Categories and
General Algebraic Structures
with Applications

Volume 14, Number 1, January 2021, 245-268.
https://doi.org/10.29252/cgasa.14.1.245

Relation between Sheffer stroke and Hilbert algebras

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Abstract. In this paper, we introduce a Sheffer stroke Hilbert algebra by giving definitions of Sheffer stroke and a Hilbert algebra. After it is shown that the axioms of Sheffer stroke Hilbert algebra are independent, it is given some properties of this algebraic structure. Then it is stated the relationship between Sheffer stroke Hilbert algebra and Hilbert algebra by defining a unary operation on Sheffer stroke Hilbert algebra. Also, it is presented deductive system and ideal of this algebraic structure. It is defined an ideal generated by a subset of a Sheffer stroke Hilbert algebra, and it is constructed a new ideal of this algebra by adding an element of this algebra to its ideal.

1 Introduction

Sheffer stroke was initially introduced by H.M. Sheffer [13]. This operation drew many researchers' attention since any Boolean operation or function is expressed via only this binary operation [10]. This also leads to reductions of axioms or formulas for many algebraic structures. Because fewer,

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Keywords: Hilbert algebra, Sheffer stroke, Sheffer stroke Hilbert algebra.
Mathematics Subject Classification [2010]: 06F05, 03G25, 03G10.
Received: 24 October 2019, Accepted: 11 January 2020.
ISSN: Print 2345-5853, Online 2345-5861.

simple and more useful axiom systems are obtained, it is easy to check some properties and notations on a new algebraic structure. Thus, many scientists want to apply such a reduction to several algebraic structures such as orthoimplication algebras [1], ortholattices [2], and Sheffer stoke basic algebras [11].

In the 1950's, Hilbert algebras, which from an algebraic counterpart of Hilbert's positive implicative propositional calculus [12], were introduced by L. Henkin and T. Skolem for studies in intuitionistic and other non-classical logics [5]. These algebraic structures can be thought as parts of the propositional logic including the implication and the distinguished element 1. Particularly, A. Diego studied Hilbert algebras, their deductive systems and various properties. They showed that Hilbert algebras form a variety [4]. Also, it was stated that Hilbert algebras are dual to positive implicative BCK-algebras [3], [7, 8]. H. Rasiowa called these algebras positive implicative algebras [12]. In 1980, J. Schmid determined maximal lattices of quotients of distributive lattices [14]. Then Idziak studied a bounded Hilbert algebra as a specific BCK-algebra with lattice operations [6].

We introduce a Sheffer Stroke Hilbert algebra and describe its a deductive systems and ideals. We prove that a Sheffer Stroke Hilbert algebra is a Hilbert algebra where $x \longrightarrow y := x|(y|y)$. After that, it is introduced a unary operation * on this structure. So, it is shown that a bounded Hilbert algebra is a Sheffer stroke Hilbert algebra. It is described an ideal generated by a subset of a Sheffer Stroke Hilbert algebra. We construct an ideal of a Sheffer stroke Hilbert algebra by adding a new element.

2 Preliminaries

In this section, we give fundamental definitions and notions about Sheffer stroke and Hilbert algebras. In what follows we recall some known concepts and results from [2].

Definition 2.1. Let $\mathcal{H} = \langle H, |\rangle$ be a groupoid. The operation $|$ is said to be *Sheffer stroke* if it satisfies the conditions

(S1) $x|y = y|x$,

(S2) $(x|x)|(x|y) = x$,

(S3) $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,

(S4) $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

Lemma 2.2. *Let $\mathcal{H} = \langle H, |\rangle$ be a groupoid. The binary relation $\leq$ defined on H as*

$$x \leq y \text{ if and only if } x|y = x|x$$

is an order on H.

Lemma 2.3. *Let $|$ be Sheffer stroke on A and $\leq$ the induced order of $\mathcal{H} = \langle H, |\rangle$. Then*
(i) *$a \leq x$ and $a \leq y$ imply $x|y \leq a|a$.*
(ii) *$x \leq y$ if and only if $y|y \leq x|x$.*
(iii) *$x \leq y$ implies $y|z \leq x|z$ for all $z \in H$.*

In order to build Sheffer stroke Hilbert algebras, we first give the definition of a Hilbert algebra.

Definition 2.4. [9] A Hilbert algebra is a triple $\mathcal{H} = \langle H, \longrightarrow, 1\rangle$, where H is a non-empty set, $\longrightarrow$ is a binary operation on H, $1 \in H$ is an element such that the following axioms are satisfied for all $x, y, z \in H$:

(a_1) $x \longrightarrow (y \longrightarrow x) = 1$,

(a_2) $(x \longrightarrow (y \longrightarrow z)) \longrightarrow ((x \longrightarrow y) \longrightarrow (x \longrightarrow z)) = 1$,

(a_3) If $x \longrightarrow y = y \longrightarrow x = 1$ then $x = y$.

Remark 2.5. [9] Let $\mathcal{H} = \langle H, \longrightarrow, 1\rangle$ be a Hilbert algebra. Then the relation $x \leq y$ if and only if $x \longrightarrow y = 1$ is a partial order on H, which is called the natural ordering of H. According to this ordering, 1 is the largest element of H.

3 Sheffer Stroke Hilbert Algebras

In this section, we introduce Sheffer Stroke Hilbert algebras and give their properties.

Definition 3.1. A Sheffer stroke Hilbert algebra is a structure $\langle H, |\rangle$ of type (2), in which H is a non-empty set and $|$ is Sheffer stroke on H such that the following identities are satisfied for all $x, y, z \in H$:

(SHa_1) $(x|((y|(z|z))|(y|(z|z))))|(((x|(y|y))|((x|(z|z))|(x|(z|z))))|((x|(y|y))|((x|(z|z))|(x|(z|z))))) = x|(x|x)$,

(SHa_2) If $x|(y|y) = y|(x|x) = x|(x|x)$ then $x = y$.

Lemma 3.2. *The axioms (SHa_1) and (SHa_2) are independent.*

Proof. Consider the groupoid $\langle \{a, b, 1\}, |\rangle$.

