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Abundant semigroups with medial idempotents

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Abstract. The effect of the existence of a medial or related idempotent in any abundant semigroup is the subject of this paper. The aim is to naturally order any abundant semigroup S which contains an ample multiplicative medial idempotent u in a way that $\mathcal{L}^*$ and $\mathcal{R}^*$ are compatible with the natural order and u is a maximum idempotent. The structure of an abundant semigroup containing an ample normal medial idempotent studied in [6] will be revisited.

1 Introduction

A partial order relation $\leqslant$ on any semigroup S with set of idempotents E is a *natural partial order* if for any $e, f \in E$:

$$e = ef = fe \quad \text{implies} \quad e \leqslant f.$$

When the natural partial order on S is compatible with the binary operation, S is said to be *naturally ordered*. The book of Blyth and Janowitz [3] con-

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tains a substantial literature on partially ordered semigroups. Most of the results of the theory concentrate on naturally ordered semigroups. Blyth, McFadden and McAlister (see [2], [4] and [5]) study the structure of several classes of naturally partially ordered regular semigroups that contain maximum idempotents with respect to the imposed orders. Much of this work and related ideas has already been transferred to the abundant case (see, for example, [12], [13], [14] and [16]). Blyth and McFadden [4] describe the structure of all regular semigroups that possess a normal medial idempotent. Subsequently, medial idempotents played a role in the structure of certain classes of regular semigroups, leading on to related work for abundant semigroups. McAlistar and McFadden [19] prove that any regular semigroup which is locally inverse and contains a medial idempotent can be naturally ordered with a maximum idempotent. The main objective of this current paper (Theorem 4.9) is to show that any abundant semigroup with an ample multiplicative medial idempotent can be naturally ordered with a maximum idempotent such that $\mathcal{L}^*$ and $\mathcal{R}^*$ are compatible in a specific way with the partial order, termed as *$\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant* in the sense described in Section 4. The approach to this result is an adaptation of that of [19]. Blyth and McFadden [5] study the structure of regular semigroups that possess normal medial idempotents. This structure is extended to the related class of abundant semigroups (see [6]). In this paper, we revisit this structure to naturally order such abundant semigroups.

In the first section, we review some concepts related to general abundant semigroups. In Section 3, we present the necessary background on medial and related idempotents. Seeking for completeness and a self-contained text, we demonstrate in this section - and perhaps elsewhere - proofs for some known (or almost known) results. The main result of the paper is contained in Section 4, where we show that any abundant semigroup with an ample multiplicative medial idempotent can be naturally ordered with a maximum idempotent such that $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant. We revisit in Section 5 the structure of abundant semigroups that possess ample normal medial idempotents and give an additional order to that presented in Section 4.

Any undefined notation and terminology be as in [15].

2 Abundant semigroups

We shall review in this section some of the basic concepts which will be used frequently throughout the paper. The main theme is abundant semigroups. The study of such semigroups was initiated by Fountain [11], though he introduced the concept earlier in [10]. The investigation of this class of semigroups relies heavily on the basic facts of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$. *The relation* $\mathcal{L}^*$ is defined on a semigroup S by:

$$\mathcal{L}^* = \{(a,b) \mid \text{for all } s,t \in S^1 \ as = at \iff bs = bt\},$$

and *the relation* $\mathcal{R}^*$ is defined dually. Evidently, $\mathcal{L}^*$ is right congruence and $\mathcal{R}^*$ is left congruence.

As a consequence of the definition we have from [11] the following result.

Lemma 2.1. *If e is an idempotent of a semigroup S, then for any $a \in S$; $a\ \mathcal{L}^*\ e$ if and only if the following two statements hold:*
(i) *$ae = a$;*
(ii) *$as = at$ implies $es = et$; for any $s,t \in S^1$.*

A dual statement of Lemma 2.1 holds for $\mathcal{R}^*$. A semigroup S is *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent. It has been practiced that abundant semigroups may be treated analogously to regular semigroups (see [9], [11] and [18]). At the same time, the class of abundant semigroups properly contains the class of regular semigroups. It is well known and easy to see that for any semigroup S

$$\mathcal{L} \subseteq \mathcal{L}^* \quad \text{and} \quad \mathcal{R} \subseteq \mathcal{R}^*$$

where $\mathcal{L}$ and $\mathcal{R}$ are the well know Green's relations on S. Equality holds when S is regular.

An abundant semigroup S with set of idempotents E is said to be *quasi-adequate* if E is a band and is *adequate* if E is a semilattice. The class of quasi-adequate (adequate) semigroups is an analogue to and properly contains the class of orthodox (inverse) semigroups. The study of adequate semigroups in general was initiated by Fountain [10], and the investigation of the class of quasi-adequate semigroups was initiated by El-Qallali and Fountain [9].

The $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) containing an element a of a semigroup S is denoted by L_a^* (R_a^*) or $L_a^*(S)$ ($R_a^*(S)$) in case of ambiguity. For an element a of an abundant semigroup S, a typical idempotent in L_a^* (R_a^*) is denoted by a^* ($a^\dagger$).

From [10] we have the following result.

Proposition 2.2. *If S is an adequate semigroup with semilattice of idempotents E, then for any elements $a, b \in S$:*
(i) *$a\ \mathcal{L}^*\ b$ ($a\ \mathcal{R}^*\ b$) if and only if $a^* = b^*$ ($a^\dagger = b^\dagger$);*
(ii) *$(ab)^* = (a^*b)^*$ and $(ab)^\dagger = (ab^\dagger)^\dagger$;*
(iii) *$(ab)^*b^* = (ab)^*$ and $a^\dagger(ab)^\dagger = (ab)^\dagger$.*

It is noticeable that for any semigroup homomophism $\phi : S \to T$, and for any $a, b \in S$:

$$a\ \mathcal{L}(S)\ b \implies a\phi\ \mathcal{L}(T)\ b\phi, \text{ and } a\ \mathcal{R}(S)\ b \implies a\phi\ \mathcal{R}(T)\ b\phi.$$

But in general

$$a\ \mathcal{L}^*(S)\ b \not\Longrightarrow a\phi\ \mathcal{L}^*(T)\ b\phi, \text{ and } a\ \mathcal{R}^*(S)\ b \not\Longrightarrow a\phi\ \mathcal{R}^*(T)\ b\phi.$$

The following example illustrates this observation.

Example 2.3 (see [7])**.** Let A be a cancellative monoid and i its only unit. Let α be the relation defined on A by the rule: $(x, y) \in \alpha$ if and only if $x = i = y$ or $x \neq i \neq y$. Then α is a congruence on A where equivalence classes are: $\{i\}$ and $A \backslash \{i\}$. Let B be the semigroup $\{0, 1\}$ under the usual multiplication. Let ψ be the semigroup isomorphism from A/α onto B. Notice that for any $a \in A \backslash \{i\}$, we have $(i, a) \in \mathcal{L}^*(A)$. However $(i\alpha)\psi = 1$ and $(a\alpha)\psi = 0$. As $(1, 0) \notin \mathcal{L}^*(B)$ then $(i\alpha, a\alpha) \notin \mathcal{L}^*(A/\alpha)$.

In [8], the following restricted class of homomorphisms is considered. A homomorphism $\phi : S \to T$ of semigroups is said to be an *admissible homomorphism* if for any elements $a, b \in S$,

$$a\ \mathcal{L}^*(S)\ b \implies a\phi\ \mathcal{L}^*(T)\ b\phi \quad \text{and} \quad a\ \mathcal{R}^*(S)\ b \implies a\phi\ \mathcal{R}^*(T)\ b\phi.$$

Recall that the natural homomorphism of A onto (A/α) of Example 2.3 is not admissible.

As a direct consequence of the following proposition any admissible homomorphic image of an abundant semigroup is abundant.

Proposition 2.4 ([8]). *Let S be an abundant semigroup and $\phi : S \to T$ be a semigroup homomorphism. Then ϕ is admissible if and only if for each $a \in S$ there are idempotents $e, f \in S$ with $e \in L_a^*$, $f \in R_a^*$ such that:*

$$a\phi \ \mathcal{L}^*(T) \ e\phi \quad \textit{and} \quad a\phi \ \mathcal{R}^*(T) \ f\phi.$$

Analogously; a congruence ρ on an abundant semigroup S is *admissible* if for any $a \in S$, and any $s, t \in S^1$

$$(as, at) \in \rho \implies (a^*s, a^*t) \in \rho \ \text{ for some } a^* \text{ (hence for any } a^*)$$

and

$$(sa, ta) \in \rho \implies (sa^\dagger, ta^\dagger) \in \rho \ \text{ for some } a^\dagger \text{ (hence for any } a^\dagger).$$

Obviously, any congruence on any regular semigroup is admissible and any cancellative congruence on any abundant semigroup is admissible. The congruence α on the cancellative monoid A of Example 2.3 is not admissible.

A subsemigroup U of a semigroup S is a $*$-*subsemigroup* of S if for any $a \in U$, there exist idempotents $e, f \in U$ such that $a \ \mathcal{L}^*(S) \ e$ and $a \ \mathcal{R}^*(S) \ f$. Clearly any $*$-subsemigroup is abundant.

The following lemma could be concluded from [8].

Lemma 2.5. *Let S be an abundant semigroup with set of idempotents E. Then for any $e \in E$ the set eSe is a $*$-subsemigroup of S.*

Proof. Obviously eSe is a subsemigroup of S. Let a be an element in eSe with $f \ \mathcal{L}^*(S) \ a$ for some $f \in E$. As $ae = a = af$, then $fe = f$ and ef is an idempotent in eSe. Moreover, $aef = af = a$ and for any $s, t \in S^1$:

$$as = at \implies fs = ft \implies efs = eft.$$

By Lemma 2.1, $ef \ \mathcal{L}^* \ a$. Together with dual argument, this gives that eSe is a $*$-subsemigroup. $\square$

The easy proof of the following corollary is omitted.

