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Separated finitely supported Cb-sets

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Abstract. The monoid Cb of name substitutions and the notion of finitely supported Cb-sets introduced by Pitts as a generalization of nominal sets. A simple finitely supported Cb-set is a one point extension of a cyclic nominal set. The support map of a simple finitely supported Cb-set is an injective map. Also, for every two distinct elements of a simple finitely supported Cb-set, there exists an element of the monoid Cb which separates them by making just one of them into an element with the empty support.

In this paper, we generalize these properties of simple finitely supported Cb-sets by modifying slightly the notion of the support map; defining the notion of 2-equivariant support map; and introducing the notions of s-separated and z-separated finitely supported Cb-sets. We show that the notions of s-separated and z-separated coincide for a finitely supported Cb-set whose support map is 2-equivariant. Among other results, we find a characterization of simple s-separated (or z-separated) finitely supported Cb-sets. Finally, we show that some subcategories of finitely supported Cb-sets with injective equivariant maps which constructed applying the defined notions are reflective.

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1 Introduction

Let $\mathbb{D}$ be a countable infinite set. A permutation π on $\mathbb{D}$ is said to be *finitary* if it changes only a finite number of elements of $\mathbb{D}$. Consider the group $G = \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$ of finitary permutations on $\mathbb{D}$, and take a set X with an action of G on it, that is, a G-set. An element $x \in X$ is said to have a *finite support* $C \subseteq \mathbb{D}$ if it is invariant (fixed) under the action of each element π of G which fixes all the elements of C (that is, if $\pi c = c$, for all $c \in C$, then $\pi x = x$).

A G-set X all of whose elements have a finite support is said to be a *nominal set*. The notion of a nominal set was introduced by Fraenkel in 1922, and developed by Mostowski in the 1930s under the name of legal sets. The legal sets were used to prove the independence of the axiom of choice from the other axioms (in Zermelo-Fraenkel set theory: ZFA).

In 2001, Gabbay and Pitts rediscovered those sets in the context of name abstraction. They called them nominal sets, and applied this notion to properly model the syntax of formal systems involving variable binding operations (see [5]).

In [9], Pitts generalized the notion of nominal sets, by first adding two elements 0, 1 to $\mathbb{D}$, then generalizing the notion of a finitary permutation to *finite substitution* and considering the monoid Cb instead of the group G. Then he defined the notion of a support for Cb-sets, sets with an action of Cb on them, and invented the notion of *finitely supported* Cb-sets as a generalization of nominal sets.

In [8], Pitts defined the support map from a nominal set X to the set of all finite subsets of $\mathbb{D}$ which takes each element of X to its least support. In [3], we showed that the support map of a simple finitely supported Cb-set is an injective map.

In this paper, we slightly modify the definition of the support map for a finitely supported Cb-set, and then consider finitely supported Cb-sets whose support maps are injective. We call them s-separated finitely supported Cb-sets. On the other hand, for every two distinct elements of a simple finitely supported Cb-set, there exists an element of the monoid Cb which separates them by making just one of them into a zero element. Generalizing this property, led us to introduce the notion of z-separated finitely supported Cb-sets. The notions of s-separated and z-separated for simple finitely supported Cb-sets are the same. This fact, then, motivates us to define the notion of 2-equivariant support maps under which z-separated finitely supported Cb-set are exactly s-separated finitely supported Cb-sets. Among other things, we find some adjoint relations between the categories of defined notions and the category (a subcategory) of finitely supported Cb-sets with injective equivariant maps between them.

2 Preliminaries

This section is devoted to giving some basic notions needed in this paper. For more information one can see [3, 4, 7, 9].

2.1 ***M*-sets** A (left) *M-set* for a monoid M with identity e is a set X equipped with a map $M \times X \to X, (m, x) \mapsto mx$, called an *action* of M on X, such that $ex = x$ and $m(m'x) = (mm')x$, for all $x \in X$ and $m, m' \in M$. An *equivariant map* from an M-set X to an M-set Y is a map $f : X \to Y$ with $f(mx) = mf(x)$, for all $x \in X, m \in M$.

An element x of an M-set X is called a *zero* (or a *fixed*) element if $mx = x$, for all $m \in M$. We denote the set of all zero elements of an M-set X by $Z(X)$.

The M-set X all of whose elements are zero is called a *discrete* M-set, or an M-set with *the identity action*.

A subset Y of an M-set X is a *sub M-set* (or *M-subset*) of Y if for all $m \in M$ and $y \in Y$ we have $my \in Y$. The subset $Z(X)$ of X is in fact a sub M-set.

An equivalence relation ρ on an M-set X is called a *congruence* on X if $x\rho x'$ implies $mx \,\rho\, mx'$, for $x, x' \in X$, $m \in M$. We denote the set of all congruences on X by $\mathrm{Con}(X)$.

For a sub M-set Y of an M-set X, the *Rees congruence* ρ_Y on X is defined by

$$x\rho_Y x' \quad \text{if and only if} \quad x, x' \in Y \quad \text{or} \quad x = x'.$$

The Rees factor of X by the sub M-set Y is denoted by X/Y.

Finally, an M-set X is called *simple* if $\mathrm{Con}(X) = \{\Delta_X, \nabla_X\}$, where $\nabla_X = X \times X$, and $\Delta_X = \{(x, x) \mid x \in X\}$ is the equality relation.

2.2 ***Cb*-sets** Let $\mathbb{D}$ be an infinite countable set, whose elements are sometimes called *atomic names* (*data values*) and $\mathrm{Perm}\mathbb{D}$ be the group of all permutations (bijection maps) on $\mathbb{D}$. A permutation $\pi \in \mathrm{Perm}\mathbb{D}$ is said to be *finite* if $\{d \in \mathbb{D} \mid \pi(d) \neq d\}$ is finite. Clearly the set $\mathrm{Perm}_{\mathrm{f}}\mathbb{D}$ of all finitary permutations is a subgroup of $\mathrm{Perm}\mathbb{D}$.

Also, we take $2 = \{0, 1\}$ with $0, 1 \notin \mathbb{D}$.

Definition 2.1. (a) A finite *substitution* is a function $\sigma : \mathbb{D} \to \mathbb{D} \cup 2$ for which $\mathrm{Dom}_{\mathrm{f}}\sigma = \{d \in \mathbb{D} \mid \sigma(d) \neq d\}$ is finite.

(b) A finite substitution satisfies *injectivity condition*, if

$$(\forall d, d' \in \mathbb{D}),\ \sigma(d) = \sigma(d') \notin 2 \Rightarrow d = d'.$$

(c) If $d \in \mathbb{D}$ and $b \in 2$, we write (b/d) for the finite substitution which maps d to b, and is the identity mapping on all the other elements of $\mathbb{D}$. Each (b/d) is called a *basic substitution.*

(d) If $d, d' \in \mathbb{D}$ then we write $(d\,d')$ for the finite substitution that transposes d and d', and keeps fixed all other elements. Each $(d\,d')$ is called a *transposition substitution.*

Definition 2.2. (a) Let Cb be the monoid whose elements are finite substitutions satisfying injectivity condition, with the monoid operation given by $\sigma \cdot \sigma' = \hat{\sigma}\sigma'$, where $\hat{\sigma} : \mathbb{D} \cup 2 \to \mathbb{D} \cup 2$ maps 0 to 0, 1 to 1, and on $\mathbb{D}$ is defined the same as σ. The identity element of Cb is the inclusion $\iota : \mathbb{D} \hookrightarrow \mathbb{D} \cup 2$.

(b) Take S to be the subsemigroup of Cb generated by basic substitutions. The members of S are of the form $\delta = (b_1/d_1) \cdots (b_k/d_k) \in S$ for some $d_i \in \mathbb{D}$ and $b_i \in 2$, and we denote the set $\{d_1, \cdots, d_k\}$ by $\mathbb{D}_\delta$.

Remark 2.3. (1) Notice that each finite permutation π on $\mathbb{D}$ can be considered as a finite substitution $\iota \circ \pi : \mathbb{D} \to \mathbb{D} \cup 2$. Doing so, throughout this paper, we consider the group $\mathrm{Perm}_{\mathrm{f}}\mathbb{D}$ as a submonoid of Cb, and denote $\iota \circ \pi$ with the same notation π.

(2) Let $d_1, \cdots, d_k \in \mathbb{D}$ and $b_1, \cdots, b_k \in 2$. Then, for all $\pi \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$ and $(b_1/d_1) \cdots (b_k/d_k) \in S$, one can compute that in Cb,

$$\pi(b_1/d_1) \cdots (b_k/d_k) = (b_1/\pi d_1) \cdots (b_k/\pi d_k)\pi,$$

and

$$(b_1/d_1) \cdots (b_k/d_k)\pi = \pi(b_1/\pi^{-1}d_1) \cdots (b_k/\pi^{-1}d_k).$$

(3) Let $d \neq d' \in \mathbb{D}$ and $b, b' \in 2$. Then

$$(b/d)(b'/d') = (b'/d')(b/d).$$

But, we see that $(1/d)(0/d) = (0/d)$ and $(0/d)(1/d) = (1/d)$, and hence $(1/d)(0/d) \neq (0/d)(1/d)$.

Theorem 2.4. [3] *For the monoid Cb, we have*

$$Cb = \mathrm{Perm_f}(\mathbb{D}) \cup \mathrm{Perm_f}(\mathbb{D})S.$$

2.3 Finitely supported Cb-sets In this subsection, basic notions about finitely supported Cb-sets, which are needed in the sequel, are given, some of which are in [9].

The following definition introduces the notion of a, so called, *support*, which is the central notion to define finitely supported Cb-sets.

Definition 2.5. (a) Suppose X is a Cb-set. A subset $C \subseteq \mathbb{D}$ *supports* an element x of X if, for every $\sigma, \sigma' \in Cb$,

$$(\sigma(c) = \sigma'(c), (\forall c \in C)) \Rightarrow \sigma x = \sigma' x.$$

If there is a finite (possibly empty) support C then we say that x is *finitely supported.*

(b) A Cb-set X all of whose elements have finite supports, is called a *finitely supported Cb-set.*

We denote the category of all Cb-sets with equivariant maps between them by Cb-**Set**, and its full subcategory of all finitely supported Cb-sets by $(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$.

Lemma 2.6. ([9], Lemma 2.4) *Suppose X is a Cb-set, $x \in X$ and $b \in 2$. Also, let C be a finite subset of $\mathbb{D}$. Then, C is a support of x if and only if*

$$(\forall d \in \mathbb{D}) \quad d \notin C \Rightarrow (b/d)x = x.$$

Remark 2.7. Let X be a Cb-set and $x \in X$.

