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The function ring functors of pointfree topology revisited

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Dedicated to George Grätzer
in recognition of his many contributions to mathematics

Abstract. This paper establishes two new connections between the familiar function ring functor $\mathfrak{R}$ on the category **CRFrm** of completely regular frames and the category **CRσFrm** of completely regular σ-frames as well as their counterparts for the analogous functor $\mathfrak{Z}$ on the category **ODFrm** of 0-dimensional frames, given by the integer-valued functions, and for the related functors $\mathfrak{R}^*$ and $\mathfrak{Z}^*$ corresponding to the bounded functions. Further it is shown that some familiar facts concerning these functors are simple consequences of the present results.

For general background, see [2] and its references.

The function ring functor given by the real-valued continuous functions on frames is considered here as

$$\mathfrak{R} : \mathbf{CRFrm} \to \mathfrak{R}\mathbf{Frm}$$

with the categories of completely regular frames and of ℓ-rings isomorphic to

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some $\mathfrak{R}L$, $L \in \mathbf{CRFrm}$, (with *all* their ℓ-ring homomorphisms) as domain and codomain, respectively, and its left adjoint $\mathfrak{K} : \mathfrak{R}\mathbf{Frm} \to \mathbf{CRFrm}$ is then provided by the familiar correspondence $A \mapsto \mathfrak{K}A$ where the latter is the (indeed completely regular) frame of archimedean kernels of A. In addition, we then have the adjunction maps

$$\lambda_A : A \to \mathfrak{R}\mathfrak{K}A,\ a \mapsto \hat{a},\ \hat{a}(p,q) = \langle (a - \mathbf{p})^+ \wedge (\mathbf{q} - a)^+ \rangle$$

where $\langle \cdot \rangle$ indicates the archimedean kernel of A generated by $\cdot$ and $\mathbf{p}$ and $\mathbf{q}$ are the elements of A corresponding to $p, q \in \mathbf{Q}$, and

$$\mu_L : \mathfrak{K}\mathfrak{R}L \to L,\ J \mapsto \bigvee \{\mathrm{coz}(\gamma) \mid \gamma \in J\},$$

with the familiar adjunction identities

$$(\mathfrak{R}\mu_L)\lambda_{\mathfrak{R}L} = \mathrm{id}_{\mathfrak{R}L} \quad \text{and} \quad \mu_{\mathfrak{K}A}(\mathfrak{K}\lambda_A) = \mathrm{id}_{\mathfrak{K}A}$$

for all $L \in \mathbf{CRFrm}$ and $A \in \mathfrak{R}\mathbf{Frm}$. It should be noted that, in this setting, all λ_A are isomorphisms: for any $A \in \mathfrak{R}\mathbf{Frm}$, $A \cong \mathfrak{R}L$ for some $L \in \mathbf{CRFrm}$, and the $\lambda_{\mathfrak{R}L}$ are isomorphisms by the adjunction identities.

On the other hand, we consider the functor Coz: $\mathbf{CRFrm} \to \mathbf{CR}\sigma\mathbf{Frm}$ with its left adjoint $\mathfrak{H} : \mathbf{CR}\sigma\mathbf{Frm} \to \mathbf{CRFrm}$ where $\mathbf{CR}\sigma\mathbf{Frm}$ is the category of completely regular σ-frames, $\mathrm{Coz}\, L$ is the sub-σ-frame of L given by its cozero elements, and $\mathfrak{H}S$ is the frame of σ-ideals of $S \in \mathbf{CR}\sigma\mathbf{Frm}$, with the obvious effects on the maps involved. Here, the adjunction maps are

$$\pi_S : S \to \mathrm{Coz}\,\mathfrak{H}S,\ a \mapsto \downarrow a = \{s \in S \mid s \le a\},$$

and

$$\eta_L : \mathfrak{H}\mathrm{Coz}\, L \to L,\ J \mapsto \bigvee J \text{ (in } L),$$

such that

$$(\mathrm{Coz}\,\eta_L)\pi_{\mathrm{Coz}\, L} = \mathrm{id}_{\mathrm{Coz}\, L} \quad \text{and} \quad \eta_{\mathfrak{H}S}(\mathfrak{H}\pi_S) = \mathrm{id}_{\mathfrak{H}S}.$$

Concerning $\mathbf{CRFrm} \to \mathbf{CR}\sigma\mathbf{Frm}$, recall that a σ-frame S is completely regular if each $a \in S$ is a countable join of elements $s \prec\!\!\prec a$ where $\prec\!\!\prec$ is the usual strong inclusion, and $\mathrm{Coz}\, L$ is well known to be of that kind.

Next, entirely parallel to the above, we shall consider the functor $\mathfrak{Z}$: $\mathbf{ODFrm} \to \mathfrak{Z}\mathbf{Frm}$ where $\mathfrak{Z}L$ is the usual ℓ-ring of integer-valued continuous functions on L and $\mathfrak{Z}\mathbf{Frm}$ the present analogue of $\mathfrak{R}\mathbf{Frm}$. Further, $\mathfrak{Z}$ has a left adjoint, also provided by the archimedean kernels and denoted by $\mathfrak{K} : \mathfrak{Z}\mathbf{Frm} \to \mathbf{ODFrm}$, based on the fact that the principal archimedean kernels of the $\mathfrak{Z}L$ are complemented because the $\mathfrak{Z}L$ satisfy the $\mathbf{Z}$-identity $\gamma \wedge (\mathbf{1} - \gamma) \leq \mathbf{0}$.

