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# $(m, n)$-Hyperideals in ordered semihypergroups

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**Abstract.** In this paper, first we introduce the notions of an $(m, n)$-hyperideal and a generalized $(m, n)$-hyperideal in an ordered semihypergroup, and then, some properties of these hyperideals are studied. Thereafter, we characterize $(m, n)$-regularity, $(m, 0)$-regularity, and $(0, n)$-regularity of an ordered semihypergroup in terms of its $(m, n)$-hyperideals, $(m, 0)$-hyperideals and $(0, n)$-hyperideals, respectively. The relations ${}_m\mathcal{I}, \mathcal{I}_n, \mathcal{H}_m^n$, and $\mathcal{B}_m^n$ on an ordered semihypergroup are, then, introduced. We prove that $\mathcal{B}_m^n \subseteq \mathcal{H}_m^n$ on an ordered semihypergroup and provide a condition under which equality holds in the above inclusion. We also show that the $(m, 0)$-regularity [$(0, n)$-regularity] of an element induce the $(m, 0)$-regularity [$(0, n)$-regularity] of the whole $\mathcal{H}_m^n$-class containing that element as well as the fact that $(m, n)$-regularity and $(m, n)$-right weakly regularity of an element induce the $(m, n)$-regularity and $(m, n)$-right weakly regularity of the whole $\mathcal{B}_m^n$-class and $\mathcal{H}_m^n$-class containing that element, respectively.

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## 1 Introduction and preliminaries

By an ordered semigroup, we mean an algebraic structure $(S, \cdot \leq)$, which satisfies the following conditions: (1) $S$ is a semigroup with respect to the multiplication "$\cdot$"; (2) $S$ is a partially ordered set by $\leq$; (3) if $a$ and $b$ are elements of $S$ such that $a \leq b$, then $ac \leq bc$ and $ca \leq cb$ for all $c \in S$. Many authors, especially Alimov [1], Clifford [2–4], Hion [13], Conrad [5], and Kehayopulu [15] studied such semigroups with some restrictions.

In 1934, Marty [21] introduced the concept of a hyperstructure and defined hypergroup. Later on several authors studied hyperstructure in various algebraic structures such as rings, semirings, semigroups, ordered semigroups, Γ-semigroups and Ternary semigroups, etc. The concept of a semihypergroup is a generalization of the concept of a semigroup and many classical notions such as of ideals, quasi-ideals and bi-ideals defined in semigroups and regular semigroups have been generalized to semihypergroups (see [8, 9] for other related notions and results on semihypergroups). In [14], Heidari and Davvaz introduced the notion of an ordered semihypergroup as a generalization of the notion of an ordered semigroup. Davvaz et al. in [6, 7, 14, 22, 23, 25, 26] studied some properties of hyperideals and bi-hyperideals in ordered semihypergroups. Lajos [16] introduced the concept of $(m, n)$-ideals in semigroups (see also [17–19]). In [12], the authors defined the notion of an $(m, n)$-quasi-hyperideal in a semihypergroup and investigated several properties of these $(m, n)$-quasi-hyperideals.

A *hyperoperation* on a non-empty set $H$ is a map $\circ : H \times H \to \mathcal{P}^*(H)$ where $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$ (the set of all non-empty subsets of $H$). In such a case, $H$ is called a *hypergroupoid*. Let $H$ be a hypergroupoid and $A$, $B$ be any non-empty subsets of $H$. Then

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b.$$

We shall write, in whatever follows, $A \circ x$ instead of $A \circ \{x\}$ and $x \circ A$ instead of $\{x\} \circ A$, for any $x \in H$. Also, for simplicity, throughout the paper, we shall write $A^n$ for $A \circ A \circ \cdots \circ A$ ($n-$ copies of $A$) for any $n \in \mathbb{Z}^+$. Also the integers $m$, $n$ will stand for positive integers throughout the paper until and unless otherwise specified. Moreover, the hypergroupoid $H$ is called a *semihypergroup* if, for all $x, y, z \in H$,

$$(x \circ y) \circ z = x \circ (y \circ z)$$

that is,

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

A non-empty subset $T$ of a semihypergroup $H$ is called a *subsemihypergroup* of $H$ if $T \circ T \subseteq T$.

Let $H$ be a non-empty set, the triplet $(H, \circ, \leq)$ is called an *ordered semihypergroup* if $(H, \circ)$ is a semihypergroup and $(H, \leq)$ is a partially ordered set such that

$$x \leq y \Rightarrow x \circ z \leq y \circ z \text{ and } z \circ x \leq z \circ y$$

for all $x, y, z \in H$. Here, if $A$ and $B$ are non-empty subsets of $H$, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.

Let $H$ be an ordered semihypergroup. For a non-empty subset $A$ of $H$, we denote $(A] = \{x \in H \mid x \leq a$ for some $a \in A\}$. A non-empty subset $A$ of $H$ is called *idempotent* if $A = (A \circ A]$. A non-empty subset $A$ of $H$ is called *left* (*right*)*-hyperideal* [7] of $H$ if $H \circ A \subseteq A (A \circ H \subseteq A)$ and $(A] \subseteq A$. A non-empty subset $J$ of $H$ is called a *hyperideal* of $H$ if $J$ is both a left hyperideal and a right hyperideal of $H$. A subsemihypergroup (non-empty subset) $B$ of an ordered semihypergroup $H$ is called a *bi-hyperideal* (*generalized bi-hyperideal*) of $H$ if $B \circ H \circ B \subseteq B$ and $(B] \subseteq B$. An ordered semihypergroup $H$ is called *regular* (*left-regular, right-regular*) [7] if for each $x \in H$, $x \in (x \circ H \circ x] (x \in (H \circ x \circ x], x \in (x \circ x \circ H])$.

**Lemma 1.1.** [7] *Let $H$ be an ordered semihypergroup and $A, B$ be any non-empty subsets of $H$. Then the following conditions hold:*

(i) $A \subseteq (A]$;
(ii) $A \subseteq B \Rightarrow (A] \subseteq (B]$;
(iii) $(A] \circ (B] \subseteq (A \circ B]$;
(iv) $((A] \circ (B]] = (A \circ B]$;
(v) $(A] \cup (B] = (A \cup B]$.

## 2 $(m, 0)$-hyperideals, $(0, n)$-hyperideals and $(m, n)$-hyperideals in ordered semihypergroups

In this section, the notions of $(m, n)$-hyperideals and generalized $(m, n)$-hyperideals in ordered semihypergroups are introduced. Moreover, important some properties of these hyperideals are studied.

**Definition 2.1.** Let $H$ be an ordered semihypergroup and $m, n$ be the positive integers. Then a subsemihypergroup (respectively, non-empty subset) $A$ of $H$ is called an (respectively, *generalized*) $(m,n)$-*hyperideal* of $H$ if

(i) $A^m \circ H \circ A^n \subseteq A$; and

(ii) $(A] \subseteq A$.

Note that in Definition 2.1, if $m = 1 = n$, then $A$ is called a (*generalized*) *bi-hyperideal* of $H$. Moreover, a (generalized) bi-hyperideal of an ordered semihypergroup $H$ is an (generalized) $(m,n)$-hyperideal of $H$ for all positive integers $m$ and $n$. It is clear that, for positive integers $m$ and $n$, the notion of (generalized) $(m,n)$-hyperideal of $H$ is a generalization of the notion of (generalized) bi-hyperideal of $H$. The following example shows that a generalized $(m,n)$-hyperideal of $H$ need not be an $(m,n)$-hyperideal and generalized bi-hyperideal of $H$.

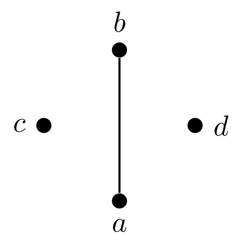
**Example 2.2.** Let $H = \{a, b, c, d\}$. Define the hyperoperation $\circ$ and order $\leq$ on $H$ as follows:

| $\circ$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a\}$ |
| $b$ | $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a\}$ |
| $c$ | $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a, b\}$ |
| $d$ | $\{a\}$ | $\{a\}$ | $\{a, b\}$ | $\{a, b, c\}$ |

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (a,b)\}.$$

The covering relation $\prec$ and the figure of $H$ are as follows:

$$\prec := \{(a,b)\}$$




Then $H$ is an ordered semihypergroup. The subset $\{a, d\}$ of $H$ is a generalized $(m,n)$-hyperideal of $H$ for all integers $m, n \geq 2$ which is neither an $(m,n)$-hyperideal nor a generalized bi-hyperideal of $H$.

**Definition 2.3.** [20] Let $H$ be an ordered semihypergroup and $m, n$ be positive integers. Then a subsemihypergroup $A$ of $H$ is called an $(m, 0)$-*hyperideal* (respectively, $(0, n)$-*hyperideal*) of $H$ if

(i) $A^m \circ H \subseteq A$ (respectively, $H \circ A^n \subseteq A$); and

(ii) $(A] \subseteq A$.