(i) Independence of (SHa_1):

$\vert$	a	b	1
a	a	b	1
b	a	b	1
1	a	b	1

Table 1: Operation table for independence of (SHa_1)

Then $|$ satisfies (SHa_2) while it does not satisfy (SHa_1) since

$$\begin{aligned} a &= a|(a|a) \\ &\neq (a|((a|(1|1))|(a|(1|1))))|(((a|(a|a))|((a|(1|1))|(a|(1|1))))|((a|(a|a))|((a|(1|1))|(a|(1|1))))) \\ &= 1|(1|1) \\ &= 1. \end{aligned}$$

(ii) Independence of (SHa_2):

Then $|$ satisfies (SHa_1) while it does not satisfy (SHa_2), because $a \neq 1$ when $a|(1|1) = 1 = 1|(a|a)$. □

Example 3.3. Consider the groupoid $\langle H, |\rangle$, where the set $H = \{0, x, y, z, t, u, v, 1\}$ and the binary operation $|$ on H has the Cayley table as

Then this structure is a Sheffer stroke Hilbert algebra.

|	a	b	1
a	1	1	1
b	1	1	1
1	1	1	1

Table 2: Operation table for independence of (SHa_2)

|	0	x	y	z	t	u	v	1
0	1	1	1	1	1	1	1	1
x	1	v	1	1	v	v	1	v
y	1	1	u	1	u	1	u	u
z	1	1	1	t	1	t	t	t
t	1	v	u	1	z	v	u	z
u	1	v	1	t	v	y	t	y
v	1	1	u	t	u	t	x	x
1	1	v	u	t	z	y	x	0

Table 3:

Lemma 3.4. *Let* $\langle H, |\rangle$ *be a Sheffer Stroke Hilbert algebra. Then*

$$x|(x|x) = y|(y|y)$$

for all $x, y \in H$.

Proof. Substituting y instead of z and x instead of y in (SHa_1), simultaneously, it is concluded that

$$
\begin{aligned}
x|(x|x) &= (x|((x|(y|y))|(x|(y|y))))|(((x|(x| \\
&\quad x))|((x|(y|y))|(x|(y|y))))|((x| \\
&\quad (x|x))|((x|(y|y))|(x|(y|y))))) \\
&= (((x|x)|(x|x))|(y|y))|(((((x|(x|x)) \\
&\quad |x)|((x|(x|x))|x))|(y|y))|((((x| \\
&\quad (x|x))|x)|((x|(x|x))|x))|(y|y))) && (S3) \\
&= (x|(y|y))|(((((x|(x|x))|((x|x)|(x| \\
&\quad x)))|((x|(x|x))|((x|x)|(x|x))))|(y| \\
&\quad y))|((((x|(x|x))|((x|x)|(x|x)))|((x \\
&\quad |(x|x))|((x|x)|(x|x))))|(y|y))) && (S2) \\
&= (x|(y|y))|((((((x|x)|(x|x))|((x|x)|x))|(((x| \\
&\quad x)|(x|x))|((x|x)|x)))|(y|y))|(((((x|x)|(x|x))| \\
&\quad ((x|x)|x))|(((x|x)|(x|x))|((x|x)|x)))|(y|y))) && (S1) \\
&= (x|(y|y))|((((x|x)|(x|x))| \\
&\quad (y|y))|(((x|x)|(x|x))|(y|y))) && (S2) \\
&= (x|(y|y))|((x|(y|y))|(x|(y|y))) && (S2)
\end{aligned}
$$

Putting $x|y$ instead of x in the above equation, we get

$$
\begin{aligned}
(x|y)|((x|y)|(x|y)) &= ((x|y)|(y|y))|(((x|y)| \\
&\quad (y|y))|((x|y)|(y|y))) \\
&= ((y|y)|(y|x))|(((y|y)| \\
&\quad (y|x))|((y|y)|(y|x))) && (S1) \\
&= y|(y|y). && (S2)
\end{aligned}
$$

Thus, it follows from (S1) that

$$
\begin{aligned}
y|(y|y) &= (x|y)|((x|y)|(x|y)) \\
&= (y|x)|((y|x)|(y|x)) \\
&= x|(x|x).
\end{aligned}
$$

□

Remark 3.5. The Sheffer Stroke Hilbert algebra $\langle H, |\rangle$ satisfies the identity $x|(x|x) = y|(y|y)$ for all $x, y \in H$, by Lemma 3.4. It means that $\langle H, |\rangle$ has an algebraic constant which will be denoted by 1.

Lemma 3.6. *Let $\langle H, | \rangle$ be a Sheffer Stroke Hilbert algebra. Then the following identities hold for all $x \in H$:*

(i) $x|(1|1) = 1$,

(ii) $1|(x|x) = x$.

Proof. (i)

$$\begin{aligned} x|(1|1) &= x|((x|(x|x))|(x|(x|x))) && (Remark\ 3.5) \\ &= ((x|x)|(x|x))|(x|x) && (S3) \\ &= (x|x)|((x|x)|(x|x)) && (S1) \\ &= 1. && (Remark\ 3.5) \end{aligned}$$

(ii)

$$\begin{aligned} 1|(x|x) &= (x|(x|x))|(x|x) && (Remark\ 3.5) \\ &= (x|x)|(x|(x|x)) && (S1) \\ &= x. && (S2) \end{aligned}$$

□

Lemma 3.7. *Let $\langle H, | \rangle$ be a Sheffer stroke Hilbert algebra. Then the relation $x \leq y$ if and only if $x|(y|y) = 1$ is a partial order on H, which will be called the natural ordering on H. With respect to this ordering, 1 is the largest element of H.*

Proof.

- *Reflective*: it is clear from Remark 3.5.
- *Antisymmetric*: Let $x \leq y$ and $y \leq x$, that is, $x|(y|y) = 1$ and $y|(x|x) = 1$. Then it follows from (SHa_2) that $x = y$.
- *Transitive*: Let $x \leq y$ and $y \leq z$, that is, $x|(y|y) = 1$ and $y|(z|z) = 1$.

$$\begin{aligned} 1 &= x|(x|x) && (Remark\ 3.5) \\ &= (x|((y|(z|z))|(y|(z|z))))|(((x|(y|y))|((x|(z|z))|(x|(z|z))))|((x|(y|y))|((x|(z|z))|(x|(z|z))))) && (SHa_1) \\ &= (x|(1|1))|((1|((x|(z|z))|(x|(z|z))))|(1|((x|(z|z))|(x|(z|z))))) && (hypothesis) \\ &= (x|(z|z)), && (Lemma\ 3.6\ (i) - (ii)) \end{aligned}$$

that is, $x \leq z$.