The adjunction maps in the present situation are

$$\kappa_A : A \to \mathfrak{Z}\mathfrak{K}A,\ a \mapsto \tilde{a},\ \tilde{a}(m) = \langle(\mathbf{1} - |\mathbf{m} - a|)^+\rangle \text{ (where } \mathbf{m} = m\mathbf{1})$$

and

$$\nu_L : \mathfrak{K}\mathfrak{Z}L \to L,\ J \mapsto \bigvee\{\mathrm{coz}(\gamma) \mid \gamma \in J\},$$

with identities analogous to the case of $\mathfrak{R}$; also, all κ_A are isomorphism here by the nature of $\mathfrak{Z}\mathbf{Frm}$.

Now, the counterpart of Coz in the present situation is the functor

$$\mathbf{S} : \mathbf{ODFrm} \to \mathbf{OD}\sigma\mathbf{Frm},\ L \mapsto \mathbf{S}L,$$

where the latter is the sub-σ-frame of L generated by its complemented elements, with formally the same left adjoint as in the earlier situation,

$$\mathfrak{H} : \mathbf{OD}\sigma\mathbf{Frm} \to \mathbf{ODFrm},$$

and the adjunction maps

$$\iota_S : S \to \mathbf{S}\mathfrak{H}S,\ a \mapsto \downarrow a,$$

and

$$\theta_L : \mathfrak{H}\mathbf{S}L \to L,\ J \mapsto \bigvee J \text{ (in } L),$$

subject to the exact analogues of the adjunction identities in the case of Coz.

The following familiar facts will be used later on:

(I) μ_L and ν_L are isomorphisms if and only if L is Lindelöf.

(II) For any Lindelöf $L \in \mathbf{CRFrm}$, its cozero elements are exactly its Lindelöf elements; similarly, for any Lindelöf $L \in \mathbf{ODFrm}$ the $a \in \mathbf{S}L$ are exactly the Lindelöf elements of L.

(III) $\eta_L : \mathfrak{H}\,\mathrm{Coz}\,L \to L$ is the Lindelöf coreflection map in $\mathbf{CRFrm}$ and the same holds for $\theta_L : \mathfrak{H}\mathbf{S}L \to L$ in $\mathbf{ODFrm}$.

Proposition 1. $\operatorname{Coz}\mathfrak{K} : \mathfrak{R}\mathbf{Frm} \to \mathbf{CR}\sigma\mathbf{Frm}$ *is a category equivalence with inverse* $\mathfrak{R}\mathfrak{H}$.

Proof. Concerning $\operatorname{Coz}\mathfrak{K}\mathfrak{R}\mathfrak{H}$, there is the composite homomorphism

$$\operatorname{Coz}\mathfrak{K}\mathfrak{R}\mathfrak{H}S \xrightarrow{\operatorname{Coz}\mu_{\mathfrak{H}S}} \operatorname{Coz}\mathfrak{H}S \xrightarrow{\pi_S^{-1}} S$$

for each $S \in \mathbf{CR}\sigma\mathbf{Frm}$, where $\operatorname{Coz}\mu_{\mathfrak{H}S}$ is an isomorphism because this holds already for $\mu_{\mathfrak{H}S}$ by (I) since $\mathfrak{H}S$ is Lindelöf, and

$$\pi_S : S \to \operatorname{Coz}\mathfrak{H}S,\ a \mapsto \downarrow a,$$

is an isomorphism by (II) as the Lindelöf elements of $\mathfrak{H}S$ are clearly the principal ideals. Hence $\operatorname{Coz}\mathfrak{K}\mathfrak{R}\mathfrak{H} \cong \mathrm{Id}$ since all maps involved here are natural in S. Similarly, for $\mathfrak{R}\mathfrak{H}\operatorname{Coz}\mathfrak{K}$, one has the analogous situation

$$\mathfrak{R}\mathfrak{H}\operatorname{Coz}\mathfrak{K}\mathfrak{R}L \xrightarrow{\mathfrak{R}\eta_{\mathfrak{K}\mathfrak{R}L}} \mathfrak{R}\mathfrak{K}\mathfrak{R}L \xrightarrow{\mathfrak{R}\mu_L} \mathfrak{R}L$$

where $\mathfrak{R}\eta_{\mathfrak{K}\mathfrak{R}L}$ is an isomorphism since this holds for $\eta_{\mathfrak{K}\mathfrak{R}L}$ by (III) given that $\mathfrak{K}\mathfrak{R}L$ is Lindelöf, and $\mathfrak{R}\mu_L$ is an isomorphism by the adjunction identities for $\mathfrak{R}$ and $\mathfrak{K}$. □

