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Tense like equality algebras

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Abstract. In this paper, first we define the notion of involutive operator on bounded involutive equality algebras and by using it, we introduce a new class of equality algebras that we called it a tense like equality algebra. Then we investigate some properties of tense like equality algebra. For two involutive bounded equality algebras and an equality homomorphism between them, we prove that the tense like equality algebra structure can be transfer by this equality homomorphism. Specially, by using a bounded involutive equality algebra and quotient structure of it, we construct a quotient tense like equality algebra. Finally, we investigate the relation between tense like equality algebras and tense MV-algebras.

1 Introduction and preliminaries

As a generalization of EQ-algebras defined in [16], Jenei introduced a new class of logical algebras in [13] and called it equality algebras, where the product operation in EQ-algebras is replaced by another binary operation smaller or equal to the original product. An equality algebra consists of two binary operations meet, equivalence and constant 1. Relations of equality algebras with the other logical algebras are studied in many works [1, 2, 10,

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11, 17, 18]. In [10, 11, 14] the authors proved that any equality algebra has a corresponding BCK-meet-semilattice and any BCK-meet-semilattice with distributivity property has a corresponding equality algebra. In [17] the authors have proved that there are relations between the equality algebras and some of other logical algebras, such as residuated lattice, MTL-algebra, BL-algebra, MV-algebra, Hertz-algebra, Heyting-algebra, Boolean-algebra, EQ-algebra, and Hoop. Moisil introduced the n-valued Łukasiewicz-Moisil algebras in [15]. After him, Chang defined the MV-algebras as algebraic structures for the infinite valued Łukasiewicz propositional calculus in [7]. The propositional calculus by adding two unary operators G, H which are called tense operators and the derived operators $F :=^{-} G^{-}$ and $P :=^{-} H^{-}$, where "$^{-}$" denotes the classical negation connective is called tense logic, for more details we refer to [5]. Tense operators for MV-algebras and Łukasiewicz-Moisil algebras were introduced in [12] and studied in [4, 8]. Tense operators in lattice effect algebras were introduced in [6]. Tense operators are in certain sense quantifiers which quantify the time dimension of the logic under consideration. The semantical interpretation of these tense operators G and H is as follows. Consider a pair $(T, \leq)$, where T is a non-void set and $\leq$ is a partial order on T. Let $x \in T$ and $f(x)$ be a formula of a given logical calculus. We say that $G(f(t))$ is valid if for any $s \geq t$ the formula $f(s)$ is valid. Analogously, $H(f(t))$ is valid if $f(s)$ is valid for each $s \leq t$. Thus the unary operators G and H constitute an algebraic counterpart of the tense operations "it is always going to be the case that" and "it has always been the case that", respectively." This concept was generalized for the so-called basic algebras in [3], where the authors have related the basic algebras with tense operators to the quantum structures which are the so-called dynamic effect algebras. Operators such as state and equality homomorphism on equality algebras are studied in [11, 18]. In Section 3, we define two unary operators that we call them tense like operators and tense like equality algebras. We study on these new algebras and their properties. The last section deals with relations of the tense like equality algebras with tense MV-algebras and vice versa.

2 Preliminaries

In this section, we give the basic definitions and results related to equality algebras and MV-algebras which we will use in the next sections. Also, we prove some properties of equality algebras.

Definition 2.1. [13] An *equality algebra* $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ is an algebra of type $(2, 2, 0)$ such that the following axioms are fulfilled, for all $x, y, z \in X$:
(E1) $\langle X, \wedge, 1\rangle$ is a $\wedge$-semilattice with the top element 1,
(E2) $x \sim y = y \sim x$,
(E3) $x \sim x = 1$,
(E4) $x \sim 1 = x$,
(E5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$,
(E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,
(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$.

The operation $\wedge$ is called *meet* (infimum) and $\sim$ is an *equality operation.* We write $x \leq y$ if and only if $x \wedge y = x$, as usual. Define the following two derived operations, the *implication* and the *equivalence operation* of the equality algebra $\langle X, \sim, \wedge, 1\rangle$ as follows,

$$x \to y = x \sim (x \wedge y) \quad \text{and} \quad x \leftrightarrow y = (x \to y) \wedge (y \to x)$$

Proposition 2.2. *[13]Let $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ be an equality algebra. Then the following statements hold, for all $x, y, z \in X$:*
(i) $x \sim y \leq x \leftrightarrow y \leq x \to y$,
(ii) $x \leq (x \sim y) \sim y$,
(iii) $x \sim y = 1$ *if and only if* $x = y$,
(iv) $x \to y = 1$ *if and only if* $x \leq y$,
(v) $x \to y = 1$ *and* $y \to x = 1$ *imply* $x = y$,
(vi) $1 \to x = x$, $x \to 1 = 1$ *and* $x \to x = 1$,
(vii) $x \leq y \to x$,
(viii) $y \leq x$ *implies* $x \leftrightarrow y = x \to y = x \sim y$.

