}})$*) gives an equivalence of symmetric monoidal* O_k*-linear categories* $(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k), \hat{\otimes}_k, k)^{\mathrm{op}}$ *and* $(\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)$ *(respectively, symmetric monoidal* k*-linear categories* $(\mathrm{Ban}(k), \hat{\otimes}_k, k)^{\mathrm{op}}$ *and* $(k\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)$*).*

In order to verify Theorem 3.2, we prepare several lemmas.

Lemma 3.3. *The correspondence* $I \rightsquigarrow O_k^I$ *(respectively,* $\mathrm{C}_0(I,k)$*) restricted to* $\mathscr{U}$ *gives a symmetric monoidal functor* $O_k^{\bullet} \colon (\mathrm{Set}, \times, 1) \to (\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)^{\mathrm{op}}$ *(respectively,* $\mathrm{C}_0(\bullet, k) \colon (\mathrm{Set}, \times, 1) \to (\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k), \hat{\otimes}_k, k)$*).*

Proof. The functoriality is obvious. We show that the functors are compatible with the symmetric monoidal structures. For $\mathscr{U}$-small sets I_0 and I_1, denote by $\mathrm{T}(I_0, I_1)$ the isomorphism $O_k^{I_0 \times I_1} \to O_k^{I_0} \hat{\otimes}_{O_k} O_k^{I_1}$ in $\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}$ (respectively, $\mathrm{C}_0(I_0,k) \hat{\otimes}_k \mathrm{C}_0(I_1,k) \to \mathrm{C}_0(I_0 \times I_1, k)$ in $\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k)$) explicitly constructed in the proof of Proposition 2.2 (Proposition 2.6). The correspondence $(I_0, I_1) \rightsquigarrow \mathrm{T}(I_0, I_1)$ gives a natural equivalence $\mathrm{T} \colon O_k^{\bullet_0 \times \bullet_1} \to O_k^{\bullet_0} \hat{\otimes}_{O_k} O_k^{\bullet_1}$ (respectively, $\mathrm{C}_0(\bullet_0, k) \hat{\otimes}_k \mathrm{C}_0(\bullet_1, k) \Rightarrow \mathrm{C}_0(\bullet_0 \times \bullet_1, k)$) with the desired coherence by the construction. □

We recall classical results on classifications of unramified Banach k-vector spaces and compact Hausdorff flat linear topological O_k-modules.

Lemma 3.4. (i) *For any unramified Banach* k*-vector space* V*, there is an isometric* k*-linear isomorphism* $\mathrm{C}_0(I,k) \to V$ *for some set* I*.*

(ii) *For any compact Hausdorff flat linear topological* O_k*-module* M*, there is a homeomorphic* O_k*-linear isomorphism* $M \to O_k^I$ *for some set* I*.*

Proof. The first assertion follows from [5] IV 3 Corollaire 1 (cf. [2] 2.5.2 Lemma 2 and the proof of [7] Proposition 10.1), and the second assertion follows from [8] Expose VII_B 0.3.8. Corollaire. □

Proof of Theorem 3.2. The linearity immediately follows from the definition of the enrichment. The localisation by $(\mathscr{C}_k, \otimes_k, k)$ is compatible with the monoidal structures, and hence it suffices to verify that $\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}$ and $\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}$ form symmetric monoidal functors by the argument on η_{Ban} and $\eta_{\mathrm{fl}}^{\mathrm{ch}}$ in the beginning of this subsection.

We construct a natural equivalence $(\bullet_0)^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \hat{\otimes}_{O_k} (\bullet_1)^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \Rightarrow (\bullet_0 \hat{\otimes}_k \bullet_1)^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}}$ with the desired coherence. For this purpose, it suffices to show that the O_k-linear homomorphism $V_0^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \otimes_{O_k} V_1^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \to (V_0 \hat{\otimes}_k V_1)^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}}$ associated to the O_k-linear pairing $(V_0^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \otimes_{O_k} V_1^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}}) \otimes_{O_k} (V_0 \otimes_{O_k} V_1) \cong (V_0^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \otimes_{O_k} V_0) \otimes_{O_k} (V_1^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \otimes_{O_k} V_1) \to k \otimes_{O_k} k \cong k$ extends to a unique homeomorphic O_k-linear isomorphism $V_0^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \hat{\otimes}_{O_k} V_1^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \to (V_0 \hat{\otimes}_k V_1)^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}}$ in $\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}$ for any unramified Banach k-vector spacesV_0 and V_1. By Lemma 3.4 (i), it is reduced to the case $V_0 = \mathrm{C}_0(I, k)$ and $V_1 = \mathrm{C}_0(J, k)$ for sets I and J. By Lemma 3.3, we have a natural homeomorphic O_k-linear isomorphism $\mathrm{C}_0(I, k)^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \hat{\otimes}_{O_k} \mathrm{C}_0(J, k)^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \cong O_k^I \hat{\otimes}_{O_k} O_k^J \cong O_k^{I \times J} \cong \mathrm{C}_0(I \times J, k)^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \cong (\mathrm{C}_0(I, k) \hat{\otimes}_k \mathrm{C}_0(J, k))^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}}$ extending the given homomorphism. Therefore $\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}$ forms a symmetric monoidal equivalence.

We construct a natural equivalence $(\bullet_0)^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}} \hat{\otimes}_k (\bullet_1)^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}} \Rightarrow (\bullet_0 \hat{\otimes}_{O_k} \bullet_1)^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}}$ with the desired coherence. For this purpose, it suffices to show that the k-linear homomorphism $M_0^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}} \otimes_k M_1^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}} \to (M_0 \hat{\otimes}_{O_k} M_1)^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}}$ associated to the O_k-linear pairing $(M_0^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}} \otimes_k M_1^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}}) \otimes_{O_k} (M_0 \otimes_{O_k} M_1) \cong (M_0^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}} \otimes_{O_k} M_0) \otimes_{O_k} (M_1^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}} \hat{\otimes}_{O_k} M_1) \to k \otimes k \cong k$ extends to a unique isometric k-linear isomorphism $M_0^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}} \hat{\otimes}_k M_1^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}} \to (M_0 \hat{\otimes}_{O_k} M_1)^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}}$ for any compact Hausdorff flat linear topological O_k-modules M_0 and M_1. By Lemma 3.4 (ii), it is reduced to the case $M_0 = O_k^{I_0}$ and $M_1 = O_k^{I_1}$ for sets I_0 and I_1. By Lemma 3.3, we have a natural isometric k-linear isomorphism $(O_k^{I_0})^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}} \hat{\otimes}_k (O_k^{I_1})^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}} \cong \mathrm{C}_0(I_0, k) \hat{\otimes}_k \mathrm{C}_0(I_1, k) \cong \mathrm{C}_0(I_0 \times I_1, k) \cong (O_k^{I_0 \times I_1})^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}} \cong (O_k^{I_0} \hat{\otimes}_{O_k} O_k^{I_1})^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}}$ extending the given homomorphism. Therefore D_{O_k} and $\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}$ are symmetric monoidal functors. □

We also have a relation to Cartesian products of topological spaces.

Corollary 3.5. (i) *For any profinite groups G_0 and G_1, the O_k-algebra isomorphism $O_k[G_0] \otimes_{O_k} O_k[G_1] \to O_k[G_0 \times G_1]$ associated to* $\mathrm{id}_{G_0 \times G_1}$ *extends to a unique homeomorphic O_k-algebra isomorphism $O_k[[G_0]] \hat{\otimes}_{O_k} O_k[[G_1]] \to O_k[[G_0 \times G_1]]$.*

(ii) *For any topological spaces* X_0 *and* X_1*, the* k*-algebra homomorphism* $\mathrm{C}_{\mathrm{bd}}(X_1, k) \otimes_k \mathrm{C}_{\mathrm{bd}}(X_0, k) \to \mathrm{C}_{\mathrm{bd}}(X_0 \times X_1, k)$, $(f_1(x_1), f_0(x_0)) \mapsto f_0(x_0) f_1(x_1)$ *extends to a unique isometric* k*-algebra isomorphism*

$$\mathrm{C}_{\mathrm{bd}}(X_1, k) \hat{\otimes}_k \mathrm{C}_{\mathrm{bd}}(X_0, k) \to \mathrm{C}_{\mathrm{bd}}(X_0 \times X_1, k).$$

Proof. The assertion (i) immediately follows from Theorem 3.2 and the assertion (ii) applied to the underlying topological spaces of G_0 and G_1. We show the assertion (ii). In the case where X_0 and X_1 are totally disconnected compact Hausdorff topological spaces, the assertion immediately follows from Proposition 2.8, because $\hat{\otimes}_k$ satisfies the universality of the coproduct. In the general case, the assertion follows from the fact that the inclusion PTop $\hookrightarrow$ Top is a right adjoint functor (cf. [4] Corollary 2.3). We note that the universality and the adjoint property are formalisable without using categories, and hence we do not have to assume the $\mathscr{U}$-smallness of X_0 and X_1. □

3.2 Hopf monoid For a symmetric monoidal category $(\mathscr{S}, \otimes, I)$, we denote by $\mathrm{Hopf}(\mathscr{S}, \otimes, I)$ the category of Hopf monoid objects in $(\mathscr{S}, \otimes, I)$ and Hopf monoid homomorphisms. As an immediate consequence of Theorem 3.2, we obtain the following:

Theorem 3.6. *The pair* $(\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}, \mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}})$ *(respectively,* $(\mathrm{D}_{\mathrm{Ban}}, k\mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}})$*) gives an equivalence of categories* $\mathrm{Hopf}(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k), \hat{\otimes}_k, k)^{\mathrm{op}}$ *and* $\mathrm{Hopf}(\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)$ *(respectively,* $\mathrm{Hopf}(\mathrm{Ban}(k), \hat{\otimes}_k, k)^{\mathrm{op}}$ *and* $\mathrm{Hopf}(k\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)$*).*

We give an explicit example of a Hopf monoid object in $(\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)$. We recall one of the simplest example of a Hopf O_k-algebra, that is, a Hopf monoid object in $(\mathscr{C}, \otimes_{O_k}, O_k)$, is the group algebra over O_k. Similarly, we construct two sorts of Hopf monoid objects in $(\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)$ by using a topological group.

Proposition 3.7. (i) *Let* G *be a* $\mathscr{U}$*-small profinite group. Then the monoid object structure of* $O_k[[G]]$ *in* $(\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)$ *extends to a unique Hopf monoid object structure in* $(\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)$ *such that* ι^{c}_G *preserves the comultiplication, the counit, and the antipode. Moreover,* $O_k[[G]]$ *is cocommutative.*

(ii) *Let* G *be a* $\mathscr{U}$*-small discrete group. Then the monoid object structure of* $\mathrm{C}_0(G, k)$ *in* $(\mathrm{Ban}^{\mathrm{ur}}_{\leq}, \hat{\otimes}_k, k)$ *extends to a unique Hopf monoid object* *structure in* $(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k), \hat{\otimes}_k, k)$ *such that* ι^{d}_G *preserves the comultiplication, the counit, and the antipode. Moreover,* $\mathrm{C}_0(G,k)$ *is cocommutative.*

Proof. (i) The uniqueness and the cocommutativity follow from the fact that the image of ι^{c}_G is dense in $O_k[[G]]$. The system $((O_k/\wp)[G/K])_{(\wp,K)}$ forms an inverse system in the under category $O_k[G]/\mathrm{Hopf}(\mathscr{C}, \otimes_{O_k}, O_k)$ with respect to the canonical projections, where $\wp$ runs through open ideals of O_k and K runs through open normal subgroups of G. By the definition of $\hat{\otimes}_{O_k}$, the systems of comultiplications, counits, and antipodes induce a comultiplication, a counit, and an antipode on $O_k[[G]]$ respectively with respect to $(\hat{\otimes}_{O_k}, O_k)$, for which $O_k[[G]]$ forms a Hopf monoid object in $(\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)$ satisfying the desired conditions again because the image of ι^{c}_G is dense in $O_k[[G]]$.