In Definition 2.3, if $m = 1 = n$, then $A$ is called a *right hyperideal* (*left hyperideal*) of $H$. Clearly, each right hyperideal (respectively, left hyperideal) of $H$ is an $(m, 0)$-hyperideal for each positive integer $m$ (respectively, $(0, n)$-hyperideal for each positive integer $n$), that is, the notion of an $(m, 0)$-hyperideal ($(0, n)$-hyperideal) of $H$ is a generalization of the notion of a right hyperideal (respectively, left hyperideal) of $H$. Conversely, an $(m, 0)$-hyperideal (respectively, $(0, n)$-hyperideal) of $H$ need not be a right hyperideal (respectively, left hyperideal) of $H$. We illustrate it by the following example.

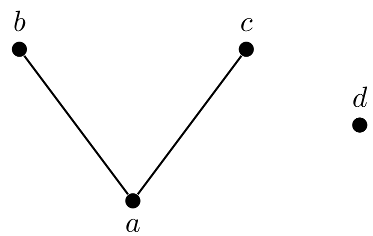
**Example 2.4.** Let $H = \{a, b, c, d\}$. Define the hyperoperation $\circ$ and order $\leq$ on $H$ as follows:

| $\circ$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a\}$ |
| $b$ | $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a\}$ |
| $c$ | $\{a\}$ | $\{a\}$ | $\{a, b\}$ | $\{a, b\}$ |
| $d$ | $\{a\}$ | $\{a\}$ | $\{a, b\}$ | $\{a\}$ |

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c)\}.$$

The covering relation $\prec$ and the figure of $H$ are as follows:

$$\prec := \{(a, b), (a, c)\}$$



Then $H$ is an ordered semihypergroup. It is easy to verify that the subset $A = \{a, d\}$ of $H$ is an $(m, 0)$-hyperideal and a $(0, n)$-hyperideal of $H$ for all integers $m, n \geq 2$, but it is neither a right hyperideal nor a left hyperideal of $H$.

**Remark 2.5.** Let $H$ be an ordered semihypergroup, $m \geq 2$ be any positive integer and $B$ be any non-empty subset of $H$. Then $(B^m \cup B \circ H \circ B^m]$ is a (generalized) bi-hyperideal of $H$. Indeed, $(B^m \cup B \circ H \circ B^m] \circ (B^m \cup B \circ H \circ B^m] \subseteq ((B^m \cup B \circ H \circ B^m) \circ (B^m \cup B \circ H \circ B^m)] = (B^m \circ B^m \cup B^m \circ B \circ H \circ B^m \cup B \circ H \circ B^m \circ B^m \cup B \circ H \circ B^m \circ B \circ H \circ B^m] \subseteq (B \circ H \circ B^m] \subseteq (B^m \cup B \circ H \circ B^m]$ and $(B^m \cup B \circ H \circ B^m] \circ H \circ (B^m \cup B \circ H \circ B^m] \subseteq (B^m \circ H \cup B \circ H \circ B^m \circ H] \circ (B^m \cup B \circ H \circ B^m] \subseteq (B^m \circ H \circ B^m \cup B^m \circ H \circ B \circ H \circ B^m \cup B \circ H \circ B^m \circ H \circ B^m \cup B \circ H \circ B^m \circ H \circ B \circ H \circ B^m] \subseteq (B \circ H \circ B^m] \subseteq (B^m \cup B \circ H \circ B^m]$.

Note that in Remark 2.5, if $m = 1$, then $(B \cup B \circ H \circ B)$ is a generalized bi-hyperideal of $H$ which is not a bi-hyperideal of $H$. Thus $(B^m \cup B \circ H \circ B^m)$ is a generalized bi-hyperideal of $H$ for each positive integer $m$.

**Theorem 2.6.** *Let $B$ be a non-empty subset of an ordered semihypergroup $H$ and let $m \geq 2$ be any positive integer. Then the following are equivalent:*
(i) *$B$ is a $(1, m)$-hyperideal of $H$;*
(ii) *$B$ is a left hyperideal of some bi-hyperideals of $H$;*
(iii) *$B$ is a bi-hyperideal of some left hyperideals of $H$;*
(iv) *$B$ is a $(0, m)$-hyperideal of some right hyperideals of $H$;*
(v) *$B$ is a right hyperideal of some $(0, m)$-hyperideals of $H$.*

*Proof.* (i) $\Rightarrow$ (ii) Let $B$ be a $(1, m)$-hyperideal of $H$. So $B \circ B \subseteq B, (B] \subseteq B$ and $B \circ H \circ B^m \subseteq B$. Therefore, $(B^m \cup B \circ H \circ B^m] \circ B = (B^m \cup B \circ H \circ B^m] \circ (B] \subseteq (B^{m+1} \cup B \circ H \circ B^{m+1}] \subseteq (B^{m+1} \cup B \circ H \circ B^m] \subseteq (B] = B$. If $b \in B$, then $h \in (B^m \cup B \circ H \circ B^m]$ such that $h \leq b$. As $h \in H$ and $B$ is a $(1, m)$-hyperideal of $H$, $h \in B$. Hence, $B$ is a left hyperideal of the bi-hyperideal $(B^m \cup B \circ S \circ B^m]$ of $H$.

(ii) $\Rightarrow$ (iii) Let $B$ be a left hyperideal of a bi-hyperideal $A$ of $H$. So $B \subseteq A$, $A \circ B \subseteq B$ and $A \circ H \circ A \subseteq A$. Therefore, $B \circ B \subseteq A \circ B \subseteq B$ and $B \circ (B \cup H \circ B] \circ B = (B] \circ (B \cup H \circ B] \circ (B] \subseteq (B \circ (B \cup H \circ B)] \circ (B] \subseteq (B \circ (B \cup H \circ B) \circ B] \subseteq (B^3 \cup B^2 \circ H \circ B^2] \subseteq (A^2 \circ B \cup A \circ A \circ H \circ A \circ B] \subseteq (A \circ B \cup A \circ A \circ B] \subseteq (B \cup A \circ B] \subseteq (B] = B$. Let $b \in B, h \in (B \cup H \circ B]$ such that $h \leq b$. As $b \in B \subseteq A, b \in A$. So $h \in A$. Thus $h \in B$. Hence, $B$ is a bi-hyperideal of the left hyperideal $(B \cup H \circ B]$ of $H$.

(iii) $\Rightarrow$ (iv) Let $B$ be a bi-hyperideal of a left hyperideal $L$ of $H$. Then $B \subseteq L$, $B \circ L \circ B \subseteq B$ and $H \circ L \subseteq L$. Therefore, $(B \cup B \circ H] \circ B^m \subseteq (B \cup B \circ H] \circ (B^m] \subseteq (B^{m+1} \cup B \circ H \circ B^m] \subseteq (B \cup B \circ (H \circ B^{m-2}) \circ L \circ B] \subseteq (B \cup B \circ H \circ L \circ B] = (B \cup B \circ (H \circ L) \circ B] \subseteq (B \cup B \circ L \circ B] = (B] = B$. Let $b \in B$, $h \in (B \cup B \circ H]$ such that $h \leq b$. As $B \subseteq L$, $b \in L$. So $h \in L$ and, thus, $h \in B$. Hence, $B$ is a $(0,m)$-hyperideal of the right hyperideal $(B \cup B \circ H]$ of $H$.

(iv) $\Rightarrow$ (v) Let $B$ be a $(0,m)$-hyperideal of a right hyperideal $R$ of $H$. So $B \subseteq R$, $R \circ B^m \subseteq B$ and $R \circ H \subseteq R$. Therefore, $B \circ (B \cup H \circ B^m] \subseteq (B] \circ (B \cup H \circ B^m] \subseteq (B^2 \cup B \circ H \circ B^m] \subseteq (B \cup R \circ H \circ B^m] \subseteq (B \cup R \circ B^m] = (B] = B$. Let $b \in B$, $h \in (B \cup H \circ B^m]$ be such that $h \leq b$. As $B \subseteq R$, $b \in R$. So $h \in R$ which implies that $h \in B$. Hence, $B$ is a right hyperideal of the $(0,m)$-hyperideal $(B \cup H \circ B^m]$ of $H$.

(v) $\Rightarrow$ (i) Let $B$ be a right hyperideal of a $(0,m)$-hyperideal $A$ of $H$. Thus $B \subseteq A$, $B \circ A \subseteq B$ and $H \circ A^m \subseteq A$. Therefore, $B \circ H \circ B^m \subseteq B \circ H \circ A^m \subseteq B \circ A \subseteq B$. Let $b \in B$, $h \in H$ be such that $h \leq b$. As $B \subseteq R$, we have $b \in R$. Therefore, $h \in R$ and, thus, $h \in B$. Hence, $B$ is a $(1,m)$-hyperideal of $H$. □

**Definition 2.7.** Let $H$ be an ordered semihypergroup, $m, n$ be positive integers and $A$ be any (generalized) $(m,n)$-hyperideal of $H$. Then $A$ is said to be a *minimal* (*generalized*) $(m,n)$-hyperideal of $H$ if for every (generalized) $(m,n)$-hyperideal $B$ of $H$, $B \subseteq A$ implies $B = A$.