Therefore, this relation is a partial order on H. Moreover, 1 is the largest element of H, since $x \leq 1$ for all $x \in H$ by Lemma 3.6 (i). □

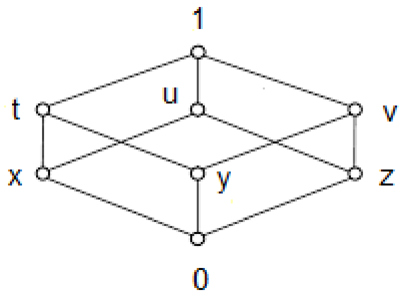

Figure 1: The Hasse diagram of the Sheffer stroke Hilbert algebra from Example 3.3.

Lemma 3.8. *Let $\langle H, |\rangle$ be a Sheffer stroke Hilbert algebra. Then the followings hold for all $x, y, z \in H$:*

(Shb$_1$) $x \leq y|(x|x)$,

(Shb$_2$) $x|((y|(z|z))|(y|(z|z))) = (x|(y|y))|((x|(z|z))|(x|(z|z)))$,

(Shb$_3$) $(x|(y|y))|(y|y) = (y|(x|x))|(x|x)$,

(Shb$_4$) $x|((y|(z|z))|(y|(z|z))) = y|((x|(z|z))|(x|(z|z)))$,

(Shb$_5$) $x \leq (x|(y|y))|(y|y)$,

(Shb$_6$) $((x|(y|y))|(y|y))|(y|y) = x|(y|y)$,

(Shb$_7$) $x|(y|y) \leq (y|(z|z))|((x|(z|z))|(x|(z|z)))$,

(Shb$_8$) If $x \leq y$, then $z|(x|x) \leq z|(y|y)$ and $y|(z|z) \leq x|(z|z)$.

Proof.

(Shb_1)

$$\begin{aligned} x|((y|(x|x))|(y|(x|x))) &= x|(((x|x)|y) \\ &\quad |((x|x)|y)) && (S1) \\ &= ((x|(x|x))| \\ &\quad (x|(x|x)))|y && (S3) \\ &= (1|1)|y && (\textit{Remark } 3.5) \\ &= y|(1|1) && (S1) \\ &= 1 && (\textit{Lemma } 3.6\ (i)), \end{aligned}$$

that is, $x \leq y|(x|x)$.

(Shb_4)

$$\begin{aligned} x|((y|(z|z))|(y|(z|z))) &= x|(((z|z)|y)|((z|z)|y)) && (S1) \\ &= ((x|(z|z))|(x|(z|z)))|y && (S3) \\ &= y|((x|(z|z))|(x|(z|z))) && (S1) \end{aligned}$$

(Shb_5)

$$\begin{aligned} & x|(((x|(y|y))|(y|y))|((x|(y|y))|(y|y))) \\ & = x|(((y|y)|(x|(y|y)))|((y|y)|(x|(y|y)))) && (S1) \\ & = ((x|(y|y))|(x|(y|y)))|(x|(y|y)) && (S3) \\ & = (x|(y|y))|((x|(y|y))|(x|(y|y))) && (S1) \\ & = 1, && (\textit{Remark } 3.5) \end{aligned}$$

that is, $x \leq (x|(y|y))|(y|y)$.

(Shb_2) From (SHa_1) and Remark 3.5, it is known that $(x|((y|(z|z))|(y|(z|z))))|(((x|(y|y))|((x|(z|z))|(x|(z|z))))|((x|(y|y))|((x|(z|z))|(x|(z|z))))) = x|(x|x) = 1$, that is,

$$x|((y|(z|z))|(y|(z|z))) \leq (x|(y|y))|((x|(z|z))|(x|(z|z))). \quad (1)$$

It is obtained from (Shb_1) that $y \leq x|(y|y)$, and from Lemma 2.3 (iii) and (Shb_4) that

$$\begin{aligned} (x|(y|y))|((x|(z|z))|(x|(z|z))) &\leq y|((x|(z|z))|(x|(z|z))) \\ &= x|((y|(z|z))|(y|(z|z))). \quad (2) \end{aligned}$$

It follows from (1) and (2) that

$$(x|(y|y))|((x|(z|z))|(x|(z|z))) = x|((y|(z|z))|(y|(z|z))).$$

(Shb$_6$) From (Shb$_5$), we have $x|(y|y) \leq ((x|(y|y))|(y|y))|(y|y)$ (3)

$$\begin{aligned}
&(((x|(y|y))|(y|y))|(y|y))|((x|(y|y))|(x|(y|y))) \\
&= x|(((((x|(y|y))|(y|y))|(y|y))|(y|y)) \\
&\quad |((((x|(y|y))|(y|y))|(y|y))|(y|y))) && (Shb_4) \\
&= (x|((((x|(y|y))|(y|y))|(y|y))|(((x|(y|y)) \\
&\quad |(y|y))|(y|y))))|((x|(y|y))|(x|(y|y))) && (Shb_2) \\
&= (((x|(y|y))|(y|y))|((x|(y|y)) \\
&\quad |(x|(y|y))))|((x|(y|y))|(x|(y|y))) && (Shb_4) \\
&= (((x|(y|y))|(x|(y|y)))|((x|(y|y)) \\
&\quad |(y|y)))|((x|(y|y))|(x|(y|y))) && (S1) \\
&= (x|(y|y))|((x|(y|y))|(x|(y|y))) && (S2) \\
&= 1, && (Remark\ 3.5)
\end{aligned}$$

that is,

$$((x|(y|y))|(y|y))|(y|y) \leq x|(y|y). \qquad (4)$$

Hence, it is concluded from (3) and (4) that $((x|(y|y))|(y|y))|(y|y) = x|(y|y)$.

(Shb$_7$)

$$\begin{aligned}
&(x|(y|y))|(((y|(z|z))|((x|(z|z))|(x|(z| \\
&z))))|((y|(z|z))|((x|(z|z))|(x|(z|z))))) \\
&= (y|(z|z))|(((x|(y|y))|((x|(z|z))|(x|(z| \\
&\quad z))))|((x|(y|y))|((x|(z|z))|(x|(z|z))))) && (Shb_4) \\
&= (y|(z|z))|((x|((y|(z|z))|(y|(z|z))))| \\
&\quad (x|((y|(z|z))|(y|(z|z))))) && (Shb_2) \\
&= x|(((y|(z|z))|((y|(z|z))|(y|(z|z))))| \\
&\quad ((y|(z|z))|((y|(z|z))|(y|(z|z))))) && (Shb_4) \\
&= 1, && (Lemma\ 3.6\ (i))
\end{aligned}$$

that is, $x|(y|y) \leq (y|(z|z))|((x|(z|z))|(x|(z|z)))$.