Corollary 2.6. *If S is adequate then for any idempotent e in S; eSe is adequate.*

An adequate semigroup S is called an *ample* semigroup if for any element a and any idempotent e in S

$$ea = a(ea)^* \quad \text{and} \quad ae = (ae)^\dagger a.$$

The following corollary is an easy consequence of Corollary 2.6.

Corollary 2.7. *If S is ample then for any idempotent e in S; eSe is ample.*

Let $\langle E \rangle$ be the semiband generated by a set of idempotents E. It is noted in [10] that a semiband may be abundant but not regular as demonstrated by the following example.

Example 2.8. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; and put

$$S = \{2^m A, 2^n B, 2^n C, 2^n D \mid m \geqslant 1, n \geqslant 0\}.$$

It is easy to see that S is a semigroup under matrix multiplication. Further S is generated by B and C and these elements are the only idempotents in S. It is routine to check that the $\mathcal{L}^*$-classes of S are: $\{2^m A, 2^n B \mid m \geqslant 1, n \geqslant 0\}$ and $\{2^n C, 2^n D \mid n \geqslant 0\}$, and the $\mathcal{R}^*$-classes are: $\{2^m A, 2^n C \mid m \geqslant 1, n \geqslant 0\}$ and $\{2^n B, 2^n D \mid n \geqslant 0\}$. Thus each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent. Therefore S is an abundant semiband. Clearly, it is not regular.

Also, not every semiband is abundant, we illustrate this by an example from [1] as follows.

Example 2.9. Choose a seven element semiband B with representation $\langle a, b \mid a^2 = a, b^2 = b, (ab)^2 = (ba)^2 \rangle$ from the list of all semibands of type two in [1, Theorem 2]. Then $B = \{a, b, ab, ba, (ab)^2, bab, aba\}$ and $E = \{a, b, (ab)^2\}$ is the set of idempotents of B. It is routine to check that neither L^*_{ab} nor R^*_{ab} has an idempotent, so B is not abundant.

The class of ample semigroups properly contains the class of inverse semigroups and it is contained properly in the class of adequate semigroups (see [10] for further details). We follow Lawson [18] in saying that a semigroup S with set of idempotents E satisfies the *regularity condition* if $\langle E\rangle$ is a regular subsemigroup. The semigroup S of Example 2.8 is an abundant semigroup which does not satisfy the regularity condition. However, if S is regular or quasi-adequate then it satisfies the regularity condition [9].

On any semigroup S, it is well known that its set of idempotents E is partially ordered by ω, where

$$e \,\omega\, f \ \text{ if and only if } \ e = ef = fe \ \text{ for any } \ e, f \in E.$$

This order is called the *natural order* on E. A partial order $\leqslant$ on S - as we recall from the introduction - is a *natural partial order* if it extends ω in the sense that

$$e \,\omega\, f \implies e \leqslant f.$$

Recall that the natural partial order on S is a *natural order* if it is compatible - on both sides - with the binary operation of S.

Let S be an abundant semigroup with set of idempotents E. *The two relations* $\leqslant_l$ *and* $\leqslant_r$ are defined for any $x, y \in S$ by $x \leqslant_l y$ if and only if

$$\text{for any } \ y^* \ \text{ there is an } \ x^* \ \text{ such that } \ x^* \,\omega\, y^* \ \text{ and } \ x = yx^*,$$

and $x \leqslant_r y$ if and only if

$$\text{for any } \ y^\dagger \ \text{ there is an } \ x^\dagger \ \text{ such that } \ x^\dagger \,\omega\, y^\dagger \ \text{ and } \ x = x^\dagger y.$$

These are natural partial order relations on S [18].

Put $\eta = \leqslant_l \cap \leqslant_r$. The relation η is a natural partial order on S. In general η is not compatible with the binary operation of S. If S is idempotent-connected (as defined in [8]) and $\langle E\rangle$ is a regular subsemigroup then η is compatible on S if and only if S is locally ample [18], that is each local submonoid eSe for $e \in E$ is an ample subsemigroup. In particular, if S is ample, so it is idempotent-connected [8], then as eSe is ample for any idempotent e in S (Corollary 2.7), the relation η is a natural order on S. The order relation η on S can be redefined (see [18]); for any $x, y \in S$:

$$x \,\eta\, y \ \text{ if and only if } \ ey = x = yf \ \text{ for some idempotents } \ e, f \in S.$$

For any $x, y \in S$ we get by the fact that S is ample the equivalence of the following two statements:

(i) There exists $e \in E$ such that $x = ey$;

(ii) There exists $f \in E$ such that $x = yf$.

Clearly, $e\ \eta\ f$ if and only if $e\ \omega\ f$ for any $e, f \in E$. So we can state for reference the following proposition without any further argument which will be in use frequently.

Proposition 2.10. *Let S be an ample semigroup with semilattice of idempotents E. Then the relation η defined for any $x, y \in S$ by the rule: $x\ \eta\ y$ if and only if $x = ey$ for some $e \in E$, is a natural order on S.*

Corollary 2.11. *Let S and η be as in Proposition 2.10. Then for any $x, y \in S$:*

$$x\ \eta\ y \quad \textit{implies} \quad x^\dagger\ \omega\ y^\dagger \quad \textit{and} \quad x^*\ \omega\ y^*.$$

Proof. For any $x, y \in S$:

$$\begin{aligned} x\ \eta\ y &\Longrightarrow ey = x = yf \quad \text{for some} \quad e, f \in E \\ &\Longrightarrow (ey)^\dagger = x^\dagger,\ x^* = (yf)^* \\ &\Longrightarrow ey^\dagger = x^\dagger,\ x^* = y^* f \\ &\Longrightarrow x^\dagger = x^\dagger y^\dagger,\ x^* = y^* x^* \\ &\Longrightarrow x^\dagger\ \omega\ y^\dagger,\ x^*\ \omega\ y^*. \end{aligned}$$

Hence the result holds. □

3 Medial and related idempotents

The concept of medial idempotents was introduced by Blyth and McFadden [5], and used in constructing classes of regular semigroups. It has attracted several authors and has been applied in investigating not only the class of regular semigroups, but also in studying the structure of classes of abundant semigroups. As examples of this approach the reader may refer to [6], [16], [17] and [20]. The subject of this section is to concentrate on some consequences of the existence of medial or related idempotents in certain class of semigroups.

Let S be a semigroup and $E(S)$ be the set of idempotents in S. For $E = E(S)$ let $\langle E\rangle$ be the semiband generated by E. From now on for the purpose of this paper an idempotent u is called a *medial idempotent* if $eue = e$ for any $e \in E$. In this case the following terminologies will be adopted which may be slightly different from the ones that appear in the literature.

(1) The medial idempotent u is *strong medial* if $eue = e$ for any $e \in \langle E\rangle$.

(2) The medial idempotent u is *normal medial* if u is strong medial and $u\langle E\rangle u$ is a semilattice [5].

(3) The medial idempotent u is *ample normal medial* if u is normal medial and uSu is ample (strong normal in the sense of [17]).

(4) The medial idempotent u is *ample (adequate) medial* if uSu is ample (adequate).

(5) The medial idempotent u is *band medial* if uEu is a band [20].

(6) The medial idempotent u is *multiplicative medial* if $uefu \in E$ for any $e, f \in E$.

(7) The medial idempotent u is *ample multiplicative medial* if u is multiplicative medial and uSu is ample.

Let S be a semigroup with set of idempotents E containing a medial idempotent u. Observe that Su is a subsemigroup of S, and for any $e \in E$, eu is an idempotent in Su. Also for any idempotent g in Su, $g \in Eu$. The following lemma is evident.

Lemma 3.1. *If u is a medial idempotent then:*
(i) *Su, uS and uSu are subsemigroups of S;*
(ii) *$E(Su) = Eu$, $E(uS) = uE$, and $E(uSu) = uEu$.*

Lemma 3.2. *If u is strong medial, then it is multiplicative medial.*

Proof. Let $e, f \in E$. Then $ef \in \langle E\rangle$. As u is strong medial $efuef = ef$. That is $uefuuefu = uefu$ so $uefu \in E$. Hence the result holds. □

Lemma 3.3. *The medial idempotent u is multiplicative medial if and only if $u\langle E\rangle u = uEu$.*

Proof. Assume u is multiplicative medial and let $x \in u\langle E\rangle u$. Choose $z \in \langle E\rangle$ such that $x = uzu$. There exist $e_1, e_2, \ldots, e_n \in E$ and $z = e_1e_2\ldots e_n$. Since $eue = e$ for any $e \in E$, and by the hypothesis condition;

$$ue_1e_2u \in E,\ ue_2e_3u \in E,\ \ldots,\ ue_{n-1}e_nu \in E.$$

Thus

$$ue_1ue_1e_2u \in E,\ ue_1ue_1e_2ue_2e_3u \in E,\ \ldots,$$
$$ue_1ue_1e_2ue_2e_3u\ldots ue_{n-1}e_nu \in E,$$

so that,

$$ue_1e_2e_3\ldots e_{n-1}e_nu \in E,\ uzu \in E,\quad \text{and}\quad x \in uEu.$$

Therefore $u\langle E\rangle u \subseteq uEu$. However it is clear that $uEu \subseteq u\langle E\rangle u$. Hence, $u\langle E\rangle u = uEu$.