Similarly, a *minimal* $(m,0)$-*hyperideal* and a *minimal* $(0,n)$-*hyperideal* of $H$ may be defined.

**Lemma 2.8.** *Let $H$ be an ordered semihypergroup, $m \geq 2$ be any positive integer and $B$ be a non-empty subset of $H$. Then $B$ is a minimal* (*generalized*) *$(m,m-1)$-hyperideal of $H$ if and only if $B$ is a minimal* (*generalized*) *bi-hyperideal of $H$.*

*Proof.* Let $H$ be an ordered semihypergroup and $B$ be a minimal $(m,m-1)$-hyperideal of $H$. Since $(B^m \circ H \circ B^{m-1}] \circ (B^m \circ H \circ B^{m-1}] \subseteq (B^m \circ H \circ B^{m-1}]$, $((B^m \circ H \circ B^{m-1}])^m \circ H \circ ((B^m \circ H \circ B^{m-1}])^{m-1}] \subseteq (B^m \circ H \circ B^{m-1}]$ and $((B^m \circ H \circ B^{m-1}]] \subseteq (B^m \circ H \circ B^{m-1}]$. Therefore, $(B^m \circ H \circ B^{m-1}]$ is a $(m,m-1)$-hyperideal of $H$ such that $(B^m \circ H \circ B^{m-1}] \subseteq B$. So by minimality of $(m,m-1)$-hyperideal $B$ of $H$, $(B^m \circ H \circ B^{m-1}] = B$. Now $B \circ B =$

$(B^m \circ H \circ B^{m-1}] \circ (B^m \circ H \circ B^{m-1}] \subseteq ((B^m \circ H \circ B^{m-1}) \circ (B^m \circ H \circ B^{m-1})] \subseteq (B^m \circ H \circ B^{m-1}] = B$ and $B \circ H \circ B = (B^m \circ H \circ B^{m-1}] \circ H \circ (B^m \circ H \circ B^{m-1}] \subseteq (B^m \circ H \circ B^{m-1}] = B$. Therefore, $B$ is bi-hyperideal of $H$. It remains to show that $B$ is a minimal bi-hyperideal of $H$, so assume that $A$ is any bi-hyperideal of $H$ contained in $B$. Therefore, $A$ is $(m, m-1)$-hyperideal of $H$. Since $B$ is a minimal $(m, m-1)$-hyperideal of $H$, $B = A$. Hence, $B$ is a minimal bi-hyperideal of $H$. For the converse, assume that $B$ is a minimal bi-hyperideal of $H$. As $B^m \circ H \circ B^{m-1} = B \circ (B^{m-1} \circ H \circ B^{m-2}) \circ B \subseteq B \circ H \circ B \subseteq B$, $B$ is an $(m, m-1)$-hyperideal of $H$. To show that $B$ is a minimal $(m, m-1)$-hyperideal of $H$, let $A$ be any $(m, m-1)$-hyperideal of $H$ such that $A \subseteq B$. As $(A^m \circ H \circ A^{m-1}] \circ (A^m \circ H \circ A^{m-1}] \subseteq ((A^m \circ H \circ A^{m-1}) \circ (A^m \circ H \circ A^{m-1})] = (A^m \circ (H \circ A^{m-1} \circ A^m \circ H) \circ A^{m-1}] \subseteq (A^m \circ H \circ A^{m-1}]$ and $(A^m \circ H \circ A^{m-1}] \circ H \circ (A^m \circ H \circ A^{m-1}] \subseteq ((A^m \circ H \circ A^{m-1}) \circ H \circ (A^m \circ H \circ A^{m-1})] = (A^m \circ (H \circ A^{m-1} \circ H \circ A^m \circ H) \circ A^{m-1}] \subseteq (A^m \circ H \circ A^{m-1}]$, $(A^m \circ H \circ A^{m-1}]$ is a bi-hyperideal of $H$. Since $B$ is a minimal bi-hyperideal of $H$ and $(A^m \circ H \circ A^{m-1}] \subseteq B$, $(A^m \circ H \circ A^{m-1}] = B$. As $(A^m \circ H \circ A^{m-1}] \subseteq A$, $B \subseteq A$. Now, as $A \subseteq B$, we have $A = B$. Hence, $B$ is a minimal $(m, m-1)$-hyperideal of $H$. □

**Theorem 2.9.** *Let $H$ be an ordered semihypergroup and $\{A_i \mid i \in I\}$ be a set of $(m, n)$-hyperideals of $H$. If $\bigcap_{i \in I} A_i \neq \emptyset$, then $\bigcap_{i \in I} A_i$ is an $(m, n)$-hyperideal of $H$.*

*Proof.* Assume that $\bigcap_{i \in I} A_i \neq \emptyset$. Let $x, y \in \bigcap_{i \in I} A_i$. Then, $x, y \in A_i$ for each $i \in I$. As for each $i \in I$, $A_i$ is an $(m, n)$-hyperideal, $x \circ y \subseteq A_i$. Therefore, $x \circ y \subseteq \bigcap_{i \in I} A_i$. Thus, $\bigcap_{i \in I} A_i$ is a subsemihypergroup of $H$. Next we show that $(\bigcap_{i \in I} A_i)^m \circ H \circ (\bigcap_{i \in I} A_i)^n \subseteq \bigcap_{i \in I} A_i$. We have

$$\begin{aligned}
&(\bigcap_{i \in I} A_i)^m \circ H \circ (\bigcap_{i \in I} A_i)^n \\
&\subseteq (A_i)^m \circ H \circ (A_i)^n \quad (\text{as } \bigcap_{i \in I} A_i \subseteq A_i, \forall i \in I) \\
&\subseteq A_i \quad (\text{as } A_i\text{'s are } (m, n)\text{-hyperideals}).
\end{aligned}$$

Thus $(\bigcap_{i \in I} A_i)^m \circ H \circ (\bigcap_{i \in I} A_i)^n \subseteq \bigcap_{i \in I} A_i$. Finally, we show that $(\bigcap_{i \in I} A_i] \subseteq \bigcap_{i \in I} A_i$. Let $a \in \bigcap_{i \in I} A_i, h \in H$ such that $h \leq a$. As $a \in A_i$ for each $i \in I$ and $A_i$'s are $(m,n)$-hyperideals, $h \in A_i$ for each $i \in I$. Therefore, $h \in \bigcap\limits_{i\in I} A_i$, as required. $\square$

**Theorem 2.10.** [20] *Let $H$ be an ordered semihypergroup. Then the following conditions hold:*

(i) *Let $\{L_i \mid i \in I\}$ be a set of $(m,0)$-hyperideals of $H$. If $\bigcap\limits_{i\in I} L_i \neq \emptyset$, then $\bigcap\limits_{i\in I} L_i$ is an $(m,0)$-hyperideal of $H$.*

(ii) *Let $\{R_i \mid i \in I\}$ be a set of $(0,n)$-hyperideals of $H$. If $\bigcap\limits_{i\in I} R_i \neq \emptyset$, then $\bigcap\limits_{i\in I} R_i$ is a $(0,n)$-hyperideal of $H$.*

Let $H$ be an ordered semihypergroup and $A$ be any non-empty subset of $H$. We denote $\mathcal{P} = \{J \mid J$ is an $(m,n)$-hyperideal of $H$ containing $A\}$. Clearly, $\mathcal{P} \neq \emptyset$ since $H \in \mathcal{P}$. Let $[A]_{m,n} = \bigcap_{J\in\mathcal{P}} J$. As $A \subseteq J$ for each $J \in \mathcal{P}$, $[A]_{m,n} \neq \emptyset$. By Theorem 1.9, $[A]_{m,n}$ is an $(m,n)$-hyperideal of $H$ containing $A$. The $(m,n)$-hyperideal $[A]_{m,n}$ is called the $(m,n)$-*hyperideal of $H$ generated by $A$*. Similarly, $[A]_{m,0}$ and $[A]_{0,n}$ are called $(m,0)$-*hyperideal* and $(0,n)$-*hyperideal of $H$ generated by $A$*, respectively.