(Shb$_8$) Let $x \leq y$, that is, $x|(y|y) = 1$. Then

$$\begin{aligned}
&(z|(x|x))|((z|(y|y))|(z|(y|y))) \\
&= z|((x|(y|y))|(x|(y|y))) && (Shb_2) \\
&= z|(1|1) && (hypothesis) \\
&= 1, && (Lemma\ 3.6\ (i))
\end{aligned}$$

that is, $z|(x|x) \leq z|(y|y)$ and

$$\begin{aligned}
&(y|(z|z))|((x|(z|z))|(x|(z|z))) \\
&= 1|(((y|(z|z))|((x|(z|z))|(x|(z|z))))| \\
&\quad ((y|(z|z))|((x|(z|z))|(x|(z|z))))) && (Lemma\ 3.6\ (ii)) \\
&= (x|(y|y))|(((y|(z|z))|((x|(z|z))|(x|(z| \\
&\quad z))))|((y|(z|z))|((x|(z|z))|(x|(z|z))))) && (hypothesis) \\
&= 1, && (Shb_7)
\end{aligned}$$

that is, $y|(z|z) \leq x|(z|z)$.

(Shb_3)

$$\begin{aligned}
&((x|(y|y))|(y|y))|(((y|(x|x))|(x|x))|((y|(x|x))|(x|x))) \\
&= ((((x|(y|y))|(y|y))|(y|y))|((((x|(y|y))|(y|y))| \\
&\quad (x|x))|(((x|(y|y))|(y|y))|(x|x))))|((((x|(y| \\
&\quad y))|(y|y))|(x|x))|(((x|(y|y))|(y|y))|(x|x))) && (Shb_2) \\
&= ((x|(y|y))|((((x|(y|y))|(y|y))|(x|x))| \\
&\quad (((x|(y|y))|(y|y))|(x|x))))|((((x|(y|y))| \\
&\quad (y|y))|(x|x))|(((x|(y|y))|(y|y))|(x|x))) && (Shb_6) \\
&= (((x|(y|y))|(y|y))|(((x|(y|y))|(x|x))| \\
&\quad ((x|(y|y))|(x|x))))|((((x|(y|y))|(y| \\
&\quad y))|(x|x))|(((x|(y|y))|(y|y))|(x|x))) && (Shb_4) \\
&= (((x|(y|y))|(y|y))|(x|x))|((((x|(y|y))| \\
&\quad (y|y))|(x|x))|(((x|(y|y))|(y|y))|(x|x))) && ((S1)-(S2)) \\
&= 1, && (Remark\ 3.5)
\end{aligned}$$

that is, $(x|(y|y))|(y|y) \leq (y|(x|x))|(x|x)$, and similarly

$$(y|(x|x))|(x|x) \leq (x|(y|y))|(y|y).$$

Hence, $(x|(y|y))|(y|y) = (y|(x|x))|(x|x)$.

□

Theorem 3.9. *Let $\langle H, |\rangle$ be a Sheffer stroke Hilbert algebra. If we define $x \longrightarrow y := x|(y|y)$, then $\langle H, \longrightarrow, 1\rangle$ is a Hilbert algebra.*

Proof. Let $\langle H, |, 1\rangle$ be a Sheffer stroke Hilbert algebra and $x \longrightarrow y := x|(y|y)$. Then

(a_1) It is concluded from (Shb_4), Remark 3.5 and Lemma 3.6 (i), respectively, that

$$\begin{aligned} x \longrightarrow (y \longrightarrow x) &= x|((y|(x|x))|(y|(x|x))) \\ &= y|((x|(x|x))|(x|(x|x))) \\ &= y|(1|1) \\ &= 1. \end{aligned}$$

(a_2) It follows from (SHa_1) and Remark 3.5, respectively, that

$$\begin{aligned} &(x \longrightarrow (y \longrightarrow z)) \longrightarrow ((x \longrightarrow y) \longrightarrow (x \longrightarrow z)) \\ &= (x|((y|(z|z))|(y|(z|z))))|(((x|(y|y))|((x|(z|z))| \\ &\quad (x|(z|z))))|((x|(y|y))|((x|(z|z))|(x|(z|z))))) \\ &= x|(x|x) \\ &= 1. \end{aligned}$$

(a_3) Let $x \longrightarrow y = y \longrightarrow x = 1$. Then $x|(y|y) = y|(x|x) = 1 = x|(x|x)$ from Remark 3.5. So, we obtain from (SHa_2) that $x = y$.

□

Example 3.10. Consider a Sheffer stroke Hilbert algebra $\langle H, |\rangle$ with $H := \{0, p, q, 1\}$, and with the Cayley table 4

$\vert$	1	p	q	0
1	0	q	p	1
p	q	q	1	1
q	p	1	p	1
0	1	1	1	1

Table 4:

Then a Hilbert algebra defined by the Sheffer stroke Hilbert algebra $\langle H, |\rangle$ has the Cayley table 5.

If a Sheffer stroke Hilbert algebra $\langle H, |\rangle$ has the least element 0, then a unary operation "*" can be defined by $x^* = x|(0|0)$ for all x in H.

$\longrightarrow$	1	p	q	0
1	1	p	q	0
p	1	1	q	q
q	1	p	1	p
0	1	1	1	1

Table 5:

Lemma 3.11. *Let* $\langle H, |\rangle$ *be a Sheffer stroke Hilbert algebra with* 0*. Then the followings hold, for all* $x \in H$

(i) $0|0 = 1$ *and* $1|1 = 0$,

(ii) $1^* = 0$ *and* $0^* = 1$,

(iii) $x|1 = x|x$,

(iv) $x^* = x|x$,

(v) $x|0 = 1$,

(vi) $(x^*)^* = x$,

(vii) $x|x^* = 1$.

Proof. (i) Since 0 is the smallest element in H, $0 \leq x$ and $0 \leq x|x$ for all $x \in H$. By Lemma 2.3 (i), it is obtained $x|(x|x) \leq 0|0$. It follows from Remark 3.5 that $0|0 = 1$, because 1 is the largest element on H. Also, $1|1 = (0|0)|(0|0) = 0$ from (S2).

(ii) It follows from Remark 3.5 and Lemma 3.6 (ii) that $0^* = 0|(0|0) = 1$ and $1^* = 1|(0|0) = 0$, respectively.

(iii) By using Lemma 2.2, we get $x|1 = x|x$ because 1 is the largest element on H, that is, $x \leq 1$ for all $x \in H$.

(iv) It follows from (i) and (iii) that $x^* = x|(0|0) = x|1 = x|x$.

(v) It is concluded from (i) and Lemma 3.6 (i) that $x|0 = x|(1|1) = 1$ for all $x \in H$.