Remark 1. As an obvious alternative of the above proof, one might note the following where $\mathbf{\Lambda}$ is the category of completely regular Lindelöf frames. Given the familiar facts that the adjunction maps λ_A and μ_L are isomorphisms for $A \in \mathfrak{R}\mathbf{Frm}$ and $L \in \mathbf{\Lambda}$ and every $\mathfrak{R}\mu_L$ is an isomorphism, $\mathfrak{R}$ induces a category equivalence $\mathbf{\Lambda} \to \mathfrak{R}\mathbf{Frm}$ with inverse induced by $\mathfrak{K}$. On the other hand, there are the functors

$$\operatorname{Coz} : \mathbf{\Lambda} \to \mathbf{CR}\sigma\mathbf{Frm},\ L \mapsto \operatorname{Coz}L,$$

and

$$\mathfrak{H} : \mathbf{CR}\sigma\mathbf{Frm} \to \mathbf{\Lambda},\ S \mapsto \mathfrak{H}S,$$

with the natural isomorphisms

$$\eta_L : \mathfrak{H}\operatorname{Coz}L \to L,\ J \mapsto \bigvee J \text{ (in } L)$$

and

$$\pi_S : S \to \operatorname{Coz}\mathfrak{H}S,\ a \mapsto \downarrow a,$$

which show that Coz provides a category equivalence with inverse given by $\mathfrak{H}$, and combining the two then proves the proposition. Maybe this two-step approach has a certain appeal, but somehow the more direct argument seemed preferable here.

The following adds some detail concerning the situation in Proposition 1.

Proposition 2. *For any* $L \in \mathbf{CRFrm}$, *the natural homomorphism* $\mathrm{Coz}\,\mu_L : \mathrm{Coz}\,\mathfrak{K}\mathfrak{R}L \to \mathrm{Coz}\,L$ *is an isomorphism.*

Proof. To begin with, note that $\mu_L : \mathfrak{K}\mathfrak{R}L \to L$ and $\eta_L : \mathfrak{H}\,\mathrm{Coz}\,L \to L$ are both the Lindelöf coreflection map in **CRFrm**, by (I) and (III) respectively, so that there exists a natural isomorphism $h_L : \mathfrak{K}\mathfrak{R}L \to \mathfrak{H}\,\mathrm{Coz}\,L$ such that $\mu_L = \eta_L h_L$ and hence $\mathrm{Coz}\,\mu_L = (\mathrm{Coz}\,\eta_L)(\mathrm{Coz}\,h_L)$; on the other hand,

$$\mathrm{Coz}\,(\mathfrak{H}\,\mathrm{Coz}\,L) = \{\downarrow c \mid c \in \mathrm{Coz}\,L\}$$

by (II), showing that $\mathrm{Coz}\,\eta_L$ is an isomorphism, and the same then holds for $\mathrm{Coz}\,\mu_L$. □

Corollary 1. (i) *For any* $L, M \in \mathbf{CRFrm}$, $\mathfrak{R}L \cong \mathfrak{R}M$ *if and only if* $\mathrm{Coz}\,L \cong \mathrm{Coz}\,M$.

(ii) *For any* $h : L \to M$ *in* **CRFrm**, $\mathfrak{R}h$ *is an isomorphism if and only if* $\mathrm{Coz}\,h$ *is an isomorphism.*

Proof. (i) For any isomorphism $\varphi : \mathfrak{R}L \to \mathfrak{R}M$, the proposition trivially provides the isomorphism

$$(\mathrm{Coz}\,\mu_M)(\mathrm{Coz}\,\mathfrak{K}\varphi)(\mathrm{Coz}\,\mu_L)^{-1} : \mathrm{Coz}\,L \to \mathrm{Coz}\,M.$$

Conversely, given any isomorphism $\sigma : \mathrm{Coz}\,L \to \mathrm{Coz}\,M$, the corresponding isomorphism $\mathfrak{R}\mathfrak{H}\sigma : \mathfrak{R}\mathfrak{H}\mathrm{Coz}\,L \to \mathfrak{R}\mathfrak{H}\mathrm{Coz}\,M$ determines an isomorphism $\mathfrak{R}L \to \mathfrak{R}M$ as follows: since the frame of reals is Lindelöf, (III) implies any $\mathfrak{R}\eta_L : \mathfrak{R}\mathfrak{H}\mathrm{Coz}\,L \to \mathfrak{R}L$ is onto and hence an isomorphism so that one obtains

$$(\mathfrak{R}\eta_M)(\mathfrak{R}\mathfrak{H}\sigma)(\mathfrak{R}\eta_L)^{-1} : \mathfrak{R}L \to \mathfrak{R}M.$$