On the other hand, if $u\langle E\rangle u = uEu$, then for any $e, f \in E$; $uefu \in u\langle E\rangle u$ and there exists $h \in E$ such that $uefu = uhu$. As $uhuuhu = uhu$ then $uefu \in E$ and u is multiplicative medial. □

Corollary 3.4. *If u is multiplicative medial then u is band medial.*

Proof. Let u be multiplicative medial. By Lemma 3.3; $u\langle E\rangle u = uEu$. Recall from Lemma 3.1 that $uEu \subseteq E$. If x and y are two elements in uEu, $x = ueu$, $y = ufu$ for some $e, f \in E$. Then $euf \in \langle E\rangle$ and $ueufu \in uEu$. So there is an idempotent z such that $u(euf)u = uzu$, so then $xy = ueuufu = u(euf)u = uzu$ and $uzu \in E$. Therefore xy is an idempotent belonging to uEu. Hence the result holds. □

If u is a strong medial idempotent, then it is easy to observe that for any $x \in \langle E\rangle$, uxu is an idempotent which is an inverse of x in $\langle E\rangle$. This clarifies the first part of the following proposition.

Proposition 3.5. *If u is strong medial then:*

(1) $\langle E\rangle$ *is a regular semigroup;*
(2) $\langle E\rangle u = Eu$;
(3) $u\langle E\rangle = uE$;
(4) $u\langle E\rangle u = uEu$.

Moreover, Eu, uE and uEu are bands.

Proof. (1) This is from the previous remark.

(2) For any $z \in \langle E \rangle$; $zuzu = zu$ so that $zu \in E$ and $zu \in Eu$. Therefore $\langle E \rangle u \subseteq Eu$. Obviously $Eu \subseteq \langle E \rangle u$. Hence $Eu = \langle E \rangle u$.

(3) and (4)] follow similarly to (2).

For any $e, f \in E$

$$(eufu)(eufu) = (euf)u(euf)u = eufu \qquad (euf \in \langle E \rangle).$$

Then Eu is a band. Likewise uE and uEu are bands. □

The following result from [4] recognizes a medial idempotent in any naturally ordered semigroup which contains a maximum idempotent.

Proposition 3.6. *If $(S, \leqslant)$ is a naturally ordered semigroup with set of idempotents E which contains a maximum idempotent u, then u is medial.*

Proof. Since for any $e \in E$, as $e \leqslant u$, then $eu \leqslant u$ and $eueu \leqslant euu = eu$. Also $e \leqslant eu$ and $eeu \leqslant eueu$, that is $eu \leqslant eueu$. Hence $eueu = eu$ so that $eue = eueeue$ and $eue \in E$. Now $eeue = eue = euee$, and by the natural order of S $eue \leqslant e$. But also $e \leqslant u$ which implies $e \leqslant eue$. Hence $eue = e$. □

Again as in [4] we have the following proposition.

Proposition 3.7. *Let $(S, \leqslant)$ be a naturally ordered semigroup with set of idempotents E containing a maximum idempotent u. Then for any $e, f \in E$:*

(i) *ue is the maximum idempotent in L_e;*

(ii) *$f \mathcal{L} e$ if and only if $uf = ue$;*

(iii) *eu is the maximum idempotent in R_e;*

(iv) *$f \mathcal{R} e$ if and only if $fu = eu$.*

Proof. By Proposition 3.6, it follows directly from the hypothesis that $ue \in E$ and $e \mathcal{L} ue$ for any $e \in E$. If $e, f \in E$ such that $e \mathcal{L} f$, then $f \leqslant u$, $f = fe \leqslant ue$, so (*i*) holds, and then (*ii*) is evident. (*iii*) and (*iv*) follow similarly. □

In what follows - for the rest of the section - let S be an abundant semigroup with set of idempotents E containing a medial idempotent u. The easy proof of the following lemma is omitted.

Lemma 3.8. *For any $a \in S$; $a^{\dagger}ua = a = aua^{*}$.*

The following technical lemma which appears in [20] will be in use frequently.

Lemma 3.9. *For any $a \in S$:*
(1) $a\ \mathcal{L}^*\ ua\ \mathcal{L}^*\ ua^*$;
(2) $a\ \mathcal{R}^*\ au\ \mathcal{R}^*\ a^\dagger u$;
(3) $ua^*u\ \mathcal{L}^*\ uau\ \mathcal{R}^*\ ua^\dagger u$.

Proof. Since $\mathcal{L}^*$ is right congruence and $\mathcal{R}^*$ is left congruence, then (3) is a direct consequence of (1) and (2). As (2) is dual to (1), it suffices to prove (1). Since for any $s, t \in S^1$

$$as = at \implies uas = uat \implies a^\dagger uas = a^\dagger uat \implies as = at.$$

Then $a\ \mathcal{L}^*\ ua$. Also

$$\begin{aligned} as = at &\implies a^*s = a^*t \implies ua^*s = ua^*t \\ &\implies aua^*s = aua^*t \implies as = at, \end{aligned}$$

and $a\ \mathcal{L}^*\ ua^*$. Hence the result holds. □

Proposition 3.10. *The subsemigroups Su and uS are $*$-subsemigroups of S.*

Proof. Let $xu \in Su$ where $x \in S$. Notice that $x^*u\ \mathcal{L}^*(S)\ xu$ and $x^*u \in Su \cap E$. By Lemma 3.9(2); $x^\dagger u\ \mathcal{R}^*(S)\ xu$ and $x^\dagger u \in Su \cap E$. Hence Su is a $*$-subsemigroup of S.

Similarly uS is a $*$-subsemigroup of S. □

The following corollary appears in [17] which is a direct consequence of Lemmas 2.5 and 3.1 and Propositions 3.5 and 3.10.

Corollary 3.11. *If u is a strong medial idempotent, then Su, uS and uSu are quasi-adequate semigroups. Moreover*

$$\begin{gathered} E(uS) = u\langle E\rangle = uE,\ E(Su) = \langle E\rangle u = Eu \\ \text{and}\ \ E(uSu) = u\langle E\rangle u = uEu. \end{gathered}$$

Proposition 3.12. *If u is a strong medial idempotent, then uSu is adequate if and only if uEu is a semilattice.*

Proof. As uSu is a $*$-subsemigroup of S (Lemma 2.5) so it is abundant and by Corollary 3.11 $E(uSu) = uEu$. Hence the result holds. □

Proposition 3.13. *If u is a normal medial idempotent, then:*

(1) *Eu is a left normal band which is a set of representatives of the $\mathcal{R}^*$-classes of S;*

(2) *uE is a right normal band which is a set of representatives of the $\mathcal{L}^*$-classes of S.*

Proof. As (2) is dual to (1) it is suffices to prove (1). Clearly $Eu \subseteq E$. By Proposition 3.5 Eu is a band. Moreover, for any $eu, fu, gu \in Eu$ $(e, f, g \in E)$

$$eufugu = euguufu = eugufu$$

(uEu is a semilattice as a consequence of Lemmas 3.2 and 3.3) and Eu is a left normal band.

Let $x \in S$ and $e, f \in E$ such that $e \mathrel{\mathcal{R}^*} x \mathrel{\mathcal{R}^*} f$. By Lemma 3.9; $x \mathrel{\mathcal{R}^*} xu$ so $e \mathrel{\mathcal{R}^*} xu$ and $ue \mathrel{\mathcal{R}^*} uxu$, where $uxu \in uSu$, and $ux^\dagger u \mathrel{\mathcal{R}^*} uxu$. The idempotents $ux^\dagger u$ and ueu are in $E(uSu)$. Notice that $ue \in E$, $ue \mathrel{\mathcal{R}} ux^\dagger u$ and $ue = (ux^\dagger u)ue$. Thus

$$ueu = (ux^\dagger u)(ueu) = ueuux^\dagger u = ueux^\dagger u = ux^\dagger u.$$

Likewise $ufu = ux^\dagger u$. Hence $ueu = ufu$.

Since $eu \mathrel{\mathcal{R}} e$, $fu \mathrel{\mathcal{R}} f$ and $eu, fu \in Eu$, then eu and fu are idempotents $\mathcal{R}^*$-related to x. Thus $eu \mathrel{\mathcal{R}} fu$. But also $eu \mathrel{\mathcal{L}} ueu = ufu \mathrel{\mathcal{L}} fu$. Therefore $eu = fu$. Hence the set Eu is as required. □

Fountain [10] provides a semigroup $H = A \cup B \cup \{v\}$ where A is the infinite cyclic semigroup generated by an element a, B is the infinite cyclic monoid generated by an element b and $b^0 = v$ where the binary operation on H extends that of A and B as follows

$$a^m b^m = a^{n+m}, \quad b^n a^m = a^{n+m}; \quad \text{for } m > 0,\ n \geqslant 0.$$

The semigroup H is adequate but is not ample, and v is an adequate medial idempotent ($vHv = H$) which is not an ample medial idempotent.

If u is an ample medial idempotent, then as described in Proposition 2.10, η is a natural order relation on uSu. For any $e \in E(uSu)$; $eu = e$ so that $e \mathrel{\eta} u$, and u is a maximum idempotent in uSu with respect to the order η. For reference we state the following result.

Proposition 3.14. *If S is an abundant semigroup with an ample medial idempotent u, then (uSu, η) is a naturally ordered ample semigroup with a maximum idempotent u.*

4 Naturally ordering an abundant semigroup

Let S be an abundant semigroup with set of idempotents E containing an ample multiplicative medial idempotent u. It follows from Section 3 that:

(1) $E(uSu) = uEu = u\langle E\rangle u$ (Lemmas 3.1 and 3.3), denote this set by E^0, it is clearly a semilattice;

(2) uSu is a $*$-subsemigroup of S (Lemma 2.5);

(3) (uSu, η) is a naturally ordered ample semigroup with a maximum idempotent u (Proposition 3.14).

The aim of this section is to naturally order S in a way to have a maximum idempotent with respect to the imposed order. This will be a generalization to abundant semigroups of the result of McAlister and McFadden [19] in regular semigroups.