(ii) The uniqueness and the cocommutativity follow from the fact that the image of ι^{d}_G is dense in $\mathrm{C}_0(G,k)$. Take a uniformiser $\varpi \in O_k$. Since $\mathrm{C}_0(G,k)^\circ$ is naturally identified with the ϖ-adic completion of $O_k[G]$, the Hopf O_k-algebra structure of $O_k[G]$ induces a comultiplication $\mathrm{C}_0(G,k)^\circ \to (\mathrm{C}_0(G,k)\hat{\otimes}_k \mathrm{C}_0(G,k))^\circ$, a counit $\mathrm{C}_0(G,k)^\circ \to O_k$, and an antipode $\mathrm{C}_0(G,k)^\circ \to \mathrm{C}_0(G,k)^\circ$, which extend to a comultiplication, a counit, and an antipode on $\mathrm{C}_0(G,k)$ respectively with respect to $(\hat{\otimes}_k, k)$, for which $\mathrm{C}_0(G,k)$ forms a Hopf monoid object in $(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k), \hat{\otimes}_k, k)$ satisfying the desired conditions again because the image of ι^{d}_G is dense in $\mathrm{C}_0(G,k)$. □

Corollary 3.8. (i) *Let* G *be a* $\mathscr{U}$*-small profinite group. Then the monoid object structure of* $\mathrm{C}(G,k)$ *extends to a unique Hopf monoid object structure in* $(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k), \hat{\otimes}_k, k)$ *such that the composite of the comultiplication and the isometric* k*-linear isomorphism* $\iota \colon \mathrm{C}(G,k)\hat{\otimes}_k\mathrm{C}(G,k) \to \mathrm{C}(G \times G, k)$ *in Corollary 3.5 (ii) coincides with the composition to the multiplication* $G \times G \to G$*, the counit is* $\iota^{\mathrm{c}}_G(1) \in O_k[[G]] \cong \mathrm{C}(G,k)^{\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}}$*, and the antipode is the involution* $\mathrm{C}(G,k) \to \mathrm{C}(G,k)$ *given as the composition to the map* $G \to G,\ g \mapsto g^{-1}$.

(ii) *Let* G *be a* $\mathscr{U}$*-small discrete group. Then the monoid object structure of* O_k^G *extends to a unique Hopf monoid object structure in* $(\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)$ *such that the composite of the comultiplication and the homeomorphic* O_k*-linear isomorphism* $O_k^G \hat{\otimes}_{O_k} O_k^G \to O_k^{G\times G}$ *in Lemma 3.3 coincides with the composition to the multiplication* $G \times G \to G$*, the counit is* $\iota^{\mathrm{d}}_G(1) \in \mathrm{C}_0(G,k) \cong (O_k^G)^{\mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}}}$*, and the antipode is the involution* $O_k^G \to O_k^G,\ (c_g)_{g\in G} \mapsto (c_{g^{-1}})_{g \in G}$.

Proof. The assertions immediately follow from Example 3.1, Lemma 3.3, Theorem 3.6, and Proposition 3.7 by the fact that the dual of the comultiplication of $O_k[[G]]$ (respectively, $\mathrm{C}_0(G,k)$) coincides with the pointwise multiplication of $\mathrm{C}(G,k)$ (respectively, O_k^G). $\square$

4 Pontryagin duality

We introduce two sorts of p-adic Pontryagin dualities in terms of functors of points. Through the duality in Theorem 3.6, we establish p-adic analogues of Fourier transform and Plancherel's theorem extending Amice transform $\mathrm{C}(\mathbb{Z}_p,k)^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \to O_k[[T]]$.

4.1 Discrete Abelian group For a Banach k-algebra A, we denote by $A^{\mathbb{G}_{\mathrm{m}/k}}$ the discrete group which shares the underlying group with $(A^\circ)^\times$. The correspondence $A \rightsquigarrow A^{\mathbb{G}_{\mathrm{m}/k}}$ restricted to $\mathscr{U}$ gives a functor $\mathbb{G}_{\mathrm{m}/k} \colon \mathrm{CAlg}(k) \to \mathrm{Ab}$, which is an analytic geometric counterpart of S^1. For any $\mathscr{U}$-small set I, the correspondence $A \rightsquigarrow (A^{\mathbb{G}_{\mathrm{m}/k}})^I$ restricted to $\mathscr{U}$ gives a functor $\mathbb{G}_{\mathrm{m}/k}^I \colon \mathrm{CAlg}(k) \to \mathrm{Ab}$. A functor $\mathscr{G} \colon \mathrm{CAlg}(k) \to \mathrm{Ab}$ is said to be a *torus* if it is naturally isomorphic to $\mathbb{G}_{\mathrm{m}/k}^I$ for some $\mathscr{U}$-small set I, and is said to be a *multiplicative analytic group over* k if it is naturally isomorphic to the kernel of a natural transformation between tori. We establish a p-adic Pontryagin duality between discrete Abelian groups and multiplicative analytic groups over k.

Example 4.1. (i) Let $n \in \mathbb{N} \setminus \{0\}$. For a Banach k-algebra A, we denote by $A^{\mu_{n/k}} \subset A^{\mathbb{G}_{\mathrm{m}/k}}$ the subgroup $\{f \in A^{\mathbb{G}_{\mathrm{m}/k}} \mid f^n = 1\}$. The correspondence $A \rightsquigarrow A^{\mu_{n/k}}$ restricted to $\mathscr{U}$ gives a functor $\mu_{n/k} \colon \mathrm{CAlg}(k) \to \mathrm{Ab}$, which is a multiplicative analytic group over k because it is the kernel of the natural transformation $\mathbb{G}_{\mathrm{m}/k} \to \mathbb{G}_{\mathrm{m}/k}$ given by the n-th power.

(ii) Let $D_{\mathbb{N}}$ denote the set $\{(n_0,n_1) \in (\mathbb{N} \setminus \{0\})^2 \mid \exists d \in \mathbb{N}, dn_0 = n_1\}$. For a Banach k-algebra A, we denote by $A^{\widehat{\mathbb{Z}}(1)_{/k}}$ the inverse limit of $(A^{\mu_{n/k}})_{n \in \mathbb{N} \setminus \{0\}}$ with respect to the system of $n_0^{-1}n_1$-th powers $A^{\mu_{n_1/k}} \to A^{\mu_{n_0/k}}$ indexed by $(n_0,n_1) \in D_{\mathbb{N}}$ equipped with the discrete topology. The correspondence $A \rightsquigarrow A^{\widehat{\mathbb{Z}}(1)_{/k}}$ restricted to $\mathscr{U}$ gives a functor

$$\widehat{\mathbb{Z}}(1)_{/k} \colon \mathrm{CAlg}(k) \to \mathrm{Ab},$$

which is a multiplicative analytic group over k because it is the kernel of the natural transformation

$$\mathbb{G}_{\mathrm{m}/k}^{\mathbb{N}\backslash\{0\}} \to \mathbb{G}_{\mathrm{m}/k}^{D_{\mathbb{N}}},\ (f_n)_{n\in\mathbb{N}} \mapsto (f_{n_0}^{-1} f_{n_1}^{n_0^{-1}n_1})_{(n_0,n_1)\in D_{\mathbb{N}}}.$$

For discrete Abelian groups G and H, we denote by $\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G,H)$ the discrete set of group homomorphisms $G \to H$ equipped with the point-wise multiplication. Let G be a $\mathscr{U}$-small discrete Abelian group. The correspondence $A \rightsquigarrow \mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, A^{\mathbb{G}_{\mathrm{m}/k}})$ restricted to $\mathscr{U}$ gives a functor $\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G,\mathbb{G}_{\mathrm{m}/k})\colon \mathrm{CAlg}(k) \to \mathrm{Ab}$. Since $\mathbb{G}_{\mathrm{m}/k}$ is a counterpart of S^1, $\mathrm{Hom}_{\mathrm{Ab}}(\bullet,\mathbb{G}_{\mathrm{m}/k})$ is a p-adic analogue of the Pontryagin dual.

Proposition 4.2. *For any $\mathscr{U}$-small discrete Abelian group G,*

$$\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G,\mathbb{G}_{\mathrm{m}/k})$$

forms a multiplicative analytic group over k.

In order to verify Proposition 4.2, we prepare several lemmata.

Lemma 4.3. *For any $\mathscr{U}$-small discrete Abelian group G, $\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G,\mathbb{G}_{\mathrm{m}/k})$ is representable by $\mathrm{C}_0(G,k)$.*

Proof. By Proposition 3.7 (ii), $\mathrm{C}_0(G,k)$ represents a functor $\widehat{G}\colon \mathrm{CAlg}(k) \to \mathrm{Ab}$. Denote by $u \in \mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, \mathrm{C}_0(G,k)^{\mathbb{G}_{\mathrm{m}/k}})$ the composite of the canonical embedding $G \hookrightarrow k[G]$ and ι_G^{d}, and by F the natural transformation $\widehat{G} \Rightarrow \mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G,\mathbb{G}_{\mathrm{m}/k})$ determined by the equality $F(\mathrm{C}_0(G,k))(\mathrm{id}_{\mathrm{C}_0(G,k)}) = u$ by Yoneda's lemma because u is a group homomorphism. We show that F is a natural equivalence. Let A be a commutative monoid object in $(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k), \hat{\otimes}_k, k)$. Then $F(A)$ is the group homomorphism

$$\mathrm{Hom}_{\mathrm{CAlg}(k)}(\mathrm{C}_0(G,k),A) \to \mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, A^{\mathbb{G}_{\mathrm{m}/k}}),\ \varphi \mapsto \varphi \circ u.$$

Since the image of u generates a dense k-subalgebra of $\mathrm{C}_0(G,k)$, $F(A)$ is injective. Let $\psi \in \mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, A^{\mathbb{G}_{\mathrm{m}/k}})$. For any $g \in G$, we have $\|\psi(g)\| = 1$ by $\psi(g) \in (A^\circ)^\times$. Therefore the k-algebra homomorphism $\varphi\colon k[G] \to A$ associated to ψ by the universality of the group algebra satisfies $\|\varphi(f)\| \leq \|\iota_G^{\mathrm{d}}(f)\|$ for any $f \in k[G]$. It implies that φ extends to a unique submetric k-algebra homomorphism $\widehat{\varphi}\colon \mathrm{C}_0(G,k) \to A$ through ι_G^{d}, because the image of ι_G^{d} is dense in $\mathrm{C}_0(G,k)$. We have $F(A)(\widehat{\varphi}) = \widehat{\varphi} \circ u = \psi$. It implies the surjectivity of $F(A)$. Thus F is a natural equivalence. $\square$

Corollary 4.4. *For any $\mathscr{U}$-small set I, $\mathbb{G}_{\mathrm{m}/k}^{I}$ is representable by* $\mathrm{C}_0(\mathbb{Z}^{\oplus I}, k)$.

Proof. The assertion immediately follows from Lemma 4.3 applied to the case $G = \mathbb{Z}^{\oplus I}$ by the universality of the direct sum, because the identity functor $\mathrm{Ab} \to \mathrm{Ab}$ is represented by $\mathbb{Z}$. □

Proof of Proposition 4.2. Denote by φ_0 the group homomorphism $\mathbb{Z}^{\oplus G} \to G$ associated to id_G, and by φ_1 the group homomorphism $\mathbb{Z}^{\oplus \ker(\varphi_0)} \to \mathbb{Z}^{\oplus G}$ associated to the inclusion $\ker(\varphi_0) \hookrightarrow \mathbb{Z}^{\oplus G}$. By Proposition 3.7 (ii),

$$\mathrm{C}_0(\varphi_0, k) \colon \mathrm{C}_0(\mathbb{Z}^{\oplus G}, k) \to \mathrm{C}_0(G, k)$$

and $\mathrm{C}_0(\varphi_1, k) \colon \mathrm{C}_0(\mathbb{Z}^{\oplus \ker(\varphi_0)}, k) \to \mathrm{C}_0(\mathbb{Z}^{\oplus G}, k)$ form Hopf monoid homomorphisms. By Lemma 4.3 and Corollary 4.4, they induce natural transformations $F_0 \colon \mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, \mathbb{G}_{\mathrm{m}/k}) \Rightarrow \mathbb{G}_{\mathrm{m}/k}^{G}$ and $F_1 \colon \mathbb{G}_{\mathrm{m}/k}^{G} \Rightarrow \mathbb{G}_{\mathrm{m}/k}^{\ker(\varphi_0)}$. We show that $(\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, \mathbb{G}_{\mathrm{m}/k}), F_0)$ satisfies the universality of the kernel of F_1.

By $\varphi_1 \circ \varphi_0 = 0$, we have $\mathrm{C}_0(\varphi_1, k) \circ \mathrm{C}_0(\varphi_0, k) = \mathrm{C}_0(0, k) = 0$, and hence $F_1 \circ F_0 = 0$. Let A be a $\mathscr{U}$-small commutative monoid object in $(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k), \hat{\otimes}_k, k)$. We show that $(\mathrm{Hom}_{\mathrm{Ab}}(G, A^{\mathbb{G}_{\mathrm{m}/k}}), F_0(A))$ satisfies the universality of the kernel of $F_1(A)$. By $F_1 \circ F_0 = 0$, we have $F_1(A) \circ F_0(A) = 0$. By the definition of F_0, $F_0(A)$ is the inclusion $\mathrm{Hom}_{\mathrm{Ab}}(G, A^{\mathbb{G}_{\mathrm{m}/k}}) \hookrightarrow (A^{\mathbb{G}_{\mathrm{m}/k}})^G$. Let $f \in \ker(F_1(A)) \subset (A^{\mathbb{G}_{\mathrm{m}/k}})^G$. For any $(g_0, g_1) \in G^2$, we have $g_0 g_1 - g_0 - g_1 \in \ker(\varphi_0)$, and hence

$$\begin{aligned} f(g_0 g_1) &= (f(g_0 g_1) f(g_0)^{-1} f(g_1)^{-1}) f(g_0) f(g_1) \\ &= F_1(f)(g_0 g_1 - g_0 - g_1) f(g_0) f(g_1) = f(g_0) f(g_1) \end{aligned}$$

by the construction of the natural isomorphism in the proof of Lemma 4.3. It implies that f is a group homomorphism $G \to A^{\mathbb{G}_{\mathrm{m}/k}}$, and hence is an element of the image of $F_0(A)$. Therefore $(\mathrm{Hom}_{\mathrm{Ab}}(G, A^{\mathbb{G}_{\mathrm{m}/k}}), F_0(A))$ satisfies the universality of the kernel of $F_1(A)$. It implies that $(\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, \mathbb{G}_{\mathrm{m}/k}), F_0)$ satisfies the universality of the kernel of F_1. □

We denote by PAb_k the category of multiplicative analytic groups over k and natural transformations. By Proposition 4.2, the correspondence $G \rightsquigarrow \mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, \mathbb{G}_{\mathrm{m}/k})$ gives a functor $\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(\bullet, \mathbb{G}_{\mathrm{m}/k}) \colon \mathrm{Ab}^{\mathrm{op}} \to \mathrm{PAb}_k$. Now we state a p-adic analogue of the Pontryagin duality.