(ii) Any $h : L \to M$ as given determines the commuting square

$$\begin{array}{ccc} \mathrm{Coz}\,\mathfrak{K}\mathfrak{R}L & \xrightarrow{\mathrm{Coz}\,\mathfrak{K}\mathfrak{R}h} & \mathrm{Coz}\,\mathfrak{K}\mathfrak{R}M \\ {\scriptstyle \mathrm{Coz}\,\mu_L}\downarrow & & \downarrow{\scriptstyle \mathrm{Coz}\,\mu_M} \\ \mathrm{Coz}\,L & \xrightarrow[\mathrm{Coz}\,h]{} & \mathrm{Coz}\,M \end{array}$$

and if $\mathfrak{R}h$ is an isomorphism this trivially makes $\mathrm{Coz}\,h$ an isomorphism by the proposition. On the other hand, given the latter, each of the following is also an isomorphism

$$\mathrm{Coz}\,\mathfrak{K}\mathfrak{R}h,\ \mathfrak{K}\mathfrak{R}h,\ \mathfrak{R}\mathfrak{K}\mathfrak{R}h,\ \mathfrak{R}h$$

the first by the above square; the second by acting $\mathfrak{H}$ on the first and then using the isomorphisms $\eta_{\mathfrak{K}\mathfrak{R}L}$ and $\eta_{\mathfrak{K}\mathfrak{R}M}$ (note (III)); the third trivially now; and the fourth because $\mathfrak{R}\mu_L$ and $\mathfrak{R}\mu_M$ are isomorphisms by the adjunction identities for $\mathfrak{R}$ and $\mathfrak{K}$. □

Remark 2. The above (ii) appeared first in [1].

Proposition 3. $\mathbf{S}\mathfrak{K} : \mathfrak{Z}\mathbf{Frm} \to \mathbf{OD}\sigma\mathbf{Frm}$ *is a category equivalence with inverse* $\mathfrak{Z}\mathfrak{H}$.

Proof. Of course, this turns out to be entirely parallel to the proof of Proposition 1, now involving the maps

$$\mathbf{S}\mathfrak{K}\mathfrak{Z}\mathfrak{H}S \xrightarrow[\mathbf{S}\nu_{\mathfrak{H}S}]{} \mathbf{S}\mathfrak{H}S \xrightarrow[\iota_S^{-1}]{} S$$

and

$$\mathfrak{Z}\mathfrak{H}\mathbf{S}\mathfrak{K}\mathfrak{Z}L \xrightarrow[\mathfrak{Z}\theta_{\mathfrak{K}\mathfrak{Z}L}]{} \mathfrak{Z}\mathfrak{K}\mathfrak{Z}L \xrightarrow[\mathfrak{Z}\nu_L]{} \mathfrak{Z}L$$

for $S \in \mathbf{OD}\sigma\mathbf{Frm}$ and $L \in \mathbf{ODFrm}$, respectively, with identically the same reasoning, where the counterpart of Coz is the functor $\mathbf{S}$ introduced earlier. Specifically, then, the following are isomorphisms: $\nu_{\mathfrak{H}S}$ by (I) since $\mathfrak{H}S$ is Lindelöf, ι_S by (II), $\theta_{\mathfrak{K}\mathfrak{Z}L}$ by (III) as $\mathfrak{K}\mathfrak{Z}L$ is Lindelöf, and $\mathfrak{Z}\nu_L$ by the adjunction identities for $\mathfrak{Z}$ and $\mathfrak{K}$. □

Remark 3. Exactly analogous to Remark 1, there is an alternative two-step argument possible here, using the equivalence of the category of 0-dimensional Lindelöf frames with $\mathfrak{Z}\mathbf{Frm}$ and with $\mathbf{OD}\sigma\mathbf{Frm}$, provided by the pairs of adjoint functors $(\mathfrak{Z}, \mathfrak{K})$ and $(\mathbf{S}, \mathfrak{H})$, respectively.

Proposition 4. *For any $L \in \mathbf{ODFrm}$, the natural homomorphism $\mathbf{S}\nu_L : \mathbf{S}\mathfrak{K}\mathfrak{Z}L \to \mathbf{S}L$ is an isomorphism.*

Proof. In the present setting, $\nu_L : \mathfrak{K}\mathfrak{Z}L \to L$ and $\theta_L : \mathfrak{H}\mathbf{S}L \to L$ are both the Lindelöf coreflection map in $\mathbf{ODFrm}$, by (I) and (III) respectively, and the same approach applied earlier to μ_L and η_L then shows that $\mathbf{S}\nu_L$ is an isomorphism. □

Corollary 2. (i) *For any $L, M \in \mathbf{ODFrm}$, $\mathfrak{Z}L \cong \mathfrak{Z}M$ if and only if $\mathbf{S}L \cong \mathbf{S}M$.*

(ii) *For any $h : L \to M$ in $\mathbf{ODFrm}$, $\mathfrak{Z}h$ is an isomorphism if and only if $\mathbf{S}h$ is an isomorphism.*

Proof. Again, this is formally the same as the proof of its counterpart for $\mathfrak{R}$, now with

$$\mathrm{Coz}\,,\ \mathfrak{R},\ \mathrm{Coz}\,\mu_L,\ \text{and}\ \eta_L : \mathfrak{H}\mathrm{Coz}\,L \to L$$

replaced by

$$\mathbf{S},\ \mathfrak{Z},\ \mathbf{S}\nu_L,\ \text{and}\ \theta_L : \mathfrak{H}\mathbf{S}L \to L$$

where $\mathfrak{Z}\theta_L$ is an isomorphism by (III).