Consider

$$T = \{(e, x, f) \in Eu \times uSu \times uE \mid uex = x = xfu\}.$$

Notice that for any $e \in Eu$, $f \in uE$ it follows that $fe = ufeu \in uSu$. In fact by the multiplicativety of u; $fe \in E^0$ and clearly T is a semigroup with respect to the binary operation defined by

$$(e, x, f)(g, y, h) = (e, xfgy, h) \text{ for any } (e, x, f) \text{ and } (g, y, h) \in T.$$

The following lemmas add more information.

Lemma 4.1. *The semigroup T is abundant.*

Proof. Let (e, a, f) be an element of T. As $a \in uSu$, there exist $a^*, a^\dagger \in E^0$ (uSu is a $*$-subsemigroup), and $uea = a$ which implies $uea^\dagger = a^\dagger$ so

$$a^\dagger = ueua^\dagger = a^\dagger ueua^\dagger = a^\dagger ea^\dagger.$$

Therefore $(e, a^\dagger, a^\dagger)$ is an idempotent in T.

Since $(e, a^\dagger, a^\dagger)(e, a, f) = (e, a^\dagger ea, f) = (e, a, f)$, for any (g, b, h) and (i, c, k) in T

$$\begin{aligned} (g, b, h)(e, a, f) = (i, c, k)(e, a, f) & \\ &\Longrightarrow (g, bhea, f) = (i, ckea, f) \\ &\Longrightarrow g = i,\ bhea = ckea \\ &\Longrightarrow g = i,\ bhea^\dagger = ckea^\dagger \\ &\Longrightarrow (g, b, h)(e, a^\dagger, a^\dagger) = (i, c, k)(e, a^\dagger, a^\dagger). \end{aligned}$$

Therefore $(e, a^\dagger, a^\dagger)\ \mathcal{R}^*\ (e, a, f)$ (dual of Lemma 2.1).

Similarly; (a^*, a^*, f) is an idempotent in T which is $\mathcal{L}^*$-related to (e, a, f). Hence the result holds. □

For T to play a role in ordering S, an order on T should be defined first. The obvious choice is to adjust the order on T by coordinates. For ordering the middle components of the elements of T we choose the natural order η (of Proposition 2.10) as considered on uSu. For the first components take the finest partial order on the set Eu and define $\leqslant_l$ on Eu to be:

$$e \leqslant_l g \ \text{ if and only if } \ e = g \ \text{ or } \ g = u; \quad e, g \in Eu.$$

Similarly, define $\leqslant_r$ on uE by:

$$f \leqslant_r h \ \text{ if and only if } \ f = h \ \text{ or } \ h = u; \quad f, h \in uE.$$

Accordingly, define $\leqslant$ on T by:

$$(e, a, f) \leqslant (g, b, h) \ \text{ if and only if } \ e \leqslant_l g,\ a\ \eta\ b,\ f \leqslant_r h.$$

for any (e, a, f) and (g, b, h) in T.

Lemma 4.2. *The relation $\leqslant$ is a natural order on the abundant semigroup T where (u, u, u) is a maximum idempotent.*

Proof. Clearly $\leqslant$ on T is a partial order. Let (e, a, f), (g, b, h) and (i, c, k) be elements in T such that $(e, a, f) \leqslant (g, b, h)$, that is $e \leqslant_l g$, $a\ \eta\ b$ and $f \leqslant_r h$. As u is multiplicative medial $fi, hi \in E^0$ and either $f = h$ so then

$fi = hi$ and - in particular - $fi\ \eta\ hi$, or $h = u$ so then $fi, u \in E^0$ and $fi\ \eta\ u$ (Proposition 3.14) so that $fiu = fi$ and $fiui = fi$. In either case $fi\ \eta\ hi$. Thus $afic\ \eta\ bhic$.

Hence $(e, a, f)(i, c, k) \leqslant (g, b, h)(i, c, k)$. Similarly $(i, c, k)(e, a, f) \leqslant (i, c, k)(g, b, h)$. Therefore the order $\leqslant$ on T is compatible.

Let (e, a, f) be an idempotent in T, that is $afea = a$, which implies $a^*fea = a^*$. As $a \in uSu$ (uSu is a $*$-subsemigroup of S) then a^* can be chosen to be in E^0. Recall that $fe \in E^0$, and thus $a^* = (a^*fea)^\dagger = a^*fea^\dagger$. But also $afea^\dagger = a^\dagger$ with $a^\dagger \in E^0$. Therefore $a^\dagger = aa^*fea^\dagger = aa^* = a$ and $a \in E^0$.

If (g, b, h) is an idempotent in T ($b \in E^0$) such that $(e, a, f)\ \omega\ (g, b, h)$, that is

$$(e, a, f)(g, b, h) = (e, a, f) = (g, b, h)(e, a, f),$$

then $e = g$, $afgb = a = bhea$, and $h = f$. Hence $ab = a$ and $a\ \eta\ b$ so that $(e, a, f) \leqslant (g, b, h)$, and the order on T is natural.

It is clear (u, u, u) is an idempotent in T and for any idempotent (e, a, f) in T ($a \in E^0$), $a\ \eta\ u$ (Proposition 3.14) and $(e, a, f) \leqslant (u, u, u)$. $\square$

To relate the order relation $\leqslant$ on T to the algebraic structure of T we introduce the *abundancy condition* of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on any naturally ordered abundant semigroup in the following sense.

The relation $\mathcal{L}^*$ is said to be *abundant on any naturally ordered abundant semigroup* $(H, \leqslant)$ if for any two elements a and b in H $a \leqslant b$ implies $a^* \leqslant b^*$, for some idempotents a^* and b^*, $\mathcal{L}^*$-related to a and b respectively.

The relation $\mathcal{R}^*$ is *abundant on* $(H, \leqslant)$ is defined similarly.

Recall from Proposition 2.10, the natural order relation η is defined on any ample semigroup M. It follows from Corollary 2.11 that $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant on (M, η).

Corollary 4.3. *The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant on $(T, \leqslant)$.*

Proof. Let (e, a, f) and (g, b, h) be elements of T such that $(e, a, f) \leqslant (g, b, h)$, then $a\ \eta\ b$, $f \leqslant_r h$. From Corollary 2.11; $a\ \eta\ b$ implies $a^*\ \eta\ b^*$ that is $a^*b^* = a^*$ ($a^*, b^* \in E^0$). As $afu = a$, then $a^*fu = a^*$, $a^*fa^* = a^*$ and $a^*fb^*a^* = a^*$ so that (b^*, a^*, f) is an idempotent in T and $(b^*, a^*, f) \leqslant (b^*, b^*, h)$. It is routine to check that $(a^*, a^*, f)\ \mathcal{L}\ (b^*, a^*, f)$. So we conclude from the proof of Lemma 4.1 that $(b^*, a^*, f)\ \mathcal{L}^*\ (e, a, f)$ and $(b^*, b^*, h)\ \mathcal{L}^*\ (g, b, h)$. Hence $\mathcal{L}^*$ is abundant on T.

By a similar argument it follows that $\mathcal{R}^*$ is abundant on T. $\square$

Let $\theta : T \to S$ be defined by $(e, a, f)\theta = eaf$. Obviously θ is a homomorphism. Since for any $x \in S$ and any $x^\dagger$, x^*; we have $x^\dagger u \in Eu$, $ux^* \in uE$ and $ux^\dagger uxu = uxu = uxux^*u$ (see Lemma 3.8). Therefore, $(x^\dagger u, uxu, ux^*) \in T$ and $(x^\dagger u, uxu, ux^*)\theta = x$. So then θ is an epimorphism. Moreover, if (e, a, f) and (g, b, h) are elements of T such that $(e, a, f)\theta = (g, b, h)\theta$, that is $eaf = gbh$, then $ueaf = ugbh$ which implies $af = bh$, and $eafu = gbhu$ which implies $ea = gb$, so $a = b$. Hence

$$(e, a, f)\theta = (g, b, h)\theta \text{ if and only if } ea = gb,\ a = b \text{ and } af = bh.$$

Lemma 4.4. *For any element (e, a, f) in T we have: a^*f, $ea^\dagger \in E$ and $a^*f\ \mathcal{L}^*\ eaf\ \mathcal{R}^*\ ea^\dagger$ in S, where a^* and $a^\dagger$ are chosen to be in E^0.*

Proof. Let $a \in uSu$ and a^* be chosen to be in E^0 so that $a^*, fu \in E^0$ and

$$a^*fa^*f = a^*fua^*f = a^*a^*fuf = a^*f,$$

therefore $a^*f \in E$. Notice that for any $s, t \in S^1$:

$$\begin{aligned} eafs = eaft &\implies ueafs = ueaft \\ &\implies afs = aft \qquad \text{(definition of T)} \\ &\implies a^*fs = a^*ft \\ &\implies aa^*fs = aa^*ft \\ &\implies eafs = eaft \end{aligned}$$

Therefore $a^*f\ \mathcal{L}^*\ eaf$. Similarly, $ea^\dagger$ is an idempotent $\mathcal{R}^*$-related to eaf. $\square$

Corollary 4.5. *The homomorphism θ is admissible.*

Proof. Let (e, a, f) and (g, b, h) be in T such that $(e, a, f)\ \mathcal{L}^*\ (g, b, h)$. Conclude from the proof of Lemma 4.1 that (a^*, a^*, f) and (b^*, b^*, h) are idempotents in T where a^* and b^* are chosen to be in E^0 and

$$(a^*, a^*, f)\ \mathcal{L}^*\ (e, a, f), \quad (g, b, h)\ \mathcal{L}^*\ (b^*, b^*, h).$$

So that $(a^*, a^*, f)\ \mathcal{L}\ (b^*, b^*, h)$. Thus

$$(a^*, a^*fb^*, h) = (a^*, a^*, f) \ \text{ and } \ (b^*, b^*ha^*, f) = (b^*, b^*, h).$$

Therefore $a^*fb^* = a^*$, $h = f$ and $b^*ha^* = b^*$ where $a^*f, b^*h \in E$ (Lemma 4.4). Hence $a^*fb^*h = a^*h = a^*f$ and $b^*ha^*f = b^*h$ and then $a^*f\ \mathcal{L}\ b^*h$.