Theorem 4.5. *The functor $\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(\bullet, \mathbb{G}_{\mathrm{m}/k})$ is fully faithful and essentially surjective.*

By the axiom of choice and the smallness of Ab, Theorem 4.5 implies that $\mathrm{Ab}^{\mathrm{op}}$ is naturally equivalent to PAb_k. However, there seems to be no natural construction of an inverse of $\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(\bullet, \mathbb{G}_{\mathrm{m}/k})$. Although the fullness and the faithfulness immediately follows from the proof of Corollary 3.8 (ii) because the maximal spectrum of O_k^G modulo the maximal ideal of O_k is canonically homeomorphic to the Stone–Čech compactification of G, we give an alternative proof constructing an "inverse correspondence". In order to verify Theorem 4.5, we prepare several notions and lemmata.

Let $\mathscr{G}$ be a multiplicative analytic group over k. We denote by $\mathscr{H}\mathrm{om}_{\mathrm{PAb}_k}(\mathscr{G}, \mathbb{G}_{\mathrm{m}/k})$ the discrete set $\mathrm{Hom}_{\mathrm{PAb}_k}(\mathscr{G}, \mathbb{G}_{\mathrm{m}/k})$ equipped with the pointwise multiplication. We note that $\mathscr{H}\mathrm{om}_{\mathrm{PAb}_k}(\mathscr{G}, \mathbb{G}_{\mathrm{m}/k})$ is not $\mathscr{U}$-small in our context, and hence is not an object of Ab. Therefore the correspondence $\mathscr{G} \rightsquigarrow \mathscr{H}\mathrm{om}_{\mathrm{PAb}_k}(\mathscr{G}, \mathbb{G}_{\mathrm{m}/k})$ does not give a functor

$$\mathscr{H}\mathrm{om}_{\mathrm{PAb}_k}(\bullet, \mathbb{G}_{\mathrm{m}/k}) \colon \mathrm{PAb}_k \to \mathrm{Ab}^{\mathrm{op}}.$$

Nevertheless, it will essentially play a role of an inverse of $\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(\bullet, \mathbb{G}_{\mathrm{m}/k})$. Indeed, it is functorial in the sense that for any natural transformation F between multiplicative analytic groups $\mathscr{G}_0$ and $\mathscr{G}_1$ over k, the map

$$\mathscr{H}\mathrm{om}_{\mathrm{PAb}_k}(F, \mathbb{G}_{\mathrm{m}/k}) \colon \mathscr{H}\mathrm{om}_{\mathrm{PAb}_k}(\mathscr{G}_1, \mathbb{G}_{\mathrm{m}/k}) \to \mathscr{H}\mathrm{om}_{\mathrm{PAb}_k}(\mathscr{G}_0, \mathbb{G}_{\mathrm{m}/k}), \ \gamma \mapsto \gamma \circ F$$

is a group homomorphism, and the correspondence $F \rightsquigarrow \mathscr{H}\mathrm{om}_{\mathrm{PAb}_k}(F, \mathbb{G}_{\mathrm{m}/k})$ preserves identities and compositions.

Let G be a $\mathscr{U}$-small discrete Abelian group. For any $g \in G$, $\iota_G^{\mathrm{d}}(g)$ is a group-like element of $\mathrm{C}_0(G,k)$ because ι_G^{d} preserves the comultiplication, and hence $\iota_G^{\mathrm{d}}(g) \in \mathbb{G}_{\mathrm{m}/k}(\mathrm{C}_0(G,k))$ yields a natural transformation $\eta_{\mathrm{Ab}}(G)(g) \colon \mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, \mathbb{G}_{\mathrm{m}/k}) \Rightarrow \mathbb{G}_{\mathrm{m}/k}$ by Lemma 4.3 and Yoneda's lemma.

Lemma 4.6. *For any $\mathscr{U}$-small discrete Abelian group G, the map*

$$\begin{array}{rcl} \eta_{\mathrm{Ab}}(G) \colon G & \to & \mathscr{H}\mathrm{om}_{\mathrm{PAb}_k}(\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, \mathbb{G}_{\mathrm{m}/k}), \mathbb{G}_{\mathrm{m}/k}) \\ g & \mapsto & \eta_{\mathrm{Ab}}(G)(g) \end{array}$$

is a group isomorphism.

Proof. The map $\eta_{\mathrm{Ab}}(G)$ is a group homomorphism because the evaluation at $\mathrm{C}_0(G,k)$ preserves the multiplication by the definition of $\mathscr{H}\mathrm{om}_{\mathrm{PAb}_k}(G, \mathbb{G}_{\mathrm{m}/k})$. For a Hopf monoid homomorphism $\varphi : \mathrm{C}_0(k) \to \mathrm{C}_0(G,k)$, we denote by

$F(\varphi)$ the natural transformation $\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, \mathbb{G}_{\mathrm{m}/k}) \Rightarrow \mathbb{G}_{\mathrm{m}/k}$ corresponding to φ by Lemma 4.3 and Corollary 4.4. The map

$$\begin{aligned} F \quad : \quad & \mathrm{Hom}_{\mathrm{Hopf}(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k), \hat{\otimes}_k, k)}(\mathrm{C}_0(\mathbb{Z}, k), \mathrm{C}_0(G, k)) \\ \to \quad & \mathrm{Hom}_{\mathrm{PAb}_k}(\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, \mathbb{G}_{\mathrm{m}/k}), \mathbb{G}_{\mathrm{m}/k}), \ \varphi \mapsto F(\varphi) \end{aligned}$$

is bijective by Yoneda's lemma. Put $H \coloneqq F^{-1} \circ \eta_{\mathrm{Ab}}(G)$. It suffices to verify the bijectivity of H.

We have $H(g)(f) = \sum_{n \in \mathbb{Z}} f(n) \iota^{\mathrm{d}}_G(g)^n$ for any $(g, f) \in G \times \mathrm{C}_0(\mathbb{Z}, k)$ by the construction. In particular, we obtain $H(g)(\iota^{\mathrm{c}}_{\mathbb{Z}}(1)) = \iota^{\mathrm{d}}_G(g)$ for any $g \in G \times \mathrm{C}_0(\mathbb{Z}, k)$. Therefore the injectivity of ι^{d}_G implies that of H. We show the surjectivity of H. Let φ be a Hopf monoid homomorphism $\mathrm{C}_0(\mathbb{Z}, k) \to \mathrm{C}_0(G, k)$. Since $\iota^{\mathrm{d}}_{\mathbb{Z}}(1)$ is a group-like element of $\mathrm{C}_0(\mathbb{Z}, k)$, $\varphi(\iota^{\mathrm{d}}_{\mathbb{Z}}(1))$ is a group-like element of $\mathrm{C}_0(G, k)$. Therefore we have

$$\begin{aligned} & \textstyle\sum_{g \in G} \varphi(\iota^{\mathrm{d}}_{\mathbb{Z}}(1))(g) \iota^{\mathrm{d}}_G(g) \otimes \iota^{\mathrm{d}}_G(g) = \varphi(\iota^{\mathrm{d}}_{\mathbb{Z}}(1)) \otimes \varphi(\iota^{\mathrm{d}}_{\mathbb{Z}}(1)) \\ & \quad = \textstyle\sum_{(g_0, g_1) \in G^2} \varphi(\iota^{\mathrm{d}}_{\mathbb{Z}}(1))(g_0) \varphi(\iota^{\mathrm{d}}_{\mathbb{Z}}(1))(g_1) \iota^{\mathrm{d}}_G(g_0) \otimes \iota^{\mathrm{d}}_G(g_1) \end{aligned}$$

in $\mathrm{C}_0(G, k) \hat{\otimes}_k \mathrm{C}_(G, k)$ for any $(g_0, g_1) \in G^2$. It implies that there is at most one $g \in G$ satisfying $\varphi(\iota^{\mathrm{d}}_{\mathbb{Z}}(1))(g) \neq 0$ by Proposition 2.6, and such a g satisfies $\varphi(\iota^{\mathrm{d}}_{\mathbb{Z}}(1))(g)^2 = \varphi(\iota^{\mathrm{d}}_{\mathbb{Z}}(1))(g)$, that is, $\varphi(\iota^{\mathrm{d}}_{\mathbb{Z}}(1))(g) = 1$. By $\varphi(\iota^{\mathrm{d}}_{\mathbb{Z}}(1)) \in \mathrm{C}_0(G, k)^{\times}$, it implies that there is a $g \in G$ such that $\varphi(\iota^{\mathrm{d}}_{\mathbb{Z}}(1)) = \iota^{\mathrm{d}}_G(g)$. For any $f \in \mathrm{C}_0(\mathbb{Z}, k)$, we have $H(g)(f) = \sum_{n \in \mathbb{Z}} f(n) \iota^{\mathrm{d}}_G(g)^n = \sum_{n \in \mathbb{Z}} f(n) \varphi(\iota^{\mathrm{d}}_{\mathbb{Z}}(1))^n = \varphi(f)$. It implies $\varphi = H(g)$. Therefore H is surjective. □

Lemma 4.7. *For any multiplicative Abelian group $\mathscr{G}$ over k, there is a $\mathscr{U}$-small Abelian group G such that $\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, \mathbb{G}_{\mathrm{m}/k})$ is isomorphic to $\mathscr{G}$.*

Proof. Since $\mathscr{G}$ is a multiplicative analytic group over k, there is a tuple (I, J, F, H) of $\mathscr{U}$-small sets I and J and natural transformations $F \colon \mathbb{G}^I_{\mathrm{m}/k} \to \mathbb{G}^J_{\mathrm{m}/k}$ and $H \colon \mathscr{G} \to \mathbb{G}^I_{\mathrm{m}/k}$ such that $(\mathscr{G}, H)$ satisfies the universality of the kernel of F. By Corollary 4.4 and Lemma 4.6, there is a unique group homomorphism

$$\varphi \colon \mathbb{Z}^{\oplus J} \to \mathbb{Z}^{\oplus I}$$

such that $\eta_{\mathrm{Ab}}(\mathbb{Z}^{\oplus I}) \circ \varphi = \mathscr{H}\mathrm{om}_{\mathrm{Ab}}(\mathscr{H}\mathrm{om}_{\mathrm{PAb}_k}(F, \mathbb{G}_{\mathrm{m}/k}), \mathbb{G}_{\mathrm{m}/k}) \circ \eta_{\mathrm{Ab}}(\mathbb{Z}^{\oplus J})$. Put $G \coloneqq \mathrm{coker}(\varphi)$. Since G is a quotient of $\mathbb{Z}^{\oplus I}$, G is $\mathscr{U}$-small. We denote by π the canonical projection $\mathbb{Z}^{\oplus I} \twoheadrightarrow G$. By the left exactness of the functor $\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(\bullet, A^{\mathbb{G}_{\mathrm{m}/k}}) \colon \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}$ for any commutative monoid object A in $(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k), \hat{\otimes}_k, k)$, $(\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, \mathbb{G}_{\mathrm{m}/k}), \mathscr{H}\mathrm{om}_{\mathrm{Ab}}(\pi, \mathbb{G}_{\mathrm{m}/k}))$ satisfies the universality of the kernel of F identified with $\mathscr{H}\mathrm{om}_{\mathrm{PAb}_k}(\varphi, \mathbb{G}_{\mathrm{m}/k})$ through the natural isomorphisms $\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(\mathbb{Z}^{\oplus I}, \mathbb{G}_{\mathrm{m}/k}) \Rightarrow \mathbb{G}^{I}_{\mathrm{m}/k}$ and

$$\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(\mathbb{Z}^{\oplus J}, \mathbb{G}_{\mathrm{m}/k}) \Rightarrow \mathbb{G}^{J}_{\mathrm{m}/k}.$$

Thus $\mathscr{G}$ is naturally isomorphic to $\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(G, \mathbb{G}_{\mathrm{m}/k})$. □