(1) If X is finitely supported, then the set $\{d \in \mathbb{D} \mid (0/d)x \neq x\}$ is in fact the least finite support of x. From now on, we call the least finite support for x *the support* for x, and denote it by $\mathrm{supp}\, x$.

(2) x is a zero element if and only if $\mathrm{supp}\, x = \emptyset$ if and only if $\delta x = x$, for all $\delta \in S$.

(3) Every non-empty finitely supported Cb-set has a zero element.

Example 2.8. (1) The set $\mathbb{D} \cup 2$ is a finitely supported Cb-set, with the *canonical action* given by evaluation; that is,

$$\forall \sigma \in Cb, \ x \in \mathbb{D} \cup 2, \ \sigma x = \hat{\sigma}(x),$$

in which $\hat{\sigma}$ is defined as in Definition 2.2(a). Also, for each $d \in \mathbb{D}$, $\operatorname{supp} d = \{d\}$, and $\operatorname{supp} 0 = \operatorname{supp} 1 = \emptyset$.

(2) Let $X = \mathbb{D}^{(k)} \cup \{0\}$, where k is a natural number, $\mathbb{D}^{(k)} = \{(d_1, \cdots, d_k) : d_i \in \mathbb{D}, d_i \neq d_j, \text{ for } i \neq j\}$, and 0 be a zero element which is not included in $\mathbb{D}^{(k)}$. Then, we see that X is a finitely supported Cb-set with the following action of Cb. Let $\sigma \in Cb$ and $x \in \mathbb{D}^{(k)}$. Then, applying Theorem 2.4, $\sigma = \pi$ or $\sigma = \pi\delta$, where $\pi \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta \in S$. For $\sigma = \pi$ or $\sigma = \pi\delta$ with $\mathbb{D}_\delta \cap \operatorname{supp} x = \emptyset$, define $\sigma x = \pi x$ and for $\sigma = \pi\delta$ with $\mathbb{D}_\delta \cap \operatorname{supp} x \neq \emptyset$, and $\sigma x = 0$. Notice that, for each element $(d_1, \cdots, d_k)$, the set $\{d_1, \cdots, d_k\}$ is the support.

(3) The set $\mathcal{P}_\mathrm{f}(\mathbb{D} \cup 2) = \{Y | \ Y \text{ is a finite subset of } \mathbb{D} \cup 2\}$ is a finitely supported Cb-set with the natural Cb-action

$$* : Cb \times \mathcal{P}_\mathrm{f}(\mathbb{D} \cup 2) \to \mathcal{P}_\mathrm{f}(\mathbb{D} \cup 2), \ \sigma * Y = \sigma Y = \{\sigma y | \ y \in Y\}.$$

Notice that $\operatorname{supp} Y = Y$.

(4) All discrete Cb-sets are clearly finitely supported Cb-sets, because of Remark 2.7(2).

Lemma 2.9. [3] *Let X be a non-empty finitely supported Cb-set and $x \in X$. Then,*

(i) *for $\delta \in S$, we have*

$$\delta x = x \text{ if and only if } \mathbb{D}_\delta \cap \operatorname{supp} x = \emptyset.$$

(ii) *for $\delta \in S$, $\operatorname{supp} \delta x \subseteq \operatorname{supp} x \setminus \mathbb{D}_\delta$.*

(iii) *for $\pi \in \mathrm{Perm_f}(\mathbb{D})$, we have $\operatorname{supp} \pi x = \pi \operatorname{supp} x$. In particular, $|\operatorname{supp} \pi x| = |\pi \operatorname{supp} x| = |\operatorname{supp} x|$.*

Remark 2.10. [8] Suppose $f : X \to Y$ is an equivariant map between finitely supported Cb-sets X and Y.

(1) If $x \in X$, then $\operatorname{supp} f(x) \subseteq \operatorname{supp} x$.

(2) If $x \in X$ and f is injective, then $\operatorname{supp} f(x) = \operatorname{supp} x$.

The following proposition is needed in the next section.

Proposition 2.11. (i) *Let A be a subset of $\mathbb{D} \cup 2$, and $\delta \in S$. Then,* $(\delta A) \setminus 2 = (A \setminus 2) \setminus \mathbb{D}_\delta$.

(ii) *Suppose X is a finitely supported Cb-set, and $x \in X$. If $\sigma \in Cb$, then* $\operatorname{supp} \sigma x \subseteq (\sigma \operatorname{supp} x) \setminus 2$.

Proof. (i) Notice that $\delta A = \{\delta(d) : d \in A\}$. Let $d \in (A \setminus 2) \setminus \mathbb{D}_\delta$. Then, $d \in A \setminus 2$ and $d \notin \mathbb{D}_\delta$. By Lemma 2.9(i), $\delta(d) = d$, and so we get that $d = \delta(d) \in (\delta A) \setminus 2$. To prove the reverse inclusion, suppose $d \in (\delta A) \setminus 2$. So, there exists $d_1 \in A$ with $d = \delta(d_1)$ and $d \notin 2$. If $d_1 \in \mathbb{D}_\delta$, then $d = \delta(d_1) \in 2$, which is impossible. Thus $d_1 \notin \mathbb{D}_\delta$ and so, by Lemma 2.9(i), $\delta(d_1) = d_1$. Now, we have $d = \delta(d_1) = d_1 \in A$ and so $d \in (A \setminus 2) \setminus \mathbb{D}_\delta$. Therefore, $(\delta A) \setminus 2 = (A \setminus 2) \setminus \mathbb{D}_\delta$.

(ii) Let $\sigma \in Cb$. Then, by Theorem 2.4, $\sigma \in \mathrm{Perm_f}(\mathbb{D})$ or $\sigma = \pi\delta$, where $\pi \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta \in S$. If $\sigma \in \mathrm{Perm_f}(\mathbb{D})$, then, by Lemma 2.9(iii), $\operatorname{supp} \sigma x = \sigma \operatorname{supp} x$. Notice that, $0, 1 \notin \operatorname{supp} x$. Let $\sigma = \pi\delta$. Then, applying Lemma 2.9(ii,iii), we have

$$\begin{aligned}\operatorname{supp} \sigma x &= \operatorname{supp} \pi\delta x = \pi \operatorname{supp} \delta x \subseteq \pi((\operatorname{supp} x) \setminus \mathbb{D}_\delta)\\ &= \pi((\delta \operatorname{supp} x) \setminus 2) = (\pi\delta \operatorname{supp} x) \setminus 2 = (\sigma \operatorname{supp} x) \setminus 2,\end{aligned}$$

where the third equality follows from (i), by replacing $A = \operatorname{supp} x$. □

Remark 2.12. [3] The sets

$$S_x \doteq \{\delta \in S \mid \delta x = x\} \quad \text{and} \quad S'_x \doteq S \setminus S_x = \{\delta \in S \mid \delta x \neq x\},$$

are two subsemigroups of S.

Theorem 2.13. ([3], Theorem 6.3) *Suppose X is an infinite finitely supported Cb-set with a unique zero element θ. If X is simple, then there exists a non-zero element x with $X = \mathrm{Perm_f}(\mathbb{D})x \cup \{\theta\}$. The converse is also true if the support of each non-zero element of X is a singleton.*

3 Separated finitely supported Cb-sets

In [8], Pitts showed that the support map of a nominal set is equivariant, where the support map is the map $\operatorname{supp} : X \to \mathcal{P}_{\mathrm{f}}(\mathbb{D})$, which takes $x \in X$

to supp x. In this section, in Definition 3.1, we modify slightly the definition of the support map for a finitely supported Cb-set, and define the notion of 2-equivariant support map. In Theorem 3.6, we characterize simple finitely supported Cb-sets in the category of finitely supported Cb-sets with 2-equivariant support maps. Also, in Subsection 3.1 we define the notions of s-separated and z-separated finitely supported Cb-sets. In Theorem 3.13, we show that a z-separated finitely supported Cb-set is exactly an s-separated finitely supported Cb-set if its support map is 2-equivariant. For this reason we first give the necessary facts about 2-equivariant support maps.

Definition 3.1. Let X be a finitely supported Cb-set and $x \in X$. Then,

(a) the map

$$\text{supp} : X \to \mathcal{P}_{\text{f}}(\mathbb{D} \cup 2), x \mapsto \text{supp}\, x$$

is called *the support map of* X.

(b) The support map is called 2-*equivariant* if $\text{supp}\,\sigma x = (\sigma\text{supp}\,x) \setminus 2$, for all $\sigma \in Cb$.

We denote the category of all finitely supported Cb-sets with 2-equivariant support maps by $(Cb\text{-}\mathbf{Set})^2_{\text{fs}}$.

Remark 3.2. Suppose X is a finitely supported Cb-set. Let $x \in X$ and $\sigma \in Cb$. Then we have the facts

(1) if $\sigma \in \text{Perm}_{\text{f}}(\mathbb{D})$, then $(\sigma\text{supp}\,x) \setminus 2 = \sigma\text{supp}\,x$. This is because $\sigma\text{supp}\,x \subseteq \mathbb{D}$, for all $\sigma \in \text{Perm}_{\text{f}}(\mathbb{D})$. So, by Lemma 2.9(iii), we get that $(\sigma\text{supp}\,x) \setminus 2 = \text{supp}\,\sigma x$.

(2) if $x \in Z(X)$, then $(\sigma\text{supp}\,x) \setminus 2 = \emptyset = \text{supp}\,\sigma x$.

Example 3.3. (1) The support maps of $\mathbb{D} \cup 2$ and $\mathbb{D} \cup \{0\}$ are 2-equivariant.

This is because taking $d \in \mathbb{D}$ and $\sigma = \pi\delta$, for $\pi \in \text{Perm}_{\text{f}}(\mathbb{D})$ and $\delta \in S$, we have if $d \in \mathbb{D}_\delta$, then $\sigma d = \pi\delta d \in 2$ and $(\sigma\text{supp}\,d) = \pi\delta\{d\} \subseteq 2$, which gives $\text{supp}\,\sigma d = \emptyset = (\sigma\text{supp}\,d) \setminus 2$. Also, if $d \notin \mathbb{D}_\delta$, then $\sigma d = \pi d$ and the result holds by Remark 3.2.

(2) The support map of $X = \mathbb{D}^{(k)} \cup \{0\}$ is 2-equivariant if and only if $k = 1$.