In particular, regarding (i), any isomorphism $\varphi : \mathfrak{Z}L \to \mathfrak{Z}M$ trivially determines the isomorphism

$$\mathbf{S}\nu_M(\mathbf{S}\mathfrak{K}\varphi)(\mathbf{S}\nu_L)^{-1} : \mathbf{S}L \to \mathbf{S}M.$$

Conversely, for any isomorphism $\sigma : \mathbf{S}L \to \mathbf{S}M$, the isomorphism $\mathfrak{Z}\mathfrak{H}\sigma : \mathfrak{Z}\mathfrak{H}\mathbf{S}L \to \mathfrak{Z}\mathfrak{H}\mathbf{S}M$ provides an isomorphism $\mathfrak{Z}L \to \mathfrak{Z}M$ since $\mathfrak{Z}\theta_L$ and $\mathfrak{Z}\theta_M$ are isomorphisms.

Regarding (ii), any $h : L \to M$ as given determines the commuting square

$$\begin{array}{ccc} \mathbf{S}\mathfrak{K}\mathfrak{Z}L & \xrightarrow{\mathbf{S}\mathfrak{K}\mathfrak{Z}h} & \mathbf{S}\mathfrak{K}\mathfrak{Z}M \\ {\scriptstyle \mathbf{S}\nu_L}\downarrow & & \downarrow{\scriptstyle \mathbf{S}\nu_M} \\ \mathbf{S}L & \xrightarrow[\mathbf{S}h]{} & \mathbf{S}M \end{array}$$

and since the downward maps are isomorphisms $\mathbf{S}h$ is an isomorphism whenever $\mathfrak{Z}h$ is. Conversely, given the former, each of the following is an isomorphism

$$\mathbf{S}\mathfrak{K}\mathfrak{Z}h,\ \mathfrak{K}\mathfrak{Z}h,\ \mathfrak{Z}\mathfrak{K}\mathfrak{Z}h,\ \mathfrak{Z}h$$

the first by the above square, the second by acting $\mathfrak{H}$ on the first and using the isomorphisms $\theta_{\mathfrak{K}\mathfrak{Z}L}$ and $\theta_{\mathfrak{K}\mathfrak{Z}M}$ (note (III)), the third trivially by acting $\mathfrak{Z}$ on the second, and the last because $\mathfrak{Z}\nu_L$ and $\mathfrak{Z}\nu_M$ are isomorphisms by the adjunction identities for $\mathfrak{Z}$ and $\mathfrak{K}$. □

Remark 4. The above (ii) appeared in [3].

For the case of $\mathfrak{R}^*$, some further entities will be used besides the present counterpart $\mathfrak{R}^*\mathbf{Frm}$ of the earlier $\mathfrak{R}\mathbf{Frm}$:

the category $\mathbf{K}_\sigma$ of compact completely regular σ-frames;

the compact coreflection in **CRFrm**; given by $\beta_L : \beta L \to L$ where $\beta L = CR\mathfrak{J}L$, the largest completely regular subframe of the ideal frame $\mathfrak{J}L$ of L, with $\beta_L(J) = \bigvee J$ (in L); and

the obvious natural isomorphism $\varrho_L : \mathfrak{R}\beta L \to \mathfrak{R}^*L$ provided by the image factorization of $\mathfrak{R}\beta_L : \mathfrak{R}\beta L \to \mathfrak{R}L$ which determines the composite

$$\tau_L : \mathfrak{K}\mathfrak{R}^*L \underset{(\mathfrak{K}\varrho_L)^{-1}}{\longrightarrow} \mathfrak{K}\mathfrak{R}\beta L \underset{\mu_{\beta L}}{\longrightarrow} \beta L,$$

an isomorphism by the nature of ϱ_L and the compactness of βL.

Now, the present analogue of Proposition 1 is

Proposition 5. *$Coz\,\mathfrak{K} : \mathfrak{R}^*\mathbf{Frm} \to \mathbf{K}_\sigma$ is a category equivalence with inverse $\mathfrak{R}^*\mathfrak{H}$.*

Proof. Of course $\mathfrak{K}\mathfrak{R}^*L$ is compact, as shown by the above τ_L, so that $\mathrm{Coz}\,\mathfrak{K}$ indeed maps $\mathfrak{R}^*\mathbf{Frm}$ into $\mathbf{K}_\sigma$. Further, $\mathrm{Coz}\,\mathfrak{K}\mathfrak{R}^*\mathfrak{H} \cong \mathrm{Coz}\,\mathfrak{H}$ because the map $\mathfrak{K}\mathfrak{R}^*\mathfrak{H}S \to \mathfrak{H}S$ provided by the adjunction maps of $\mathfrak{R}$ and $\mathfrak{K}$ is an isomorphism for any $S \in \mathbf{K}_\sigma$ because $\mathfrak{H}S$ is compact by the compactness of S, and since $\mathrm{Coz}\,\mathfrak{H}S \cong S$ as in the case involving $\mathfrak{R}$ it follows that $\mathrm{Coz}\,\mathfrak{K}\mathfrak{R}^*\mathfrak{H} \cong \mathrm{Id}$. Similarly, $\mathfrak{R}^*\mathfrak{H}\,\mathrm{Coz}\,\mathfrak{K} \cong \mathfrak{R}^*\mathfrak{K}$ because $\mathfrak{H}\,\mathrm{Coz}\,\mathfrak{K} \cong \mathfrak{K}$, and since $\mathfrak{R}^*\mathfrak{K} \cong \mathrm{Id}$ by the definition of $\mathfrak{R}^*\mathbf{Frm}$ it follows that $\mathfrak{R}^*\mathfrak{H}\,\mathrm{Coz}\,\mathfrak{K} \cong \mathrm{Id}$ as well. □