**Theorem 2.11.** *Let $H$ be an ordered semihypergroup and $A$ be a non-empty subset of $H$. Then*

$$[A]_{m,n} = (\textstyle\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n]$$

*for any positive integers $m,n$.*

*Proof.* Clearly $(\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n] \neq \emptyset$. Now we have

$$\begin{aligned}
&(\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n] \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n] \\
&\subseteq ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)] \\
&= ((\bigcup_{i=1}^{m+n} A^i) \circ (\bigcup_{i=1}^{m+n} A^i) \cup (\bigcup_{i=1}^{m+n} A^i) \circ A^m \circ H \circ A^n \cup (A^m \circ H \circ A^n) \circ (\bigcup_{i=1}^{m+n} A^i) \\
&\cup (A^m \circ H \circ A^n) \circ (A^m \circ H \circ A^n)] \\
&\subseteq ((\bigcup_{i=1}^{m+n} A^i) \circ (\bigcup_{i=1}^{m+n} A^i) \cup A^m \circ H \circ A^n] \qquad (1).
\end{aligned}$$

Let $x \in (\bigcup_{i=1}^{m+n} A^i) \circ (\bigcup_{i=1}^{m+n} A^i)$. Then, $x \in z_1 \circ z_2$ for some $z_1, z_2 \in$

$\bigcup_{i=1}^{m+n} A^i$. Then, $z_1 = A^p, z_2 = A^q$ for some $1 < p, q \leq m+n$. There are two cases arising. If $p+q \leq m+n$, then $z_1 \circ z_2 \subseteq \bigcup_{i=1}^{m+n} A^i$. If $m+n \leq p+q$, then $z_1 \circ z_2 \subseteq A^m \circ H \circ A^n$. Therefore, in both cases $z_1 \circ z_2 \subseteq \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n$. As $x \in z_1 \circ z_2$, $x \in \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n$. Thus, $(\bigcup_{i=1}^{m+n} A^i) \circ (\bigcup_{i=1}^{m+n} A^i) \subseteq \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n$. Therefore, from (1), $(\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n] \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n] \subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n]$. Hence, $(\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n]$ is a subsemihypergroup of $H$ containing $A$. We have

$$
\begin{aligned}
&((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^m \circ H \\
&= (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^{m-1} \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n] \circ H \\
&\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^{m-1} \circ (\bigcup_{i=1}^{m+n} A^i \circ H \cup A^m \circ H \circ A^n \circ H] \\
&\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^{m-1} \circ (A \circ H] \\
&= (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^{m-2} \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n] \circ (A \circ H] \\
&\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^{m-2} \circ (\bigcup_{i=1}^{m+n} A^i \circ A \circ H \cup A^m \circ H \circ A^n \circ A \circ H] \\
&\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^{m-1} \circ (A^2 \circ H] \\
&\vdots \\
&= (A^m \circ H].
\end{aligned}
$$

Similarly, $H \circ ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^n \subseteq (H \circ A^n]$. Therefore, we have

$$
\begin{aligned}
&((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^m \circ H \circ ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^n \\
&\subseteq (A^m \circ H \circ A^n] \\
&\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n].
\end{aligned}
$$

Also $((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n]] \subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n]$. Therefore, $(\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n]$ is an $(m,n)$-hyperideal of $H$ containing $A$. It follows that $[a]_{m,n} \subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n]$. For the reverse inclusion, let $x \in (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n]$, that is, there exist $z \in \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n$

such that $x \leq z$. If $z \in \bigcup_{i=1}^{m+n} A^i$, then $z = A^p$ for some $1 \leq p \leq m+n$. Therefore, $x \in [A]_{m,n}$. If $z \in A^m \circ H \circ A^n$, then

$$A^m \circ H \circ A^n \subseteq ([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n \subseteq [A]_{m,n}.$$

Therefore, $z \in [A]_{m,n}$ implies $x \in [A]_{m,n}$. Hence, $[A]_{m,n} = (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n]$, as required. □

**Theorem 2.12.** [20] *Let $H$ be an ordered semihypergroup and $A$ be any non-empty subset of $H$. Then:*

(i) $[A]_{m,0} = (\bigcup\limits_{i=1}^{m} A^i \cup A^m \circ H]$;

(ii) $[A]_{0,n} = (\bigcup\limits_{i=1}^{n} A^i \cup H \circ A^n]$.

**Theorem 2.13.** *Let $H$ be an ordered semihypergroup and $A$ be a non-empty subset of $H$. Then*

$$(([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n] = (A^m \circ H \circ A^n]$$

*for any positive integers $m, n$.*

*Proof.* We have

$([A]_{m,n})^m \circ H$

$= ((\bigcup\limits_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^m \circ H$

$= ((\bigcup\limits_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^{m-1} \circ (\bigcup\limits_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n] \circ H$

$\subseteq (\bigcup\limits_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^{m-1} \circ (\bigcup\limits_{i=1}^{m+n} A^i \circ H \cup A^m \circ H \circ A^n \circ H]$

$\subseteq (\bigcup\limits_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^{m-1} \circ (A \circ H]$

$= (\bigcup\limits_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^{m-2} \circ (\bigcup\limits_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n] \circ (A \circ H]$

$\subseteq (\bigcup\limits_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^{m-2} \circ (\bigcup\limits_{i=1}^{m+n} A^i \circ A \circ H \cup A^m \circ H \circ A^n \circ A \circ H]$

$\subseteq (\bigcup\limits_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n])^{m-1} \circ (A^2 \circ H]$

$\vdots$
$= (A^m \circ H]$.

Similarly, $H \circ ([A]_{m,n})^n \subseteq H \circ A^n$. Therefore, $(([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n] \subseteq (A^m \circ H \circ A^n]$. The reverse inclusion is obvious, that is, $(A^m \circ H \circ A^n] \subseteq (([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n]$. Hence, $(([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n] = (A^m \circ H \circ A^n]$. □

**Theorem 2.14.** [20] *Let $H$ be an ordered semihypergroup and $A$ be a non-empty subset of $H$. Then*
(i) $(([A]_{m,0})^m \circ H] = (A^m \circ H]$ *for any positive integer $m$.*
(ii) $(H \circ ([A]_{(0,n)})^n] = (H \circ A^n]$ *for any positive integer $n$.*

## 3 $(m, n)$-regularity in ordered semihypergroups

In this section, we characterize $(m, n)$-regular, $(m, 0)$-regular and $(0, n)$-regular ordered semihypergroup in terms of its $(m, n)$-hyperideals, $(m, 0)$-hyperideals and $(0, n)$-hyperideals.

**Definition 3.1.** Let $H$ be an ordered semihypergroup and $m, n$ be non-negative integers. An element a of $H$ is said to be an $(m, n)$-*regular* element if $a \in (a^m \circ H \circ a^n]$. The ordered semihypergroup $H$ is said to be $(m, n)$-*regular* if each element of $H$ is $(m, n)$-regular, equivalently, for each subset $A$ of $H$ we have $A \subseteq (A^m \circ H \circ A^n]$. Here, $A^0 \circ H = H \circ A^0 = H$.

It is clear from Definition 3.1 that, for each non-negative integers $m$ and $n$ every $(m, n)$-regular ordered semihypergroup is $(r, s)$-regular ($r \leq m, s \leq n$ are non-negative integers). In particular, for any positive integers $m$ and $n$, an $(m, n)$-regular ordered semihypergroup is regular. Indeed, $a \in (a^m \circ H \circ a^n] \subseteq (a \circ H \circ a]$. On the other hand, for each positive integer $m$, an $(m, 0)$-regular ordered semihypergroup need not be a regular ordered semihypergroup.

**Proposition 3.2.** *Let $H$ be an $(m, n)$-regular ordered semihypergroup and $A$ be a generalized $(m, n)$-hyperideal of $H$ for any positive integers $m, n$. Then $A$ is an $(m, n)$-hyperideal of $H$.*

*Proof.* Let $a,b \in A$. Since $H$ is an $(m,n)$-regular ordered semihypergroup, there exist $x,y \in H$ such that $a \leq a^m \circ x \circ a^n, b \leq b^m \circ y \circ b^n$. Therefore, $a \circ b \leq a^m \circ x \circ a^n \circ b^m \circ y \circ b^n = a^m \circ (x \circ a^n \circ b^m \circ y) \circ b^n \subseteq A^n \circ H \circ A^m \subseteq A$ whence $a \circ b \subseteq (A] = A$. Thus $A$ is a subsemihypergroup of $H$. Hence, $A$ is an $(m,n)$-hyperideal of $H$. □

**Theorem 3.3.** *Let $H$ be an ordered semihypergroup and $m$,$n$ be non-negative integers. The set of all $(m,0)$-hyperideals, $(0,n)$-hyperideals, and $(m,n)$-hyperideals will be denoted by $I_{(m,0)}, I_{(0,n)}$ and $I_{(m,n)}$, respectively. Then, we have*

(i) *$H$ is $(m,0)$-regular if and only if $I_{(m,0)}$ is $(m,0)$-regular;*

(ii) *$H$ is $(0,n)$-regular if and only if $I_{(0,n)}$ is $(0,n)$-regular;*

(iii) *$H$ is $(m,n)$-regular if and only if $I_{(m,n)}$ is $(m,n)$-regular.*

*Proof.* (i) When $m = 0$, the statement holds trivially because $H$ is the only $(0,0)$-hyperideal of $H$. So, let $m \neq 0$ and $A \in I_{(m,0)}$. Therefore $(A^m \circ H] \subseteq A$. As $S$ is $(m,0)$-regular, $A \subseteq (A^m \circ H]$. Thus, $A = (A^m \circ H]$. Since $H \in I_{(m,0)}$, $A$ is a $(m,0)$-regular element of $I_{(m,0)}$. Hence $I_{(m,0)}$ is $(m,0)$-regular. For the converse, assume that $I_{(m,0)}$ is $(m,0)$-regular. Take any $a \in S$. As $[a]_{m,0}$ is in $I_{(m,0)}$ and $I_{(m,0)}$ is $(m,0)$-regular, there exists $B \in I_{(m,0)}$ such that $[a]_{m,0} = ([a]_{m,0})^m \circ B \subseteq ([a]_{m,0})^m \circ H \subseteq (([a]_{m,0})^m \circ H]$. By Theorem 2.14, $([a]_{m,0})^m \circ H] = (a^m \circ H]$. As $\{a\} \subseteq [a]_{m,0}$, we have $a \in (a^m \circ H]$. Hence $H$ is $(m,0)$-regular.