Recall Example 2.8(2) where X is a finitely supported Cb-set. By part (1), if $k = 1$ then the support map of X is 2-equivariant. Conversely, we show that if $k > 1$ then the support map of X is not 2-equivariant. Let $k > 1$

and $x = (d_1, \cdots, d_k)$. Then $\operatorname{supp} x = \{d_1, \cdots, d_k\}$, and taking $\delta = (0/d_1)$ we get $\delta x = 0$, and so $\operatorname{supp} \delta x = \emptyset$, while

$$(\delta \operatorname{supp} x) \setminus 2 = \{0, d_2, \cdots, d_k\} \setminus 2 = \{d_2, \cdots, d_k\}.$$

Notice that, by part (2) of the above example, the support map of $X = \mathbb{D}^{(k)} \cup \{0\}$, for $k > 1$, is not 2-equivariant.

Theorem 3.4. *Let X be a finitely supported Cb-set and x a non-zero element of X. Then, the support map of X is 2-equivariant if and only if*

$$\forall \delta \in S'_x, (\operatorname{supp} x) \setminus \mathbb{D}_\delta \subseteq (\operatorname{supp} \delta x) \qquad (*)$$

where S'_x is defined as in 2.12.

Proof. Let X have a 2-equivariant support map and $\delta \in S'_x$. Then, by Definition 3.1(b) and Proposition 2.11(i), we get that

$$\operatorname{supp} \delta x = (\delta \operatorname{supp} x) \setminus 2 = (\operatorname{supp} x) \setminus \mathbb{D}_\delta.$$

Conversely suppose $(*)$ holds and $\sigma \in Cb$. Applying Definition 3.1(b), we must show that $\operatorname{supp} \sigma x = (\sigma \operatorname{supp} x) \setminus 2$. By Theorem 2.4, we have the cases

Case (1): If $\sigma \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$, then, by Lemma 2.9(iii), we get $\operatorname{supp} \sigma x = \sigma(\operatorname{supp} x)$.

Case (2): Suppose $\sigma = \pi\delta$ where $\pi \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$ and $\delta \in S$. Now, if $\delta \in S_x$, then, by Lemma 2.9(i), $\mathbb{D}_\delta \cap \operatorname{supp} x = \emptyset$. So, $\delta(\operatorname{supp} x) = \operatorname{supp} x$ and $\delta x = x$. Therefore,

$$\operatorname{supp} \sigma x = \operatorname{supp} \pi x = \pi(\operatorname{supp} x) = \pi\delta(\operatorname{supp} x).$$

If $\delta \in S'_x$, then, applying the assumption and Lemma 2.9(ii), we have $\operatorname{supp} \delta x = (\operatorname{supp} x) \setminus \mathbb{D}_\delta$. Thus by Proposition 2.11 and Lemma 2.9(iii),

$$\begin{aligned}&\operatorname{supp} \sigma x = \operatorname{supp} \pi\delta x = \pi(\operatorname{supp} \delta x)\\ &= \pi((\operatorname{supp} x) \setminus \mathbb{D}_\delta) = \pi((\delta \operatorname{supp} x) \setminus 2) = (\sigma \operatorname{supp} x) \setminus 2.\end{aligned}$$

□

Corollary 3.5. *Let X be a finitely supported Cb-set, and $x \in X$. Then, the support map of X is 2-equivariant if and only if $(\operatorname{supp} x) \setminus \mathbb{D}_\delta = (\operatorname{supp} \delta x)$, for all $\delta \in S'_x$.*

Proof. This follows from Theorem 3.4 and Lemma 2.9(ii). □

Theorem 3.6. *Let X be an infinite finitely supported Cb-set with 2-equivariant support map, and a unique zero element θ. Then, the following statements are equivalent:*

(i) *X is simple ;*
(ii) *$X = \mathrm{Perm_f}(\mathbb{D})x \cup \{\theta\}$ with $|\mathrm{supp}\, x| = 1$;*
(iii) *X is isomorphic to $\mathbb{D} \cup \{0\}$.*

Proof. (i)⇒(ii) Let X be simple and $x \in X \setminus \{\theta\}$. Then, applying Theorem 2.13, we have $X = \mathrm{Perm_f}(\mathbb{D})x \cup \{\theta\}$, and so, by Theorem 5.6 and Corollary 5.5 of [3], $\mathrm{supp}\, \delta x = \emptyset$, for all $\delta \in S'_x$. We show that $\mathrm{supp}\, x$ is singleton. On the contrary, let $|\mathrm{supp}\, x| > 1$ and $d \in \mathrm{supp}\, x$. Then, by Lemma 2.9(i), $(0/d) \in S'_x$. Now, since the support map of X is 2-equivariant, we get that $\mathrm{supp}\,(0/d)x = (\mathrm{supp}\, x) \setminus \{d\} \neq \emptyset$, which is a contradiction.

(ii)⇒(i) Follows by Theorem 2.13.

(ii)⇔(iii) This holds by Corollary 5.7 of [3]. □

3.1 Stabilizer separated and zero-separated finitely supported Cb-sets

In Theorem 6.4(iv) of [3], we showed that the support map of a simple finitely supported Cb-set is an injective (one-one) map. In other words, every two distinct elements have different least supports. Also, for every two distinct elements of a simple finitely supported Cb-set, there exists an element of the monoid Cb which makes just one of them into a zero element. In this subsection, we generalize these properties and define the notions of s-separated and z-separated finitely supported Cb-sets (see Definition 3.7). In Theorem 3.10, we show that s-separated finitely supported Cb-sets are exactly ones with injective support maps on non-zero elements. A characterization of simple finitely supported Cb-sets is given in Theorem 3.14. In Theorem 3.13, it is shown that the notions of z-separated and s-separated finitely supported Cb-sets are the same in the category of finitely supported Cb-sets with 2-equivariant support maps.

Definition 3.7. Let X be a finitely supported Cb-set. Then,

(a) X is called *stabilizer-separated* or briefly s-*separated* if for every two non-zero elements $x \neq x' \in X$, we have $S_{x'} \setminus S_x \neq \emptyset$, or $S_x \setminus S_{x'} \neq \emptyset$. In other words, for every $x \neq x' \in X \setminus Z(X)$, there exists some $\delta \in S$ with $(\delta\, x \neq x$ and $\delta\, x' = x')$ or $(\delta\, x = x$ and $\delta\, x' \neq x')$.

(b) X is *zero-separated* or briefly z-*separated*, if for all non-zero elements $x \neq x' \in X$, there exists $\delta \in S$ with ($\delta x \in Z(X)$ and $\delta x' \notin Z(X)$) or ($\delta x \notin Z(X)$ and $\delta x' \in Z(X)$).

Example 3.8. The set $\mathbb{D} \cup 2$ is a z-separated (s-separated) finitely supported Cb-set. To see this, for all $d \neq d'$, it is sufficient to take $\delta = (0/d)$.

Lemma 3.9. *Any finitely supported Cb-set X whose support map is injective over non-zero elements, is* s-*separated.*

Proof. Let $x \neq x' \in X \setminus Z(X)$. Then, since the support map of X is injective, we get $\operatorname{supp} x \neq \operatorname{supp} x'$, and so there exists some $d \in \mathbb{D}$, with $d \in \operatorname{supp} x \setminus \operatorname{supp} x'$ or $d \in \operatorname{supp} x' \setminus \operatorname{supp} x$. Assuming $d \in \operatorname{supp} x \setminus \operatorname{supp} x'$, we show that $S_{x'} \setminus S_x \neq \emptyset$. For the case $d \in \operatorname{supp} x' \setminus \operatorname{supp} x$, it is similarly proved that $S_x \setminus S_{x'} \neq \emptyset$. Let $\delta = (0/d)$. Then, by Lemma 2.9(i), $\delta\, x = (0/d)\, x \neq x$ and $\delta\, x' = (0/d)\, x' = x'$, which means $(0/d) \in S_{x'} \setminus S_x$. □

In the following theorem, we show that the converse of the above lemma is also true, that is, distinct non-zero elements of an s-separated finitely supported Cb-set have different least supports.

Theorem 3.10. *Let X be a finitely supported Cb-set. Then, X is* s-*separated if and only if the support map of X over non-zero elements is an injective map.*

Proof. Let X be s-separated and $x \neq x'$ be two non-zero elements of X. Then, we must show $\operatorname{supp} x \neq \operatorname{supp} x'$. Since X is s-separated, applying Definition 3.7(a) we get $S'_x \cap S_{x'} \neq \emptyset$ or $S'_{x'} \cap S_x \neq \emptyset$. Assuming $\delta \in S'_x \cap S_{x'}$, we show that $\operatorname{supp} x \neq \operatorname{supp} x'$. The other case is proved similarly. Since $\delta \in S'_x \cap S_{x'}$, by Lemma 2.9(i), we get $\mathbb{D}_\delta \cap \operatorname{supp} x \neq \emptyset$ and $\mathbb{D}_\delta \cap \operatorname{supp} x' = \emptyset$. Therefore, $\operatorname{supp} x \neq \operatorname{supp} x'$.

Conversely, let the support map of X over non-zero elements be injective. Then, by Lemma 3.9, we get the result. □

Theorem 3.11. *Let X be an* s-*separated finitely supported Cb-set and x, x' be two distinct non-zero elements of X. Then, $|\operatorname{supp} x| = |\operatorname{supp} x'|$ if and only if $\mathrm{Perm}_\mathrm{f}(\mathbb{D})x = \mathrm{Perm}_\mathrm{f}(\mathbb{D})x'$ if and only if $Cb\, x = Cb\, x'$.*

Proof. We prove the non-trivial parts. Let $|\operatorname{supp} x| = |\operatorname{supp} x'|$. Then, we show that $\mathrm{Perm}_{\mathrm{f}}(\mathbb{D})x = \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})x'$. Since X is s-separated, by Theorem 3.10, $\operatorname{supp} x \neq \operatorname{supp} x'$. We have the following two cases:

Case (1): Suppose $\operatorname{supp} x \cap \operatorname{supp} x' \neq \emptyset$. Let

$$\operatorname{supp} x = \{d_1, \cdots, d_k, d''_1, \cdots, d''_l\}$$

and

$$\operatorname{supp} x' = \{d'_1, \cdots, d'_k, d''_1, \cdots, d''_l\}.$$

Take $\pi = (d_1\ d'_1) \cdots (d_k\ d'_k) \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$. Then, by Lemma 2.9(iii),

$$\begin{aligned} \operatorname{supp} \pi x &= \pi \operatorname{supp} x \\ &= \pi\{d_1, \cdots, d_k, d''_1, \cdots, d''_l\} \\ &= \{d'_1, \cdots, d'_k, d''_1, \cdots, d''_l\} \\ &= \operatorname{supp} x'. \end{aligned}$$

Case (2): Suppose $\operatorname{supp} x \cap \operatorname{supp} x' = \emptyset$. Take $\operatorname{supp} x = \{d_1, \cdots, d_k\}$ and $\operatorname{supp} x' = \{d'_1, \cdots, d'_k\}$. In this case, similar to Case (1), applying Lemma 2.9(iii), we have

$$\operatorname{supp} \pi x = \pi \operatorname{supp} x = \pi\{d_1, \cdots, d_k\} = \{d'_1, \cdots, d'_k\} = \operatorname{supp} x',$$

where $\pi = (d_1\ d'_1) \cdots (d_k\ d'_k) \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$.