Remark 5. As in the previous two cases, the present result can also be obtained by a natural two-step argument: for the category **K** of compact completely regular frames, one has the two equivalences $\mathbf{K} \cong \mathfrak{R}^*\mathbf{Frm}$ and $\mathbf{K} \cong \mathbf{K}_\sigma$ produced by the pairs of adjoint functors $(\mathfrak{R}^*, \mathfrak{K})$ and $\mathrm{Coz}\,, \mathfrak{H})$, respectively.

Next, regarding the present analogue of Proposition 2, the above isomorphism $\tau_L : \mathfrak{K}\mathfrak{R}^*L \to \beta L$ immediately implies

Proposition 6. *For any* $L \in \mathbf{CRFrm}$*, the natural homomorphism* $Coz\,\tau_L$: $Coz\,\mathfrak{K}\mathfrak{R}^*L \to Coz\,\beta L$ *is an isomorphism.*

Remark 6. (i) By way of comparison with Proposition 2, it may be worth noting that $\mathrm{Coz}\,\beta L$ is the compact coreflection of $\mathrm{Coz}\,L$ in $\mathbf{CR}\sigma\mathbf{Frm}$ with coreflection map $\mathrm{Coz}\,\beta_L : \mathrm{Coz}\,\beta L \to \mathrm{Coz}\,L$: clearly, for arbitrary $S \in \mathbf{CR}\sigma\mathbf{Frm}$, that map is

$$\mathrm{Coz}\,\beta\mathfrak{H}S \underset{\mathrm{Coz}\,\beta_{\mathfrak{H}S}}{\longrightarrow} \mathrm{Coz}\,\mathfrak{H}S \underset{(\pi_S)^{-1}}{\longrightarrow} S\,;$$

on the other hand, in **CRFrm**,

$$\beta\mathfrak{H}\,\mathrm{Coz}\,L \underset{\beta_{\mathfrak{H}\mathrm{Coz}L}}{\longrightarrow} \mathfrak{H}\,\mathrm{Coz}\,L \underset{\eta_L}{\longrightarrow} L$$

is readily seen to be the compact coreflection map, providing a natural isomorphism $\beta\mathfrak{H}\,\mathrm{Coz}\,L \to \beta L$ which then implies the claim.

(ii) Concerning τ_L, it should be noted that $\beta_L\tau_L = \mu_L\mathfrak{K}i_L$ for the identical embedding $i_L : \mathfrak{R}^*L \to \mathfrak{R}L$, and that the existence of an isomorphism $\mathfrak{K}\mathfrak{R}^*L \to \beta L$ which satisfies this is a familiar fact, but this simple way of presenting it seems to be new.

Corollary 3. (i) *For any* $L, M \in \mathbf{CRFrm}$*,* $\mathfrak{R}^*L \cong \mathfrak{R}^*M$ *if and only if* $Coz\,\beta L \cong Coz\,\beta M$*.*

(ii) *For any* $h : L \to M$ *in* **CRFrm***,* $\mathfrak{R}^*h$ *is an isomorphism if and only if* $Coz\,\beta h$ *is an isomorphism.*

Proof. (i) For any isomorphism $\varphi : \mathfrak{R}^*L \to \mathfrak{R}^*M$, the isomorphism $\tau_M(\mathfrak{K}\varphi)(\tau_L)^{-1}$: $\beta L \to \beta M$ induces an isomorphism $\mathrm{Coz}\,\beta L \to \mathrm{Coz}\,\beta M$. Conversely, given any isomorphism $\sigma : \mathrm{Coz}\,\beta L \to \mathrm{Coz}\,\beta M$, the corresponding isomorphism $\mathfrak{H}\sigma : \mathfrak{H}\,\mathrm{Coz}\,\beta L \to \mathfrak{H}\,\mathrm{Coz}\,\beta M$ shows $\beta L \cong \beta M$, given the isomorphisms $\eta_{\beta L}$

and $\eta_{\beta M}$ by (III), and this in turn implies $\mathfrak{R}^*L \cong \mathfrak{R}^*M$ by the natural isomorphism $\varrho_L : \mathfrak{R}\beta L \to \mathfrak{R}^*L$.

