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Frankl's Conjecture for a subclass of semimodular lattices

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This article is dedicated to Prof. George A. Grätzer

Abstract. In this paper, we prove Frankl's Conjecture for an upper semimodular lattice L such that $|J(L) \setminus A(L)| \leq 3$, where $J(L)$ and $A(L)$ are the set of join-irreducible elements and the set of atoms respectively. It is known that the class of planar lattices is contained in the class of dismantlable lattices and the class of dismantlable lattices is contained in the class of lattices having breadth at most two. We provide a very short proof of the Conjecture for the class of lattices having breadth at most two. This generalizes the results of Joshi, Waphare and Kavishwar [9, Theorem 2.4] as well as Czédli and Schmidt [6, Theorem 1].

1 Introduction

In 1979, Peter Frankl conjectured the following, known as the Union-Closed Sets Conjecture or Frankl's Conjecture.

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Union-Closed Sets Conjecture 1.1. *Let $\mathcal{F}$ be a collection of subsets of a finite set X such that $F \cup G \in \mathcal{F}$ holds for all $F, G \in \mathcal{F}$, that is, $\mathcal{F}$ is a union-closed family. If $|\mathcal{F}| \geq 2$ then there is an element x in X such that at least $|\mathcal{F}|/2$ members $F \in \mathcal{F}$ satisfy $x \in F$.*

Poonen [10] formulated the Conjecture in the language of lattice theory and is equivalent to the Union-Closed Sets Conjecture.

Frankl's Conjecture 1.2 (Poonen [10], Stanley [13])**.** *In every finite lattice L with $|L| \geq 2$, there is a nonzero join-irreducible element j (that is $j = a \vee b \Rightarrow j = a$ or $j = b$) such that $|\{x \in L \colon j \leq x\}| \leq |L|/2$.*

Many partial results have been obtained by using lattice theoretic methods which solve the conjecture for special classes of lattices; see Abe [2], Abe and Nakano [3] and Czédli and Schmidt [6]. Recently, Henning Bruhn and Oliver Schaudt [5] wrote a nice survey on the journey of the Union-Closed Sets Conjecture which lists around 50 articles related to this Conjecture. However, it is unknown whether Frankl's Conjecture is true for upper semimodular lattices and this case is supposed to be the difficult one in the lattice theoretic version of the Conjecture. Czédli and Schmidt [6] proved this conjecture for large semimodular lattices.

Czédli and Schmidt [6] proved the Conjecture for planar upper semimodular lattice. In [4], Baker et al. proved that every planar lattice is dismantlable. Rival [11] proved that a dismantlable lattice is of breadth at most two.

Now, we provide the most trivial proof of the result which states that every lattice of breadth two satisfies the Conjecture. In fact, this extends the result of Joshi et al. [9, Theorem 2.4] which states that every dismantlable lattice satisfies Frankl's Conjecture. Note that in a lattice L, if $1 = j_1 \vee j_2$ for some $j_1, j_2 \in J(L)$, then L is not necessarily of breadth two. Recently, statement (A) of Theorem 1.3 is proved in [1, Corollary 1.11]. Here we provide a different proof of statement (A). In fact, we prove:

Theorem 1.3. *Let L be a finite lattice with $|L| \geq 2$.*

(A) *If the greatest element 1 of L is join-irreducible or is the join of two join-irreducibles, then L satisfies Frankl's Conjecture.*

(B) *If L is semimodular and $|J(L) \setminus A(L)| \leq 3$, then L satisfies Frankl's Conjecture again.*

We feel that the method adopted to prove this result is effective for the case $|J(L) \setminus A(L)| \leq 3$ and is difficult to extend for the general case.

Throughout this paper, all lattices are assumed to be finite. For undefined notions and terminology, the reader is referred to Grätzer [7].

2 Frankl's Conjecture

Now, we begin with the necessary definitions and terminology.