Therefore, in each case we get $\operatorname{supp} \pi x = \operatorname{supp} x'$. Now, by Theorem 3.10, $\pi x = x'$, and so $\mathrm{Perm}_{\mathrm{f}}(\mathbb{D})x = \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})x'$.

Assuming $Cb\, x = Cb\, x'$, we show that $\mathrm{Perm}_{\mathrm{f}}(\mathbb{D})x = \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})x'$. Since $Cb\, x = Cb\, x'$, there exist $\sigma, \sigma' \in Cb$ with $x = \sigma x'$ and $x' = \sigma' x$. By Theorem 2.4, $\sigma, \sigma' \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D}) \cup \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})S$. Now, we have the following cases:

(a) $x = \pi x'$ and $x' = \pi' x$, for some $\pi, \pi' \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$;

(b) $x = \pi x'$ and $x' = \pi' \delta' x$, for some $\pi', \pi \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$ and $\delta' \in S'_x$;

(c) $x = \pi \delta x'$ and $x' = \pi' x$, for some $\pi', \pi \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$ and $\delta \in S'_{x'}$;

(d) $x = \pi \delta x'$ and $x' = \pi' \delta' x$, for some $\pi, \pi' \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$, $\delta \in S'_{x'}$ and $\delta' \in S'_x$.

We show that only (a) occures. In case (b), by Lemma 2.9, we have

$$|\operatorname{supp} x| = |\operatorname{supp} \pi x'| = |\operatorname{supp} x'| = |\operatorname{supp} \pi' \delta' x| = |\operatorname{supp} \delta' x| < |\operatorname{supp} x|,$$

which is impossible. Similarly, case (c) does not occure. Also, in case (d), we have $x = \pi\delta x' = \pi\delta\pi'\delta' x$. Now, by Remark 2.3(2), $\delta\pi' = \pi'\delta''$, and so, by Lemma 2.9,

$$|\mathrm{supp}\, x| = |\mathrm{supp}\, \pi\delta\pi'\delta' x| = |\mathrm{supp}\, \pi\pi'\delta''\delta'\, x| = |\mathrm{supp}\, \delta''\delta'\, x| < |\mathrm{supp}\, x|,$$

which is again impossible. □

In the following proposition, we show that in a cyclic finitely supported Cb-set with an additional property, the notions of z-separated and s-separated are the same. We first notice that a cyclic finitely supported Cb-set is of the form Cbx, also, recall from [3], that for every cyclic finitely supported Cb-set Cbx, we have

$$Cbx = \mathrm{Perm_f}(\mathbb{D})x \cup \mathrm{Perm_f}(\mathbb{D})S'_x x.$$

Proposition 3.12. *Let $X = Cbx$ be a cyclic finitely supported Cb-set with* $\mathrm{supp}\, \delta x = \emptyset$*, for all $\delta \in S'_x$. Then, X is* s-*separated if and only if X is* z-*separated.*

Proof. Suppose X is s-separated and $z \neq z'$ are two non-zero elements of X. Thus $\mathrm{supp}\, z \neq \mathrm{supp}\, z'$ and so there exists some $d \in (\mathrm{supp}\, z) \setminus \mathrm{supp}\, z'$ or $d \in (\mathrm{supp}\, z') \setminus \mathrm{supp}\, z$. Assuming $d \in \mathrm{supp}\, z \setminus \mathrm{supp}\, z'$, we prove that X is z-separated. The other case is proved similarly. Since $d \in (\mathrm{supp}\, z)\setminus \mathrm{supp}\, z'$, by Remark 2.5, we get $(0/d)z \neq z$ and $(0/d)z' = z'$. Applying the assumption, we have $\mathrm{supp}\,(0/d)z = \emptyset$ and $(0/d)z' = z'$. So $(0/d)z \in Z(X)$ and $(0/d)z' \notin Z(X)$.

Conversely, suppose X is z-separated and $z \neq z'$. Thus, there exists some $\delta \in S$ with ($\delta z \in Z(X)$ and $\delta z' \notin Z(X)$) or ($\delta z' \in Z(X)$ and $\delta z \notin Z(X)$). Assuming $\delta z \in Z(X)$ and $\delta z' \notin Z(X)$, we show that X is z-separated. The other case is proved similarly. Notice that, since $\delta z' \notin Z(X)$, we get $\mathrm{supp}\, \delta z' \neq \emptyset$, and so, by the assumption, $\delta \notin S'_{z'}$. Also, since $\delta z \in Z(X)$ and z is a non-zero element, we get $\delta z \neq z$. Thus, $\delta \in S'_z$ and so $\delta \in S'_z \setminus S'_{z'}$, which means X is s-separated. □

Now, in the following theorem, we show that the notions of z-separated and s-separated finitely supported Cb-sets in the category of finitely supported Cb-sets with 2-equivariant support maps are the same.

Theorem 3.13. *Let X be a finitely supported Cb-set with 2-equivariant support map. Then, X is* s*-separated if and only if X is* z*-separated.*

Proof. Let X be z-separated and $x \neq x'$ two non-zero elements of X. Then, there exists $\delta \in S$ with ($\delta x \in Z(X)$ and $\delta x' \notin Z(X)$) or ($\delta x \notin Z(X)$ and $\delta x' \in Z(X)$). We show that $\operatorname{supp} x \neq \operatorname{supp} x'$ and so, by Theorem 3.10, X is s-separated. Assuming $\delta x \in Z(X)$ and $\delta x' \notin Z(X)$, we prove the result. The other case is proved similarly. Since $\delta x \in Z(X)$ and the support map of X is 2-equivariant, we have $\emptyset = \operatorname{supp} \delta x = (\operatorname{supp} x) \setminus \mathbb{D}_\delta$. So $\operatorname{supp} x \subseteq \mathbb{D}_\delta$. On the other hand, notice that $\delta x' \notin Z(X)$. Thus, by the assumption, we get that $\operatorname{supp} x' \setminus \mathbb{D}_\delta = \operatorname{supp} \delta x' \neq \emptyset$, and so there exists $d \in \operatorname{supp} x' \setminus \mathbb{D}_\delta$, which means that $d \in \operatorname{supp} x'$ and $d \notin \mathbb{D}_\delta$. Since $\operatorname{supp} x \subseteq \mathbb{D}_\delta$, $d \notin \operatorname{supp} x$. Therefore, $\operatorname{supp} x \neq \operatorname{supp} x'$.

To prove the converse, suppose X is s-separated and $x \neq x'$ are two non-zero elements of X. Thus $\operatorname{supp} x \neq \operatorname{supp} x'$ and so there exists some $d \in \operatorname{supp} x \setminus \operatorname{supp} x'$ or $d \in \operatorname{supp} x' \setminus \operatorname{supp} x$. Assuming $d \in \operatorname{supp} x \setminus \operatorname{supp} x'$, we prove that X is z-separated. The other case is proved similarly. To show this claim, take $\delta \in S$ with $\mathbb{D}_\delta = \operatorname{supp} x'$. Then, $d \in (\operatorname{supp} x) \setminus \mathbb{D}_\delta$. So $\operatorname{supp} \delta x' = \operatorname{supp} x' \setminus \mathbb{D}_\delta = \emptyset$ and $\operatorname{supp} \delta x = \operatorname{supp} x \setminus \mathbb{D}_\delta \neq \emptyset$. Therefore, $\delta x' \in Z(X)$ and $\delta x \notin Z(X)$. □

Theorem 3.14. *Let X be an* s*-separated* (z*-separated*) *finitely supported Cb-set with a unique zero element θ. Then, X is simple if and only if $X = \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})x \cup \{\theta\}$, where $x \in X \setminus Z(X)$.*

Proof. First, notice that, applying Theorem 5.6 and Corollary 5.5 of [3], $X = \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})x \cup \{\theta\}$ is cyclic and $\operatorname{supp} \delta x = \emptyset$, for all $\delta \in S'_x$. Therefore, for $X = \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})x \cup \{\theta\}$, by Lemma 3.12, the notions z-separated and s-separated are the same.

Now, let X be a finitely supported Cb-set with a unique zero element θ, and x a non-zero element of X. Then, by Theorem 6.7 of [3], X is simple if and only if $X = \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})x \cup \{\theta\}$ and the support map of X is injective. So, by Theorem 3.10, X is simple if and only if $X = \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})x \cup \{\theta\}$. □

4 Some reflective subcategories of the category of finitely supported Cb-sets with injective equivariant maps

Let us denote the category of all finitely supported Cb-sets with injective equivariant maps between them by **Inj**-$(Cb\text{-}\mathbf{Set})_{\text{fs}}$, and its full subcategory of all finitely supported Cb-sets with unique zero elements by **Inj**-$(Cb\text{-}\mathbf{Set})^{\theta}_{\text{fs}}$.

In this section, we show that **Inj**-$(Cb\text{-}\mathbf{Set})^{\theta}_{\text{fs}}$ is a reflective subcategory of **Inj**-$(Cb\text{-}\mathbf{Set})_{\text{fs}}$. We also construct some reflective subcategories of **Inj**-$(Cb\text{-}\mathbf{Set})^{\theta}_{\text{fs}}$ using z-separated and s-separated finitely supported Cb-sets introduced in the last section.

4.1 Inj-$(Cb\text{-}\mathbf{Set})^{\theta}_{\text{fs}}$ is a reflective subcategory of Inj-$(Cb\text{-}\mathbf{Set})_{\text{fs}}$.

To define the reflector functor, given a finitely supported Cb-set X, we construct the Rees factor of X by its sub Cb-set $Z(X)$.