(ii) Any $h : L \to M$ as given determines the commuting square

$$\begin{array}{ccc} \operatorname{Coz}\mathfrak{K}\mathfrak{R}^*L & \xrightarrow{\operatorname{Coz}\mathfrak{K}\mathfrak{R}^*h} & \operatorname{Coz}\mathfrak{K}\mathfrak{R}^*M \\ {\scriptstyle \operatorname{Coz}\tau_L}\downarrow & & \downarrow{\scriptstyle \operatorname{Coz}\tau_M} \\ \operatorname{Coz}\beta L & \xrightarrow[\operatorname{Coz}\beta h]{} & \operatorname{Coz}\beta M \end{array}$$

where the downward maps are isomorphisms by the proposition, and if $\mathfrak{R}^*h$ is an isomorphism this trivially implies the same for $\operatorname{Coz}\beta h$. On the other hand, given the latter, each of the following is also an isomorphism

$$\operatorname{Coz}\mathfrak{K}\mathfrak{R}^*h,\ \mathfrak{K}\mathfrak{R}^*h,\ \mathfrak{R}\mathfrak{K}\mathfrak{R}^*h,\ \mathfrak{R}\beta h,\ \mathfrak{R}^*h$$

the first by the above square, the second by acting $\mathfrak{H}$ on the first and using the isomorphisms $\eta_{\mathfrak{K}\mathfrak{R}^*L}$ and $\eta_{\mathfrak{K}\mathfrak{R}^*M}$ (note (III)), the third trivially now, the fourth by the natural isomorphism $\tau_L : \mathfrak{K}\mathfrak{R}^*L \to \beta L$, and the last by the natural isomorphism $\varrho_L : \mathfrak{R}\beta L \to \mathfrak{R}^*L$ and its version for M. □

Remark 7. The above (ii) corresponds to a result in [1].

Finally, the situation regarding $\mathfrak{Z}^*$ can certainly be treated by modifying the arguments used above for $\mathfrak{R}^*$, now involving the compact 0-dimensional σ-frames, the compact coreflection map $\zeta_L : \zeta L \to L$ in the category **ODFrm**, and the natural isomorphism $\mathfrak{Z}\zeta L \to \mathfrak{Z}^*L$ determined by the image factorization of $\mathfrak{Z}\zeta_L : \mathfrak{Z}\zeta L \to \mathfrak{Z}L$. This will produce the exact counterparts for $\mathfrak{Z}^*$ of the above result for $\mathfrak{R}^*$, to be left as an exercise. Instead, we shall use an interesting alternative approach based on the familiar functor **B** from **ODFrm** to the category **BAlg** of Boolean algebras, taking each $L \in$ **ODFrm** to the Boolean algebra $\mathbf{B}L$ of its complemented elements, and its familiar left adjoint $\mathfrak{J} : \mathbf{BAlg} \to \mathbf{ODFrm}$, $A \mapsto \mathfrak{J}A$, the ideal frame of A, with the adjunction maps

$$\zeta_L : \mathfrak{J}\mathbf{B}L \to L,\ J \mapsto \bigvee J \text{ (in } L)$$

and

$$\delta_A : A \to \mathbf{B}\mathfrak{J}A,\ a \mapsto \downarrow a = \{s \in A \mid s \le a\}.$$

Note that ζ_L is in fact the compact coreflection map in **ODFrm** and $\mathfrak{J}\mathbf{B}L$ is often denoted ζL. Further. $\varrho_L^0 : \mathfrak{Z}\zeta L \to \mathfrak{Z}^* L$ and

$$\tau_L^0 = \nu_{\zeta L}(\mathfrak{K}\varrho_L^0)^{-1} : \mathfrak{K}\mathfrak{Z}^* L \to \zeta L$$

will be the present analogues of the earlier isomorphisms ϱ_L and τ_L.

Now, the relevant results are as follows.

Proposition 7. $\mathbf{B}\mathfrak{K} : \mathfrak{Z}^*\mathbf{Frm} \to \mathbf{BAlg}$ *is a category equivalence with inverse* $\mathfrak{Z}\mathfrak{J}$.

Proof. Concerning $\mathbf{B}\mathfrak{K}\mathfrak{Z}\mathfrak{J}$, there is the composite homomorphism

$$\mathbf{B}\mathfrak{K}\mathfrak{Z}\mathfrak{J}A \xrightarrow[\mathbf{B}\nu_{\mathfrak{J}A}]{} \mathbf{B}\mathfrak{J}A \xrightarrow[(\delta_A)^{-1}]{} A$$

for any $A \in \mathbf{BAlg}$, where the adjunction map $\nu_{\mathfrak{J}A} : \mathfrak{K}\mathfrak{Z}\mathfrak{J}A \to \mathfrak{J}A$ is an isomorphism by the compactness of $\mathfrak{J}A$, and since $\delta_A : A \to \mathbf{B}\mathfrak{J}A$ is obviously an isomorphism this provides the isomorphism $(\delta_A)^{-1}$, showing in all that $\mathbf{B}\mathfrak{K}\mathfrak{Z}\mathfrak{J} \cong \mathrm{Id}$. Similarly, $\mathfrak{Z}\mathfrak{J}\mathbf{B}\mathfrak{K} \cong \mathrm{Id}$ because each map in the sequence

$$\mathfrak{Z}\mathfrak{J}\mathbf{B}\mathfrak{K}\mathfrak{Z}^* L \xrightarrow[\mathfrak{Z}\zeta_{\mathfrak{K}\mathfrak{Z}^* L}]{} \mathfrak{Z}\mathfrak{K}\mathfrak{Z}^* L \xrightarrow[\mathfrak{Z}\tau_L^0]{} \mathfrak{Z}\zeta L \xrightarrow[\varrho_L^0]{} \mathfrak{Z}^* L$$

is an isomorphism for any L: the first by the compactness of $\mathfrak{K}\mathfrak{Z}^* L$ and the other two obviously. □