By Lemma 4.3, Lemma 4.6, and Lemma 4.7, we obtain the following:

Corollary 4.8. *Every multiplicative analytic group $\mathscr{G}$ over k is representable by $\mathrm{C}_0(G, k)$ for a $\mathscr{U}$-small Abelian group G isomorphic to $\mathscr{H}\mathrm{om}_{\mathrm{PAb}_k}(\mathscr{G}, \mathbb{G}_{\mathrm{m}/k})$.*

Proof of Theorem 4.5. The fullness and faithfulness immediately follow from Lemma 4.6. The essential surjectivity precisely coincides with the assertion of Lemma 4.7. □

By Theorem 4.5, we obtain the following:

Corollary 4.9. *The category PAb_k forms an Abelian category with respect to a natural* Ab*-enrichment, and admits all $\mathscr{U}$-small colimits and all $\mathscr{U}$-small limits.*

Remark 4.10. The functor $\mathrm{C}_0(\bullet, k) \colon \mathrm{Ab} \to \mathrm{Hopf}(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k), \hat{\otimes}_k, k)$ does not preserve $\mathscr{U}$-small limits. For example, the limit of $(\mathbb{Z}/n\mathbb{Z})_{n \in \mathbb{N} \setminus \{0\}}$ in Ab with respect to the canonical projections is $\widehat{\mathbb{Z}}$ equipped with the discrete topology, while the colimit of $(\mu_{n/k})_{n \in \mathbb{N} \setminus \{0\}} \cong (\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_{\mathrm{m}/k}))_{n \in \mathbb{N} \setminus \{0\}}$ in PAb_k is the functor $\mu_{\infty/k} \colon \mathrm{CAlg}(k) \to \mathrm{Ab}$ which assigns the torsion group of $A^{\mathbb{G}_{\mathrm{m}/k}}$ to each commutative monoid object A in $(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k), \hat{\otimes}_k, k)$. The natural transformation $\mu_{\infty/k} \Rightarrow \mathscr{H}\mathrm{om}_{\mathrm{Ab}}(\widehat{\mathbb{Z}}, \mathbb{G}_{\mathrm{m}/k})$ given by the universality of the colimit of $(\mu_{n/k})_{n \in \mathbb{N} \setminus \{0\}}$ is not a natural isomorphism, because $\iota^{\mathrm{d}}_{\widehat{\mathbb{Z}}} \colon \widehat{\mathbb{Z}} \hookrightarrow \mathrm{C}_0(\widehat{\mathbb{Z}}, k)^{\mathbb{G}_{\mathrm{m}/k}}$ assigns to $1 \in \widehat{\mathbb{Z}}$ a $\mathbb{Z}$-torsionfree element. It implies that $\mu_{\infty/k}$ is not a multiplicative analytic group over k and does not admit a maximal multiplicative analytic subgroup over k.

4.2 Profinite Abelian group Let A be a compact Hausdorff flat linear topological O_k-algebra. Then $A^{\times} \subset A$ is the intersection of the preimages of the multiplicative groups of finite quotients, and hence is a closed subset. Therefore it forms a profinite group. We denote by $A^{\mathbb{G}_{\mathrm{m}/O_k}}$ the discrete group which shares the underlying group with $A^{\times}$.

The correspondence $A \rightsquigarrow A^{\mathbb{G}_{\mathrm{m}/O_k}}$ restricted to $\mathscr{U}$ gives a functor $\mathbb{G}_{\mathrm{m}/O_k} \colon \mathrm{CAlg}(O_k) \to \mathrm{Ab}$, which is a formal geometric counterpart of S^1. For any $\mathscr{U}$-small set I, the correspondence $A \rightsquigarrow (A^{\mathbb{G}_{\mathrm{m}/O_k}})^I$ restricted to $\mathscr{U}$ gives a functor $\mathbb{G}_{\mathrm{m}/O_k}^I \colon \mathrm{CAlg}(O_k) \to \mathrm{Ab}$. A functor $\mathscr{G} \colon \mathrm{CAlg}(O_k) \to \mathrm{Ab}$ is said to be a *torus* if it is naturally isomorphic to $\mathbb{G}_{\mathrm{m}/O_k}^I$ for some $\mathscr{U}$-small set I, and is said to be a *multiplicative formal group over* O_k if it is naturally isomorphic to the kernel of a natural transformation between tori . We establish a p-adic Pontryagin duality between profinite Abelian groups and multiplicative formal groups over O_k in a parallel way to the p-adic Pontryagin duality in §4.1.

Example 4.11. (i) Let $n \in \mathbb{N} \setminus \{0\}$. For a compact Hausdorff flat linear topological O_k-algebra A, we denote by $A^{\mu_{n/O_k}} \subset A^{\mathbb{G}_{\mathrm{m}/O_k}}$ the subgroup $\{f \in A^{\mathbb{G}_{\mathrm{m}/O_k}} \mid f^n = 1\}$. The correspondence $A \rightsquigarrow A^{\mu_{n/O_k}}$ restricted to $\mathscr{U}$ gives a functor $\mu_{n/O_k} \colon \mathrm{CAlg}(O_k) \to \mathrm{Ab}$, which is a multiplicative formal group over O_k by a completely similar similar argument in Example 4.1 (i).

(ii) Let $D_{\mathbb{N}}$ denote the set introduced in Example 4.1 (ii). For a compact Hausdorff flat linear topological O_k-algebra A, we denote by $A^{\widehat{\mathbb{Z}}(1)_{/O_k}}$ the inverse limit of $(A^{\mu_{n/O_k}})_{n \in \mathbb{N} \setminus \{0\}}$ with respect to the system of $n_0^{-1} n_1$-th powers $A^{\mu_{n_1/O_k}} \to A^{\mu_{n_0/O_k}}$ indexed by $(n_0, n_1) \in D_{\mathbb{N}}$ equipped with the discrete topology. The correspondence $A \rightsquigarrow A^{\widehat{\mathbb{Z}}(1)_{/O_k}}$ restricted to $\mathscr{U}$ gives a functor $\widehat{\mathbb{Z}}(1)_{/O_k} \colon \mathrm{CAlg}(O_k) \to \mathrm{Ab}$, which is a multiplicative formal group over O_k by a completely similar similar argument in Example 4.1 (ii).

For profinite Abelian groups G and H, we denote by $\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, H)$ the discrete Abelian group whose underlying set is the set of continuous group homomorphisms $G \to H$ and whose operation is the pointwise multiplication. Let G be a $\mathscr{U}$-small profinite Abelian group. The correspondence $A \rightsquigarrow \mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, A^{\times})$ restricted to $\mathscr{U}$ gives a functor

$$\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k}) \colon \mathrm{CAlg}(O_k) \to \mathrm{Ab}.$$

We note that this convention is misleading because $\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k})$ is not a functor which assigns the discrete Abelian group of continuous group homomorphism $G \to A^{\mathbb{G}_{\mathrm{m}/O_k}}$ to each commutative monoid object A in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$, but employ it in order to formulate a p-adic Pontryagin duality in a parallel way to the p-adic Pontryagin duality in §4.1. Since $\mathbb{G}_{\mathrm{m}/O_k}$ is a counterpart of S^1, $\mathrm{Hom}_{\mathrm{PAb}}(\bullet, \mathbb{G}_{\mathrm{m}/O_k})$ is a p-adic analogue of the Pontryagin dual.

Proposition 4.12. *For any $\mathscr{U}$-small profinite Abelian group G,*

$$\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k})$$

forms a multiplicative formal group over O_k.

In order to verify Proposition 4.12, we prepare several lemmata.

Lemma 4.13. *For any $\mathscr{U}$-small profinite Abelian group G,*

$$\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k})$$

is representable by $O_k[[G]]$.

Proof. By Proposition 3.7 (i), $O_k[[G]]$ represents a functor $\widehat{G} \colon \mathrm{CAlg}(O_k) \to \mathrm{Ab}$. Denote by $u \in \mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, O_k[[G]]^{\times})$ the composite of the canonical embedding $G \hookrightarrow O_k[G]$ and ι_G^{c}, and by F the natural transformation $\widehat{G} \Rightarrow \mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k})$ determined by the equality $F(O_k[[G]])(\mathrm{id}_{O_k[[G]]}) = u$ by Yoneda's lemma because u is a continuous group homomorphism. For any $\mathscr{U}$-small commutative monoid object A in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$, $F(A)$ is the map $\mathrm{Hom}_{\mathrm{CAlg}(O_k)}(O_k[[G]], A) \to \mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, A^{\times})$, $\varphi \mapsto \varphi \circ u$, which is bijective by the universality of the Iwasawa algebra. Therefore F is a natural equivalence. □

Let I be a set. We denote by $\widehat{\mathbb{Z}}^{\hat{\oplus} I}$ the profinite completion of the direct sum $\widehat{\mathbb{Z}}^{\oplus I}$ of I-copies of the underlying discrete group of $\widehat{\mathbb{Z}}$. For each $i \in I$, we denote by $\iota_{I,i} \colon \widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}^{\hat{\oplus} I}$ the composite of the i-th canonical embedding $\widehat{\mathbb{Z}} \hookrightarrow \widehat{\mathbb{Z}}^{\oplus I}$ and the canonical embedding $\widehat{\mathbb{Z}}^{\oplus I} \hookrightarrow \widehat{\mathbb{Z}}^{\hat{\oplus} I}$.

Lemma 4.14. *Let I be a set. For any profinite group G, the map*

$$\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(\widehat{\mathbb{Z}}^{\hat{\oplus} I}, G) \to G^I, \ \chi \mapsto (\chi(\iota_{I,i}(1)))_{i \in I}$$

is a group isomorphism, where G^I is equipped with the pointwise multiplication.

Proof. Since every subgroup of $\widehat{\mathbb{Z}}$ of finite index is closed, $\iota_{I,i}$ is continuous for any $i \in I$. Therefore the assertion immediately follows from the universality of the completion and the direct sum, because the forgetful functor $\mathrm{PAb} \to \mathrm{Ab}$ is represented by $\widehat{\mathbb{Z}}$. $\square$

Corollary 4.15. *For any $\mathscr{U}$-small set I, $\mathbb{G}_{\mathrm{m}/O_k}^I$ is representable by $O_k[[\widehat{\mathbb{Z}}^{\hat{\oplus} I}]]$.*

Proof. The assertion immediately follows from Lemma 4.13 applied to the case $G = \widehat{\mathbb{Z}}^{\hat{\oplus} I}$ and Lemma 4.14. $\square$

Proof of Proposition 4.12. For a set I and a profinite group H, denote by $\varphi_{I,H}$ the group isomorphism $\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(\widehat{\mathbb{Z}}^{\hat{\oplus} I}, H) \to H^I$ in Lemma 4.14. Denote by φ_0 the group homomorphism $\varphi_{G,G}^{-1}(\mathrm{id}_G) \colon \widehat{\mathbb{Z}}^{\hat{\oplus} G} \to G$, and by φ_1 the composite of the group homomorphism $\varphi_{\ker(\varphi),\ker(\varphi)} \colon \mathbb{Z}^{\hat{\oplus} \ker(\varphi_0)} \to \ker(\varphi_0)$ and the inclusion $\ker(\varphi_0) \hookrightarrow \mathbb{Z}^{\hat{\oplus} G}$. By Proposition 3.7 (i),

$$O_k[[\varphi_0]] \colon O_k[[\widehat{\mathbb{Z}}^{\hat{\oplus} G}]] \to O_k[[G]]$$

and $O_k[[\varphi_1]] \colon O_k[[\widehat{\mathbb{Z}}^{\hat{\oplus} \ker(\varphi_0)}]] \to O_k[[\widehat{\mathbb{Z}}^{\hat{\oplus} G}]]$ form Hopf monoid homomorphisms. By Lemma 4.13 and Corollary 4.15, they induce natural transformations $F_0 \colon \mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k}) \Rightarrow \mathbb{G}_{\mathrm{m}/O_k}^G$ and $F_1 \colon \mathbb{G}_{\mathrm{m}/O_k}^G \Rightarrow \mathbb{G}_{\mathrm{m}/O_k}^{\ker(\varphi_0)}$. Since φ_0 is a continuous surjective group homomorphism between compact Hausdorff topological groups, $\ker(\varphi_0) \subset \widehat{\mathbb{Z}}^{\hat{\oplus} G}$ is closed and the group isomorphism $\widehat{\mathbb{Z}}^{\hat{\oplus} G} / \ker(\varphi_0) \to G$ induced by φ_0 is a homeomorphism. Therefore $(\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k}), F_0)$ satisfies the universality of the kernel of F_1 by a completely similar argument in the second paragraph in the proof of Proposition 4.2. $\square$

We denote by Ab_{O_k} the category of multiplicative formal groups over O_k and natural transformations. By Proposition 4.12, the correspondence $G \rightsquigarrow \mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k})$ gives a functor $\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(\bullet, \mathbb{G}_{\mathrm{m}/O_k}) \colon \mathrm{PAb}^{\mathrm{op}} \to \mathrm{Ab}_{O_k}$. Now we state a p-adic analogue of the Pontryagin duality.