Remark 4.1. Let X be a finitely supported Cb-set. Consider the Rees factor on X by the sub Cb-set $Z(X)$, of all zero elements of X. Then

$$X/Z(X) = \{Z(X), \{x\} : x \in X - Z(X)\}$$

and for $x \in X \setminus Z(X)$, we have $\operatorname{supp} x \neq \emptyset$.

Lemma 4.2. *For a a finitely supported Cb-set, $X/Z(X)$ is a finitely supported Cb-set with a unique zero element.*

Proof. Define the action on $X/Z(X)$ by

$$\sigma *_X a = \begin{cases} a, & \text{if } a = Z(X) \\ Z(X), & \text{if } a = \{x\}, x \in X \setminus Z(X),\ \operatorname{supp} \sigma x = \emptyset \\ \{\sigma x\}, & \text{if } a = \{x\}, x \in X \setminus Z(X),\ \operatorname{supp} \sigma x \neq \emptyset \end{cases}$$

for $\sigma \in Cb$ and $a \in X/Z(X)$. It is really an action, because $\iota *_X a = a$, for all $a \in X/Z(X)$. Also, if $\sigma_1, \sigma_2 \in Cb$, then $(\sigma_1\sigma_2) *_X a = \sigma_1 *_X (\sigma_2 *_X a)$. This is because, if $a = Z(X)$ or $a = \{x\}$, $x \in X - Z(X)$ with $\sigma_1, \sigma_2 \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$, then the result holds. If $a = \{x\}$, $x \in X - Z(X)$ with $\sigma_1 \notin \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$ or $\sigma_2 \notin \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$, then we have the following cases:

Case (1): Let $\sigma_1 \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$. If $\sigma_2 = \pi_2\delta_2$ with $\operatorname{supp} \sigma_2 x = \emptyset$, then $\sigma_1 *_X (\sigma_2 *_X a) = Z(X)$. On the other hand,

$$\operatorname{supp} \sigma_1\sigma_2 x = \operatorname{supp} \sigma_1\sigma_2 x = \sigma_1 \operatorname{supp} \sigma_2 x = \emptyset,$$

and so $(\sigma_1\sigma_2) *_X a = Z(X)$.

Case (2): Let $\sigma_1 = \pi_1\delta_1$ and $\sigma_2 = \pi_2\delta_2$. Then,

$$(\sigma_1\sigma_2) *_X a = (\pi_1\delta_1\pi_2\delta_2) *_X a, \quad \text{and} \quad \sigma_1 *_X (\sigma_2 *_X a) = \pi_1\delta_1 *_X (\pi_2\delta_2 *_X a).$$

Applying Remark 2.3(2), $\delta_1\sigma_2 x = \delta_1\pi_2\delta_2 x = \pi_2\delta_1'\delta_2 x$. Now, we have the following subcases:

Subcase (a): Suppose $\operatorname{supp}\delta_2 x = \emptyset$. Applying Lemma 2.9(ii), we have $\operatorname{supp}\delta_1\sigma_2 x \subseteq \pi_2[(\operatorname{supp}\delta_2 x) \setminus \mathbb{D}_{\delta_1'}]$. Thus $\operatorname{supp}\delta_1\sigma_2 x = \emptyset$, and so

$$(\sigma_1\sigma_2) *_X a = Z(X) = \sigma_1 *_X (\sigma_2 *_X a).$$

Subcase (b): Let $\operatorname{supp}\delta_2 x \neq \emptyset$. Then, $\pi_1\delta_1 *_X (\pi_2\delta_2 *_X a) = \pi_1\delta_1 *_X (\{\pi_2\delta_2 x\})$, and $(\pi_1\delta_1\pi_2\delta_2) *_X \{x\} = (\pi_1\pi_2\delta_1'\delta_2) *_X \{x\}$.
Notice that, $|\operatorname{supp}\delta_1'\delta_2 x| = |\operatorname{supp}\delta_1\pi_2\delta_2 x|$. Thus $\operatorname{supp}\delta_1'\delta_2 x = \emptyset$ if and only if $\operatorname{supp}\delta_1\pi_2\delta_2 x = \emptyset$. Therefore, $\sigma_1 *_X (\sigma_2 *_X a) = (\sigma_1\sigma_2) *_X a$.

Finally, we show that all elements of $X/Z(X)$ have a finite support, and so $X/Z(X)$ is a finitely supported Cb-set. Let $a \in X/Z(X)$. Then, $a = Z(X)$ or $a = \{x\}$ with $x \in X \setminus Z(X)$. If $a = Z(X)$, then it is clear that a is a zero element, and so, by Remark 2.7(2), $\operatorname{supp} a = \emptyset$. If $a = \{x\}$ with $x \in X \setminus Z(X)$, then, by Lemma 2.6 we show that $\operatorname{supp} x$ is a finite support of a. To prove this, let $d \notin \operatorname{supp} x$. Then, $(0/d)x = x$ and so, applying the definition of the action $*_X$ on $X/Z(X)$, we get that

$$(0/d)a = (0/d)\{x\} = \{(0/d)x\} = \{x\} = a.$$

$\square$

Theorem 4.3. *The inclusion functor* $\mathbf{Inj}\text{-}(Cb\text{-}\mathbf{Set})^{\theta}_{\mathrm{fs}} \hookrightarrow \mathbf{Inj}\text{-}(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$ *has a left adjoint* $L : \mathbf{Inj}\text{-}(Cb\text{-}\mathbf{Set})_{\mathrm{fs}} \to \mathbf{Inj}\text{-}(Cb\text{-}\mathbf{Set})^{\theta}_{\mathrm{fs}}$.

Proof. Take X to be a finitely supported Cb-set. Define $L(X) = X/Z(X)$. By Lemma 4.2, $L(X)$ is a finitely supported Cb-set with a unique zero element $Z(X)$. Suppose $g : X \to Y$ is an injective equivariant map between finitely supported Cb-sets X, Y. We show that $L(g) : X/Z(X) \to Y/Z(Y)$ defined by

$$L(g)(a) = \begin{cases} \{g(x)\} & \text{if } a = \{x\}, x \in X - Z(X) \\ Z(Y) & \text{if } a = Z(X) \end{cases}$$

is an injective equivariant map. Notice that, if $a = \{x\}$ with $x \in X - Z(X)$, then, applying Remark 2.10(2), we get that $\operatorname{supp} g(x) = \operatorname{supp} x \neq \emptyset$, and so $L(g)(a) = \{g(x)\} \in Y \setminus Z(Y)$. Also, since g is injective, $L(g)$ is injective. To prove that $L(g)$ is equivariant, let $\sigma \in Cb$. Then, by Theorem 2.4, we have $\sigma \in \mathrm{Perm_f}(\mathbb{D})$ or $\sigma = \pi\delta$ with $\pi \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta \in S$.

If $a = Z(X)$, then

$$\begin{aligned} L(g)(\sigma *_X a) &= L(g)(Z(X)) = Z(Y) = \sigma *_Y Z(Y) \\ &= \sigma *_Y L(g)(Z(X)) = \sigma *_Y L(g)(a). \end{aligned}$$

If $a = \{x\}$ with $x \in X - Z(X)$, then we have the following cases;

Case (1): If $\sigma \in \mathrm{Perm_f}(\mathbb{D})$ or $\sigma = \pi\delta$ with $\operatorname{supp} \delta x \neq \emptyset$, then we have $\sigma x \in X \setminus Z(X)$, and so

$$L(g)(\sigma *_X a) = L(g)(\{\sigma x\}) = \{g(\sigma x)\} = \{\sigma g(x)\} = \sigma *_Y L(g)(a).$$

Case (2): Suppose $\sigma = \pi\delta$ with $\operatorname{supp} \delta x = \emptyset$. Since g is equivariant, by Remark 2.10, we get that $\operatorname{supp} \delta g(x) = \operatorname{supp} g(\delta x) \subseteq \operatorname{supp} \delta x = \emptyset$. Thus

$$\begin{aligned} L(g)(\sigma *_X a) &= L(g)(Z(X)) = Z(Y) \\ &= \sigma *_Y \{g(x)\} = \sigma *_Y L(g)(a). \end{aligned}$$

Checking the properties of $L(id_X) = id_{L(X)}$ and $L(g_2 g_1) = L(g_2)L(g_1)$, where $g_1 : X \to Y$ and $g_2 : Y \to Z$ are two injective equivariant maps between finitely supported Cb-sets, is clear.

Now, we define the reflection arrow $r_X : X \to L(X)$ by

$$r_X(x) = \begin{cases} \{x\}, & \text{if } \operatorname{supp} x \neq \emptyset \\ Z(X), & \text{if } \operatorname{supp} x = \emptyset. \end{cases}$$

Then r_X is equivariant. To see this, taking $\sigma \in Cb$, we consider the following cases:

Case (1): If $\sigma \in \mathrm{Perm_f}(\mathbb{D})$, then since $\operatorname{supp} \sigma x = \sigma \operatorname{supp} x$, for the case $\operatorname{supp} x = \emptyset$, we get $\sigma *_X r_X(x) = \sigma *_X Z(X) = Z(X) = r_X(\sigma x)$, and for the case $\operatorname{supp} x \neq \emptyset$, we get $\sigma *_X r_X(x) = \sigma *_X \{x\} = \{\sigma x\} = r_X(\sigma x)$.

Case (2): Suppose $\sigma = \pi\delta$ and $\operatorname{supp} x = \emptyset$. By Proposition 2.11(ii), we have $\operatorname{supp} \sigma x \subseteq \sigma(\operatorname{supp} x) \setminus 2 = \emptyset$. So

$$\sigma *_X r_X(x) = \sigma *_X Z(X) = Z(X) = r_X(\sigma x).$$

Case (3): Suppose $\sigma = \pi\delta$ with $\operatorname{supp} x \neq \emptyset$. If $\operatorname{supp} \sigma x = \emptyset$, then

$$\sigma *_X r_X(x) = \sigma *_X \{x\} = Z(X) = r_X(\sigma x).$$

If $\operatorname{supp} \sigma x \neq \emptyset$, then $\sigma *_X r_X(x) = \sigma *_X \{x\} = \{\sigma x\} = r_X(\sigma x)$.