By Lemma 4.4

$$(e, a, f)\theta = eaf\ \mathcal{L}^*\ a^*f\ \mathcal{L}\ b^*h\ \mathcal{L}^*\ gbh = (g, b, h)\theta.$$

Similarly

$$(e, a, f)\ \mathcal{R}^*\ (g, b, h) \implies (e, a, f)\theta\ \mathcal{R}^*\ (g, b, h)\theta.$$

Hence the result holds. □

Let ρ be the kernel of θ, that is for any two elements (e, a, f) and (g, b, h) in T:

$$(e, a, f)\ \rho\ (g, b, h) \ \text{ if and only if } \ eaf = gbh$$

Obviously ρ is a congruence on T. Since θ is epimorphism, T/ρ is isomorphic to S. Similar to Corollary 4.5, we have the following.

Lemma 4.6. *The congruence ρ is admissible.*

Proof. Let (e, a, f), (g, s, h) and (i, t, j) be elements in T. Then

$$\begin{aligned}
&((e, a, f)(g, s, h), (e, a, f)(i, t, j)) \in \rho \\
&\qquad \implies eafgsh = eafitj \\
&\qquad \implies a^*fgsh = a^*fitj \qquad \text{(Lemma 4.4)} \\
&\qquad \implies ((a^*, a^*, f)(g, s, h), (a^*, a^*, f)(i, t, j)) \in \rho.
\end{aligned}$$

Similarly

$$\begin{aligned}
&(g, s, h)(e, a, f), (i, f, j)(e, a, f)) \in \rho \\
&\qquad \implies ((g, s, h)(e, a^\dagger, a^\dagger), (i, t, j)(e, a^\dagger, a^\dagger)) \in \rho.
\end{aligned}$$

Hence the result holds. □

A *closed bracelet modulo* ρ is a finite subset A of T consisting of $2(n+1)$ (for n a positive integer) elements

$$\{a_1, a_2, \ldots, a_n, a_{n+1}, b_1, b_2, \ldots, b_n, b_{n+1}\}$$

satisfying:

$$a_1 \equiv b_1 \leqslant a_2 \equiv b_2 \leqslant \ldots \leqslant a_{n-1} \equiv b_{n-1} \leqslant a_n \equiv b_n \leqslant a_{n+1} \equiv b_{n+1} \leqslant a_1,$$

where $\equiv$ denotes equivalence module ρ, provided that $b_i \neq a_{i+1}$; for $i = 1, 2, \ldots, n$ and $b_{n+1} \neq a_1$. In this case n is said to be the *length of the closed bracelet.* Notice that $n = m-1$ where m is the number of $\leqslant$'s in A. We may denote this closed bracelet modulo ρ by $\mathcal{A}$. A *sub-closed bracelet* of $\mathcal{A}$ is a subset

$$B = \{c_1, d_1, c_2, d_2, \ldots, c_{m-1}, d_{m-1}, c_m, d_m\}$$

of A such that B itself can be set up as a closed bracelet modulo ρ. The use of bracelets in contexts of this kind was introduced in [3].

Lemma 4.7. *All the elements of every closed bracelet modulo ρ belong to the same ρ-class.*

Proof. The proof is by induction. Consider the closed bracelet $\mathcal{A}$ modulo ρ of length n, where

$$a_i = (e_i, x_i, f_i),\ b_i = (g_i, y_i, h_i); \quad i = 1, \ldots, n+1.$$

Recall that two elements (e, x, f) and (g, y, h) in T are ρ-equivalent if and only if $exf = gyh$, and this is so if and only if

$$ex = gy,\ xf = yh \ \text{ and } \ x = y.$$

It follows from of the order of T that

$$x_1 = y_1 \ \eta \ x_2 = y_2 \ \eta \ \ldots \ \eta \ x_{n-1} = y_{n-1} \ \eta \ x_n = y_n \ \eta \ x_{n+1} = y_{n+1} \ \eta \ x_1.$$

As η is a partial order we conclude that all the elements in the closed bracelet $\mathcal{A}$ have their middle components x_i, y_i equal. So we may write these elements in the following form

$$a_i = (e_i, x, f_i),\ b_i = (g_i, x, h_i).$$

To show that the statement of the lemma is true for any closed bracelet modulo ρ of length $n = 1$, consider a subset $\{a_1, a_2, b_1, b_2\}$ of T such that

$$a_1 \equiv b_1 \leqslant a_2 \equiv b_2 \leqslant a_1$$

where

$$b_1 \neq a_2,\ b_2 \neq a_1,\ a_i = (e_i, x, f_i),\ b_i = (g_i, x, h_i) \text{ for } i = 1, 2.$$

There are three cases to be considered in the comparison $b_2 \leqslant a_1$ (the case where $b_2 = a_1$ is excluded).

***Case* 1** $(e_1 = u = f_1)$**:** Here $x = e_1 x f_1 = g_1 x h_1$ while $g_1 \leqslant_l e_2$, $h_1 \leqslant_r f_2$. Since $b_1 \neq a_2$, there are only three subcases. The first of these is $e_2 = u = f_2$ and this immediately implies $a_1 = a_2$. The second is $g_1 = e_2$ and $f_2 = u$ which leads to $e_2 x = g_1 x = e_1 x$ $(b_1 \equiv a_1)$ and $x f_1 = x f_2$ $(f_1 = u = f_2)$. Hence $a_2 \equiv a_1$. The last subcase is $e_2 = u$ and $h_1 = f_2$ and is completely analogous to the preceding. Therefore, in these subcases, all the elements a_1, a_2, b_1, b_2 are ρ-related.

***Case* 2** $(e_1 = u,\ h_2 = f_1)$**:** We have in this case the following relations

$$x = e_1 x = g_1 x, \qquad g_1 \leqslant_l e_2.$$

There are precisely two ways in which $g_1 \leqslant_l e_2$, namely $g_1 = e_2$ or $e_2 = u$.

(i) If $g_1 = e_2$, then

$$x = e_1 x = g_1 x = e_2 x = g_2 x \quad (a_1 \equiv b_1, a_2 \equiv b_2).$$

(ii) If $e_2 = u$, then

$$x = e_1 x,\ x = e_2 x = g_2 x \quad (a_2 \equiv b_2).$$

Either way $e_1 x = g_2 x$. From the assumption we have $x f_1 = x h_2$. Therefore $a_1 \equiv b_2$. Hence also in this case all the elements a_1, a_2, b_1, b_2 are ρ-related.

***Case* 3** $(g_2 = e_1, f_1 = u)$**:** This is completely analogous to the case 2.

Therefore, the lemma is true for any closed bracelet modulo ρ of length $n = 1$. For the induction hypothesis, assume the statement of the lemma is true for any sub-closed bracelet of length $n-1$ of the closed bracelet $\mathcal{A}$ modulo ρ of length n. In the comparison $b_{n+1} \leqslant a_1$ where $b_{n+1} \neq a_1$ we have - as before - three cases to be considered.

Case* i** $(e_1 = u,\ f_1 = u)$:*** Here

$$x = e_1 x f_1 = g_1 x h_1 \quad (a_1 \equiv b_1).$$

From $b_1 \leqslant a_2$, it follows that $g_1 \leqslant_l e_2$ and $h_1 \leqslant_r f_2$. In this case, as $b_1 \neq a_2$ we have three subcases:

(i) $e_2 = u$ and $f_2 = u$: this implies $a_1 = a_2$;

(ii) $g_1 = e_2$ and $f_2 = u$: this leads to

$$e_2 x = g_1 x = e_1 x \quad (b_1 \equiv a_1)$$

but also $x f_2 = x u = x f_1$. thus $a_2 \equiv a_1$;

(iii) $e_2 = u$ and $h_1 = f_2$: this is an analogue of (ii).

These subcases lead to $a_1 \equiv a_2$, so we have the sub-closed bracelet

$$a_1 \equiv b_2 \leqslant \ldots \leqslant a_n \equiv b_n \leqslant a_{n+1} \equiv b_{n+1} \leqslant a_1$$

of length n-1. By the induction hypothesis all the elements

$$a_1, b_2, \ldots, a_{n-1}, b_{n-1}, a_n, b_n, a_{n+1}, b_{n+1}$$

belong to the same ρ-class. But also $a_2 \equiv a_1 \equiv b_1$. Thus all the elements of the closed bracelet $\mathcal{A}$ are ρ-related. Hence the statement of the lemma is true in this case.

Case* ii** $(e_1 = u,\ h_{n+1} = f_1)$:*** We have - in this case - the following relations

$$\begin{array}{cccccc}
x = e_1 x = g_1 x & (a_1 \equiv b_1) & \text{and} & g_1 \leqslant_l e_2 & (b_1 \leqslant a_2) \\
e_2 x = g_2 x & (a_2 \equiv b_2) & \text{and} & g_2 \leqslant_l e_3 & (b_2 \leqslant a_3) \\
\vdots & \vdots & & \vdots & \vdots \\
e_n x = g_n x & (a_n \equiv b_n) & \text{and} & g_n \leqslant_l e_{n+1} & (b_n \leqslant a_{n+1}) \\
e_{n+1} x = g_{n+1} x & (a_{n+1} \equiv b_{n+1}) & \text{and} & g_{n+1} \leqslant_l e_1 & (b_{n+1} \leqslant a_1).
\end{array}$$

When $g_1 \leqslant_l e_2$, either $g_1 = e_2$ and so

$$x = e_1 x = g_1 x = e_2 x = g_2 x$$

or $e_2 = u$ so then $x = e_2 x = g_2 x$ and $e_1 x = x = e_2 x$. In either case:

$$x = e_1 x = g_1 x = e_2 x = g_2 x.$$

Similarly, $g_2 \leqslant_l e_3$ implies

$$x = e_1 x = g_1 x = e_2 x = g_2 x = e_3 x = g_3 x.$$

It follows by a simple induction argument that

$$x = e_1 x = g_1 x = e_2 x = g_2 x = \cdots = e_n x = g_n x = e_{n+1} x = g_{n+1} x,$$

that is $g_{n+1} x = e_1 x$. As $h_{n+1} = f_1$, then $x h_{n+1} = x f_1$ and $b_{n+1} \equiv a_1$. Thus $a_{n+1} \equiv a_1$.