(ii) On the similar lines to (i), we may prove (ii).

(iii) If $m = n = 0$, then the statement is true because $I_{(0,0)} = \{H\}$. If $m \neq 0$ and $n = 0$ or $m = 0$ and $n \neq 0$, then the statement follows by (i) and (ii), respectively. So, let $m \neq 0, n \neq 0$ and $A \in I_{(m,n)}$. Therefore $(A^m \circ H \circ A^n] \subseteq A$. As $H$ is $(m,n)$-regular, $A \subseteq (A^m \circ H \circ A^n]$. Thus, $A = (A^m \circ H \circ A^n]$. Since $H \in I_{(m,n)}$, $A$ is an $(m,n)$-regular element of $I_{(m,n)}$. Hence, $I_{(m,n)}$ is $(m,n)$-regular. For the converse, assume that $I_{(m,n)}$ is $(m,n)$-regular and $a \in H$. As $[a]_{m,n}$ is in $I_{(m,n)}$and $I_{(m,n)}$ is $(m,n)$-regular, there exists $B \in I_{(m,n)}$ such that $[a]_{m,n} = ([a]_{m,n})^m \circ B \circ ([a]_{m,n})^n \subseteq ([a]_{m,n})^m \circ H \circ ([a]_{m,n})^n \subseteq (([a]_{m,n})^m \circ H \circ ([a]_{m,n})^n]$. By Theorem 4.1, we have $(([a]_{m,n})^m \circ H \circ ([a]_{m,n})^n] = (a^m \circ H \circ a^n]$. As $\{a\} \subseteq [a]_{m,n}$, $a \in (a^m \circ H \circ a^n]$. This implies that $a$ is an $(m,n)$-regular element of $H$. Hence, $H$ is $(m,n)$-regular. □

**Lemma 3.4.** [20] *Let $H$ be an ordered semihypergroup. If the sets of all $(m,0)$-hyperideals and $(0,n)$-hyperideals are denoted by $I_{(m,0)}$ and $I_{(0,n)}$ respectively, then*

(i) *$H$ is $(m,0)$-regular if and only if $R = (R^m \circ H]$ $(\forall R \in I_{(m,0)})$, where $m$ is any positive integer;*

(ii) *$H$ is $(0,n)$-regular if and only if $L = (H \circ L^n]$ $(\forall L \in I_{(0,n)})$, where $n$ is any positive integer.*

**Theorem 3.5.** *Let $H$ be an ordered semihypergroup and $m,n$ be non-negative integers. The set of all $(m,n)$-hyperideals will be denoted by $I_{(m,n)}$. Then $H$ is $(m,n)$-regular if and only if $A = (A^m \circ H \circ A^n]$ for all $A \in I_{(m,n)}$.*

*Proof.* If $m = n = 0$, then the statement is true because $I_{(0,0)} = \{H\}$. If $m \neq 0$ and $n = 0$ or $m = 0$ and $n \neq 0$, then the statement follows by Lemma 3.4. So, let $m \neq 0, n \neq 0$ and $A \in I_{(m,n)}$. Then, by definition of $(m,n)$-regularity, we have $A \subseteq (A^m \circ H \circ A^n]$ and, by definition of $(m,n)$-hyperideal, we have $(A^m \circ H \circ A^n] \subseteq (A] = A$. Hence, $A = (A^m \circ H \circ A^n]$.

For the converse, assume that $A = (A^m \circ H \circ A^n]$ for each $A \in I_{(m,n)}$. Take any $a \in H$, so $[a]_{m,n} \in I_{(m,n)}$. From Theorem 4.1 and by the assumption, $[a]_{m,n} = (([a]_{m,n})^m \circ H \circ [a]_{m,n})^n] = (a^m \circ H \circ a^n]$. As $\{a\} \subseteq [a]_{m,n}$, $a \in (a^m \circ H \circ a^n]$. Hence, $H$ is $(m,n)$-regular. □

**Theorem 3.6.** *Let $H$ be an ordered semihypergroup and $m$, $n$ be non-negative integers. Then, $H$ is $(m,n)$-regular if and only if $L \cap R = (R^m \circ L^n]$ for each $(m,0)$-hyperideal $R$ and for each $(0,n)$-hyperideal $L$ of $H$.*

*Proof.* The statement is trivially true for $m = 0 = n$. If $m = 0$ and $n \neq 0$ or $m \neq 0$ and $n = 0$, then the result follows by Lemma 3.4. So, let $m \neq 0$, $n \neq 0$, $R$ be any $(m,0)$-hyperideal and $L$ be any $(0,n)$-hyperideal of $H$. Therefore $(R^m \circ L^n] \subseteq (R^m \circ H] \subseteq (R] = R$ and $(R^m \circ L^n] \subseteq (H \circ L^n] \subseteq (L] = L$. Therefore, $(R^m \circ L^n] \subseteq R \cap L$. As $H$ is $(m,n)$-regular, we have

$(R \cap L)$
$\subseteq ((R \cap L)^m \circ H \circ (R \cap L)^n]$
$\subseteq (R^m \circ H \circ L^n]$
$\subseteq (R^m \circ H \circ L^{n-1} \circ (L^m \circ H \circ L^n]]$ (as $H$ is $(m,n)$-regular)
$= (R^m \circ H \circ L^{n-1} \circ L^m \circ H \circ L^n]$ (by Lemma 1.1)
$\subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n] \circ H \circ L^n]$ (as $H$ is $(m,n)$-regular)
$\subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n] \circ (H \circ L^n]]$ (as $H \circ L^n \subseteq (H \circ L^n])$

$$
\begin{aligned}
&\subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n \circ H \circ L^n]] \quad \text{(by Lemma 1.1)}\\
&\subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ L^m \circ H \circ L^n \circ H \circ L^n] \quad \text{(by Lemma 1.1)}\\
&\subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n] \circ H \circ L^n \circ H \circ L^n]\\
&\vdots\\
&\subseteq (R^m \circ H \circ L^{n-1} \circ \underbrace{L^{m-1} \circ L^{m-1} \circ \cdots \circ L^{m-1}}_{n-1\text{-times}} \circ (L^m \circ H \circ L^n)\\
&\circ \underbrace{H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n}_{n-1\text{-times}}]\\
&= (R^m \circ H \circ L^{n-1} \circ (L^{m-1})^{n-1} \circ L^m \circ \underbrace{H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n}_{n\text{-times}}]\\
&= (R^m \circ H \circ (L^{n-1} \circ L^{mn-m-n+1} \circ L^m) \circ \underbrace{H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n}_{n\text{-times}}]\\
&= (R^m \circ (H \circ L^{mn}) \circ \underbrace{H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n}_{n\text{-times}}]\\
&\subseteq (R^m \circ H \circ \underbrace{H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n}_{n\text{-times}}]\\
&\subseteq (R^m \circ \underbrace{H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n}_{n\text{-times}}]\\
&\subseteq (R^m \circ (H \circ L^n)^n]\\
&\subseteq (R^m \circ L^n].
\end{aligned}
$$

Therefore, $L \cap R = (R^m \circ L^n]$.