Finally, we prove the universal property of r_X. Let Y be a finitely supported Cb-set with a unique zero element θ and $f : X \to Y$ be an injective equivariant map. Define the map $\bar{f} : X/Z(X) \to Y$ by

$$\bar{f}(a) = \begin{cases} f(x) & \text{if } a = \{x\}, x \in X - Z(X) \\ \theta & \text{if } a = Z(X). \end{cases}$$

We show that $\bar{f}$ is equivariant. If $a = Z(X)$, then $\sigma \bar{f}(a) = \theta = \bar{f}(\sigma *_X a)$. Now, suppose $a = \{x\}$ with $x \in X - Z(X)$. If $\sigma \in \mathrm{Perm_f}(\mathbb{D})$ or $\sigma = \pi\delta$ with $\operatorname{supp} \delta x \neq \emptyset$, then

$$\sigma \bar{f}(a) = \sigma f(x) = f(\sigma x) = \bar{f}(\sigma *_X a).$$

Suppose $\sigma = \pi\delta$ with $\operatorname{supp} \delta x = \emptyset$. Since f is equivariant, by Remark 2.10(1), we get that $\operatorname{supp} f(\delta x) \subseteq \operatorname{supp} \delta x$, and so $f(\delta x) = \theta$. On the other hand, since $\sigma *_X a = Z(X)$, we get that

$$\sigma \bar{f}(a) = \sigma f(x) = f(\sigma x) = \theta = \bar{f}(\sigma *_X a).$$

Also, $\bar{f} r_X(x) = f(x)$, for all $x \in X$. This is because, if $\operatorname{supp} x \neq \emptyset$, then $\bar{f} r_X(x) = \bar{f}(\{x\}) = f(x)$, and if $\operatorname{supp} x = \emptyset$, then $r_X(x) = Z(X)$ and so $\bar{f} r_X(x) = \bar{f}(Z(X)) = \theta = f(x)$. To show uniqueness, suppose $\bar{\bar{f}} r_X = f$. If $a = Z(X)$, then $\bar{\bar{f}}(Z(X)) = \theta = \bar{f}(Z(X))$. Let $a = \{x\}$ with $x \in X - Z(X)$. Then $\operatorname{supp} x \neq \emptyset$, and so

$$\bar{\bar{f}}(a) = \bar{\bar{f}}(r_X(x)) = f(x) = \bar{f}(a)$$

as required. □

4.2 Inj-$(Cb$-Set$)^2_{\mathrm{fs}}$ is a reflective subcategory of Inj-$(Cb$-Set$)_{\mathrm{fs}}$

Let us denote the category of finitely supported Cb-sets equipped with 2-equivariant support maps and injective equivariant maps between them by **Inj**-$(Cb$-**Set**$)^2_{\mathrm{fs}}$. We show that it is a reflective subcategory of **Inj**-$(Cb$-**Set**$)_{\mathrm{fs}}$.

Let X be a finitely supported Cb-set. Consider the Cb-set $Cb \times X$ with the action $\sigma_1(\sigma, x) = (\hat{\sigma_1}\sigma, x)$, for $x \in X$ and $\sigma \in Cb$. Here, we define a congruence relation on $Cb \times X$, which makes it into a finitely supported Cb-set.

Remark 4.4. Notice that, the Cb-set $Cb \times X$ is not finitely supported. To prove this, on the contrary, we assume that $Cb \times X$ is a finitely supported Cb-set. Then taking C to be a finite support for $(\iota, x) \in Cb \times X$, applying Lemma 2.6, we get $(0/d)(\iota, x) = (\iota, x)$, for all $d \notin C$, and hence we have $(\widehat{(0/d)}\iota, x) = (\iota, x)$. This implies that $\widehat{(0/d)}\iota = \iota$, which contradicts $\widehat{(0/d)}\iota(d) = 0 \neq d = \iota(d)$.

Lemma 4.5. *Let X be a finitely supported Cb-set. Then, the relation $\sim_2$ defined on $Cb \times X$ by*

$$(\sigma, x) \sim_2 (\sigma', x') \Leftrightarrow \quad \sigma x = \sigma' x' \text{ and } (\sigma \operatorname{supp} x) \setminus 2 = (\sigma' \operatorname{supp} x') \setminus 2,$$

is a congruence relation on the Cb-set $Cb \times X$.

Proof. First notice that $\sim_2$ is clearly an equivalence relation on $Cb \times X$. Let us denote the equivalence class of (σ, x) by x_σ. To show that it is a congruence relation, let $x_\sigma = x'_{\sigma'}$. Then, $(\sigma \operatorname{supp} x) \setminus 2 = (\sigma' \operatorname{supp} x') \setminus 2$ and $\sigma x = \sigma' x'$. So, for all $\sigma_1 \in Cb$, we have

$$(\hat{\sigma_1}\sigma \operatorname{supp} x) \setminus 2 = \sigma_1 [(\sigma \operatorname{supp} x) \setminus 2] = \sigma_1[(\sigma' \operatorname{supp} x') \setminus 2] = (\hat{\sigma_1}\sigma' \operatorname{supp} x') \setminus 2,$$

and $\hat{\sigma_1}\sigma x = \hat{\sigma_1}\sigma' x'$. Therefore, $x_{\hat{\sigma_1}\sigma} = x'_{\hat{\sigma_1}\sigma'}$. □

Lemma 4.6. *Let X be a finitely supported Cb-set, $x \in X$, and $\sigma \in Cb$. Then* $\operatorname{supp} x_\sigma = (\sigma \operatorname{supp} x) \setminus 2$.

Proof. First, we show that $(\sigma(\operatorname{supp} x)) \setminus 2$ is a finite support for x_σ. Let $d \notin (\sigma \operatorname{supp} x) \setminus 2$ and $b \in 2$. Then, applying Lemma 2.6, we show that $(b/d)x_\sigma = x_\sigma$. In other words, we prove $x_{\widehat{(b/d)}\sigma} = x_\sigma$. Notice that, by Proposition 2.11(ii), $\operatorname{supp} \sigma x \subseteq (\sigma \operatorname{supp} x) \setminus 2$. Now, since $d \notin (\sigma \operatorname{supp} x) \setminus 2$, we get $(b/d)((\sigma \operatorname{supp} x) \setminus 2) = (\sigma \operatorname{supp} x) \setminus 2$ and $d \notin \operatorname{supp} \sigma x$. Therefore, $(\widehat{(b/d)}\sigma \operatorname{supp} x) \setminus 2 = (\sigma \operatorname{supp} x) \setminus 2$ and $\widehat{(b/d)}\sigma x = \sigma x$. This implies that $\operatorname{supp} x_\sigma \subseteq (\sigma \operatorname{supp} x) \setminus 2$.

Now, to prove the reverse inclusion, we define a map $g : (Cb \times X)/\sim_2 \to \mathcal{P}_f(\mathbb{D} \cup 2)$ as $g(x_\sigma) = (\sigma \operatorname{supp} x) \setminus 2$. Notice that g is well-defined and equivariant. To see this, suppose $x_\sigma = x'_{\sigma'}$. Then, $(\sigma \operatorname{supp} x) \setminus 2 = (\sigma' \operatorname{supp} x') \setminus 2$. To prove that g is equivariant, let $\sigma_1 \in Cb$. Then

$$\begin{aligned} g(\sigma_1 x_\sigma) &= g(x_{\hat{\sigma_1}\sigma}) \\ &= (\hat{\sigma_1}\sigma \operatorname{supp} x) \setminus 2 \\ &= \sigma_1((\sigma \operatorname{supp} x) \setminus 2) \\ &= \sigma_1 g(x_\sigma). \end{aligned}$$

Finally, we see that

$$\begin{aligned}(\sigma\,\mathrm{supp}\,x)\setminus 2 &= \mathrm{supp}\,[(\sigma\,\mathrm{supp}\,x)\setminus 2]\\ &= \mathrm{supp}\,g(x_{\sigma})\\ &\subseteq \mathrm{supp}\,(x_{\sigma}),\end{aligned}$$

where the inclusion is because of the fact that g is equivariant. □

Theorem 4.7. *The inclusion functor* $\mathbf{Inj}$-$(Cb$-$\mathbf{Set})^2_{\mathrm{fs}} \hookrightarrow \mathbf{Inj}$-$(Cb$-$\mathbf{Set})_{\mathrm{fs}}$ *has a left adjoint* $K : \mathbf{Inj}$-$(Cb$-$\mathbf{Set})_{\mathrm{fs}} \to \mathbf{Inj}$-$(Cb$-$\mathbf{Set})^2_{\mathrm{fs}}$.

Proof. Let X be a finitely supported Cb-set and take $K(X) = (Cb\times X)/\sim_2$. Then, as a corollary of Lemma 4.6, we get that $(Cb \times X)/\sim_2$ is a finitely supported Cb-set. We show that the support map of $K(X)$ is 2-equivariant. Let $\delta \in S'_{x_\sigma}$. Then, by Lemma 4.6, we have

$$\begin{aligned}&\mathrm{supp}\,\delta x_{\sigma} = \mathrm{supp}\,x_{\hat{\delta}\sigma} = (\hat{\delta}\sigma\mathrm{supp}\,x)\setminus 2\\ &= [(\sigma\mathrm{supp}\,x)\setminus 2]\setminus \mathbb{D}_{\delta} = (\mathrm{supp}\,x_{\sigma})\setminus \mathbb{D}_{\delta},\end{aligned}$$

where the third equality is because of Proposition 2.11(i). Now, applying Theorem 3.4, we get that the support map of $K(X)$ is 2-equivariant. Therefore, $K(X)$ is in the category $\mathbf{Inj}$-$(Cb$-$\mathbf{Set})^2_{\mathrm{fs}}$.

Now, given an injective equivariant map $g : X \to Y$ between finitely supported Cb-sets, we define $K(g) : K(X) \to K(Y)$ by $K(g)(x_{\sigma}) = (g(x))_{\sigma}$. Notice that $K(g)$ is an injective equivariant map. It is injective, since so is g, and so, by Remark 2.10(2), $\mathrm{supp}\,x = \mathrm{supp}\,g(x)$, for all $x \in X$, and thus

$$\begin{aligned}&x_{\sigma} = x'_{\sigma'}\\ &\Leftrightarrow \sigma x = \sigma' x', \quad \text{and} \quad (\sigma\mathrm{supp}\,x)\setminus 2 = (\sigma'\mathrm{supp}\,x')\setminus 2\\ &\Leftrightarrow \sigma g(x) = g(\sigma x) = g(\sigma' x') = \sigma' g(x'),\ \text{and}\\ &(\sigma\mathrm{supp}\,g(x))\setminus 2 = (\sigma\mathrm{supp}\,x)\setminus 2 = (\sigma'\mathrm{supp}\,x')\setminus 2 = (\sigma'\mathrm{supp}\,g(x'))\setminus 2\\ &\Leftrightarrow K(g)(x_{\sigma}) = K(g)(x'_{\sigma'}).\end{aligned}$$

Also, $K(g)$ is equivariant, since for $\sigma_1 \in Cb$, we have

$$\sigma_1 K(g)(x_{\sigma}) = \sigma_1 (g(x))_{\sigma} = (g(x))_{\hat{\sigma_1}\sigma} = K(g)(x_{\hat{\sigma_1}\sigma}) = K(g)(\sigma_1 x_{\sigma}).$$

Finally, the proofs of $K(id_X) = id_{K(X)}$ and $K(g_2 g_1) = K(g_2)K(g_1)$ are straightforward.