Theorem 4.16. *The functor $\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(\bullet, \mathbb{G}_{\mathrm{m}/O_k})$ is fully faithful and essentially surjective.*

By the axiom of choice and the smallness of PAb, Theorem 4.16 implies that $\mathrm{PAb}^{\mathrm{op}}$ is naturally equivalent to Ab_{O_k}, but there is an issue on an inverse similar to that for $\mathscr{H}\mathrm{om}_{\mathrm{Ab}}(\bullet, \mathbb{G}_{\mathrm{m}/k})$. Although the fullness and the faithfulness immediately follows from the proof of Corollary 3.8 (i) because the Berkovich spectrum of $\mathrm{C}(G,k)$ is canonically homeomorphic to G by [1] 9.2.7 Corollary, we give an alternative proof constructing an "inverse correspondence". For this purpose, we prepare several notions and lemmata.

Let $\mathscr{G}$ be a multiplicative formal group over O_k. We denote by

$$\mathscr{H}\mathrm{om}_{\mathrm{Ab}_{O_k}}(\mathscr{G}, \mathbb{G}_{\mathrm{m}/O_k})$$

the set $\mathrm{Hom}_{\mathrm{Ab}_{O_k}}(\mathscr{G}, \mathbb{G}_{\mathrm{m}/O_k})$ equipped with the pointwise multiplication and the topology generated by all subsets of the form $\{g \in \mathrm{Hom}_{\mathrm{Ab}_{O_k}}(\mathscr{G}, \mathbb{G}_{\mathrm{m}/O_k}) \mid g(A)(\varphi) \in U\}$ for some tuple (A, φ, U) of a commutative monoid object A in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$, a $\varphi \in \mathscr{G}(A)$, and an open subset $U \subset A$. It actually forms a topological group by the definition of the pointwise multiplication. By a reason similar to that for $\mathscr{H}\mathrm{om}_{\mathrm{PAb}_k}(\bullet, \mathbb{G}_{\mathrm{m}/k})$, the correspondence $\mathscr{G} \rightsquigarrow \mathscr{H}\mathrm{om}_{\mathrm{Ab}_{O_k}}(\mathscr{G}, \mathbb{G}_{\mathrm{m}/O_k})$ does not give a functor $\mathscr{H}\mathrm{om}_{\mathrm{Ab}_{O_k}}(\bullet, \mathbb{G}_{\mathrm{m}/O_k}) \colon \mathrm{Ab}_{O_k} \to \mathrm{Ab}^{\mathrm{op}}$, but is functorial in the sense that for any natural transformation F between multiplicative formal groups $\mathscr{G}_0$ and $\mathscr{G}_1$ over O_k, the map

$$\mathscr{H}\mathrm{om}_{\mathrm{Ab}_{O_k}}(F, \mathbb{G}_{\mathrm{m}/O_k}) \colon \mathscr{H}\mathrm{om}_{\mathrm{Ab}_{O_k}}(\mathscr{G}_1, \mathbb{G}_{\mathrm{m}/O_k})$$

$$\to \mathscr{H}\mathrm{om}_{\mathrm{Ab}_{O_k}}(\mathscr{G}_0, \mathbb{G}_{\mathrm{m}/O_k}),\ \gamma \mapsto \gamma \circ F$$

is a group homomorphism, and the correspondence $F \rightsquigarrow \mathscr{H}\mathrm{om}_{\mathrm{Ab}_{O_k}}(F, \mathbb{G}_{\mathrm{m}/O_k})$ preserves identities and compositions.

Let G be a $\mathscr{U}$-small profinite Abelian group. For any $g \in G$, $\iota_G^{\mathrm{c}}(g)$ is a group-like element of $O_k[[G]]$ because g is a group-like element of $O_k[G]$, and hence $\iota_G^{\mathrm{c}}(g) \in O_k[[G]]^{\times}$ determines a natural transformation $\eta_{\mathrm{PAb}}(G)(g) \colon \mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k}) \Rightarrow \mathbb{G}_{\mathrm{m}/O_k}$ by Lemma 4.13 and Yoneda's lemma.

Lemma 4.17. *For any $\mathscr{U}$-small profinite Abelian group G, the map*

$$\begin{array}{rcl} \eta_{\mathrm{PAb}}(G) \colon G & \to & \mathscr{H}\mathrm{om}_{\mathrm{Ab}_{O_k}}(\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k}), \mathbb{G}_{\mathrm{m}/O_k}) \\ g & \mapsto & \eta_{\mathrm{PAb}}(G)(g) \end{array}$$

is a homeomorphic group isomorphism.

Proof. The map $\eta_{\mathrm{PAb}}(G)$ is a group homomorphism because the evaluation at $O_k[[G]]$ preserves the multiplication by the definition of $\mathscr{H}\mathrm{om}_{\mathrm{Ab}_{O_k}}(G, \mathbb{G}_{\mathrm{m}/O_k})$. For any pair (A, φ) of a commutative monoid object A in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$ and a $\varphi \in \mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, A^\times)$, φ corresponds to a continuous O_k-algebra homomorphism $\tilde{\varphi} \colon O_k[[G]] \to A$ by Lemma 4.13, and the preimage of the open subset $\{g \in \mathrm{Hom}_{\mathrm{Ab}_{O_k}}(\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k}), \mathbb{G}_{\mathrm{m}/O_k}) \mid g(A)(\varphi) \in U\}$ by $\eta_{\mathrm{PAb}}(G)$ coincides with $\{g \in G \mid \eta_{\mathrm{PAb}}(G)(g)(A)(\varphi) \in U\} = \{g \in G \mid \tilde{\varphi}(\iota_G^{\mathrm{c}}(g)) \in U\}$, which is open by the continuity of $\tilde{\varphi}$ and ι_G^{c}, for any open subset $U \subset A$. Therefore $\eta_{\mathrm{Ab}}(G)$ is continuous.

For a Hopf monoid homomorphism $\varphi : O_k[[\widehat{\mathbb{Z}}]] \to O_k[[G]]$, we denote by $F(\varphi)$ the natural transformation $\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k}) \Rightarrow \mathbb{G}_{\mathrm{m}/O_k}$ corresponding to φ by Lemma 4.13 and Corollary 4.15. The map

$$\begin{array}{rcl} F \colon \mathrm{Hom}_{\mathrm{Hopf}(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)}(O_k[[\widehat{\mathbb{Z}}]], O_k[[G]]) & \to & \mathrm{Hom}_{\mathrm{Ab}_{O_k}}(\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k}), \mathbb{G}_{\mathrm{m}/O_k}) \\ \varphi & \mapsto & F(\varphi) \end{array}$$

is bijective by Yoneda's lemma. Put $H \coloneqq F^{-1} \circ \eta_{\mathrm{PAb}}(G)$. It suffices to verify the bijectivity of H. We have $H(g)(\iota_{\widehat{\mathbb{Z}}}^{\mathrm{c}}(n)) = \iota_G^{\mathrm{c}}(g)^n$ for any $(g, n) \in G \times \mathbb{Z}$ by the construction. In particular, we obtain $H(g)(\iota_{\widehat{\mathbb{Z}}}^{\mathrm{c}}(1)) = \iota_G^{\mathrm{c}}(g)$ for any $g \in G \times O_k[[\widehat{\mathbb{Z}}]]$. Therefore the injectivity of ι_G^{c} implies that of H.

We show the surjectivity of H. Let φ be a Hopf monoid homomorphism $O_k[[\widehat{\mathbb{Z}}]] \to O_k[[G]]$. Since $\iota_{\widehat{\mathbb{Z}}}^{\mathrm{c}}(1)$ is a group-like element of $O_k[[\widehat{\mathbb{Z}}]]$, $\varphi(\iota_{\widehat{\mathbb{Z}}}^{\mathrm{c}}(1))$ is a group-like element of $O_k[[G]]$. Let $G_0 \subset G$ be an open subgroup. Denote by $\pi_{G_0} \colon O_k[[G]] \to O_k[[G/G_0]]$ the canonical projection. Then $\pi_{G_0}(\varphi(\iota_{\widehat{\mathbb{Z}}}^{\mathrm{c}}(1)))$ is a group-like element of $O_k[G/G_0]$, and hence there is a unique $g_{G_0} \in G/G_0$ such that $\pi_{G_0}(\varphi(\iota_{\widehat{\mathbb{Z}}}^{\mathrm{c}}(1))) = \iota_{G/G_0}^{\mathrm{c}}(g_{G_0})$. By the uniqueness, $(g_{G_0})_{G_0}$ forms an element of the profinite completion of G, and hence corresponds to a unique $g \in G$. It implies $\varphi(\iota_{\widehat{\mathbb{Z}}}^{\mathrm{c}}(1)) = \iota_G^{\mathrm{c}}(g)$. For any $n \in \mathbb{Z}$, we have $H(g)(\iota_{\widehat{\mathbb{Z}}}^{\mathrm{c}}(n)) = \iota_G^{\mathrm{c}}(g)^n = \varphi(\iota_{\widehat{\mathbb{Z}}}^{\mathrm{c}}(1))^n = \varphi(\iota_{\widehat{\mathbb{Z}}}^{\mathrm{c}}(n))$. It implies $\varphi = H(g)$ because the image of $\mathbb{Z}$ is dense in $\widehat{\mathbb{Z}}$. Therefore H is surjective.

For any open subgroup $G_0 \subset G$, the composite of ι_G^{c} and π_{G_0} gives a continuous group homomorphism $\varphi \colon G \to O_k[[G/G_0]]^\times$, and the image of G_0 by H coincides with the open subset

$$\left\{ g \in \mathrm{Hom}_{\mathrm{Ab}_{O_k}}(\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k}), \mathbb{G}_{\mathrm{m}/O_k}) \middle| g(O_k[[G/G_0]])(\varphi) \in O_k \iota_{G/G_0}^{\mathrm{c}}(1) \right\}$$

by $\varphi(g) = \pi_{G_0}(\iota_G^{\mathrm{c}}(g)) = \iota_{G/G_0}^{\mathrm{c}}(gG_0)$. Therefore H is open. □

Lemma 4.18. *For any multiplicative formal group $\mathscr{G}$ over O_k, there is a $\mathscr{U}$-small profinite Abelian group G such that $\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k})$ is isomorphic to $\mathscr{G}$.*

Proof. The assertion follows from a completely parallel argument in the proof of Lemma 4.7 by Corollary 4.15 and Lemma 4.17. □

By Lemma 4.13, Lemma 4.17, and Lemma 4.18, we obtain the following:

Corollary 4.19. *Every multiplicative formal group $\mathscr{G}$ over O_k is representable by $O_k[[G]]$ for a $\mathscr{U}$-small profinite Abelian group G isomorphic to $\mathscr{H}\mathrm{om}_{\mathrm{Ab}_{O_k}}(\mathscr{G}, \mathbb{G}_{\mathrm{m}/O_k})$.*

Proof of Theorem 4.16. The fullness and faithfulness immediately follow from Lemma 4.17. The essential surjectivity precisely coincides with the assertion of Lemma 4.18. □

Corollary 4.20. *The category Ab_{O_k} forms an Abelian category with respect to a natural* Ab*-enrichment, and admits all $\mathscr{U}$-small colimits and all $\mathscr{U}$-small limits.*

Proof. By Theorem 4.16 and the classical Pontryagin duality, the assertion immediately follows from the elementary fact that the full subcategory of Ab consisting of $\mathscr{U}$-small torsion Abelian groups admits all $\mathscr{U}$-small colimits and all $\mathscr{U}$-small limits. □

Remark 4.21. The functor $O_k[[\bullet]] \colon \mathrm{PAb} \to \mathrm{Hopf}(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$ preserves $\mathscr{U}$-small limits unlike $\mathrm{C}_0(\bullet, k)$ (cf. Remark 4.10). Therefore for any functor $\mathscr{G} \colon \mathrm{CAlg}(O_k) \to \mathrm{Grp}$, the set Σ of multiplicative formal subgroups of $\mathscr{G}$ over O_k forms a partially ordered set with respect to inclusions, which corresponds to a $\mathscr{U}$-small diagram in PAb by Theorem 4.16, and the image of its limit in PAb by $O_k[[\bullet]]$ gives a Hopf monoid in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$, which represents a multiplicative formal subgroup of $\mathscr{G}$ over O_k satisfying the universality of the colimit of Σ. It implies that $\mathscr{G}$ admits a unique maximal multiplicative formal subgroup over O_k.

Following the traditional convention of a representation of an algebraic group, we define the notion of a representation of formal group schemes of a certain type. Let $\mathscr{G}$ be a functor $\mathrm{CAlg}(O_k) \to \mathrm{Grp}$ represented by a commutative Hopf monoid object A in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$. A *representation of* $\mathscr{G}$ is a right A-comodule object in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$. A representation of $\mathscr{G}$ is said to be *irreducible* if it admits precisely two quotients as representations of $\mathscr{G}$. By Theorem 3.2, the irreducibility of a representation of $\mathscr{G}$ is equivalent to the irreducibility of its dual as a Banach left $A^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}}$-module in the sense that it admits precisely two closed $A^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}}$-submodules.