Conversely, assume that $L \cap R = (R^m \circ L^n]$ for each $(m, 0)$-hyperideal $R$ and for each $(0, n)$-hyperideal $L$ of $H$. Let $a \in S$. As $[a]_{m,0}$ is an $(m, 0)$-hyperideal and $H$ is a $(0, n)$-hyperideal of $H$, we have

$$
\begin{aligned}
[a]_{m,0} &= [a]_{m,0} \cap H = (([a]_{m,0})^m \circ H^n]\\
&\subseteq (([a]_{m,0})^m \circ H] = (a^m \circ H] \qquad \text{(by Theorem 2.14)}
\end{aligned}
$$

Similarly, $[a]_{0,n} \subseteq (H \circ a^n]$. As $(a^m \circ H]$ and $(H \circ a^n]$ are an $(m, 0)$-hyperideal and $(0, n)$-hyperideal of $H$, by hypothesis we get

$$
\begin{aligned}
\{a\} &\subseteq [a]_{m,0} \cap [a]_{0,n} \subseteq (a^m \circ H] \cap (H \circ a^n]\\
&= (((a^m \circ H])^m \circ ((H \circ a^n])^n] \qquad \text{(by hypothesis)}\\
&\subseteq (a^m \circ H \circ a^n].
\end{aligned}
$$

Hence, $H$ is $(m, n)$-regular. □

**Theorem 3.7.** *Let $H$ be an ordered semihypergroup and $m$, $n$ be positive integers (either $m \geq 2$ or $n \geq 2$). Then, the following are equivalent:*

(i) *Each* $(m,n)$*-hyperideal of* $H$ *is idempotent;*

(ii) *For each* $(m,n)$*-hyperideals* $A$, $B$ *of* $H$, $A \cap B \subseteq (A^m \circ B^n]$*;*

(iii) $[a]_{m,n} \cap [b]_{m,n} \subseteq (([a]_{m,n})^m \circ ([b]_{m,n})^n]$ $\forall a, b \in H$*;*

(iv) $[a]_{m,n} \subseteq (([a]_{m,n})^m \circ ([a]_{m,n})^n]$ $\forall a \in H$*;*

(v) $H$ *is* $(m,n)$*-regular.*

*Proof.* (i) ⇒ (ii) Assume that each $(m,n)$-hyperideal of $H$ is idempotent. Let $A$ and $B$ be any $(m,n)$-hyperideals of $H$. As $A \cap B$ is an $(m,n)$-hyperideal of $H$, we have

$$\begin{aligned} A \cap B &= ((A \cap B)^2] = ((A \cap B) \circ ((A \cap B])^2] \\ &= ((A \cap B)^3] = \cdots = ((A \cap B)^{m+n}] \\ &= ((A \cap B)^m \circ (A \cap B)^n] \subseteq (A^m \circ B^n]. \end{aligned}$$

(ii) ⇒ (iii) and (iii) ⇒ (iv) are obvious.

(iv) ⇒ (v) Take any $a \in A$. Then, by (iv), we have

$$\begin{aligned} &[a]_{m,n} \subseteq (([a]_{m,n})^m \circ ([a]_{m,n})^n] \\ &\subseteq (([a]_{m,n})^m \circ ([a]_{m,n})^{n-1} \circ (([a]_{m,n})^m \circ ([a]_{m,n})^n]] \\ &= (([a]_{m,n})^m \circ ([a]_{m,n})^{n-1} \circ ([a]_{m,n})^m \circ ([a]_{m,n})^n] \quad \text{(by Lemma 1.1)} \\ &\subseteq (([a]_{m,n})^m \circ H \circ ([a]_{m,n})^n] \\ &= (((([a]_{m,n})^m \circ H] \circ ([a]_{m,n})^n] \quad \text{(by Lemma 1.1)} \\ &= ((a^m \circ H] \circ ([a]_{m,n})^n] \quad \text{(by Theorem 4.1)} \\ &= (a^m \circ H \circ ([a]_{m,n})^n] \quad \text{(by Lemma 1.1)} \\ &= (a^m \circ (H \circ ([a]_{m,n})^n]] \quad \text{(by Lemma 1.1)} \\ &= (a^m \circ (H \circ a^n]] \quad \text{(by Theorem 4.1)} \\ &= (a^m \circ H \circ a^n] \quad \text{(by Lemma 1.1)} \end{aligned}$$

As $\{a\} \subseteq [a]_{m,n},\ a \in (a^m \circ H \circ a^n]$. Hence $H$ is $(m,n)$-regular.

(v) ⇒ (i) Take any $(m,n)$-hyperideal $A$ of $H$. As $H$ is $(m,n)$-regular and $A$ is an $(m,n)$-hyperideal, $A = (A^m \circ H \circ A^n]$. Now

$$(A \circ A] = ((A^m \circ H \circ A^n] \circ (A^m \circ H \circ A^n)] \subseteq (A^m \circ H \circ A^n] = A$$

and

$$\begin{aligned} A &= (A^m \circ H \circ A^n] = (((A^m \circ H \circ A^n])^m \circ H \circ A^n] \\ &= (\underbrace{(A^m \circ H \circ A^n] \circ \cdots \circ (A^m \circ H \circ A^n]}_{m\text{-times}} \circ H \circ A^n] \\ &= ((A^m \circ H \circ A^n] \circ (A^m \circ H \circ A^n] \circ \end{aligned}$$

$$\underbrace{(A^m \circ H \circ A^n] \circ \cdots \circ (A^m \circ H \circ A^n]}_{(m-2)\text{-times}} \circ H \circ A^n]$$
$$\subseteq ((A^m \circ H \circ A^n] \circ (A^m \circ H \circ A^n] \circ H \circ H \circ A^n]$$
$$\subseteq ((A^m \circ H \circ A^n] \circ (A^m \circ H \circ A^n] \circ H \circ A^n]$$
$$\subseteq ((A^m \circ H \circ A^n] \circ (A^m \circ H \circ A^n] \circ (H \circ A^n]]$$
$$\subseteq ((A^m \circ H \circ A^n] \circ (A^m \circ H \circ A^n \circ H \circ A^n]]$$
$$\subseteq ((A^m \circ H \circ A^n] \circ (A^m \circ H \circ A^n]]$$
$$= (A \circ A].$$

Therefore, $A = (A \circ A]$. Hence, each $(m,n)$-hyperideal of $H$ is an idempotent. □

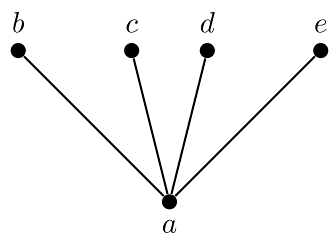
The following example shows that the condition $m \geq 2$ or $n \geq 2$ in Theorem 3.7 is necessary.

**Example 3.8.** [24] Let $H = \{a, b, c, d, e\}$. Define a hyperoperation $\circ$ on $H$ by the table

| $\circ$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a\}$ |
| $b$ | $\{a\}$ | $\{a,b\}$ | $\{a\}$ | $\{a,d\}$ | $\{a\}$ |
| $c$ | $\{a\}$ | $\{a,e\}$ | $\{a,c\}$ | $\{a,c\}$ | $\{a,e\}$ |
| $d$ | $\{a\}$ | $\{a,b\}$ | $\{a,d\}$ | $\{a,d\}$ | $\{a,b\}$ |
| $e$ | $\{a\}$ | $\{a,e\}$ | $\{a\}$ | $\{a,c\}$ | $\{a\}$ |

and the order $\leq$ on $H$ as $\leq := \{(a,a), (b,b), (c,c), (d,d), (e,e), (a,b), (a,c), (a,d), (a,e)\}$. The covering relation $\prec$ and the figure of $H$ are as

$$\prec := \{(a,b), (a,c), (a,d), (a,e)\}$$




Now, $(H, \circ, \leq)$ is a regular ordered semihypergroup. One may easily check that $A = \{a, e\}$ is a bi-hyperideal of $H$, but $A \neq (A^2]$.

## 4 Relations $\mathcal{I}_n, {}_m\mathcal{I}, \mathcal{H}_m^n$ and $\mathcal{B}_m^n$ on ordered semihypergroups

In this section, the relations $\mathcal{I}_n, {}_m\mathcal{I}, \mathcal{H}_m^n$ and $\mathcal{B}_m^n$ on an ordered semihypergroup are introduced. Then, some related properties of these relations are studied.

**Definition 4.1.** Let $H$ be an ordered semihypergroup and $m, n$ be positive integers. We define the relations $\mathcal{I}_n,\ {}_m\mathcal{I},\ \mathcal{H}_m^n$ and $\mathcal{B}_m^n$ as

$$\begin{aligned}
\mathcal{I}_n &= \{(a,b) \in S \times S \mid [a]_{0,n} = [b]_{0,n}\};\\
{}_m\mathcal{I} &= \{(a,b) \in S \times S \mid [a]_{m,0} = [b]_{m,0}\};\\
\mathcal{H}_m^n &= {}_m\mathcal{I} \cap \mathcal{I}_n;\\
\mathcal{B}_m^n &= \{(a,b) \in S \times S \mid [a]_{m,n} = [b]_{m,n}\}.
\end{aligned}$$

Clearly, all the relations defined above are equivalence relations on $H$.