Thus we have the sub-closed bracelet

$$a_{n+1} \equiv b_1 \leqslant a_2 \equiv b_2 \leqslant \ldots \leqslant a_{n-1} \equiv b_{n-1} \leqslant a_n \equiv b_n \leqslant a_{n+1}$$

of length $n - 1$. By the induction hypothesis all the elements

$$a_{n+1}, b_1, a_2, \ldots, a_n, b_n$$

are ρ-related. But also $a_1 \equiv a_{n+1} \equiv b_{n+1}$. Hence all the elements of the closed bracelet $\mathcal{A}$ are ρ-related and the statement of the lemma is true in this case.

***Case* iii** $(g_{n+1} = e_1,\ f_1 = u)$**:** This is completely analogous to the previous case.

Hence the result holds. $\square$

The main part of the proof of Lemma 4.7 is essentially the same as that in [19]. Its presentation here is for completing the text.

An *open bracelet modulo* ρ is a finite subset

$$\{\bar{x}, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, \bar{y}\}$$

of T with $a_i = (e_i, x_i, f_i)$, $b_i = (g_i, y_i, h_i)$ for $i = 1, \dots, n$, $\bar{x} = (e, x, f)$ and $\bar{y} = (g, y, h)$ satisfying

$$\bar{x} \leqslant a_1 \equiv b_1 \leqslant a_2 \equiv b_2 \leqslant \dots \leqslant a_n \equiv b_n \leqslant \bar{y}$$

In this $\bar{x}$ is called the *initial clasp* and $\bar{y}$ the *terminal clasp*.

The following corollary is a result similar to that of [3, Theorem 6.1]. We present it here with its direct proof.

Corollary 4.8. *The abundant semigroup T/ρ can be partially ordered in such a way that the natural map $T \to T/\rho$ is isotone.*

Proof. Define $\leqslant_\rho$ on T/ρ for any two elements $x\rho, y\rho$ $(x, y \in T)$ in T/ρ by $x\rho \leqslant_\rho y\rho$ if and only if there are $2n$ (n a positive integer) elements $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ in T such that

$$x \leqslant a_1 \equiv b_1 \leqslant a_2 \equiv b_2 \leqslant \dots \leqslant a_n \equiv b_n \leqslant y.$$

That is, there is an open bracelet modulo ρ with initial clasp x and terminal clasp y.

It is obvious that $x\rho \leqslant_\rho x\rho$ for any $x\rho \in T/\rho$ so the relation $\leqslant_\rho$ on T/ρ is reflexive. Equally clear is that it is transitive. Suppose $x\rho \leqslant_\rho y\rho$ and $y\rho \leqslant_\rho x\rho$. Then there are elements

$$a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_k, d_1, \dots, d_k$$

in T such that

$$x \leqslant a_1 \equiv b_1 \leqslant a_2 \equiv \dots a_n \equiv b_n \leqslant y$$

and

$$y \leqslant c_1 \equiv d_1 \leqslant c_2 \equiv \dots \leqslant c_k \equiv d_k \leqslant x.$$

Notice that if for some positive integer t, $b_t = a_{t+1}$, then we may delete the elements b_t, a_{t+1} from the sequence and renumber the rest accordingly. Thus clearly we can assume without loss of generality that for $i = 1, \dots, n$ and $j = 1, \dots, k$

$$b_i \neq a_{i+1},\ d_j \neq c_{j+1},\ x \neq a_1,\ b_n \neq y,\ y \neq c_1,\ d_k \neq x.$$

We then have a closed bracelet modulo ρ. It follows by Lemma 4.7 that $x \equiv y$ and $x\rho = y\rho$. Hence the relation $\leqslant_\rho$ on T/ρ is partial order.

As for any $x, y \in T$; $x \leqslant y$ (in T) implies $x\rho \leqslant_\rho y\rho$ in T/ρ then the natural map $T \to T/\rho$ is isotone. $\square$

Armed with these results we can naturally order S as required. The following theorem completes the process.

Theorem 4.9. *The abundant semigroup S with an ample multiplicative medial idempotent u can be naturally ordered in such a way that u is the maximum idempotent.*

Proof. For any $x, y \in S$; choose $\bar{x}, \bar{y}$ in T so that $\bar{x}\theta = x$ and $\bar{y}\theta = y$. Order S by δ, where $x\ \delta\ y$ if and only there exists an open bracelet modulo ρ

$$\bar{x} \leqslant a_1 \equiv b_1 \leqslant a_2 \ldots a_n \equiv b_n \leqslant \bar{y}.$$

It is obvious that δ is reflexive. If $x, y \in S$ so that $x\ \delta\ y$ and $y\ \delta\ x$ where $\bar{x}\theta = x$, and $\bar{y}\theta = y$ for some $\bar{x}, \bar{y} \in T$, then we have the following two open bracelets

$$\bar{x} \leqslant a_1 \equiv b_1 \leqslant a_2 \ldots a_n \equiv b_n \leqslant \bar{y}$$

and

$$\bar{y} \leqslant c_1 \equiv d_1 \leqslant c_2 \ldots c_m \equiv d_m \leqslant \bar{x}.$$

By the same argument as in Corollary 4.8 we get $\bar{x} \equiv \bar{y}$, that is, $\bar{x}\theta = \bar{y}\theta$. Thus $x = y$ and δ is symmetric. The transitivity of δ is evident. Therefore δ is partial order.

For the compatibility of δ on S, let $x\ \delta\ y$ in S and $z \in S$ where $\bar{z}\theta = z$ and $\bar{z} \in T$. For any $1 \leqslant i \leqslant n$ we have $a_i \equiv b_i$, that is $a_i\theta = b_i\theta$ so $a_i\theta\bar{z}\theta = b_i\theta\bar{z}\theta$ and $a_i\bar{z} \equiv b_i\bar{z}$. Since the order $\leqslant$ on T is compatible (Lemma 4.2), we have

$$\bar{x}\bar{z} \leqslant a_1\bar{z} \equiv b_1\bar{z} \leqslant a_2\bar{z} \equiv \ldots\ldots \leqslant a_n\bar{z} \equiv b_n\bar{z} \leqslant \bar{y}\bar{z}.$$

Therefore

$$xz\ \delta\ yz.$$

Similarly $zx\ \delta\ yz$. Hence the order δ is compatible on S.

For the order δ to be natural, let e and f be idempotents in S. We may put

$$\bar{e} = (eu, ueu, ue) \quad \text{and} \quad \bar{f} = (fu, ufu, uf)$$

and $\bar{e}, \bar{f}$ are idempotents in T such that $\bar{e}\theta = e$, $\bar{f}\theta = f$. If $ef = e = fe$, then

$$ufuueu = ufufeu = ufeu = ueu,$$

so that $ueu \ \eta \ ufu$. Choose $\bar{h} = (fu, ueu, uf)$, then $\bar{h}$ is an idempotent in T and $\bar{h} \leqslant \bar{f}$. Also

$$\bar{h}\theta = fuueuuf = fueuf = fufefuf = fef = e = \bar{e}\theta.$$

Hence $\bar{e} \equiv \bar{h} \leqslant \bar{f}$ and $e \ \delta \ f$ in S. Therefore, the order δ on S is natural and the semigroup (S, δ) is naturally ordered. Finally, let k be an idempotent in S. Then $\bar{k} = (ku, uku, uk)$ is an idempotent in T where $\bar{k}\theta = k$; put $\bar{u} = (u, u, u)$ then $\bar{u}$ is the maximum idempotent in T (see the proof of Lemma 4.2) and $\bar{k} \leqslant \bar{u}$. By the definition of the order on S this implies $k \ \delta \ u$ in S so that u is the maximum idempotent in S. Hence the result holds. □

As uSu is ample, consider the natural order relation η of Proposition 2.10 on uSu. The order relation δ on S (as stated in the proof of Theorem 4.9) has the following property.

Corollary 4.10. *For any $x, y \in uSu$*

$$x \ \delta \ y \quad \textit{implies} \quad x \ \eta \ y.$$

Proof. Let $x, y \in uSu$ such that $x \ \delta \ y$. Then for some $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in T we have

$$\bar{x} \leqslant a_1 \equiv b_1 \leqslant a_2 \equiv \ldots a_n \equiv b_n \leqslant \bar{y},$$

where we may put $\bar{x} = (x^\dagger, x, x^*)$, $\bar{y} = (y^\dagger, y, y^*)$ ($x^\dagger, x, x^*, y^\dagger, y, y^*$ are in uSu) and $a_i = (e_i, x_i, f_i)$, $b_i = (g_i, y_i, h_i)$ for $i = 1, \ldots, n$. Notice that

$$\bar{x} \leqslant a_1 \ \text{ implies } \ x \ \eta \ x_1 \quad \text{ and } \quad a_1 \equiv b_1 \ \text{ implies } \ x_1 = y_1.$$

The process will continue so we have

$$x \ \eta \ x_1, \ x_1 = y_1, \ y_1 \ \eta \ x_2, \ \ldots, \ x_n = y_n, \ y_n \ \eta \ y.$$

Hence $x \ \eta \ y$. □

The easy proof of the following corollary is omitted.