The left regular $A^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}}$-module forms a "universal family" of irreducible unitary Banach k-linear representations of the discrete group $\mathscr{G}(O_k)$ in the sense that for any irreducible unitary Banach k-linear representation V of $\mathscr{G}(O_k)$ which is "analytic" in the sense that it admits a continuous O_k-linear homomorphism $\rho \colon V^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \to V^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \hat{\otimes}_{O_k} A$ for which $V^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}}$ forms a representation of $\mathscr{G}$ such that $(\mathrm{id}_{V^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}}} \otimes g)(\rho(\mu))(v) = \mu(gv)$ for any $(g, \mu, v) \in \mathscr{G}(O_k) \times V^{\mathrm{D}_{\mathrm{Ban}}^{\mathrm{ur}}} \times V$, there is a submetric k-linear homomorphism $\pi \colon A^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}} \to V$ with dense image such that $g\pi(1) = \pi(g)$ for any $g \in \mathscr{G}(O_k) = \mathrm{Hom}_{\mathrm{CAlg}(O_k)}(A, O_k) \subset A^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{ch}}}$.

Let $\mathscr{G}$ be a multiplicative formal group over O_k. By Corollary 4.19, $\mathscr{G}$ is represented by a commutative Hopf monoid object A in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$. Therefore the notion of a representation of $\mathscr{G}$ makes sense as long as we fix A, which is unique up to Hopf monoid isomorphism. At least, the following holds for any choice of such an A:

Theorem 4.22. *Every representation of a multiplicative formal group $\mathscr{G}$ finitely generated over O_k is completely reducible, that is, admits a homeomorphic O_k-linear isomorphism to the direct product of finitely many irreducible representations preserving the action of $\mathscr{G}$.*

Proof. By Corollary 4.19, there is a $\mathscr{U}$-small profinite Abelian group G such that $\mathscr{G}$ is represented by $O_k[[G]]$. It suffices to show that every $O_k[[G]]$-comodule object in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$ admits a $O_k[[G]]$-comodule isomorphism to the direct product of finitely many $O_k[[G]]$-comodule objects in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$ admitting precisely two quotients as $O_k[[G]]$-subcomodule objects in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$. By Theorem 3.2, it suffices to show that every $\mathrm{C}(G, k)$-module object V in $(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k), \hat{\otimes}_k, k)$ of finite dimension admits a $\mathrm{C}(G, k)$-module isomorphism to the direct sum of finitely many irreducible $\mathrm{C}(G, k)$-module objects in $(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k), \hat{\otimes}_k, k)$. It immediately follows from the fact that idempotents generates a dense k-subalgebra of $\mathrm{C}(G, k)$. □

4.3 Fourier transform We denote by $\mathbb{A}^1_{O_k}$ the forgetful functor $\mathrm{CAlg}(O_k) \to \mathrm{Set}$. For a functor $\mathscr{X} \colon \mathrm{CAlg}(O_k) \to \mathrm{Set}$, we denote by $\mathrm{H}^0(\mathscr{X}, \mathbb{A}^1_{O_k})$ the set of natural transformations $\mathscr{X} \Rightarrow \mathbb{A}^1_{O_k}$ equipped with the pointwise O_k-algebra structure. When $\mathscr{X}$ is representable by a commutative monoid object A in $(\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)$, then $\mathrm{H}^0(\mathscr{X}, \mathbb{A}^1_{O_k})$ is naturally isomorphic to the underlying O_k-algebra of A. Therefore if $\mathscr{X}$ is an affine formal scheme over $\mathrm{Spf}(O_k)$ represented by a commutative compact Hausdorff flat linear topological O_k-algebra, then $\mathrm{H}^0(\mathscr{X}, \mathbb{A}^1_{O_k})$ can be regarded as the underlying O_k-algebra of global sections. Therefore $\mathrm{H}^0(\mathscr{X}, \mathbb{A}^1_{O_k})$ is a generalised notion of a ring of functions on $\mathscr{X}$.

Let G be a $\mathscr{U}$-small profinite Abelian group. We denote by $\widehat{G}$ the composite of $\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, \mathbb{G}_{\mathrm{m}/O_k})$ and the forgetful functor $\mathrm{Ab} \to \mathrm{Set}$. Let $f \in \mathrm{C}(G,k)^{\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}}$. For a commutative monoid object A in $(\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)$, we denote by $\hat{f}(A) \colon \widehat{G}(A) \to A$ the group homomorphism which assigns to each $\chi \in \widehat{G}(A) = \mathscr{H}\mathrm{om}_{\mathrm{PAb}}(G, A^{\times})$ a unique $h \in A$ satisfying $f(\mu \circ \chi) = \mu(h)$ for any $\mu \in A^{\mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}}}$, which exists by the bijectivity of $\eta^{\mathrm{ch}}_{\mathrm{fl}}(A) \colon A \to A^{\mathrm{D}^{\mathrm{ch}}_{\mathrm{fl}} \mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}}$. The correspondence $A \rightsquigarrow \hat{f}(A)$ gives a natural transformation $\hat{f} \colon \widehat{G} \Rightarrow \mathbb{A}^1_{O_K}$. We denote by $\mathscr{F}_G$ the ring homomorphism $\mathrm{C}(G,k)^{\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}} \to \mathrm{H}^0(\widehat{G}, \mathbb{A}^1_{O_K}),\ f \mapsto \hat{f}$. It gives a nontrivial conversion connecting two function rings $\mathrm{C}(G,k)$ and $\mathrm{H}^0(\widehat{G}, \mathbb{A}^1_{O_k})$, and is a p-adic analogue of Fourier transform. We note that we do not need to take the first dual in the Archimedean setting, because every Hilbert space admits a canonical isomorphism to its first dual. We show a counterpart of Plancherel's theorem.

Theorem 4.23. *For any $\mathscr{U}$-small profinite Abelian group G, $\mathscr{F}_G$ is a ring isomorphism.*

Proof. We denote by $\delta \colon G \to \mathrm{C}(G,k)^{\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}}$ the map which assigns the delta function $\delta_g \colon \mathrm{C}(G,k) \to k,\ f \mapsto f(g)$ concentrated at g to each $g \in G$. The assertion immediately follows from the fact that the map $\mathrm{H}^0(\widehat{G}, \mathbb{A}^1_{O_k}) \to \mathrm{C}(G,k)^{\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}}$ which assigns to each natural transformation $h \colon \widehat{G} \Rightarrow \mathbb{A}^1_{O_k}$ the submetric k-linear homomorphism $\mathrm{C}(G,k) \to k,\ f \mapsto h(\mathrm{C}(G,k)^{\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}})(\delta)(f)$ gives the inverse transform by the construction. □

We explain the relation to Iwasawa theory. For this purpose, we recall several classical notions. The group homomorphism $\mathbb{Z} \to O_k[[T]]^{\times},\ c \mapsto (1+T)^c$ extends to a unique continuous group homomorphism $\mathbb{Z}_p \to O_k[[T]]^{\times}$, which extends to a unique continuous O_k-algebra homomorphism $O_k[[\mathbb{Z}_p]] \to O_k[[T]]$ by the universality of Iwasawa algebra through $\iota^{\mathrm{c}}_{\mathbb{Z}_p}$, which is a homeomorphic isomorphism and is called *Iwasawa's isomorphism*. The evaluation map $\mathrm{C}(\mathbb{Z}_p, k) \times \mathbb{Z}_p \to k$ extends to a unique continuous O_k-bilinear pairing $\mathrm{C}(\mathbb{Z}_p, k) \times O_k[[\mathbb{Z}_p]] \to k$, which induces a homeomorphic O_k-algebra isomorphism $O_k[[\mathbb{Z}_p]] \to \mathrm{C}(\mathbb{Z}_p, k)^{\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}}$. Combining these two homeomorphic O_k-algebra isomorphisms, we obtain a homeomorphic O_k-algebra isomorphism $\mathrm{C}(\mathbb{Z}_p, k)^{\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}} \to O_k[[T]]$, which is called *Amice transform*. We will observe in Theorem 4.24 that $\mathscr{F}$ is an extension of Amice transform.

The O_k-algebra $O_k[[T]]$ forms a commutative cocommutative Hopf monoid object in $(\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)$ with respect to the comultiplication $O_k[[T]] \to O_k[[T]] \hat{\otimes}_{O_k} O_k[[T]]$, $T \mapsto ((1+T) \otimes (1+T)) - (1 \otimes 1)$, the counit $O_k[[T]] \to O_k$, $T \mapsto 0$, and the antipode $O_k[[T]] \to O_k[[T]]$, $T \mapsto (1+T)^{-1} - 1$. The functor $U_{1/O_k} \colon \mathrm{CAlg}(O_k) \to \mathrm{Ab}$ represented by $O_k[[T]]$ is naturally equivalent to the unitary group of dimension 1, that is, the functor which assigns to each commutative monoid object A in $(\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}, \hat{\otimes}_{O_k}, O_k)$ the unit group $\{f \in A^{\times} \mid \lim_{n \to \infty} (f-1)^n = 0\}$ equipped with the discrete topology. Indeed, the evaluation at $1 + T \in O_k[[T]]$ gives a natural isomorphism between them. By the definition, U_{1/O_k} is an affine formal group scheme over $\mathrm{Spf}(O_k)$. In particular, $\mathrm{H}^0(U_{1/O_k}, \mathbb{A}^1_{O_k})$ is canonically isomorphic to the underlying O_k-algebra of $O_k[[T]]$. Therefore Amice transform can be canonically regarded as an O_k-algebra isomorphism $\mathrm{C}(\mathbb{Z}_p, k)^{\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}} \to \mathrm{H}^0(U_{1/O_k}, \mathbb{A}^1_{O_k})$.

The multiplicative formal group $\mathscr{H}\mathrm{om}_{\mathrm{PAb}}(\mathbb{Z}_p, \mathbb{G}_{\mathrm{m}/O_k})$ is also naturally equivalent to the unitary group of dimension 1. Indeed, the evaluation at $1 \in \mathbb{Z}_p$ gives a natural isomorphism between them. Therefore $\mathscr{F}_{\mathbb{Z}_p}$ also can be canonically regarded as an O_k-algebra isomorphism $\mathrm{C}(\mathbb{Z}_p, k)^{\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}} \to \mathrm{H}^0(U_{1/O_k}, \mathbb{A}^1_{O_k})$. Since Iwasawa's isomorphism $O_k[[\mathbb{Z}_p]] \to O_k[[1+T]]$ forms a Hopf monoid homomorphism which sends $\iota^{\mathrm{c}}_{\mathbb{Z}_p}(1)$ to $1+T$, we immediately obtain the following:

Theorem 4.24. *The ring homomorphism $\mathscr{F}_{\mathbb{Z}_p}$ and Amice transform coincides with each other as ring homomorphisms* $\mathrm{C}(\mathbb{Z}_p, k)^{\mathrm{D}^{\mathrm{ur}}_{\mathrm{Ban}}} \to \mathrm{H}^0(U_{1/O_k}, \mathbb{A}^1_{O_k})$

5 Example

We constructed examples of Hopf monoid objects derived from groups in Proposition 3.7 and Corollary 3.8. Now we survey several examples of Hopf monoid objects which are not derived from groups. They are helpful to grasp the computation of the dual Hopf monoid object.

5.1 General linear group Let $n \in \mathbb{N}$. Put $I_n \coloneqq \{(i,j) \in \mathbb{N}^2 \mid i,j < n\}$. We denote by U_{n/O_k} the completion of the general linear group scheme over $\mathrm{Spf}(O_k)$. Then it is an affine formal group scheme over $\mathrm{Spf}(O_k)$ represented by the Hopf monoid object $O_k[[T_{i,j} \mid (i,j) \in I_n]]$ with $\mathscr{U}$-small indeterminates in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$. The dual of the corresponding Hopf monoid object forms a cocommutative Hopf monoid object A_n in $(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k), \hat{\otimes}_k, k)$. The left regular A_n-module forms a universal family of irreducible unitary Banach k-linear representations of $U_{n/O_k}(O_k)$ in the sense in §4.2.