**Lemma 4.2.** *Let $H$ be an ordered semihypergroup and $a, b \in H$ be ${}_m\mathcal{I}$-related (respectively, $\mathcal{I}_n$-related). Then, $(a^m \circ H] = (b^m \circ H]$ (respectively, $(H \circ a^n] = (H \circ b^n]$).*

*Proof.* Suppose that $(a,b) \in {}_m\mathcal{I}$. Then, by definition, $[a]_{m,0} = [b]_{m,0}$, i.e. $(\bigcup_{i=1}^m a^i \cup a^m \circ H] = (\bigcup_{i=1}^m b^i \cup b^m \circ H]$. Therefore, $\{a\} \subseteq (\bigcup_{i=1}^m b^i \cup b^m \circ H]$ and $\{b\} \subseteq (\bigcup_{i=1}^m a^i \cup a^m \circ H]$. Thus, $(a^m \circ H] \subseteq (((\bigcup_{i=1}^m b^i \cup b^m \circ H])^m \circ H] = ((b^m \circ H]] = (b^m \circ H]$ (by Theorem 4.1). Similarly, from $\{b\} \subseteq (\bigcup_{i=1}^m a^i \cup a^m \circ H]$, we have $(b^m \circ H] \subseteq (a^m \circ H]$. Hence $(a^m \circ H] = (b^m \circ H]$. Similarly, we may show that $(a,b) \in \mathcal{I}_n$ implies $(H \circ a^n] = (H \circ b^n]$. □

**Lemma 4.3.** *Let $H$ be an ordered semihypergroup and $a, b \in H$ be $\mathcal{H}_m^n$-related. Then, $(a^m \circ H] = (b^m \circ H]$, $(H \circ a^n] = (H \circ b^n]$ and $(a^m \circ H \circ a^n] = (b^m \circ H \circ b^n]$.*

*Proof.* Suppose that $(a,b) \in \mathcal{H}_m^n$. Then, by definition, $(a,b) \in {}_m\mathcal{I}$ and $(a,b) \in \mathcal{I}_n$. By Lemma 4.1, $(a^m \circ H] = (b^m \circ H]$ and $(H \circ a^n] = (H \circ b^n]$. Therefore, we have $(a^m \circ H \circ a^n] = ((a^m \circ H] \circ a^n] = ((b^m \circ H] \circ a^n] = (b^m \circ H \circ a^n] = (b^m \circ (H \circ a^n]] = (b^m \circ (H \circ b^n]] = (b^m \circ H \circ b^n]$. □

**Lemma 4.4.** *Let $H$ be an ordered semihypergroup. Then, $\mathcal{B}_m^n \subseteq \mathcal{H}_m^n$.*

*Proof.* Let $(a, b) \in \mathcal{B}_m^n$. Then, $[a]_{m,n} = [b]_{m,n}$, i.e. $(\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n] = (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n]$. So $a^i \subseteq (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n]$ and $b^i \subseteq (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n]$ for each $i \in \{1, 2, \ldots, m+n\}$. It follows that $\bigcup_{i=1}^{m} a^i \subseteq (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n]$ and $\bigcup_{i=1}^{m} b^i \subseteq (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n]$. Now $(a^m \circ H] \subseteq (((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n])^m \circ H]$ and $(b^m \circ H] \subseteq (((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n])^m \circ H]$. Therefore, by Theorem 4.1, $(a^m \circ H] \subseteq (b^m \circ H]$ and $(b^m \circ H] \subseteq (a^m \circ H]$. Now

$$
\begin{aligned}
&[a]_{m,0} = (\bigcup_{i=1}^{m} a^i \cup a^m \circ H] \\
&\subseteq ((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n] \cup a^m \circ H] \ \big(\text{since } \bigcup_{i=1}^{m} a^i \subseteq (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n]\big) \\
&\subseteq ((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n] \cup (a^m \circ H]] \ \ (\text{as } a^m \circ H \subseteq (a^m \circ H]) \\
&\subseteq ((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n] \cup (b^m \circ H]] \ \ (\text{as } (a^m \circ H] \subseteq (b^m \circ H]) \\
&= ((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n \cup b^m \circ H]] \ \ (\text{by Lemma 1.1}) \\
&= (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n \cup b^m \circ H] \ \ (\text{by Lemma 1.1}) \\
&\subseteq (\bigcup_{i=1}^{m} b^i \cup b^m \circ H \cup b^m \circ H \circ b^n \cup b^m \circ H] \ \ (\text{since } \bigcup_{i=1}^{m+n} b^i \subseteq \bigcup_{i=1}^{m} b^i \cup b^m \circ H) \\
&= (\bigcup_{i=1}^{m} b^i \cup b^m \circ H] \ \ (\text{as } b^m \circ H \circ b^n \subseteq b^m \circ H) \\
&= [b]_{m,0};
\end{aligned}
$$

and

$$
\begin{aligned}
&[b]_{m,0} \\
&= (\bigcup_{i=1}^{m} b^i \cup b^m \circ H] \\
&\subseteq ((\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m] \cup b^m \circ H] \ \big(\text{since } \bigcup_{i=1}^{m} b^i \subseteq (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n]\big) \\
&\subseteq ((\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m] \cup (b^m \circ H]] \ \ (\text{as } b^m \circ H \subseteq (b^m \circ H]) \\
&\subseteq ((\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m] \cup (a^m \circ H]] \ \ (\text{as } (b^m \circ H] \subseteq (a^m \circ H])
\end{aligned}
$$

$$= ((\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m \cup a^m \circ H]] \text{ (by Lemma 1.1)}$$
$$= (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m \cup a^m \circ H] \text{ (by Lemma 1.1)}$$
$$\subseteq (\bigcup_{i=1}^{m} a^i \cup a^m \circ H \cup a^m \circ H \circ a^n \cup a^m \circ H] \text{ (since } \bigcup_{i=1}^{m+n} a^i \subseteq \bigcup_{i=1}^{m} a^i \cup a^m \circ H)$$
$$\subseteq (\bigcup_{i=1}^{m} a^i \cup a^m \circ H] \text{ (as } a^m \circ H \circ a^n \subseteq a^m \circ H)$$
$$= [a]_{m,0}.$$

Therefore, $[a]_{m,0} = [b]_{m,0}$. Similarly, one can show that $[a]_{0,n} = [b]_{0,n}$. Thus, $(a,b) \in \mathcal{H}_m^n$. Hence, $\mathcal{B}_m^n \subseteq \mathcal{H}_m^n$. □

**Theorem 4.5.** *Let $H$ be an $(m,n)$-regular ordered semihypergroup. Then, $\mathcal{B}_m^n = \mathcal{H}_m^n$.*

*Proof.* Let $(a,b) \in \mathcal{H}_m^n$. Therefore, by Lemma 4.2, $(a^m \circ H \circ a^n] = (b^m \circ H \circ b^n]$. As $S$ is $(m,n)$-regular, $a \in (a^m \circ H \circ a^m]$ and $b \in (b^m \circ H \circ b^n]$. So $a^i \subseteq (a^m \circ H \circ a^m]$ for each $i \in \{1, 2, \ldots, m+n\}$, it follows that $\bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^m]$. Thus, $[a]_{m,n} = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m] = (a^m \circ H \circ a^m]$ and similarly $[b]_{m,n} = (b^m \circ H \circ b^n]$. Thus, $[a]_{m,n} = [b]_{m,n}$, i.e. $(a,b) \in \mathcal{B}_m^n$. This implies that $\mathcal{H}_m^n \subseteq \mathcal{B}_m^n$. Hence, by Lemma 4.3, $\mathcal{B}_m^n = \mathcal{H}_m^n$. □

**Lemma 4.6.** *If $B_x$ and $B_y$ are two $(m,n)$-regular $\mathcal{B}_m^n$-classes contained in the same $\mathcal{H}_m^n$-class of ordered semihypergroup $H$, then $B_x = B_y$.*

*Proof.* As $x$ and $y$ are $(m,n)$-regular elements of $H$, $x \in (x^m \circ H \circ x^n]$ and $y \in (y^m \circ H \circ y^n]$, $\{x\}^i \subseteq (x^m \circ H \circ x^n]$ and $\{y\}^i \subseteq (y^m \circ H \circ y^n]$ for each $i \in \{1, 2, \ldots, m+n\}$. It follows that $\bigcup_{i=1}^{m+n} x^i \subseteq (x^m \circ H \circ x^m]$ and $\bigcup_{i=1}^{m+n} y^i \subseteq (y^m \circ H \circ y^m]$. Therefore, $[x]_{m,n} = (x^m \circ H \circ x^n]$ and $[y]_{m,n} = (y^m \circ H \circ y^n]$. Since $x$ and $y$ are contained in the same $\mathcal{H}_m^n$-class, by Lemma 4.2, $(x^m \circ H \circ x^n] = (y^m \circ H \circ y^n]$. So $[x]_{m,n} = [y]_{m,n}$. Therefore, $x\mathcal{B}_m^n y$. Hence, $B_x = B_y$. □

## 5 $(m,0)$-regularity $[(0,n)$-regularity$]$ and $(m,n)$-right weakly regularity of a $\mathcal{B}_m^n$-class, $\mathcal{Q}_m^n$-class and $\mathcal{H}_m^n$-class

In this section, the $(m,0)$-regular, $(0,n)$-regular, $(m,n)$-regular and $(m,n)$-right weakly regular class of the relations $\mathcal{H}_m^n$ and $\mathcal{B}_m^n$ are studied.