(vi) It is obtained from (iv) and (S2) that $(x^*)^* = (x|x)|(x|x) = x$.

(vii) (iv) and Remark 3.5 imply that $x|x^* = x|(x|x) = 1$.

□

Theorem 3.12. *Let $\langle H, \longrightarrow, 1\rangle$ be a Hilbert algebra with* 0*. If we define $x|y := x \longrightarrow y^*$, then $\langle H, |\rangle$ is a Sheffer stroke Hilbert algebra.*

Proof. Let $\langle H, \longrightarrow, 1\rangle$ be a bounded Hilbert algebra and $x|y := x \longrightarrow y^*$. Then
(SHa$_1$): It follows from (a$_1$) and Lemma 3.11 (vi) that

$$\begin{aligned}
&(x|((y|(z|z))|(y|(z|z))))|(((x|(y|y))|((x|(z|z))|\\
&(x|(z|z))))|((x|(y|y))|((x|(z|z))|(x|(z|z)))))\\
&= x \longrightarrow (y \longrightarrow z)) \longrightarrow ((x \longrightarrow y) \longrightarrow (x \longrightarrow z))\\
&= 1\\
&= x \longrightarrow x\\
&= x|(x|x).
\end{aligned}$$

(SHa$_2$): Let $x|(y|y) = y|(x|x) = x|(x|x)$, i. e., $x \longrightarrow y = y \longrightarrow x = x \longrightarrow x = 1$. Then (a$_3$) implies that $x = y$. □

Example 3.13. Consider a bounded Hilbert algebra $\langle H, \longrightarrow, 1\rangle$ with the set $H = \{0, x, y, z, t, u, v, 1\}$ and the binary operation $\longrightarrow$ on H defined as follows:

$\longrightarrow$	0	x	y	z	t	u	v	1
0	1	1	1	1	1	1	1	1
x	v	1	v	v	1	1	v	1
y	u	u	1	u	1	u	1	1
z	t	t	t	1	t	1	1	1
t	z	u	v	z	1	u	v	1
u	y	t	y	v	t	1	v	1
v	x	x	t	u	t	u	1	1
1	0	x	y	z	t	u	v	1

Table 6:

Then the Sheffer stroke Hilbert algebra defined by this Hilbert algebra $\langle H, \longrightarrow, 1\rangle$ is the Sheffer stroke Hilbert algebra in Example 3.3.

Proposition 3.14. *Let $\langle H, |\rangle$ be a Sheffer stroke Hilbert algebra and $\leq$ be a natural ordering induced by this algebra. Then $\langle H, \leq\rangle$ is a join-semilattice with the largest element* 1*, where $x \vee y = (x|(y|y))|(y|y)$.*

Proof. It follows from (Shb_3) and (Shb_5) that $x \leq (x|(y|y))|(y|y)$ and $y \leq (y|(x|x))|(x|x) = (x|(y|y))|(y|y)$. Thus, $(x|(y|y))|(y|y)$ is an upper bound for x and y. Suppose $x, y \leq z$. Then it is concluded

$$(x|(y|y))|(y|y) \leq (z|(y|y))|(y|y) = (y|(z|z))|(z|z) = 1|(z|z) = z$$

by applying twice (Shb_8), (Shb_3) and Lemma 3.6 (ii). Therefore, $x \vee y = (x|(y|y))|(y|y)$ is the least upper bound (supremum) of x and y. □

Proposition 3.15. *Let $\langle H, |\rangle$ be a Sheffer stroke Hilbert algebra with* 0 *and $\leq$ be natural ordering of this algebra. Then $\langle H, \leq\rangle$ is a meet-semilattice with the smallest element* 0*, in which $x \wedge y = ((x|x) \vee (y|y))|((x|x) \vee (y|y))$.*

Proof. By using Lemma 2.3, Proposition 3.14 and (S2), we have $((x|x) \vee (y|y))|((x|x) \vee (y|y)) \leq (x|x)|(x|x) = x$ and $((x|x) \vee (y|y))|((x|x) \vee (y|y)) = ((y|y) \vee (x|x))|((y|y) \vee (x|x) \leq (y|y)|(y|y) = y$. Hence, $((x|x) \vee (y|y))|((x|x) \vee (y|y)$ is a lower bound for x and y. Let $z \leq x, y$. Then, it is obtained from Proposition 3.14, (S2) and Lemma 2.3 that

$$\begin{aligned} z &= (z|z)|(z|z) \\ &= ((z|z) \vee (y|y))|((z|z) \vee (y|y) \\ &\leq ((x|x) \vee (y|y))|((x|x) \vee (y|y). \end{aligned}$$

Therefore, $x \wedge y = ((x|x) \vee (y|y))|((x|x) \vee (y|y)$ is the greatest lower bound (infimum) for x and y. □

Corollary 3.16. *If $\langle H, |\rangle$ is a Sheffer stroke Hilbert algebra with* 0*, then $(H, \vee, \wedge, 0, 1)$ forms a lattice.*

4 On Ideals of Sheffer Stroke Hilbert Algebras

In this section, we introduce ideal and deductive system of Sheffer Stroke Hilbert algebras. Assume that $\langle H, |\rangle$ is a Sheffer stroke Hilbert algebra.

Definition 4.1. A subset S of H is called a deductive system of H if it satisfies:

(i) $1 \in S$,

(ii) $x \in S$ and $x|(y|y) \in S$ imply $y \in S$.

Definition 4.2. A non-empty subset I of H is called an ideal if
(SSHI1) $0 \in I$,
(SSHI2) $(x|(y|y))|(x|(y|y)) \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Example 4.3. Consider a Sheffer stroke Hilbert algebra in Example 3.10. Then H itself, $\{0\}$, $I_1 = \{0, p\}$ and $I_2 = \{0, q\}$ are ideals of H.

Theorem 4.4. *Let I be a subset of H such that $0 \in I$. Then I is an ideal of H if and only if $x \leq y$ and $y \in I$ imply $x \in I$ for all $x \in H$.*

Proof. ($\Rightarrow$) Suppose that I is an ideal of H. Let $y \in I$ and $x \leq y$, that is, $x|(y|y) = 1$. Then we get $(x|(y|y))|(x|(y|y)) = 1|1 = 0 \in I$ from Lemma 3.11 (i). It follows from (SSHI2) that $x \in I$ for all $x \in H$.