Remark 8. Again, there is a natural two-step version of this proof, showing in this case that the category of compact 0-dimensional frames is equivalent to $\mathfrak{Z}^*\mathbf{Frm}$ as well as to **BAlg**, using the pairs of adjoint functors $(\mathfrak{Z}, \mathfrak{K})$ and $(\mathbf{B}, \mathfrak{J})$. It might be added here that the latter equivalence is, of course, the pointfree version of the classical Stone Duality.

In the following, $\nu_L^* = \nu_L \mathfrak{K} j_L : \mathfrak{K}\mathfrak{Z}^* L \to L$ for any $L \in \mathbf{ODFrm}$, where $j_L : \mathfrak{Z}^* L \to \mathfrak{Z}L$ is the identical embedding.

Proposition 8. *For any* $L \in \mathbf{ODFrm}$*, the natural homomorphism* $\mathbf{B}\nu_L^* : \mathbf{B}\mathfrak{K}\mathfrak{Z}^* L \to \mathbf{B}L$ *is an isomorphism.*

Proof. For any $a \in \mathbf{B}L$ and its characteristic function $\chi_a \in \mathfrak{Z}^* L$, $\nu_L^*(\langle \chi_a \rangle) = \mathrm{coz}(\chi_a) = a$ where $\langle \cdot \rangle$ indicates the archimedian kernel in $\mathfrak{Z}^* L$ generated by $\cdot$, and since $\langle \chi_a \rangle \in \mathbf{B}\mathfrak{K}\mathfrak{Z}^* L$ (as noted earlier) it follows that $\mathbf{B}\nu_L^*$ is onto. On the other hand, since ν_L^* is obviously dense, $\mathbf{B}\nu_L^*$ is also one-one and therefore an isomorphism. □

Corollary 4. (i) *For any* $L, M \in \mathbf{ODFrm}$, $\mathfrak{z}^*L \cong \mathfrak{z}^*M$ *if and only if* $\mathbf{B}L \cong \mathbf{B}M$.

(ii) *For any* $h : L \to M$ *in* $\mathbf{ODFrm}$, $\mathfrak{z}^*h$ *is an isomorphism if and only if* $\mathbf{B}h$ *is an isomorphism.*

Proof. (i) By the proposition, any isomorphism $\varphi : \mathfrak{z}^*L \to \mathfrak{z}^*M$ determines the isomorphism $(\mathbf{B}\nu_M^*)(\mathbf{B}\mathfrak{K}\varphi)(\mathbf{B}\nu_L^*)^{-1} : \mathbf{B}L \to \mathbf{B}M$. Conversely, any isomorphism $\sigma : \mathbf{B}L \to \mathbf{B}M$ determines the isomorphism $\varrho_M^0(\mathfrak{z}\mathfrak{J}\sigma)(\varrho_L^0)^{-1} : \mathfrak{z}^*L \to \mathfrak{z}^*M$.

(ii) Any $h : L \to M$ as given determines the commuting square

$$\begin{array}{ccc} \mathbf{B}\mathfrak{K}\mathfrak{z}^*L & \xrightarrow{\mathbf{B}\mathfrak{K}\mathfrak{z}^*h} & \mathfrak{B}\mathfrak{K}\mathfrak{z}^*M \\ {\scriptstyle \mathbf{B}\nu_L^*}\downarrow & & \downarrow{\scriptstyle \mathbf{B}\nu_M^*} \\ \mathbf{B}L & \xrightarrow[\mathbf{B}h]{} & \mathbf{B}M \end{array}$$

where the downward maps are isomorphisms by the proposition so that $\mathbf{B}h$ is trivially an isomorphism whenever $\mathfrak{z}^*h$ is. Conversely, if $\mathbf{B}h$ is an isomorphism then each of the following is an isomorphism as well

$$\mathbf{B}\mathfrak{K}\mathfrak{z}^*h, \ \mathfrak{K}\mathfrak{z}^*h, \ \mathfrak{z}\mathfrak{K}\mathfrak{z}^*h, \ \mathfrak{z}^*h$$

the first by the above square, the second by acting $\mathfrak{J}$ on the first and using the isomorphisms $\zeta_{\mathfrak{K}\mathfrak{z}^*L}$ and $\zeta_{\mathfrak{K}\mathfrak{z}^*M}$, the third then trivially, and the last by the isomorphism $\varrho_L^0\mathfrak{z}\tau_L^0 : \mathfrak{z}\mathfrak{K}\mathfrak{z}^*L \to \mathfrak{z}^*L$ and its version for M. □

Remark 9. The above (ii) was originally proved in [3].

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