**Lemma 5.1.** *An $\mathcal{H}_m^n$-class $H$ of an ordered semihypergroup is $(m,0)$-regular $[(0,n)$-regular$]$ if it contains an $(m,0)$-regular $[(0,n)$-regular$]$ element.*

*Proof.* Let $a$ be an $(m,0)$-regular element and $c$ be an element of $\mathcal{H}_m^n$-class $H$. This implies $[b]_{m,0} = [a]_{m,0}$ and $a \in (a^m \circ H]$. Therefore, $\{a\}^i \subseteq (a^m \circ H]$ for each $i \in \{1,2,\ldots,m\}$. Then $\bigcup_{i=1}^m a^i \subseteq (a^m \circ H]$ implies $(\bigcup_{i=1}^m a^i] \subseteq ((a^m \circ H]] = (a^m \circ H]$. Thus, $[b]_{m,0} = [a]_{m,0} = (\bigcup_{i=1}^m a^i \cup a^m \circ H] = (\bigcup_{i=1}^m a^i] \cup (a^m \circ H] = (a^m \circ H]$. By Lemma 4.2, $(a^m \circ H] = (b^m \circ H]$. This implies that $[b]_{m,0} \subseteq (b^m \circ H]$. Hence, $b \in (b^m \circ H]$. So $b$ is an $(m,0)$-regular element of $\mathcal{H}_m^n$-class $H$. Hence, the $\mathcal{H}_m^n$-class $H$ is $(m,0)$-regular. The dual statement follows on the similar lines. □

**Lemma 5.2.** *An $\mathcal{H}_m^n$-class $H$ of an ordered semihypergroup is $(m,n)$-regular if it contains an $(m,n)$-regular element.*

*Proof.* The proof is similar to the proof of Lemma 5.1. □

**Lemma 5.3.** *A $\mathcal{B}_m^n$-class $B$ of an ordered semihypergroup is $(m,n)$-regular if it contains an $(m,n)$-regular element.*

*Proof.* Let $a \in B$ be an $(m,n)$-regular element and $b \in B$. Then, $a \in (a^m \circ H \circ a^n]$ so that $\{a\}^i \subseteq (a^m \circ H \circ a^n]$ for each $i \in \{1,2,\ldots,m+n\}$, so $\bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^n]$ implies $(\bigcup_{i=1}^m a^i] \subseteq ((a^m \circ H \circ a^n]] = (a^m \circ H \circ a^n]$. Since $a,b \in B$, $[b]_{m,n} = [a]_{m,n} = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n] = (\bigcup_{i=1}^{m+n} a^i] \cup (a^m \circ H \circ a^n] = (a^m \circ H \circ a^n]$. By Lemmas 4.3 and 4.2, we have $(a^m \circ H \circ a^n] = (b^m \circ H \circ b^n]$. This implies that $[b]_{m,n} \subseteq (b^m \circ H \circ b^n]$. So $b \in (b^m \circ H \circ b^n]$. Thus, $b$ is an $(m,n)$-regular element of $B$. Hence, $B$ is $(m,n)$-regular. □

**Definition 5.4.** Let $H$ be an ordered semihypergroup and $m,n$ be positive integers. An element $a$ of $H$ is said to be an $(m,n)$-*right weakly regular* element if $a \in (a^m \circ H \circ a^n \circ H]$. The ordered semihypergroup $H$ is said to be $(m,n)$-*right weakly regular* if each element of $H$ is $(m,n)$-right weakly regular, equivalently, for each subset $A$ of $H$, $A \subseteq (A^m \circ H \circ A^n \circ H]$.

**Lemma 5.5.** *A $\mathcal{B}_m^n$-class $B$ of an ordered semihypergroup $H$ is $(m,n)$-right weakly regular if it contains an $(m,n)$-right weakly regular element.*

*Proof.* Let $a \in B$ be an $(m,n)$-right weakly regular element and $b \in B$. Then, $a \in (a^m \circ H \circ a^n \circ H]$. This implies that $\{a\}^i \subseteq (a^m \circ H \circ a^n \circ H]$

for each $i \in \{1, 2, \ldots, m+n\}$, so $\bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^n \circ H]$ implies $(\bigcup_{i=1}^{m} a^i] \subseteq ((a^m \circ H \circ a^n \circ H]] = (a^m \circ H \circ a^n \circ H]$. So, $(a^m \circ H \circ a^n] \subseteq (((a^m \circ H \circ a^n \circ H]) \circ H \circ ((a^m \circ H \circ a^n \circ H])] \subseteq (a^m \circ H \circ a^n \circ H]$. Since $a, b \in B$, $[b]_{m,n} = [a]_{m,n} = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n] = (\bigcup_{i=1}^{m+n} a^i] \cup (a^m \circ H \circ a^n] \subseteq (\bigcup_{i=1}^{m+n} a^i] \cup (a^m \circ H \circ a^n \circ H] = (a^m \circ H \circ a^n \circ H]$ (since $(\bigcup_{i=1}^{m+n} a^i] \subseteq (a^m \circ H \circ a^n \circ H]$). By Lemmas 4.3 and 4.2, $(a^m \circ H \circ a^n] = (b^m \circ H \circ b^n]$. This implies that $[b]_{m,n} \subseteq (a^m \circ H \circ a^n \circ H] = ((a^m \circ H \circ a^n] \circ H] = ((b^m \circ H \circ b^n] \circ H] = (b^m \circ H \circ b^n \circ H]$. So $b \in (b^m \circ H \circ b^n \circ H]$. Thus, $b$ is an $(m, n)$-right weakly regular element of $B$. Hence, $B$ is $(m, n)$-right weakly regular. □

**Corollary 5.6.** *An ordered semihypergroup $H$ is $(m, n)$-regular ($(m, n)$-right weakly regular) if and only if each $\mathcal{B}_m^n$-class of $H$ contains an $(m, n)$-regular ($(m, n)$-right weakly regular) element.*

**Lemma 5.7.** *An $\mathcal{H}_m^n$-class $H$ of an ordered semihypergroup is $(m, n)$-right weakly regular if it contains an $(m, n)$-right weakly regular element.*

*Proof.* Let $a$ be an $(m, n)$-right weakly regular element and $b$ be an element of $\mathcal{H}_m^n$-class $H$. Then, $a \in (a^m \circ H \circ a^n \circ H]$. This gives that $\{a\}^i \subseteq (a^m \circ H \circ a^n \circ H]$ for each $i \in \{1, 2, \ldots, m+n\}$, and so $\bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^n \circ H]$ implies $(\bigcup_{i=1}^{m+n} a^i] \subseteq ((a^m \circ H \circ a^n \circ H]] = (a^m \circ H \circ a^n \circ H]$. Therefore, $(a^m \circ H] \subseteq ((a^m \circ H \circ a^n \circ H] \circ H] = (a^m \circ H \circ a^n \circ H \circ H] \subseteq (a^m \circ H \circ a^n \circ H]$. Since $a, b \in H$, $[b]_{m,0} = [a]_{m,0} = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H] = (\bigcup_{i=1}^{m+n} a^i] \cup (a^m \circ H] = (a^m \circ H] \subseteq (a^m \circ H \circ a^n \circ H]$. So, by Lemma 4.2, $(a^m \circ H \circ a^n] = (b^m \circ H \circ b^n]$. This implies that $[b]_{m,0} \subseteq (a^m \circ H \circ a^n \circ H] = ((a^m \circ H \circ a^n] \circ H] = ((b^m \circ H \circ b^n] \circ H] = (b^m \circ H \circ b^n \circ H]$. Therefore, $b \in (b^m \circ H \circ b^n \circ H]$ and thus, $b$ is an $(m, n)$-right weakly regular element of $\mathcal{H}_m^n$-class $H$. Hence, $H$ is $(m, n)$-right weakly regular. □

**Corollary 5.8.** *An ordered semihypergroup $H$ is (respectively, $(m, 0)$-regular, $(0, n)$-regular, $(m, n)$-regular) $(m, n)$-right weakly regular if and only if each $\mathcal{H}_m^n$-class of $H$ contains a (respectively, $(m, 0)$-regular, $(0, n)$-regular, $(m, n)$-regular) $(m, n)$-right weakly regular element.*

## 6 Conclusion

The main purpose of the present paper is to introduce the equivalence relations ${}_m\mathcal{I}, \mathcal{I}_n, \mathcal{B}_m^n$ and $\mathcal{H}_m^n$ on an ordered semihypergroup and enhance the understanding of different classes of ordered semihypergroups ($(m,n)$-regular, $(m,0)$-regular, $(0,n)$-regular, $(m,n)$-right weakly regular) by considering the structural influence of the equivalence relations ${}_m\mathcal{I}, \mathcal{I}_n, \mathcal{B}_m^n$, and $\mathcal{H}_m^n$. In particular, if we take $m = 1 = n$, the equivalence relations ${}_m\mathcal{I}, \mathcal{I}_n$ and $\mathcal{H}_m^n$ are reduced to the equivalence relations $\mathcal{R}, \mathcal{L}$ and $\mathcal{H}$ in ordered semihypergroup, respectively, which mimic the definition of the usual Green's relations $\mathcal{R}, \mathcal{L}$ and $\mathcal{H}$ in plain semihypergroups [11]. Also when we take $m = 1 = n$ in Theorems 1.9, 1.11, 4.1, 3.6, and 4.2, and Lemmas 4.1, 4.2, 4.3, 4.3, 5.1, and 5.2, then we obtain all the results for bi-hyperideals in an ordered semihypergroup and some characterizations of regular ordered semihypergroups, which is the main application of the results presented in this paper.

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