($\Leftarrow$) Let I be a subset of H such that $0 \in I$. Assume that $x \leq y$ and $y \in I$ imply $x \in I$ for all $x \in H$. Then we have from Lemma 3.11 (i) that $(x|(y|y))|(x|(y|y)) = 1|1 = 0 \in I$ and $y \in I$ imply $x \in I$ for all $x \in H$. Thus, I is an ideal of H. □

Corollary 4.5. *Let I be an ideal of H and $y \in I$. If $y|y \leq x|x$, then $x \in I$ for all $x \in H$.*

Proof. It follows from Lemma 2.3 (ii), (S2), and Theorem 4.4. □

For a non-empty subset I of H, define

$$C(I) = \{x|x : x \in I\}.$$

Theorem 4.6. *I is an ideal of H if and only if $C(I)$ is a deductive system of H.*

Proof. Assume that I is an ideal of H. Then, $0 \in I$ from (SSHI1), and thus we have $0|0 = 1 \in C(I)$ from Lemma 3.11 (i).
Let $x \in C(I)$ and $x|(y|y) \in C(I)$ for all $x, y \in H$. Then there exist $a, b \in I$ such that $x = a|a$ and $x|(y|y) = b|b$. By using (S1) and (S2), we obtain

$$\begin{aligned} ((y|y)|(a|a))|((y|y)|(a|a)) &= ((a|a)|(y|y))|((a|a)|(y|y)) \\ &= (x|(y|y))|(x|(y|y)) \\ &= (b|b)|(b|b) \\ &= b \in I. \end{aligned}$$

It follows from (SSHI2) that $y|y \in I$. Hence, we get from (S2) that $y = (y|y)|(y|y) \in C(I)$.

Conversely, suppose that $C(I)$ is a deductive system of H. By the fact that $1 \in C(I)$ and $1 = 0|0$ from Lemma 3.11 (i), we obtain $0 \in I$. Let $(x|(y|y))|(x|(y|y)) \in I$ and $y \in I$ for all $x, y \in H$. Then

$$(y|y)|((x|x)|(x|x) = x|(y|y) = ((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))) \in C(I)$$

and $y|y \in C(I)$ from (S1)-(S2). Thus, it is concluded from Definition 4.1 (ii) that $x|x \in C(I)$, that is, $x \in I$. □

Example 4.7. Consider the ideal $I_1 = \{0, p\}$ of H in Example 4.3. Since $0|0 = 1$ and $p|p = q$, $C(I_1) = \{q, 1\}$ is a deductive system of H directly by Definition 4.1.

Observation 1 Suppose $\mathcal{I}$ is a non-empty family of ideals of H. Then $I = \cap\mathcal{I}$ is also an ideal of H.

Definition 4.8. Let A be a subset of H. The least ideal containing A is called the ideal generated by A and it is denoted by $\langle A \rangle$.

Observation 2 Let A and B be subsets of H. The following conditions hold:

(i) $\langle\{0\}\rangle = \{0\}$ and $\langle\emptyset\rangle = \{0\}$,

(ii) $\langle H \rangle = H$ and $\langle\{1\}\rangle = H$,

(iii) $A \subseteq B$ implies $\langle A \rangle \subseteq \langle B \rangle$,

(iv) $x \leq y$ implies $\langle\{x\}\rangle \subseteq \langle\{y\}\rangle$,

(v) If A is an ideal of H, then $\langle A \rangle = A$.

Observation 3 For any natural number n we define $x^n|(y|y)$ recursively as follows: $x^1|(y|y) = x|(y|y)$ and $x^{n+1}|(y|y) = x|((x^n|(y|y))|(x^n|(y|y)))$. By ($Shb_5$) and induction
(Shb_9)

$$\begin{aligned} &z|((x_n|(...(x_2|((x_1|(y|y))|(x_1|(y|y))))...))| \\ &(x_n|(...(x_2|((x_1|(y|y))|(x_1|(y|y))))...))) \\ &= x_n|((...(x_1|((z|(y|y))|(z|(y|y))))...)| \\ &\quad (...(x_1|((z|(y|y))|(z|(y|y))))...)). \end{aligned}$$

As a special case of (Shb_9)
(Shb_{10}) $z|((x^n|(y|y))|(x^n|(y|y))) = x^n|((z|(y|y))|(z|(y|y)))$.

Now let $a, y, x_1, ..., x_n$ be elements of a Sheffer stroke Hilbert algebra H. Then, by using (Shb_4), (Shb_6), (Shb_4) and Remark 3.5, respectively, we get

$$(((x_n|((...(x_1|((y|(a|a))|(y|(a|a))))...)|(...(x_1|((y|(a|a))|(y|(a|a))))...)))|(a|a))|(a|a))|((x_n|((...(x_1|((y|(a|a))|(y|(a|a))))...)|(...(x_1|((y|(a|a))|(y|(a|a))))...)))|(x_n|((...(x_1|((y|(a|a))|(y|(a|a))))...)|(...(x_1|((y|(a|a))|(y|(a|a))))...)))) = 1$$

which implies that

$$(((x_n|((...(x_1|((y|(a|a))|(y|(a|a))))...)|(...(x_1|((y|(a|a))|(y|(a|a))))...))) |(a|a))|(a|a)) \leq (x_n|((...(x_1|((y|(a|a))|(y|(a|a))))...)|(...(x_1|((y|(a|a)) |(y|(a|a))))...))).$$

The reverse inequality follows from (Shb_5). Hence, we obtain
(Shb_{11})

$$\begin{aligned}&(((x_n|((...(x_1|((y|(a|a))|(y|(a|a))))...)|(...(x_1|((y|(a|a))|(y|(a|a))))...)))|(a|a))|(a|a))\\&= (x_n|((...(x_1|((y|(a|a))|(y|(a|a))))...)|(...(x_1|((y|(a|a))|(y|(a|a))))...))).\end{aligned}$$

Theorem 4.9. *If A is a non-empty subset of H, then $\langle A\rangle = \{x \in H : (a_n|a_n)|((...(((a_1|a_1)|x)|((a_1|a_1)|x))...)|((a_1|a_1)|x)|((a_1|a_1)|x))...)) = 1$ $for\ some\ a_1, ..., a_n \in A\}$.*

Proof. Denote

$$U = \{x \in H : (a_n|a_n)|((...(((a_1|a_1)|x)|((a_1|a_1)|x))...)|(...(((a_1|a_1)|x)|((a_1|a_1)|x))...)) = 1 \ for\ some\ a_1, ...a_n \in A\}.$$