Definition 2.1. Let L be a lattice. By $a \prec b$, we mean there is no c such that $a < c < b$. An element a is an *upper cover*(*a lower cover*) of b if $b \prec a$ ($a \prec b$). A lower cover of a is denoted by a^-. A nonzero element p of L is an *atom* if $0 \prec p$. Dually, an element d of L is a *dual atom* if $d \prec 1$. The set of atoms in L is denoted by $A(L)$.

As usual, $J(L)$ stands for the set of nonzero join-irreducible elements of L. So an element $x \in L$ belongs to $J(L)$ if and only if it has the unique lower cover, denoted by x^-. Dually, the set of nonzero meet-irreducible elements is denoted by $M(L)$.

A lattice L is said to be *large*, if $|L| > 5 \cdot 2^{m-3}$, where $m = |J(L)|$.

We define $x' = \bigvee\{j^- : x \geq j \in J(L)\}$ for $x > 0$. It is called the *derivation* of x. On the other hand, if $x > 0$ then we can define the meet of all lower covers of x. We write x_+ for it.

Note that in a finite lattice L, every nonzero element is a join of join-irreducible elements of L. We say that a set $U \subseteq J(L)$ is an *irredundant set* of x, if $x = \bigvee U$ and $x > \bigvee(U \setminus \{a\})$ for $a \in U$.

The *breadth* of a lattice L is the least positive integer m such that any $\bigvee_{i=1}^{n} x_i,\ x_i \in L,\ n \geq m$, is always a join of m of the x_i's.

Lemma 2.2 (Stern [14, Corollary 6.5.3, p.254])**.** *In an upper semimodular lattice L, $x_+ \leq x'$ for every nonzero $x \in L$.*

Proof of Theorem 1.3. (A) If the greatest element 1 is join-irreducible then nothing to prove.

Now, assume that $1 = j_1 \vee j_2$ be an irredundant representation, $j_1, j_2 \in J(L)$. If $|[j_1)| \leq |L|/2$ then nothing to prove. Suppose $|[j_1)| > |L|/2$. Then $|L \setminus [j_1)| < |L|/2$. Clearly, $[j_2) \setminus \{1\} \subsetneqq L \setminus [j_1)$, as j_1^-, the unique lower cover of j_1, is in $L \setminus [j_1)$ but $j_1^- \notin [j_2)$. This proves that $|[j_2)| \leq |L|/2$. This proves that every lattice of at most breadth two satisfies the conjecture.

(B): Now, assume that L is a upper semimodular lattice with $|J(L) \setminus A(L)| \leq 3$.

We consider the following two cases.

Case I: Suppose there is an irredundant representation of 1 that contains an atom, say, p. Let U be an irredundant set of 1 such that $p \in U$. Then $1' = \bigvee\{j_i^- : j_i \in J(L) \setminus A(L)\}$, because $p^- = 0$ for p in $A(L)$. Clearly, $1' \leq 1$. If $1' = 1$, then $1 = 1' = \bigvee\{j_i^- : j_i \in J(L) \setminus A(L)\} \leq \bigvee\{j_i : j_i \in J(L) \setminus A(L)\}$. Hence $1 = \bigvee\Big(J(L) \setminus A(L)\Big)$, a contradiction to the fact that U is an irredundant set of 1 containing an atom p. Hence $1' < 1$. Therefore there is $j \in J(L)$ such that $j \not\leq 1'$. Let $x \in L$ such that $j \leq x \leq 1$. By Lemma 2.2, we have $x_+ \leq x'$. Therefore $j \not\leq x_+$, otherwise $j \leq x_+ \leq x' \leq 1'$, a contradiction. But then there is $y(x)$ such that $y(x) \prec x$ and $j \vee y(x) = x$. This defines a 1-1 correspondence from $[j)$ to $L \setminus [j)$. Hence $|[j)| \leq |L \setminus [j)|$. In this case we are done.