Corollary 4.11. *For any $e, f \in E(uSu)$,*

$$e \ \delta \ f \quad \textit{if and only if} \quad e \ \omega \ f.$$

To introduce another property of δ, consider the elements x_i for $i = 1, \ldots, 5$ in T such that:

$$x_1 \leqslant x_2 \equiv x_3 \leqslant x_4 \equiv x_5 \quad \text{where} \quad x_i = (e_i, ua_iu, f_i) \quad (a_i \in S).$$

It follows from the order of T and the ρ-equivalence in T that:

(1) $e_1 \leqslant_l e_2$, $ua_1u \ \eta \ ua_2u$ and $ue_1ua_1u = ua_1u$;

(2) $e_2ua_2u = e_3ua_3u$ and $ua_2u = ua_3u$;

(3) $e_3 \leqslant_l e_4$, $ua_3u \ \eta \ ua_4u$ and $ue_3ua_3u = ua_3u$;

(4) $e_4ua_4u = e_5ua_5u$ and $ua_4u = ua_5u$.

These imply correspondingly that:

(i) $e_1 \leqslant_l e_2$, $ua_1^\dagger u \ \eta \ ua_2^\dagger u$ (Corollary 2.11) and $ue_1ua_1^\dagger u = ua_1^\dagger u$;

(ii) $e_2ua_2^\dagger u = e_3ua_3^\dagger u$ and $ua_2^\dagger u = ua_3^\dagger u$;

(iii) $e_3 \leqslant_l e_4$, $ua_3^\dagger u \ \eta \ ua_4^\dagger u$ and $ue_3ua_3^\dagger u = ua_3^\dagger u$;

(iv) $e_4ua_4^\dagger u = e_5ua_5^\dagger u$ and $ua_4^\dagger u = ua_5^\dagger u$.

Therefore, we have the following.

(a) $ue_1ua_1^\dagger u = ua_1^\dagger u$ and $ua_1^\dagger uua_2^\dagger u = ua_1^\dagger u$. This implies that $g_1 = (e_1, ua_1^\dagger u, ua_2^\dagger u)$ is an idempotent in T which is $\mathcal{R}$-related to $(e_1, ua_1^\dagger u, ua_1^\dagger u)$. Then recall from the proof of Lemma 4.1 that $x_1 \ \mathcal{R}^* \ (e_1, ua_1^\dagger u, ua_1^\dagger u)$. Hence $g_1 \ \mathcal{R}^* \ x_1$. Clearly $g_1 \leqslant g_2$ where $g_2 = (e_2, ua_2^\dagger u, ua_2^\dagger u)$, and g_2 is an idempotent in T which is $\mathcal{R}^*$-related to x_2.

(b) $e_2ua_2^\dagger u = e_3ua_3^\dagger u$. As $ua_3^\dagger u \ \eta \ ua_4^\dagger u$ then $e_3ua_3^\dagger u = e_3ua_3^\dagger uua_4^\dagger u$ and $g_3 = (e_3, ua_3^\dagger u, ua_4^\dagger u)$ is an idempotent in T, $g_2 \equiv g_3$, and

$$g_3 \ \mathcal{R} \ (e_3, ua_3^\dagger u, ua_3^\dagger u) \ \mathcal{R}^* \ x_3$$

and $g_3 \leqslant g_4$ where $g_4 = (e_4, ua_4^\dagger u, ua_4^\dagger u)$. Recall that $g_4 \ \mathcal{R}^* \ x_4$.

(c) By the same procedure as above

$$e_4ua_4^\dagger u = e_5ua_5^\dagger u, \quad ua_4^\dagger u = ua_5^\dagger u,$$
$$g_5 = (e_5, ua_5^\dagger u, ua_5^\dagger u), \quad g_5 \ \mathcal{R}^* \ x_5, \quad g_4 \equiv g_5.$$

Hence, we have the open bracelet modulo ρ

$$g_1 \leqslant g_2 \equiv g_3 \leqslant g_4 \equiv g_5$$

where:

$$g_i = (e_i, ua_i^\dagger u, ua_{i+1}^\dagger u); \quad i = 1, 3 (x_i \leqslant x_{i+1} i = 1, 3),$$
$$g_i = (e_i, ua_i^\dagger u, ua_i^\dagger u); \quad i = 2, 4, 5 (x_i \equiv x_{i+1} i = 2, 4),$$

and $g_i \in E(T)$ with $g_i \ \mathcal{R}^* \ x_i$.

Dually there exists a subset of idempotents $\{h_1, h_2, h_3, h_4, h_5\}$ of T such that $h_i \ \mathcal{L}^* \ x_i$, where:

$$h_i = (ua_{i+1}^* u, ua_i^* u, f_i); \quad i = 1, 3 (x_1 \leqslant x_2, x_3 \leqslant x_4),$$
$$h_i = (ua_i^* u, ua_i^* u, f_i); \quad i = 2, 4, 5 (x_2 \equiv x_3, x_4 \equiv x_5),$$

and $h_1 \leqslant h_2 \equiv h_3 \leqslant h_4 \equiv h_5$ is open bracelet modulo ρ.

By a simple induction we capture the following result.

Corollary 4.12. *If $\{x_i \mid i = 1, 2, \ldots, n\}$ is a subset of T such that*

$$x_1 \leqslant x_2 \equiv x_3 \leqslant x_4 \equiv \ldots \leqslant x_{n-1} \equiv x_n$$

where $x_i = (e_i, ua_iu, f_i)$, for $i = 1, 2, \ldots, n$ ($a_i \in S$). Then:

(1) *there is a subset $\{g_1, g_2, \ldots, g_n\}$ of idempotents in T such that*

$$g_1 \leqslant g_2 \equiv g_3 \leqslant \ldots \leqslant g_{n-1} \equiv g_n$$

where $g_i \ \mathcal{R}^* \ x_i$ for $i = 1, 2, \ldots, n-1$,

$$g_i = \begin{cases} (e_i, ua_i^\dagger u, ua_{i+1}^\dagger u) & \text{if } \ x_i \leqslant x_{i+1} \\ (e_i, ua_i^\dagger u, ua_i^\dagger u) & \text{if } \ x_i \equiv x_{i+1} \end{cases}$$

and $g_n = (e_n, ua_n^\dagger u, ua_n^\dagger u)$;

(2) *there is a subset* $\{h_1, h_2, \ldots, h_n\}$ *of idempotents in* T *such that*

$$h_1 \leqslant h_2 \equiv h_3 \leqslant \ldots h_{n-1} \equiv h_n$$

where $h_i \ \mathcal{L}^* \ x_i$ *for* $i = 1, 2, \ldots, n-1$

$$h_i = \begin{cases} (ua^*_{i+1}u, ua^*_i u, f_i) & \text{if } \ x_i \leqslant x_{i+1} \\ (ua^*_i u, ua^*_i u, f_i) & \text{if } \ x_i \equiv x_{i+1} \end{cases}$$

and $h_n = (ua^*_n u, ua^*_n u, f_n)$.

Corollary 4.13. *The relations* $\mathcal{L}^*$ *and* $\mathcal{R}^*$ *are abundant on* (S, δ).

Proof. Let $a, b \in S$ such that $a \ \delta \ b$. Then there is an open bracelet modulo ρ

$$\bar{a} \equiv x_1 \leqslant x_2 \equiv x_3 \leqslant \ldots \leqslant x_{n-1} \equiv x_n \leqslant \bar{b}$$

where $\bar{a}\theta = a$ and $\bar{b}\theta = b$.

We may write - without loss of generality - $\bar{a} = (a^\dagger u, uau, ua^*)$ and $\bar{b} = (b^\dagger u, ubu, ub^*)$. By Corollary 4.12 there is a subset of idempotents $\{g_0, g_1, g_2, \ldots, g_n, g_{n+1}\}$ in T such that

$$g_0 \equiv g_1 \leqslant g_2 \equiv g_3 \leqslant \ldots \leqslant g_{n-1} = g_n \leqslant g_{n+1}, \quad g_0 \ \mathcal{R}^* \ \bar{a}, \quad g_{n+1} \ \mathcal{R}^* \ \bar{b}$$

where $g_0 = (a^\dagger u, ua^\dagger u, ua^\dagger u)$ and $g_{n+1} = (b^\dagger u, ub^\dagger u, ub^\dagger u)$. So then $g_0\theta \ \delta \ g_{n+1}\theta$ in S. As θ is admissible (Corollary 4.5) we have $g_0\theta \ \mathcal{R}^* \ \bar{a}\theta$ and $g_{n+1}\theta \ \mathcal{R}^* \ \bar{b}\theta$, where $g_0\theta = a^\dagger u$ and $g_{n+1}\theta = b^\dagger u$.

Then $a^\dagger u \ \mathcal{R}^* \ a$, and $b^\dagger u \ \mathcal{R}^* \ b$ (see Lemma 3.9) where $a^\dagger u \ \delta \ b^\dagger u$. Hence $\mathcal{R}^*$ is abundant on (S, δ). Similarly $\mathcal{L}^*$ is abundant on (S, δ). □

We conclude the section by the following result.

Proposition 4.14. *Let* S *be an abundant semigroup containing an idempotent* u *such that* uSu *is an ample subsemigroup of* S. *The idempotent* u *is multiplicative medial if and only if there is a natural order* $\leqslant$ *on* S *such that* u *is a maximum idempotent and for any* $x, y \in uSu$,

$$x \leqslant y \quad \text{implies} \quad x \ \eta \ y$$

where η *is the order relation on* uSu *as defined in Proposition 2.10.*

Proof. If u is multiplicative medial, then - as uSu is ample - the result follows from the proof of Theorem 4.9 and Corollary 4.10.