Now, we define the reflection arrow $r_X : X \to K(X)$ by $r_X(x) = x_\iota$. It is equivariant, because

$$\sigma r_X(x) = \sigma x_\iota = x_{\hat{\sigma}\iota} = x_\sigma = r_X(\sigma x),$$

for $\sigma \in Cb$. To prove the universal property of r_X, suppose Y is a finitely supported Cb-set with 2-equivariant support map, and $f : X \to Y$ is an injective equivariant map. Define $\bar{f} : (Cb \times X)/\sim_2 \to Y$ by $\bar{f}(x_\sigma) = \sigma f(x)$, for all $x_\sigma \in (Cb \times X)/\sim_2$. To see that $\bar{f}$ is well-defined, let $x_\sigma = x'_{\sigma'}$. Then, $(\sigma\,(\operatorname{supp} x)) \setminus 2 = (\sigma'\,(\operatorname{supp} x')) \setminus 2$ and $\sigma x = \sigma' x'$. Now, since f is an equivariant map, we get

$$\sigma f(x) = f(\sigma x) = f(\sigma' x') = \sigma' f(x').$$

To show that $\bar{f}$ is equivariant, let $\sigma_1 \in Cb$. Then,

$$\sigma_1 \bar{f}(x_\sigma) = \sigma_1 \sigma f(x) = f(\sigma_1 \sigma x) = \bar{f}(x_{\sigma_1 \sigma}) = \bar{f}(\sigma_1 x_\sigma).$$

Notice that, since f is injective, $\bar{f}$ is also injective. Also, for all $x \in X$, we have $\bar{f} r_X(x) = \bar{f}(x_\iota) = \iota f(x) = f(x)$. Further, $\bar{f}$ is unique. This is because, if $\bar{\bar{f}} r_X = f$, then

$$\begin{aligned} \bar{\bar{f}}(x_\sigma) &= \bar{\bar{f}}(\sigma x_\iota) = \sigma \bar{\bar{f}}(x_\iota) = \sigma(\bar{\bar{f}} r_X(x)) \\ &= \sigma f(x) = \sigma \bar{f} r_X(x) = \bar{f}(\sigma x_\iota) = \bar{f}(x_\sigma). \end{aligned}$$

□

Denoting by $\mathbf{Inj}\text{-}(Cb\text{-}\mathbf{Set})^{2\theta}_{\text{fs}}$, the full subcategory of $\mathbf{Inj}\text{-}(Cb\text{-}\mathbf{Set})^{\theta}_{\text{fs}}$, consisted of all finitely supported Cb-sets with unique zero elements, as a corollary of Theorem 4.7, we conclude the following result.

Corollary 4.8. *The inclusion functor* $\mathbf{Inj}\text{-}(Cb\text{-}\mathbf{Set})^{2\theta}_{\text{fs}} \hookrightarrow \mathbf{Inj}\text{-}(Cb\text{-}\mathbf{Set})^{\theta}_{\text{fs}}$ *has a left adjoint.*

4.3 zsep-Inj-$(Cb$-Set$)^{\theta}_{\text{fs}}$ is reflective in Inj-$(Cb$-Set$)^{\theta}_{\text{fs}}$

Let us denote the full subcategory of $\mathbf{Inj}\text{-}(Cb\text{-}\mathbf{Set})^{\theta}_{\text{fs}}$ consisting of all z-separated finitely supported Cb-sets equipped with unique zero elements by **zsep-Inj**-$(Cb$-**Set**$)^{\theta}_{\text{fs}}$. We show that it is a reflective subcategory.

We first define a congruence relation on a finitely supported Cb-set which makes it into a z-separated finitely supported Cb-set.

Lemma 4.9. *Suppose X is a finitely supported Cb-set, and $x, x' \in X$. Define*

$$x \sim_z x' \Leftrightarrow (\forall \delta \in S)\ (\delta x \in Z(X) \Leftrightarrow \delta x' \in Z(X)).$$

Then $\sim_z$ is a z-congruence on X.

Proof. The relation $\sim_z$ is clearly an equivalence relation on X. Suppose $\sigma \in Cb$ and $x \sim_z x'$. We show that $\sigma x \sim_z \sigma x'$. Notice that, by Theorem 2.4, $\sigma \in \mathrm{Perm_f}\mathbb{D}$ or $\sigma \in \mathrm{Perm_f}\mathbb{D}S$.

Case (1): Let $\sigma = \pi \in \mathrm{Perm_f}\mathbb{D}$. Then, for all $\delta \in S$ we have

$$\begin{aligned} \delta\pi x \in Z(X) &\Leftrightarrow \pi\delta' x \in Z(X) \quad \text{(by Remark 2.3(2))} \\ &\Leftrightarrow \delta' x \in Z(X) \\ &\Leftrightarrow \delta' x' \in Z(X) \quad \text{(since, by the assumption, } x \sim_z x') \\ &\Leftrightarrow \pi\delta' x' \in Z(X) \\ &\Leftrightarrow \delta\pi x' \in Z(X) \quad \text{(by Remark 2.3(2)).} \end{aligned}$$

Case (2): Let $\sigma \in \mathrm{Perm_f}\mathbb{D}S$. Then $\sigma = \pi_1\delta_1$, where $\pi_1 \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta_1 \in S$. For all $\delta \in S$, we have

$$\begin{aligned} \delta\pi_1\delta_1 x \in Z(X) &\Leftrightarrow \pi_1\delta'\delta_1 x \in Z(X) \quad \text{(by Remark 2.3(2))} \\ &\Leftrightarrow \delta'\delta_1 x \in Z(X) \\ &\Leftrightarrow \delta'\delta_1 x' \in Z(X) \quad \text{(by the assumption, } x \sim_z x') \\ &\Leftrightarrow \pi_1\delta'\delta_1 x' \in Z(X) \\ &\Leftrightarrow \delta\pi_1\delta_1 x' \in Z(X) \quad \text{(by Remark 2.3(2)).} \end{aligned}$$

□

Remark 4.10. Let X be a finitely supported Cb-set, and $x \in X$. Then,

(1) The set $(\mathrm{supp}\, x)$ is a finite support for $[x]_{\sim_z}$. To prove this, let $d \notin \mathrm{supp}\, x$. Then, applying Remark 2.7(1), we get $(0/d)x = x$ and so $(0/d)[x]_{\sim_z} = [(0/d)x]_{\sim_z} = [x]_{\sim_z}$. Thus, by Lemma 2.6, we get the result.

(2) If $\theta_1 \neq \theta_2 \in Z(X)$, then $[\theta_1]_{\sim_z} = [\theta_2]_{\sim_z}$ and so $Z(X/\sim_z)$, the set of zero elements of $X/\sim_z$, is singleton.

(3) $[x]_{\sim_z} \in Z(X/\sim_z)$ if and only if $x \in Z(X)$. To show this, if $x \in Z(X)$, then by (1), $\mathrm{supp}\, x = \emptyset$ is a support of $[x]_{\sim_z}$ and so, by Remark 2.7(3), $[x]_{\sim_z} \in Z(X/\sim_z)$. Now let $[x]_{\sim_z} \in Z(X/\sim_z)$. Then, by (2), we have $[x]_{\sim_z} = [\theta]_{\sim_z}$, where $\theta \in Z(X)$. Now, we have $\delta x \in Z(X)$, for all $\delta \in S$. Take $\delta = (0/d)$ with $d \notin \mathrm{supp}\, x$. Then, $x = (0/d)x \in Z(X)$.

Lemma 4.11. *If X is a non-discrete finitely supported Cb-set, then $X/\sim_z$ is a* z*-separated finitely supported Cb-set with a unique zero element.*

Proof. Let $[x]_{\sim_z} \neq [x']_{\sim_z}$ be two non-zero elements in $X/\sim_z$. Then, $(x, x') \notin \sim_z$, and so there exists $\delta \in S$ with ($\delta x \in Z(X)$ and $\delta x' \notin Z(X)$) or ($\delta x \notin Z(X)$ and $\delta x' \in Z(X)$). Notice that, by Remark 4.10(3), we have $x, x' \notin Z(X)$. Assuming $\delta x \in Z(X)$ and $\delta x' \notin Z(X)$, we show that $\delta[x]_{\sim_z} \in Z(X/\sim_z)$ and $\delta[x']_{\sim_z} \notin Z(X/\sim_z)$. The other case is proved similarly. By Remark 4.10(3), we get $[\delta x]_{\sim_z} \in Z(X/\sim_z)$ and $[\delta x']_{\sim_z} \notin Z(X/\sim_z)$. Therefore, $\delta[x]_{\sim_z} = [\delta x]_{\sim_z} \in Z(X/\sim_z)$, and $\delta[x']_{\sim_z} = [\delta x']_{\sim_z} \notin Z(X/\sim_z)$. Also, by Remark 4.10(2), $X/\sim_z$ has a unique zero element $[\theta]_{\sim_z}$, where $\theta \in Z(X)$. □

Remark 4.12. Let X be a finitely supported Cb-set with a unique zero element. Then, X is z-separated if and only if for all distinct elements $x, x' \in X$, there exists $\delta \in S$ with ($\delta x \in Z(X)$ and $\delta x' \notin Z(X)$) or ($\delta' x \in Z(X)$ and $\delta x \notin Z(X)$) if and only if for all distinct elements $x, x' \in X$ we have $(x, x') \notin \sim_z$ if and only if $[x]_{\sim_z} = \{x\}$, for all $x \in X$ if and only if $\sim_z = \Delta_X$.

Theorem 4.13. *The category* **zsep-Inj**-$(Cb$-$\mathbf{Set})^{\theta}_{\text{fs}}$ *is a reflective subcategory of* **Inj**-$(Cb$-$\mathbf{Set})^{\theta}_{\text{fs}}$.