Case II: Suppose no irredundant representation of 1 contains an atom. Let U be an irredundant set of 1 and $p \notin U$ for every $p \in A(L)$. By the assumption $|J(L) \setminus A(L)| \leq 3$ and by part (A), we have only possibility $U = \{j_1, j_2, j_3\}$, where $j_k \in J(L) \setminus A(L)$, $k = 1, 2, 3$.

Subcase II(a): Let $U = \{j_1, j_2, j_3\} = J(L) \setminus A(L)$. Then $1 = j_1 \vee j_2 \vee j_3$, as $p \notin U$ for any $p \in A(L)$.

Without loss of generality, choose $j_1 \in J(L) \setminus A(L)$ and let U_x be some irredundant set of x. Note that an element may have more than one irredundant sets.

We prove that for every $x \in [j_1)$, if at least one irredundant set U_x of x contains some atom $p \in L$ such that $p < j_1$ or $j_1 \in U_x$, then we are through.

For this, assume that $x \in [j_1)$ with U_x be its irredundant set such that U_x contains either an atom $p < j_1$ or j_1. In case $p \in U_x$, put $y(x) = \bigvee(U_x \setminus \{p\})$. Then $x = p \vee y(x) \leq j_1 \vee y(x) \leq x$. Thus $x = j_1 \vee y(x)$ in either the case. This proves that $x \mapsto y(x)$ is a 1-1 correspondence from $[j_1)$ to $L \setminus [j_1)$. Thus in this case, j_1 is the required join-irreducible element, and we are done.

Now, assume that there is some x with $x \in [j_1)$ and no irredundant set U_x of x contains j_1 or an atom p with $p < j_1$.

Let U_x be any irredundant set of x with the above property.

Further, if both of j_2 and j_3 are in U_x, then $j_1 \leq x = \bigvee(U_x \setminus \{j_2, j_3\}) \vee j_2 \vee j_3$. This gives $1 = j_1 \vee j_2 \vee j_3 \leq \bigvee(U_x \setminus \{j_2, j_3\}) \vee j_2 \vee j_3$. Note that $(U_x \setminus \{j_2, j_3\}) \subseteq A(L)$, as $J(L) \setminus A(L) = \{j_1, j_2, j_3\}$. Then 1 has an irredundant representation which contains at least one atom, a contradiction.

Thus without loss of generality, assume that $j_2 \in U_x$ and $j_3 \notin U_x$. In this case, $j_1 \leq x = \bigvee(U_x \setminus \{j_2\}) \vee j_2$. But again, $1 = j_1 \vee j_2 \vee j_3 \leq \bigvee(U_x \setminus \{j_2\}) \vee j_2 \vee j_3$. Since $j_1, j_3 \notin U_x$, we have $U_x \setminus \{j_2\} \subseteq A(L)$. Therefore 1 has an irredundant representation which contains at least one atom, again a contradiction.

Hence $j_1, j_2, j_3 \notin U_x$. Then $x = q_1 \vee \cdots \vee q_m$ for $q_i \in A(L)$ and for some $m \in \mathbb{N}$. Since $j_1 \leq x$, we have $1 = j_1 \vee j_2 \vee j_3 \leq x \vee j_2 \vee j_3 \leq j_2 \vee j_3 \vee q_1 \vee \cdots \vee q_m$, again a contradiction to the fact that 1 has an irredundant representation which contains an atom. Thus we are done in this case also.

This completes the proof. □

Essentially, Czédli and Schmidt [6] proved:

Lemma 2.3. *If L is a large semimodular lattice, then $|J(L) \setminus A(L)| \leq 1$.*

As an immediate consequence of Theorem 1.3 and Lemma 2.3, we have the following result of Czédli and Schmidt [6].

Corollary 2.4. *Let L be a large semimodular lattice. Then L satisfies Frankl's Conjecture.*

Now, we need the following definition of an adjunct operation to prove that Frankl's Conjecture is true for adjunct of lattices.