When $n = 0$, then U_{n/O_k} is the constant presheaf associated to the trivial group, and A_n is canonically isomorphic to $O_k[[\{1\}]] \cong O_k$. When $n = 1$, then U_{n/O_k} is canonically naturally equivalent to the unitary group of dimension 1, which is represented by $O_k[[\mathbb{Z}_p]]$ by the argument in §4.3, and hence A_n is canonically isomorphic to $\mathrm{C}(\mathbb{Z}_p, k)$ as a Hopf monoid object in $(\mathrm{Ban}_{\leq}^{\mathrm{ur}}(k), \hat{\otimes}_k, k)$. We study an explicit presentation of A_n for the general case. Put $\Delta_n \coloneqq \{i \in \mathbb{N} \mid i < n\}$ and $J_n \coloneqq I_n \setminus \{(i,i) \mid i \in \Delta_n\}$. Since the O_k-linear homomorphism

$$\begin{array}{rcl} O_k^{\mathbb{N}^{\Delta_n} \times \mathbb{N}^{J_n}} & \to & O_k[[T_{i,j} \mid (i,j) \in I_n]] \\ (c_{a,b})_{(a,b) \in \mathbb{N}^{\Delta_n} \times \mathbb{N}^{J_n}} & \mapsto & \displaystyle\sum_{(a,b) \in \mathbb{N}^{\Delta_n} \times \mathbb{N}^{J_n}} c_{a,b} \left(\prod_{i \in \Delta_n} T_{i,i}^{a(i)} \left(\prod_{(i,j) \in J_n} T_{i,j}^{b(i,j)} \right) \right) \end{array}$$

is a homeomorphic isomorphism, the underlying Banach k-vector space of A_n can be canonically identified with $\mathrm{C}_0(\mathbb{N}^{\Delta_n} \times \mathbb{N}^{J_n}, k)$. We study its O_k-algebra structure through the presentation.

The projection $O_k^{\mathbb{N}^{\Delta_n} \times \mathbb{N}^{J_n}} \twoheadrightarrow O_k^{\mathbb{N}^{\Delta_n}}$, $(c_{a,b})_{(a,b) \in \mathbb{N}^{\Delta_n} \times \mathbb{N}^{J_n}} \mapsto (c_{(a,0)})_{a \in \mathbb{N}^{\Delta_n}}$ induces an isometric k-linear homomorphism $\mathrm{C}_0(\mathbb{N}^{\Delta_n}, k) \hookrightarrow A_n$. The projection corresponds to the embedding $U_{1/O_k}^n \Rightarrow U_{n/O_k}$ into the multiplicative formal subgroup of diagonal matrices. Since U_{1/O_k} is represented by $O_k[[\mathbb{Z}_p]]$ by Lemma 4.13 and §4.3, U_{1/O_k}^n is represented by $O_k[[\mathbb{Z}_p^n]]$ by Remark 4.21. Therefore $\mathrm{C}_0(\mathbb{N}^{\Delta_n}, k)$ regarded as a closed k-subalgebra of A_n is canonically identified with $\mathrm{C}(\mathbb{Z}_p^n, k)$. Thus the embedding of the completion of the maximal torus corresponds to the embedding of the commutative O_k-algebra of continuous functions on the weight space $\mathbb{Z}_p^n$. We describe A_n in terms of difference operators on $\mathrm{C}(\mathbb{Z}_p^n, k)$.

For each $i \in \Delta_n$, we denote by $\delta_i \colon \Delta_n \to \mathbb{Z}_p$ the characteristic function of $\{i\}$, and by $\frac{\partial}{\partial \kappa_i}$ the difference operator $\mathrm{C}(\mathbb{Z}_p^n, k) \to \mathrm{C}(\mathbb{Z}_p^n, k)$, $f(\kappa) \mapsto f(\kappa + \delta_i) - f(\kappa)$. For an $(i,a) \in \Delta_n \times \mathbb{N}$, we abbreviate $(\frac{\partial}{\partial \kappa_i})^a$ to $\frac{\partial^a}{\partial \kappa_i^a}$. For an $a \in \mathbb{N}^{\Delta_n}$, we abbreviate $\prod_{i \in \Delta_n} \frac{\partial^{a(i)}}{\partial \kappa_i^{a(i)}}$ to $\frac{\partial^a}{\partial \kappa^a}$. For an element f of a $\mathbf{Q}$-algebra A and an $a \in \mathbb{N}$, we abbreviate the binomial coefficient $\begin{pmatrix} f \\ a \end{pmatrix} \in A$ to $B_A(f,a)$. For an $(a, \kappa) \in (\mathbb{N}^{\Delta_n})^2$, we abbreviate $\prod_{i \in \Delta_n} (-1)^{a(i) - \kappa(i)} B_{\mathbf{Q}}(a(i), \kappa(i)) \in O_k$ to $C(a, \kappa)$, where $B_{\mathbf{Q}}(a(i), \kappa(i))$ is an integer and hence gives an element of O_k even if $\mathrm{ch}(k) = p$.

By the definition, the homeomorphic O_k-linear isomorphism $O_k^{\mathbb{N}^{\Delta_n}} \to O_k[[\mathbb{Z}_p^n]]$ assigns to each $(c_a)_{a \in \mathbb{N}^{\Delta_n}} \in O_k^{\mathbb{N}^{\Delta_n}}$ the convergent sum

$$\sum_{a \in \mathbb{N}^{\Delta_n}} \prod_{i \in \Delta_n} c_a (\iota_{\mathbb{Z}_p^n}^{\mathrm{c}}(\delta_i) - 1)^{a(i)}.$$

In particular, for any $a \in \mathbb{N}^{\Delta_n}$, the delta function $\mathbb{N}^{\Delta_n} \to O_k$ centred at a corresponds to the essentially finite sum

$$\prod_{i \in \Delta_n} (\iota_{\mathbb{Z}_p^n}^{\mathrm{c}}(\delta_i) - 1)^{a(i)} = \sum_{\kappa \in \mathbb{N}^{\Delta_n}} C(a, \kappa) \iota_{\mathbb{Z}_p^n}(\kappa),$$

where $\mathbb{N}^{\Delta_n}$ is regarded as a subset of $\mathbb{Z}_p^n$ through the natural identification $\mathbb{Z}_p^n \cong \mathbb{Z}_p^{\Delta_n}$. Therefore every $f \in \mathrm{C}(\mathbb{Z}_p^n, k)$ corresponds to

$$(\sum_{\kappa \in \mathbb{N}^{\Delta_n}} C(a, \kappa) f(\kappa))_{a \in \mathbb{N}^{\Delta_n}} = (\frac{\partial^a f}{\partial \kappa^a}(0))_{a \in \mathbb{N}^{\Delta_n}} \in \mathrm{C}_0(\mathbb{N}^{\Delta_n}, k).$$

In the case $n = 1$ and $\mathrm{ch}(k) = 0$, the inverse correspondence $\mathrm{C}_0(\mathbb{N}^{\Delta_n}, k) \to \mathrm{C}(\mathbb{Z}_p^n, k)$ is the isometric k-linear homomorphism which assigns to each $(c_a)_{a \in \mathbb{N}} \in \mathrm{C}_0(\mathbb{N}, k) \cong \mathrm{C}_0(\mathbb{N}^{\Delta_n}, k)$ the continuous function $f(\kappa)$ given as the convergent sum $\mathbb{Z}_p^n \cong \mathbb{Z}_p \to k$, $\kappa \mapsto \sum_{a=0}^{\infty} c_a B_k(\kappa, a)$ by Mahler's theorem (cf. Theorem 1 in [3]).

For an $i \in \Delta_n$, we denote by $\kappa_i \colon \mathbb{Z}_p^n \to k$ the composite of the $(i+1)$-st projection $\mathbb{Z}_p^n \twoheadrightarrow \mathbb{Z}_p$ and the inclusion $\mathbb{Z}_p \hookrightarrow k$. For a finite subset $U \subset$

$\mathbb{N}^{\Delta_n} \times \mathbb{N}^{J_n}$, we denote by $E_U \in A_n \cong \mathrm{C}_0(\mathbb{N}^{\Delta_n} \times \mathbb{N}^{J_n}, k)$ the characteristic function of U. For an $(i,j) \in J_n$, we denote by $\delta_{(i,j)} \in \mathbb{N}^{\Delta_n} \times \mathbb{N}^{J_n} \cong \mathbb{N}^{\Delta_n \sqcup J_n}$ the characteristic function $\Delta_n \sqcup J_n \to \mathbb{N}$ of $\{(i,j)\}$, and put $E_{i,j} \coloneqq E_{\{\delta_{(i,j)}\}}$. For an $(i,j) \in I_n$, we define $X_{i,j}$ as $T_{i,j}$ in the case $(i,j) \in J_n$, and as $1 + T_{i,i}$ in the case $(i,j) \notin J_n$. Then the comultiplication of $O_k^{\mathbb{N}^{\Delta_n} \times \mathbb{N}^{J_n}} \cong O_k[[T_{i,j} \mid (i,j) \in I_n]]$ sends $X_{i,j}$ to $\sum_{h=0}^{n-1} X_{i,h} \otimes X_{h,j}$ for any $(i,j) \in I_n$. The descriptions of the correspondence $\mathrm{C}(\mathbb{Z}_p^n, k) \to \mathrm{C}_0(\mathbb{N}^{\Delta_n}, k)$ and the comultiplication of $O_k^{\mathbb{N}^{\Delta_n} \times \mathbb{N}^{J_n}}$ allow us to compute the multiplication on A_n in the following explicit way:

Proposition 5.1. (i) *For any* $((i,j),m) \in J_n \times \mathbb{N}$, $E_{i,j}^m = m! E_{\{m\delta_{(i,j)}\}}$.

(ii) *For any* $((i,j),(j',i')) \in J_n^2$ *satisfying* $j \neq j'$, $E_{i,j} E_{j',i'} = E_{\{\delta_{(i,j)} + \delta_{(j',i')}\}}$.

(iii) *For any* $(i,j,i') \in \Delta_n^3$ *satisfying* $(i,j),(j,i'),(i,i') \in J_n$, $E_{i,j} E_{j,i'} = E_{i,i'} + E_{\{\delta_{(i,j)} + \delta_{(j,i')}\}}$.

(iv) *For any* $(i,j) \in J_n$, $E_{i,j} E_{j,i} = \kappa_i + E_{\{\delta_{(i,j)} + \delta_{(j,i)}\}}$.

(v) *For any* $(f,b) \in \mathrm{C}(\mathbb{Z}_p^n, k) \times \mathbb{N}^{J_n}$,

$$
\begin{aligned}
f E_{\{b\}} &= \sum_{a \in \mathbb{N}^{\Delta_n}} \left(\left(\prod_{(i,j) \in \Delta_n} \left(1 + \frac{\partial}{\partial \kappa_i} \right)^{b(i,j)} \right) \frac{\partial^a f}{\partial \kappa^a} \right) (0) E_{\{a+b\}} \\
E_{\{b\}} f &= \sum_{a \in \mathbb{N}^{\Delta_n}} \left(\left(\prod_{(i,j) \in \Delta_n} \left(1 + \frac{\partial}{\partial \kappa_j} \right)^{b(i,j)} \right) \frac{\partial^a f}{\partial \kappa^a} \right) (0) E_{\{a+b\}},
\end{aligned}
$$

where $\mathbb{N}^{\Delta_n}$ *is regarded as a submonoid of* $\mathbb{N}^{\Delta_n} \times \mathbb{N}^{J_n}$ *through the zero extension.*

In particular, for any $(f,(i,j)) \in \mathrm{C}(\mathbb{Z}_p^n, k) \times J_n$, we have

$$
f E_{i,j} = \sum_{a \in \mathbb{N}^{\Delta_n}} \left(\frac{\partial^a}{\partial \kappa^a}\left(f + \frac{\partial f}{\partial \kappa_i}\right)\right)(0) E_{\{a + \delta_{(i,j)}\}}
$$

and $E_{i,j} f = \sum_{a \in \mathbb{N}^{\Delta_n}} (\frac{\partial^a}{\partial \kappa^a}(f + \frac{\partial f}{\partial \kappa_j}))(0) E_{\{a + \delta_{(i,j)}\}}$ by Proposition 5.1 (v).

By the results above, we obtain explicit presentations of the commutators. For any $((i,j),(j',i')) \in J_n^2$ satisfying $j \neq j'$ and $i \neq i'$, we have $E_{i,j} E_{j',i'} - E_{j',i'} E_{i,j} = 0$. For any $(i,j,i') \in \Delta_n^3$ satisfying $i \neq i'$, we have

$E_{i,j}E_{j,i'} - E_{j,i'}E_{i,j} = E_{i,i'}$. For any $(i,j) \in J_n$, we have $E_{i,j}E_{j,i} - E_{j,i}E_{i,j} = \kappa_i - \kappa_j$. For any $(f,(i,j)) \in \mathrm{C}(\mathbb{Z}_p^n, k) \times J_n$, we have $fE_{i,j} - E_{i,j}f = \sum_{a \in \mathbb{N}^{\Delta_n}} (\frac{\partial^a}{\partial \kappa^a}(\frac{\partial f}{\partial \kappa_i} - \frac{\partial f}{\partial \kappa_j}))(0) E_{\{a+\delta_{(i,j)}\}}$.