We first prove that U is an ideal of H. Since A is non-empty, there exists $a \in A$. Then, from Lemma 3.11 (v) and (S1), $(a|a)|0 = 0|(a|a) = 1$ and so $0 \in U$. Let $(x|(y|y))|(x|(y|y)) \in U$ and $y \in U$. Then there exist $a_i \in A$ $(i = 1, ..., n)$ and $b_j \in A$ $(j = 1, ..., m)$ such that
(Shb_{12})

$$(a_n|a_n)|((...(((a_1|a_1)|((x|(y|y))|(x|(y|y))))|((a_1|a_1)|((x|(y|y))|(x|(y|y)))))...)|(...(((a_1|a_1)|((x|(y|y))|(x|(y|y))))|((a_1|a_1)|((x|(y|y))|(x|(y|y)))))...)) = 1$$

and

$$(b_m|b_m)|((...(((b_1|b_1)|y)|((b_1|b_1)|y))...)|(...(((b_1|b_1)|y)|((b_1|b_1)|y))...)) = 1.$$

Hence, from (S1) and (Shb_9), we have
$y|y \leq (a_n|a_n)|((...(((a_1|a_1)|x)|((a_1|a_1)|x))...)|(...(((a_1|a_1)|x)|((a_1|a_1)|x))...))$.
By applying (Shb_8) and (S1), we have

$$\begin{aligned} 1 &= (b_m|b_m)|((...(((b_1|b_1)|y)|((b_1|b_1)|y)) \\ &\quad ...)|(...(((b_1|b_1)|y)|((b_1|b_1)|y))...)) \\ &\leq (b_m|b_m)|((...(((b_1|b_1)|((a_n|a_n)|((...(((a_1|a_1)|x)|((a_1|a_1)|x))...) \\ &\quad |(...(((a_1|a_1)|x)|((a_1|a_1)|x))...))))|((b_1|b_1)|((a_n|a_n)|((...(((a_1| \\ &\quad a_1)|x)|((a_1|a_1)|x))...)|(...(((a_1|a_1)|x)|((a_1|a_1)|x))...)))))...)| \\ &\quad (...(((b_1|b_1)|((a_n|a_n)|((...(((a_1|a_1)|x)|((a_1|a_1)|x))...)|(...(((a_1 \\ &\quad |a_1)|x)|((a_1|a_1)|x))...))))|((b_1|b_1)|((a_n|a_n)|((...(((a_1|a_1)|x)| \\ &\quad ((a_1|a_1)|x))...)|(...(((a_1|a_1)|x)|((a_1|a_1)|x))...)))))...)) \end{aligned}$$

and hence

$$\begin{aligned} &(b_m|b_m)|((...(((b_1|b_1)|((a_n|a_n)|((...(((a_1|a_1)|x)|((a_1 \\ &|a_1)|x))...)|(...(((a_1|a_1)|x)|((a_1|a_1)|x))...))))|((b_1|b_1) \\ &|((a_n|a_n)|((...(((a_1|a_1)|x)|((a_1|a_1)|x))...)|(...(((a_1|a_1) \\ &|x)|((a_1|a_1)|x))...)))))...)|(...(((b_1|b_1)|((a_n|a_n)|((... \\ &(((a_1|a_1)|x)|((a_1|a_1)|x))...)|(...(((a_1|a_1)|x)|((a_1|a_1) \\ &|x))...))))|((b_1|b_1)|((a_n|a_n)|((...(((a_1|a_1)|x)|((a_1|a_1) \\ &|x))...)|(...(((a_1|a_1)|x)|((a_1|a_1)|x))...)))))...)) = 1. \end{aligned}$$

This demonstrates that $x \in U$. Therefore, U is an ideal of H. It is clear that $A \subseteq U$. Let V be any ideal containing A, and $x \in U$. Then $(a_n|a_n)|((...(((a_1|a_1)|x)|((a_1|a_1)|x))...)|(...(((a_1|a_1)|x)|((a_1|a_1)|x))...)) = 1$ for some $a_1, ..., a_n \in A$.

$$\begin{aligned} 1 &= (a_n|a_n)|(((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x) \\ &\quad ...)|(...((a_1|a_1)|x)...)))|((a_{n-1}|a_{n-1})| \\ &\quad ((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))) \\ &= (a_n|a_n)|(((((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x) \\ &\quad ...)|(...((a_1|a_1)|x)...)))|((a_{n-1}|a_{n-1})| \\ &\quad ((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...))))| \\ &\quad (((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1 \\ &\quad |a_1)|x)...)))|((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x) \\ &\quad ...)|(...((a_1|a_1)|x)...)))))|((((a_{n-1}|a_{n-1})| \\ &\quad ((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))| \\ &\quad ((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1 \\ &\quad |a_1)|x)...))))|(((a_{n-1}|a_{n-1})|((...((a_1|a_1) \\ &\quad |x)...)|(...((a_1|a_1)|x)...)))|((a_{n-1}|a_{n-1})| \\ &\quad ((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))))) \qquad (S2) \end{aligned}$$

which implies that

$$\begin{aligned} &((a_n|a_n)|(((((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x) \\ &...)))|((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...))))| \\ &(((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))|((a_{n-1}| \\ &a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))))|((((a_{n-1}|a_{n-1})| \\ &((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))|((a_{n-1}|a_{n-1})|((...((a_1| \\ &a_1)|x)...)|(...((a_1|a_1)|x)...))))|(((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x) \\ &...)|(...((a_1|a_1)|x)...)))|((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1| \\ &a_1)|x)...)))))))|((a_n|a_n)|(((((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x)...)|(... \\ &((a_1|a_1)|x)...)))|((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x) \\ &...))))|(((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))|((a_{n-1} \\ &|a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))))|((((a_{n-1}|a_{n-1})| \\ &((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))|((a_{n-1}|a_{n-1})|((...((a_1|a_1) \\ &|x)...)|(...((a_1|a_1)|x)...))))|(((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1 \\ &|a_1)|x)...)))|((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...))))))) \\ &= 1|1 \\ &= 0 \in V. \end{aligned}$$

From $a_n \in A \subseteq V$, (S1), (SSHI2) and (S2), we obtain $((a_{n-1}|a_{n-1})|((...$ $((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))|((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1|$

$a_1)|x)...))) \in V$. Since

$$((a_{n-1}|a_{n-1})|(((a_{n-2}|a_{n-2})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))|((a_{n-2}|a_{n-2})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))))|((a_{n-1}|a_{n-1})|(((a_{n-2}|a_{n-2})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))|((a_{n-2}|a_{n-2})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...))))) = ((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))|((a_{n-1}|a_{n-1})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...))) \in V$$

and $a_{n-1} \in A \subseteq V$, we get from (S1), (SSHI2), and (S2) that $((a_{n-2}|a_{n-2})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...)))|((a_{n-2}|a_{n-2})|((...((a_1|a_1)|x)...)|(...((a_1|a_1)|x)...))) \in V$.