Definition 2.5 (Thakare et al. [15])**.** If L_1 and L_2 are two disjoint finite lattices and (a, b) is a pair of elements in L_1 such that $a < b$ and $a \not\prec b$. Define the partial order $\leq$ on $L = L_1 \cup L_2$ with respect to the pair (a, b) as follows. For $x, y \in L$, we say $x \leq y$ in L if either $x, y \in L_1$ and $x \leq y$ in L_1; or $x, y \in L_2$ and $x \leq y$ in L_2; or $x \in L_1$, $y \in L_2$ and $x \leq a$ in L_1; or $x \in L_2$, $y \in L_1$ and $b \leq y$ in L_1.

It is easy to see that L is a lattice containing L_1 and L_2 as sublattices. The procedure of obtaining L in this way is called an *adjunct operation of* L_2 *to* L_1. The pair (a, b) is called *an adjunct pair* and L is an *adjunct* of L_2 to L_1 with respect to the adjunct pair (a, b) and we write $L = L_1]_a^b L_2$.

We place the Hasse diagrams of L_1, L_2 side by side in such a way that the greatest element 1_{L_2} of L_2 is at the lower position than b and the least element 0_{L_2} of L_2 is at the higher position than a. Then add the coverings $1_{L_2} \prec b$ and $a \prec 0_{L_2}$, as shown in Figure 1, to obtain the Hasse diagram of $L = L_1]_a^b L_2$.

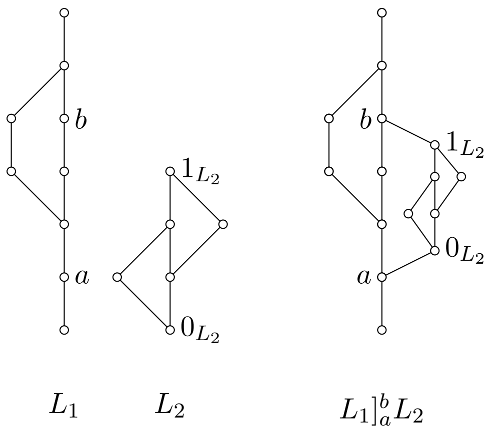

Figure 1: Adjunct of two lattices L_1 and L_2

Note that the adjunct operation preserves all the covering relations of the individual lattices L_1 and L_2. Also if $x, y \in L_2$, then $a \prec 0_{L_2} \leq x \wedge y$. Hence $x \wedge y \neq 0$ in $L = L_1]_a^b L_2$. Moreover, $J(L) \subseteq J(L_1) \cup J(L_2)$ and if $b \in J(L_1)$ then $b \notin J(L)$.

Definition 2.6. Let P and Q be disjoint posets. Let $P \cup Q$ be the union with the inherited order on P and Q such that $p < q$ for all $p \in P$ and $q \in Q$. Then it forms a poset called the *linear sum* of P and Q denoted by $P \oplus Q$.

Theorem 2.7. *Let L_1 be a lattice satisfying Frankl's Conjecture with two incomparable join-irreducible elements $j_1, j_2 \in J(L_1)$, that is, $|[j_i)| \leq |L_1|/2$*

for $i = 1, 2$. Then $L = L_1]_a^b L_2$ as well as $L = L_2 \oplus L_1$ satisfies Frankl's Conjecture for any lattice L_2.

Proof. Let $L = L_1]_a^b L_2$ with (a, b) as an adjunct pair. By the assumption, there exist two incomparable join-irreducible elements j_1, j_2 of L_1 such that $|[j_i)| \leq |L_1|/2$ for $i = 1, 2$.

Without loss of generality, assume that $j_2 = a$. Then j_1 and j_2 are incomparable join-irreducible elements of L also and $x \notin [j_1)$ for every $x \in L_2$. In this case, $|[j_1)| \leq |L|/2$ and we are through.

Now, if $j_1 = b$ or $j_2 = b$, then j_1 or j_2 is not in $J(L)$. Without loss of generality, assume that $j_2 = b$. Then clearly $j_1 \in J(L)$ and $j_1 \not\leq a$, as j_1, j_2 are incomparable. By the definition of an adjunct, we have $x \notin [j_1)$ for every $x \in L_2$. Therefore $|[j_1)| \leq |L|/2$. Thus $L = L_1]_a^b L_2$ satisfies Frankl's Conjecture.