Proof. We show that F : **Inj**-$(Cb$-$\mathbf{Set})^{\theta}_{\text{fs}} \to$ **zsep-Inj**-$(Cb$-$\mathbf{Set})^{\theta}_{\text{fs}}$ is a left adjoint of the inclusion functor **zsep-Inj**-$(Cb$-$\mathbf{Set})^{\theta}_{\text{fs}} \hookrightarrow$ **Inj**-$(Cb$-$\mathbf{Set})^{\theta}_{\text{fs}}$. Let X be a finitely supported Cb-set. Define $F(X) = X/\sim_z$, where $\sim_z$ is the congruence relation given in Lemma 4.9. Notice that, by Lemma 4.11, $X/\sim_z$ is a z-separated finitely supported Cb-set with a unique zero element. Also, since the morphisms in **Inj**-$(Cb$-$\mathbf{Set})^{\theta}_{\text{fs}}$ are injective, F is a functor.

Now, we take the canonical epimorphism $r_X : X \to F(X)$, $r_X(x) = [x]_{\sim_z}$, to be the reflection arrow. To prove its universal property, let $Y \in$ **zsep-Inj**-$(Cb$-$\mathbf{Set})^{\theta}_{\text{fs}}$ and $f : X \to Y$ is an injective equivariant map. We define $\bar{f} : X/\sim_z \to Y$ by $\bar{f}([x]_{\sim_z}) = f(x)$, for $[x]_{\sim_z} \in X/\sim_z$. Then, $\bar{f}$ is an equivariant map and $\bar{f} r_x = f$. It is well-defined, since $[x]_{\sim_z} = [x']_{\sim_z}$ implies $\delta x \in Z(X)$ if and only if $\delta x' \in Z(X)$, for all $\delta \in S$, and then, since f is injective and equivariant, $\delta f(x) = f(\delta x) \in Z(Y)$ if and only if $\delta f(x') = f(\delta x') \in Z(Y)$. Therefore, $f(x) \sim'_z f(x')$, but Y is a z-separated finitely supported Cb-set with a unique zero element, and, by Remark 4.12, we have $\sim'_z = \Delta_Y$. This gives $f(x) = f(x')$. Also, since f is injective and equivariant, so is $\bar{f}$. Further, $\bar{f}r_X(x) = \bar{f}([x]_{\sim_z}) = f(x)$, for all $x \in X$. $\square$

4.4 ssep-Inj-(Cb-Set)$^{2\theta}_{\text{fs}}$ is reflective in Inj-(Cb-Set)$^{2\theta}_{\text{fs}}$ Let us denote the full subcategory of **Inj**-(Cb-**Set**)$^{2\theta}_{\text{fs}}$ consisting of all s-separated finitely supported Cb-sets equipped with unique zero elements by **ssep-Inj**-(Cb-**Set**)$^{2\theta}_{\text{fs}}$. We show that it is a reflective subcategory.

We first define a congruence relation on a finitely supported Cb-set with 2-equivariant support map which makes it into an s-separated finitely supported Cb-set.

Lemma 4.14. *Let X be a finitely supported Cb-set with the 2-equivariant support map. Then the relation $\approx$ on X defined by*

$$x \approx_s x' \text{ if and only if } \operatorname{supp} x = \operatorname{supp} x',$$

is a congruence on X. Furthermore, $\approx_s = \sim_z$.

Proof. The relation $\approx_s$ is clearly an equivalence relation. To prove that it is a congruence, let $x, x' \in X$ with $x \approx_s x'$ and $\sigma \in Cb$. Then, $\operatorname{supp} x = \operatorname{supp} x'$ and, by Theorem 2.4, $\sigma \in \mathrm{Perm_f}(\mathbb{D})$ or $\sigma = \pi\delta$, where $\pi \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta \in S$. Let $\sigma = \pi \in \mathrm{Perm_f}(\mathbb{D})$ or $\sigma = \pi\delta$ with $\mathbb{D}_\delta \cap \operatorname{supp} x = \emptyset$. Then, by Lemma 2.9, $\sigma x = \pi x$ and so

$$\operatorname{supp} \sigma x = \operatorname{supp} \pi x = \pi \operatorname{supp} x = \pi \operatorname{supp} x' = \operatorname{supp} \pi x' = \operatorname{supp} \sigma x'.$$

Now, if $\sigma = \pi\delta$ with $\mathbb{D}_\delta \cap \operatorname{supp} x \neq \emptyset$, then we show that $\operatorname{supp} \delta x = \operatorname{supp} \delta x'$. Applying Corollary 3.5,

$$\operatorname{supp} \delta x = \operatorname{supp} x \setminus \mathbb{D}_\delta = \operatorname{supp} x' \setminus \mathbb{D}_\delta = \operatorname{supp} \delta x'.$$

Therefore, for all $\sigma \in Cb$, we have $\sigma x \approx_s \sigma x'$.

Furthermore, $\approx_s = \sim_z$. For, if $(x, x') \in \approx_s$, then $\operatorname{supp} x = \operatorname{supp} x'$. Also, for $\delta \in S$ such that $\delta x \in Z(X)$, by Corollary 3.5, we get

$$\emptyset = \operatorname{supp} \delta x = \operatorname{supp} x \setminus \mathbb{D}_\delta = \operatorname{supp} x' \setminus \mathbb{D}_\delta = \operatorname{supp} \delta x',$$

and so $\delta x' \in Z(X)$. Similarly, if $\delta x' \in Z(X)$, then $\delta x \in Z(X)$.

Now, assuming $(x,x') \notin \approx_s$, we show that $(x,x') \notin \sim_z$ and so $\sim_z \subseteq \approx_s$. Since $\operatorname{supp} x \neq \operatorname{supp} x'$, there exists some $d \in \operatorname{supp} x \setminus \operatorname{supp} x'$ or some $d \in \operatorname{supp} x' \setminus \operatorname{supp} x$. Assuming $d \in \operatorname{supp} x \setminus \operatorname{supp} x'$, we prove the result. The other case is proved similarly. Take $\delta \in S$ such that $\mathbb{D}_\delta = \operatorname{supp} x'$. Thus we have $d \in (\operatorname{supp} x) \setminus \mathbb{D}_\delta$, and so $\operatorname{supp} \delta x' = \operatorname{supp} x' \setminus \mathbb{D}_\delta = \emptyset$ and $\operatorname{supp} \delta x = \operatorname{supp} x \setminus \mathbb{D}_\delta \neq \emptyset$. Therefore, $\delta x' \in Z(X)$ and $\delta x \notin Z(X)$, which means that $(x,x') \notin \sim_z$. □

Remark 4.15. Let X be a finitely supported Cb-set equipped with the 2-equivariant support map, and $x \in X$. Then,

(1) $\operatorname{supp} [x]_{\approx_s} = \operatorname{supp} x$. To show this equality, notice that, we have $\operatorname{supp} [x]_{\approx_s} \subseteq \operatorname{supp} x$. To prove the reverse inclusion, let $d \notin \operatorname{supp} [x]_{\approx_s}$. Then, $[x]_{\approx_s} = (0/d)[x]_{\approx_s} = [(0/d)x]_{\approx_s}$ and so $((0/d)x, x) \in \approx_s$. Thus $\operatorname{supp} (0/d)x = \operatorname{supp} x$. Now, applying Remark 2.7(1), $d \notin \operatorname{supp} x$.

(2) For $x \in Z(X)$, we have $[x]_\approx = Z(X)$. This is because

$$\begin{array}{rcl} [x]_\approx & = & \{x' \in X : x' \approx_s x\} \\ & = & \{x' \in X : \operatorname{supp} x' = \operatorname{supp} x\} \\ & = & \{x' \in X : \operatorname{supp} x' = \emptyset\} \\ & = & \{x' \in X : x' \in Z(X)\} \\ & = & Z(X). \end{array}$$

Corollary 4.16. *Let X be a finitely supported Cb-set with the 2-equivariant support map. Then,*

(i) *$X/\approx_s$ is an* s*-separated finitely supported Cb-set with a unqiue zero element.*

(ii) *$X/\approx_s$ is a* z*-separated finitely supported Cb-set with a unique zero element.*

Proof. (i) Let $[x]_{\approx_s} \neq [x']_{\approx_s}$ be non-zero elements of $(X/\approx_s)$. Then, $(x,x') \notin \approx_s$ and so $\operatorname{supp} x \neq \operatorname{supp} x'$. Applying Remark 4.15(1), we have $\operatorname{supp} [x]_{\approx_s} = \operatorname{supp} x$ and $\operatorname{supp} [x']_{\approx_s} = \operatorname{supp} x'$. So $\operatorname{supp} [x]_{\approx_s} \neq \operatorname{supp} [x']_{\approx_s}$. Also, by Remark 4.15(2), $X/\approx_s$ has a unique zero element.

(ii) This follows from (i) and Theorem 3.13. □

Theorem 4.17. *The full subcategory* **ssep-Inj**-$(Cb\text{-}\mathbf{Set})^{2\theta}_{\mathrm{fs}}$ *of the category* **Inj**-$(Cb\text{-}\mathbf{Set})^{2\theta}_{\mathrm{fs}}$ *is reflective.*

Proof. Define the functor $H : \mathbf{Inj}\text{-}(Cb\text{-}\mathbf{Set})^{2\theta}_{\rm fs} \to \mathbf{ssep\text{-}Inj}\text{-}(Cb\text{-}\mathbf{Set})^{2\theta}_{\rm fs}$, by $H(X) = X/\approx_s$, where X is a finitely supported Cb-set with the 2-equivariant support map and $\approx_s$ is the congruence relation given in Lemma 4.14. By Corollary 4.17, $X/\approx_s$ is an s-separated finitely supported Cb-set with a unique zero element. Let $\delta \in S'_{[x]_{\approx_s}}$. Then, since the support map of X is 2-equivariant, by Remark 4.15, we get that

$$\delta[x]_{\approx_s} = [\delta x]_{\approx_s} = \delta x = (\mathrm{supp}\, x) \setminus \mathbb{D}_\delta = (\mathrm{supp}\, [x]_{\approx_s}) \setminus \mathbb{D}_\delta,$$

which means that $X/\approx_s$ is an s-separated finitely supported Cb-set with 2-equivariant support map. Also, since morphisms in $\mathbf{Inj}\text{-}(Cb\text{-}\mathbf{Set})^{2\theta}_{\rm fs}$ are injective, H is a functor. Now, by Lemma 4.14, since $\approx_s = \sim_z$, the rest of the proof is similar to the proof for Theorem 4.13. □

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