Repeating the above procedure (n-2) times, it follows that $x \in V$. This proves $U \subseteq V$, so $U = \langle A \rangle$. □

If $A = \{a_1, ..., a_n\}$, $\langle \{a_1, ..., a_n\} \rangle = \langle a_1, ..., a_n \rangle$ is denoted for the sake of convenience. The following corollary follows from Theorem 4.9.

Corollary 4.10. $\langle a \rangle = \{x \in H : x|(a|a)^n = 1 \ for \ some \ natural \ number \ n\}$ *for any* $a \in H$.

The following theorem shows how to generate an ideal by a given ideal and an element.

Theorem 4.11. *Let* I *be an ideal of* H *and* $a \in H$. *Then* $\langle I \cup \{a\} \rangle = \{x \in H : (x|(a|a)^n)|(x|(a|a)^n) \in I \ for \ some \ natural \ number \ n\}$.

Proof. Denote by

$$U = \{x \in H : (x|(a|a)^n)|(x|(a|a)^n) \in I \ for \ some \ natural \ number \ n\}.$$

Since $(a|(a|a)^n)|(a|(a|a)^n) = 1|1 = 0 \in I$, then $a \in U$. Let $x \in I$. Since

$$\begin{aligned} x|x &\leq (a|a)|((x|x)|(x|x)) \\ &= (a|a)|x \\ &= x|(a|a) \\ &= ((x|(a|a))|(x|(a|a)))|((x|(a|a))|(x|(a|a))), \end{aligned}$$

we have $(x|(a|a))|(x|(a|a)) \in I$ from Corollary 3.16 thus $x \in U$. Hence $I \cup \{a\} \subseteq U$. To prove that U is an ideal, let $(x|(y|y))|(x|(y|y)) \in U$ and $y \in U$. Then, there are natural numbers n and m such that

(Shb$_{13}$) $(((x|(y|y))|(x|(y|y)))|(a|a)^n)|(((x|(y|y))|(x|(y|y)))|(a|a)^n) \in I$.

(Shb$_{14}$) $(y|(a|a)^m)|(y|(a|a)^m) \in I$, respectively.

(Shb$_{13}$) and (Shb$_{14}$) imply that $(((x|(y|y))|(x|(y|y)))|(a|a)^n)|(((x|(y|y))|(x|(y|y)))|(a|a)^n) = u$ and $(y|(a|a)^m)|(y|(a|a)^m) = v$ for some $u, v \in I$. From (S2),

(Shb$_{15}$) $((x|(y|y))|(x|(y|y)))|(a|a)^n = u|u$ and

(Shb$_{16}$) $y|(a|a)^m = v|v$.

From (Shb$_{15}$), we know that $y|y \leq (u|u)|((x|(a|a)^n)|(x|(a|a)^n))$, which implies from (Shb$_4$), (Shb$_8$) and (Shb$_{16}$) that

$v|v = y|(a|a)^m \leq (u|u)|((x|(a|a)^{m+n})|(x|(a|a)^{m+n}))$. Hence

$$(v|v)|(((u|u)|((x|(a|a)^{m+n})|(x|(a|a)^{m+n})))|((u|u)|((x|(a|a)^{m+n})|(x|(a|a)^{m+n}))))\\ = 1.$$

By $u, v \in I$, Observation 2 (v) and Theorem 4.9, $(x|(a|a)^{m+n})|(x|(a|a)^{m+n}) \in I$ thus $x \in U$. Clearly, $0 \in U$. Thus, U is an ideal of H.

Finally, let V be an ideal of H containing I and a. If $x \in U$, then there exists a natural number n such that $(x|(a|a)^n)|(x|(a|a)^n) \in I \subseteq V$. Thus, we have

$$(((x|(a|a)^{n-1})|(x|(a|a)^{n-1}))|(a|a))|(((x|(a|a)^{n-1})|(x|(a|a)^{n-1}))|(a|a))\\ = (x|(a|a)^n)|(x|(a|a)^n) \in V.$$

Combining $a \in V$ and using (SSHI2), we obtain $(x|(a|a)^{n-1})|(x|(a|a)^{n-1}) \in V$. Repeating the above procedure (n-1) times, we get $x \in V$. Therefore, $U \subseteq V$, that is, U is the least ideal containing I and a, that is, $\langle I \cup \{a\}\rangle = U$. □

Example 4.12. Consider the ideal $I_2 = \{0, q\}$ of H from Example 4.3. Then $\langle I_2 \cup \{p\}\rangle = \{0, p, q, 1\} = H$ is an ideal of H because

$$(0|(p|p)^n)|(0|(p|p)^n) = 1|1 = 0 \in I_2 \text{ for all } n \in \mathbb{N},$$
$$(p|(p|p))|(p|(p|p)) = 1|1 = 0 \in I_2 \text{ for } n = 1,$$
$$(q|(p|p))|(q|(p|p)) = q \in I_2 \text{ for } n = 1 \text{ and}$$
$$(1|(p|p))|(1|(p|p)) = q \in I_2 \text{ for } n = 1.$$

5 Conclusion

In this study, we have introduced a Sheffer Stroke Hilbert algebra, and studied deductive systems and ideals on these algebras. By presenting definitions and notions about Sheffer stroke operation, we defined a Sheffer stroke Hilbert algebra and gave its properties. Then we showed that a Sheffer Stroke Hilbert algebra is a Hilbert algebra, where the binary operation $\longrightarrow$ is defined by $x \longrightarrow y := x|(y|y)$. If a Sheffer stroke Hilbert algebra has 0, we introduce a unary operation * on this structure. By presenting its properties, we showed that a bounded Hilbert algebra is a Sheffer stroke Hilbert algebra. It is defined an ideal generated by a subset of a Sheffer Stroke Hilbert algebra and proved some of its properties. Finally, we showed that how another ideal of a Sheffer stroke Hilbert algebra is constructed by adding an element of this algebra to its ideal.

In our future works, we want to study Sheffer stroke Hilbert algebras with supremum and get more results in Sheffer stroke Hilbert algebras.

Acknowledgement

The authors would like to express their sincere thanks to the referee for their valuable suggestions and comments. This study is partially funded by Ege University Scientific Research Projects Directorate with the Project Number 20772.

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