Conversely let $(S, \leqslant)$ be a naturally ordered abundant semigroup with set of idempotents E, and u be a maximum idempotent such that uSu is an ample subsemigroup of S. Moreover suppose for any $x, y \in uSu$ that

$$x \leqslant y \quad \text{implies} \quad x \; \eta \; y.$$

Then by Proposition 3.6, u is medial and by Lemma 2.5, uSu is a $*$-subsemigroup of S. Let $e, f \in E$; clearly $uefu \in uSu$. By Lemma 3.9, $uef \; \mathcal{L}^* \; u(ef)^*$ so that $uefu \; \mathcal{L}^* \; u(ef)^*u$. As u is medial, then $u(ef)^*u$ is an idempotent and $u(ef)^*u \in E(uSu)$. Since $uefu \leqslant u$ (as $e \leqslant u$, $f \leqslant u$ and $\leqslant$ is compatible), then $uefu \leqslant u(ef)^*u$ and by the hypothesis $uefu \; \eta \; u(ef)^*u$. In particular, there exists an idempotent g in uSu such that

$$gu(ef)^*u = uefu.$$

Hence $uefu$ is an idempotent and u is multiplicative medial as required. □

5 Another order for an abundant semigroup

For any abundant semigroup S with set of idempotents E containing a strong medial idempotent u such that uSu is ample, $\langle E \rangle$ is a regular subsemigroup (Proposition 3.5) and $E(uSu) = u\langle E \rangle u = uEu$ (Corollary 3.11). Thus uEu is a semilattice. As mentioned in [17], we do not need to impose the following two conditions:

(1) $u\langle E \rangle u$ is a semilattice (see Proposition 3.12);

(2) $\langle E \rangle$ is regular

in the hypothesis of [6, theorem 4.5] as they follow from the premises.

Let S be an abundant semigroup with set of idempotents E containing a strong medial idempotent u such that uSu is ample. Thus u is ample multiplicative medial (Lemma 3.2) and the semigroup S can be naturally ordered in such a way that u is a maximum idempotent (Theorem 4.9) such that $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant (Corollary 4.13). The objective of this section is to impose a natural order on S which does not coincide with the one produced in Section 4.

Let S be an abundant semigroup where $E(S) = E$ and S contains a strong medial idempotent u such that uSu is ample. We identify uEu by E^0 and uSu by S^0. Consider

$$W = \{(e, a, f) \in Eu \times S^0 \times uE \mid e \ \mathcal{L} \ a^\dagger, \ f \ \mathcal{R} \ a^*; \ a^\dagger, a^* \in E^0\}.$$

Then W is a semigroup with the binary operation defined for any two elements (e, a, f) and (g, b, h) by the rule:

$$(e, a, f)(g, b, h) = (e(afg)^\dagger, afgb, (fgb)^* h).$$

As $a, b, fg \in S^0$, then $(afg)^\dagger$ and $(fgb)^*$ are chosen to be in E^0. In fact, we have from [6] that W is an abundant semigroup containing a strong medial idempotent $\bar{u} = (u, u, u)$ such that $\bar{u}W\bar{u}$ is ample.

Impose the order $\leqslant_w$ on W defined for any elements (e, a, f) and (g, b, h) in W by the rule

$$(e, a, f) \leqslant_w (g, b, h) \quad \text{if and only if} \quad e \ \omega \ g, \ a \ \eta \ b, \ f \ \omega \ h.$$

It is evident that $\leqslant_w$ is partial order. In fact this order is compatible, for if $(e, a, f) \leqslant_w (g, b, h)$ and (i, c, j) in W then $e \ \omega \ g$, $a \ \eta \ b$, and $f \ \omega \ h$. Since $fi, hi \in E^0$ and $f = fh$, then $fi = fhi = fhii = hifi$ (uE is right normal from Proposition 3.13). Hence $fi \ \omega \ hi$ and $fi \ \eta \ hi$. Thus $afi \ \eta \ bhi$. By Corollary 2.11

$$(afi)^\dagger \ \eta \ (bhi)^\dagger \ \text{ and } \ (afi)^\dagger (bhi)^\dagger = (afi)^\dagger \qquad ((afi)^\dagger, (bhi)^\dagger \in E^0).$$

Notice that:

$$\begin{aligned} e(afi)^\dagger &= e(afi)^\dagger (bhi)^\dagger = eg(afi)^\dagger (bhi)^\dagger \qquad (e \ \omega \ g) \\ &= e(afi)^\dagger g(bhi)^\dagger \qquad (Eu \ \text{ is left normal - Proposition 3.13}), \end{aligned}$$

and

$$g(bhi)^\dagger e(afi) = ge(bhi)^\dagger (afi)^\dagger = e(afi)^\dagger.$$

Therefore $e(afi)^\dagger \ \omega \ g(bhi)^\dagger$. Also $fi \ \eta \ hi$ implies $fic \ \eta \ hic$ and again by Corollary 2.11

$$(fic)^* \ \eta \ (hic)^*.$$

As $(fic)^*j = j(fic)^*j$ (uE is right normal) then

$$(hic)^*(fic)^*j = (hic)^*j(fic)^*j.$$

That is

$$(fic)^*j = (hic)^*j(fic)^*j,$$

and clearly

$$(fic)^*j(hic)^*j = (fic)^*(hic)^*j = (fic)^*j.$$

Hence $(fic)^*j \ \omega \ (hic)^*j$. However $afi \ \eta \ bhi$ so $afic \ \eta \ bhic$. Therefore,

$$(e(afi)^\dagger, afic, (fic)^*j) \leqslant_w (g(bhi)^\dagger, bhic, (hic)^*j).$$

That is $(e, a, f)(i, c, j) \leqslant_w (g, b, h)(i, c, j)$. Similarly;

$$(i, c, j)(e, a, f) \leqslant_w (i, c, j)(g, b, h)$$

and the order $\leqslant_w$ on W is compatible.

To see that the order is natural, let (e, a, f) and (g, b, h) be idempotents in W. Then as in [6], or as can be verified directly, $a, b \in E^0$. If $(e, a, f)(g, b, h) = (e, a, f) = (g, b, h)(e, a, f)$ then

$$e(afg) = e = g(bhe), \quad afgb = a = bhea, \quad (fgb)h = f = (hea)f.$$

Notice that $ge = e = eg$. Then $e \ \omega \ g$. Similarly $f \ \omega \ h$. As a, fg, he are all in E^0 and

$$(afg)b = a = b(hea)$$

then $a \ \eta \ b$ (in uSu). Hence $(e, a, f) \leqslant_w (g, b, h)$ in W and the order on W is natural. In conclusion, we have the following.

Proposition 5.1. *The semigroup $(W, \leqslant_w)$ is a naturally ordered abundant semigroup.*

Corollary 5.2. *The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant on $(W, \leqslant_w)$.*

Proof. Recall from [6], that for any (e, a, f) in W, each of $(e, a^\dagger, a^\dagger)$ and (a^*, a^*, f) are idempotents in W, and

$$(a^*, a^*, f) \ \mathcal{L}^* \ (e, a, f) \ \mathcal{R}^* \ (e, a^\dagger, a^\dagger).$$

If $(e,a,f) \leqslant_w (g,b,h)$ in W, then $e\ \omega\ g$, $f\ \omega\ h$ and $a\ \eta\ b$. However $a\ \eta\ b$ implies $a^\dagger\ \eta\ b^\dagger$ and $a^*\ \eta\ b^*$ $(a^\dagger, b^\dagger \in E^0)$ (by Corollary 2.11). Then

$$(e, a^\dagger, a^\dagger) \leqslant_w (g, b^\dagger, b^\dagger) \quad \text{and} \quad (a^*, a^*, f) \leqslant_w (b^*, b^*, h).$$

Hence the result holds. $\square$

For any $x \in S$; $x^*, x^\dagger \in E$ and $xu \in Su$. Also Su is a $*$-subsemigroup of S (Proposition 3.10) and $x^\dagger u \in R^*_{xu}(S) \cap Eu$ (Lemma 3.9(2)). Similarly $ux \in uS$ and $ux^* \in L^*_{ux}(S) \cap uE$.

The two idempotents $x^\dagger u$ and ux^* are uniquely determined by x (see [6]), and $ux^\dagger u\ \mathcal{R}^*\ uxu$ so that $ux^\dagger u = (uxu)^\dagger$, also $x^\dagger u\ \mathcal{L}\ ux^\dagger u$ so $x^\dagger u \in L_{(uxu)^\dagger}$. Similarly, ux^* belongs to $R_{(uxu)^*}$.

Therefore $(x^\dagger u, uxu, ux^*) \in W$ and $\theta : S \to W$ defined by $x\theta = (x^\dagger u, uxu, ux^*)$ is an isomorphism [6].

Define an order $\leqslant_S$ on S for any two elements x and y in S by:

$$x \leqslant_S y \text{ (in } S) \quad \text{if and only if} \quad x\theta \leqslant_w y\theta \text{ (in } W).$$

Since θ is an isomorphism the order $\leqslant_w$ on W is a natural order in which $\mathcal{L}^*$and $\mathcal{R}^*$ are abundant. Thus we have the following proposition.

Proposition 5.3. *If S is an abundant semigroup containing a strong medial idempotent u such that uSu is ample, then S is naturally ordered with respect to $\leqslant_S$, where for any $x, y \in S$;*

$$x \leqslant_S y \ \text{ if and only if } \ x^\dagger u\ \omega\ y^\dagger u,\ \ uxu\ \eta\ uyu,\ \text{ and } ux^*\ \omega\ uy^*,$$

such that $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant.

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