Clearly, $L = L_2 \oplus L_1$ satisfies Frankl's Conjecture for any lattice L_2. □

Theorem 2 of [12] states that Frankl's Conjecture is true for the class of lattices satisfying the dual covering property, a more general class than the class of lower semimodular lattices. A careful observation of the proof reveals that, a stronger version of theorem is true. For the sake of completeness, we provide the proof of it.

Theorem 2.8 ([12]). *Let L be a lattice with the greatest element 1 as a join-reducible element and satisfies the dual covering property. Then L satisfies Frankl's Conjecture with two incomparable elements $j_1, j_2 \in J(L)$, that is, $|[j_i)| \leq |L|/2$ for $i = 1, 2$.*

Proof. Since 1 is a join-reducible element, there are at least two dual atoms, say d_1, d_2. Hence there are two join-irreducible elements such that $j_i \leq d_i$ for $i = 1, 2$ and $j_k \not\leq d_i$ for $i \neq k$. Clearly, j_1, j_2 are incomparable. From the proof of Theorem 2 of [12], it is clear that these join-irreducible elements j_1, j_2 serves the purpose. □

In view of Theorem 2.7 and Theorem 2.8, we have the following corollary.

Corollary 2.9. *Let L_1 be a lattice satisfying the dual covering property with 1 as a join-reducible element. Then $L = L_1]_a^b L_2$ satisfies Frankl's Conjecture for any lattice L_2 with (a, b) as an adjunct pair.*

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References

[1] Abdollahi, A., Woodroofe, R., and Zaimi, G., *Frankl's Conjecture for subgroup lattices*, Electron. J. Combin. 24(3) (2017), P3.25.

[2] Abe, T., *Strong semimodular lattices and Frankl's Conjecture*, Algebra Universalis 44 (2000), 379-382.

[3] Abe, T. and Nakano, B., *Lower semimodular types of lattices: Frankl's Conjecture holds for lower quasi-modular lattices*, Graphs Combin. 16 (2000), 1-16.

[4] Baker, K.A., Fishburn, P.C., and Roberts, F.S., *Partial orders of dimension 2*, Networks 2 (1972), 11-28.

[5] Bruhn, H. and Schaudt, O., *The journey of the Union-Closed Sets Conjecture*, Graphs Combin. 31 (2015), 2043-2074.

[6] Czédli, G. and Schmidt, E.T., *Frankl's conjecture for large semimodular and planar semimodular lattices*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. 47 (2008), 47-53.

[7] Grätzer, G., "General Lattice Theory", Birkhäuser, 1998.

[8] Hunh, A.P., *Schwach distributive Verbdnde-I*, Acta Sci. Math. (Szeged) 33 (1972), 297-305.

[9] Joshi, V., Waphare, B.N., and Kavishwar, S.P., *A proof of Frankl's Union-Closed Sets Conjecture for dismantlable lattices*, Algebra Universalis 76 (2016), 351-354.

[10] Poonen, B., *Union-closed families*, J. Combin. Theory Ser. A. 59 (1992), 253-268.

[11] Rival, I., *Combinatorial inequalities for semimodular lattices of breadth two*, Algebra Universalis 6 (1976), 303-311.

[12] Shewale, R.S., Joshi, V., and Kharat, V.S., *Frankl's conjecture and the dual covering property*, Graphs Combin. 25(1) (2009), 115-121.

[13] Stanley, R.P., "Enumerative Combinatorics", Vol I. Wadsworth & Brooks/Cole Advanced Books & Software, 1986.

[14] Stern, M., "Semimodular Lattices", Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1999.

[15] Thakare, N.K., Pawar, M.M., and Waphare, B.N., *A structure theorem for dismantlable lattices and enumeration*, Period. Math. Hungar. 45(1-2) (2002), 147-160.

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