Suppose $\mathrm{ch}(k) = 0$ in the following in this subsection. We denote by $\mathrm{Lie}(U_{n/O_k})$ the Lie algebra $\bigoplus_{(i,j) \in I_n} k \frac{\partial}{\partial T_{i,j}}$ of U_{n/O_k} over k, and by $\tilde{A}_n$ the universal enveloping algebra of $\mathrm{Lie}(U_{n/O_k})$. By the computation above, the inclusion $\mathrm{Lie}(U_{n/O_k}) \hookrightarrow A_n$ preserves the commutators, and hence extends to a k-algebra homomorphism $\varphi_n \colon \tilde{A}_n \to A_n$. The following gives a description of A_n as a completion of $\tilde{A}_n$ with respect to a seminorm:

Theorem 5.2. *The image of* $O_k[B_{\tilde{A}_n}(\frac{\partial}{\partial T_{i,i}}, m) \mid (m,i) \in \mathbb{N} \times \Delta_n][\frac{1}{m!}\frac{\partial^m}{\partial T_{i,j}^m} \mid (m,(i,j)) \in \mathbb{N} \times J_n] \subset \tilde{A}_n$ *by* φ_n *is a dense* O_k*-subalgebra of* A_n°.

In order to verify Theorem 5.2, we prepare several lemmata. For a $d \in \mathbb{N}$, we put $\mathbb{N}^{I_n}_{<d} \coloneqq \{b \in \mathbb{N}^{I_n} \mid \sum_{(i,j) \in I_n} b(i,j) < d\}$.

Lemma 5.3. *Let* $a \in \mathbb{N}^{J_n}$. *Put* $d \coloneqq \sum_{(i,j) \in J_n} a(i,j)$. *Then the product of* $(\frac{1}{a(i,j)!}E_{i,j}^{a(i,j)})_{(i,j) \in J_n}$ *with respect to any fixed ordering on* J_n *is contained in* $E_{\{a\}} + \bigoplus_{b \in \mathbb{N}^{I_n}_{<d}} O_k E_{\{b\}}$.

Proof. The assertion immediately follows from the computation of the multiplication above and the submetry of the comultiplication of A_n. □

Lemma 5.4. *For any* $(f,i,\epsilon) \in \mathrm{C}(\mathbb{Z}_p^n, k) \times \Delta_n \times (0,\infty)$, *there exists an* $m \in \mathbb{N}$ *such that* $\|(\frac{\partial}{\partial \kappa_i})^{p^m} f\| < \epsilon$.

Proof. By $0 < |p| < 1$, it suffices to verify that for any $(f,i) \in \mathrm{C}(\mathbb{Z}_p, k) \times \Delta_n$, there exists an $m \in \mathbb{N}$ such that $\|(\frac{\partial}{\partial \kappa_i})^{p^m} f\| \leq \|pf\|$. Since $\mathbb{Z}_p^n$ is compact and f is continuous, there is an $m \in \mathbb{N}$ such that for any $(a_0, a_1) \in (\mathbb{Z}_p^n)^2$, $a_0 - a_1 \leq p^m \mathbb{Z}_p^n$ implies $|f(a_0) - f(a_1)| \leq \|pf\|$. Denote by $\langle \bullet, \bullet \rangle$ the canonical pairing $O_k[[\mathbb{Z}_p^n]] \times \mathrm{C}(\mathbb{Z}_p^n, k) \to k$. For any $\kappa \in \mathbb{Z}_p^n$, we have

$$\begin{aligned} & \frac{\partial^{p^m} f}{\partial \kappa_i^{p^m}}(\kappa) = \langle (\iota^{\mathrm{c}}_{\mathbb{Z}_p^n}(\delta_i) - 1)^{p^m} \iota^{\mathrm{c}}_{\mathbb{Z}_p^n}(\kappa), f \rangle \\ = \quad & (f(\kappa + p^m \delta_i) - f(\kappa)) + \sum_{a=1}^{p^m - 1} (-1)^{p^m - a} B_{\mathbf{Q}}(p^m, a) f(\kappa + a\delta_i), \end{aligned}$$

and hence $|((\frac{\partial}{\partial \kappa_i})^{p^m} f)(\kappa)| \leq \|pf\|$. □

Lemma 5.5. *For any* $(f,i) \in \mathrm{C}(\mathbb{Z}_p^n, k) \times \Delta_n$, *there exists an* $\tilde{f} \in \mathrm{C}(\mathbb{Z}_p, k)$ *such that* $(1 + \frac{\partial}{\partial \kappa_i})\tilde{f} = f$ *and* $\|\tilde{f}\| \leq \|f\|$.

Proof. By Lemma 5.4, $\sum_{a=0}^{\infty}(-1)^a \frac{\partial^a f}{\partial \kappa_i^a}$ uniformly converges to an $\tilde{f} \in \mathrm{C}(\mathbb{Z}_p^n, k)$. By the submetry of $\frac{\partial}{\partial \kappa_i}$, we have $(1 + \frac{\partial}{\partial \kappa_i})\tilde{f} = f$ and $\|\tilde{f}\| \leq \|f\|$. □

Lemma 5.6. *The image of* $O_k[B_{\tilde{A}_n}(\frac{\partial}{\partial T_{i,i}}, m) \mid (m,i) \in \mathbb{N} \times \Delta_n] \subset \tilde{A}_n$ *by* φ_n *is a dense* O_k*-subalgebra of* $\mathrm{C}(\mathbb{Z}_p^n, O_k)$.

Proof. By Corollary 3.5 (i), it is reduced to the case $n = 1$. By Mahler's theorem (cf. Theorem 1 in [3]), we have $\varphi_1(T_{1,1}) = \kappa_1$, and $\{B_{\mathrm{C}(\mathbb{Z}_p,k)}(\kappa_1, a) \mid a \in \mathbb{N}\}$ generates a dense O_k-submodule of $\mathrm{C}(\mathbb{Z}_p, O_k)$. Therefore the assertion follows from the fact that φ_1 is a k-algebra homomorphism. □

Proof of Theorem 5.2. We denote by $A \subset A_n^\circ$ the image. It suffices to verify $E_{\{a+b\}} \in A$ for any $(a,b) \in \mathbb{N}^{\Delta_n} \times \mathbb{N}^{J_n}$. Put $d := \sum_{i \in \Delta_n} a(i) + \sum_{(i,j) \in J_n} b(i,j)$. We show $E_{\{a+b\}} \in A$ by induction on d. If $d = 0$, then $E_{\{a+b\}} = E_{\{0\}} = 1 \in A$. Suppose $d > 0$. If $a \neq 0$, then we have $E_{\{b\}} \in A$ by the induction hypothesis, and hence $E_{\{a+b\}} \in A$ by Proposition 5.1 (v), Lemma 5.5, and the bijectivity of the correspondence $\mathrm{C}(\mathbb{Z}_p^n, k) \to \mathrm{C}_0(\mathbb{N}^{\Delta_n}, k)$, $f \mapsto (\frac{\partial^a f}{\partial \kappa^a})_{a \in \mathbb{N}^{\Delta_n}}$. If $a = 0$, then we have $E_{\{a+b\}} = E_{\{b\}} \in A$ by the induction hypothesis and Lemma 5.3. □

By Theorem 5.2, we obtain an explicit description of universal families (cf. §4.2) of irreducible unitary Banach k-linear representations the discrete groups of O_k-valued points of closed formal subgroup schemes of U_{n/O_k} using completions of universal enveloping algebras.

5.2 Quantum group Continuing from §5.1, suppose $\mathrm{ch}(k) = 0$ throughout this subsection. We fix a $q \in O_k$ satisfying $|q - 1| < 1$. By Theorem 5.2, we obtain an expression of A_2° as the closure of $\mathrm{C}(\mathbb{Z}_p^2, k)[\frac{1}{m!}E_{i,j}^m \mid (m,(i,j)) \in \mathbb{N} \times J_2]$. We consider an analogous description for the q-deformation $U_{2,q/O_k}$ of U_{2/O_k}. Following the traditional formulation of the q-deformation of the commutators of the Hopf O_k-algebra $O_k[T_{i,j} \mid (i,j) \in I_2]$ representing GL_{2/O_k}, we define the q-deformation $O_k[[T_{i,j} \mid (i,j) \in I_2]]_q$ of $O_k[[T_{i,j} \mid (i,j) \in I_2]]$ as the Hopf monoid object in $(\mathscr{C}_{\mathrm{fl}}^{\mathrm{ch}}, \hat{\otimes}_{O_k}, O_k)$ whose underlying Hopf monoid object is the quotient of the compact Hausdorff flat linear topological O_k-algebra freely generated by the $\mathscr{U}$-small set

$\{T_{q,i,j} \mid (i,j) \in I_2\}$ of non-commutative indeterminates by the two-sided closed ideal generated by

$$\begin{aligned} &\left\{(1+T_{q,i,i})T_{q,i',j} - q^{(-1)^i}T_{q,i',j}(1+T_{q,i,i})\middle|(i,(i',j)) \in \Delta_2 \times J_2\right\} \\ \cup \ &\left\{T_{q,0,0}T_{q,1,1} - T_{q,1,1}T_{q,0,0} - (q-q^{-1})T_{q,0,1}T_{q,1,0},\ T_{q,0,1}T_{q,1,0} - T_{q,1,0}T_{q,0,1}\right\} \end{aligned}$$

and whose comultiplication is characterised by the properties that it sends $1 + T_{q,i,i}$ to $((1+T_{q,i,i}) \otimes (1+T_{q,i,i})) + (T_{q,i,j} \otimes T_{q,j,i})$ and $T_{q,i,j}$ to $((1+T_{q,i,i}) \otimes T_{q,i,j}) + (T_{q,i,j} \otimes (1+T_{q,j,j}))$ for any $(i,j) \in J_2$. We denote its dual by $A_{2,q}$. When $q = 1$, then $A_{2,q}$ is canonically isomorphic to A_2 in $\mathrm{Hopf}(\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k), \hat{\otimes}_k, k)$, and hence $A_{2,q}$ is a q-deformation of A_2. Since $T_{q,0,0}$ and $T_{q,1,1}$ commute with each other modulo the two-sided ideal generated by $\{T_{q,1,2}, T_{q,2,1}\}$, $A_{q,2}$ forms a $\mathrm{C}(\mathbb{Z}_p^2, k)$-algebra in a way similar to A_2.

We denote by $O_k[[T_{q,i,j} \mid (i,j) \in I_2]]$ the compact Hausdorff flat linear topological O_k-module $\prod_{a \in \mathbb{N}^{I_2}} O_k T_{q,0,0}^{a(0,0)} T_{q,1,1}^{a(1,1)} T_{q,0,1}^{a(0,1)} T_{q,1,0}^{a(1,0)}$. By the definition of commutators, $(T_{q,0,0}^{a(0,0)} T_{q,1,1}^{a(1,1)} T_{q,0,1}^{a(0,1)} T_{q,1,0}^{a(1,0)})_{a \in \mathbb{N}^{I_2}}$ is O_k-linearly independent, and the inclusion

$$\bigoplus_{a \in \mathbb{N}^{I_2}} O_k T_{q,0,0}^{a(0,0)} T_{q,1,1}^{a(1,1)} T_{q,0,1}^{a(0,1)} T_{q,1,0}^{a(1,0)} \hookrightarrow O_k[[T_{i,j} \mid (i,j) \in I_2]]_q$$

extends to a unique isomorphism $O_k[[T_{q,i,j} \mid (i,j) \in I_2]] \to O_k[[T_{i,j} \mid (i,j) \in I_2]]_q$ in $\mathscr{C}^{\mathrm{ch}}_{\mathrm{fl}}$. Therefore it induces an isomorphism $A_{2,q} \to \mathrm{C}_0(\mathbb{N}^{\Delta_2} \times \mathbb{N}^{J_2}, k)$ in $\mathrm{Ban}^{\mathrm{ur}}_{\leq}(k)$ in the same way as the isomorphism $A_2 \to \mathrm{C}_0(\mathbb{N}^{\Delta_2} \times \mathbb{N}^{J_2}, k)$.

For a finite subset $U \subset \mathbb{N}^{\Delta_2} \times \mathbb{N}^{J_2}$, we denote by $E_{q,U} \in A_{2,q} \cong \mathrm{C}_0(\mathbb{N}^{\Delta_2} \times \mathbb{N}^{J_2}, k)$ the characteristic function of U. For an $(i,j) \in J_2$, we put $E_{q,i,j} \coloneqq E_{q,\{\delta_{(i,j)}\}}$. Then we obtain the completely same results for $A_{2,q}$ as Proposition 5.1, Lemma 5.3, and Lemma 5.6 under the careful computation of the vanishing of the contribution of $T_{q,1,1}T_{q,0,0} - T_{q,0,0}T_{q,1,1} = (q - q^{-1})T_{q,0,1}T_{q,1,0}$. By a reasoning completely similar to Theorem 5.2, we obtain the following:

Theorem 5.7. *The O_k-subalgebra*

$$O_k[B_{\mathrm{C}(\mathbb{Z}_p^2,k)}(\kappa_i, m) \mid (m,i) \in \mathbb{N} \times \Delta_2][\frac{1}{m!}E_{q,i,j}^m \mid (m,(i,j)) \in \mathbb{N} \times J_2] \subset A_{2,q}$$

is dense in $A_{2,q}^{\circ}$.

We note that although A_2 and $A_{2,q}$ admits the same presentation of topological generators of the closed unit discs, the O_k-algebra structure heavily depends on q.

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