Proposition 2.3. *[11]In any equality algebra $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ the following statements hold, for all $x, y \in X$:*
(i) $y \leq x \sim (x \wedge y)$,
(ii) *if* $x \leq y$, *then* $x \leq x \sim y$,

(iii) $x \sim y \leq x \sim (x \wedge y)$,
(iv) $x \leq (x \sim (x \wedge y)) \sim y$,
(v) $x \to y = x \to (x \wedge y)$,
(vi) $x \to y \leq (z \wedge x) \to (y \wedge z)$.

Definition 2.4. [17] Let $\mathcal{E} = \langle X, \sim, \wedge, 1 \rangle$ be an equality algebra. Then it is called *commutative* if, for any $x, y \in X$,

$$(x \to y) \to y = (y \to x) \to x.$$

At last we bring the definition and some useful properties of MV-algebras from [12].

Definition 2.5. [12] An MV-*algebra* $\langle X, \oplus, -, 0 \rangle$ is a set X equipped with a binary operation $\oplus$, a unary operation $-$, and a distinguished constant 0 that, for any $x, y, z \in X$, satisfying
(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(MV2) $x \oplus y = y \oplus x$,
(MV3) $x \oplus 0 = x$,
(MV4) $x^{--} = x$,
(MV5) $x \oplus 0^- = 0^-$,
(MV6) $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$.

Let X be an MV-algebra such that $1 = 0^-$. For any $x, y \in X$, define the operations $\ominus$ and $\odot$ on X by

$$x \odot y = (x^- \oplus y^-)^- \quad \text{and} \quad x \ominus y = x \odot y^-. \tag{2.1}$$

Now, we define a partial order $\leq$ on MV-algebra $\langle X, \oplus, -, 0 \rangle$ by $x \leq y$ if and only if $x^- \oplus y = 1$, for any $x, y \in X$.

Lemma 2.6. *[9]Let $\langle X, \oplus, -, 0 \rangle$ be an MV-algebra and $\leq$ be the natural order. Then, for any $x, y, z \in X$, the following statements hold:*
(i) $1^- = 0$,
(ii) $x \oplus x^- = 1$,
(iii) $x \wedge y = x \odot (x^- \oplus y)$ *is a meet on* $\langle X, \oplus, -, 0 \rangle$,
(iv) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$,
(v) $x \leq y$ *if and only if* $y^- \leq x^-$,
(vi) *if* $x \leq y$, *then* $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$.

3 Tense like equality algebras

In this section, we introduce an operator on involutive equality algebras which we call tense like operator. Some essential properties of tense like operators and some examples are given. Then, using it we introduce a new class of equality algebras which we called it a tense like equality algebra.

An equality algebra $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ is called *bounded* if there exists an element $0 \in X$ such that $x \geq 0$, for all $x \in X$. In a bounded equality algebra $\mathcal{E}$, the *negation* on $\mathcal{E}$ is a map $' : X \to X$ defined by $x' = x \to 0 = x \sim 0$, for all $x \in X$. If $(x')' = x$, for all $x \in X$, then $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ is called *involutive*. Some results related to involutive equality algebras are obtained in [17]. We say that, for any $x, y \in X$, the negation " $'$" on $\mathcal{E}$ satisfies

$$\text{if } x \leq y, \text{ then } \quad y' \leq x' \qquad \textbf{(N)}.$$

A map $F : X \to X$ is called *increasing* (*decreasing*), if, $x \leq y$, then $F(x) \leq F(y)$ ($F(y) \leq F(x)$).

Definition 3.1. Let $\mathcal{E} = \langle X, \sim, \wedge, 0, 1\rangle$ be a bounded involutive equality algebra and $G : X \to X$ be a map. We call G an *involutive operator* if $G(x') = G(x)'$, for all $x \in X$.

Example 3.2. (i) Let $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ be a linearly ordered equality algebra and G be an identity map on X. Then G is an involutive and increasing operator.

(ii) Let $X = \{0, a, b, 1\}$ be a set such that $0 < a < b < 1$. Then, for any $x, y \in X$, define the operations $\wedge$ and $\sim$ on X by

$x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	a	a
b	a	a	1	b
1	0	a	b	1

Then, by routine calculations, we can see that $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ is an equality algebra. Now, define $G_1, G_2 : X \to X$ by

x	0	a	b	1
$G_1(x)$	0	0	1	1
$G_2(x)$	0	a	b	1

Thus, it is easy to see that G_i $(1 \leq i \leq 2)$ is an involutive and increasing operator on X.

(iii) Let $X = \{0, a, b, c, d, 1\}$ be a set such that $0 < a < b < c < d < 1$. Then, for any $x, y \in X$, define the operations $\wedge$ and $\sim$ on X by

$x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	a	c	d	a	1	d
1	0	a	b	c	d	1

By routine calculations, we can see that $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ is an equality algebra. Now, for $1 \leq i \leq 4$, define $G_i : X \to X$ as

x	0	a	b	c	d	1
$G_1(x)$	0	d	c	b	d	1
$G_2(x)$	0	c	d	a	b	1
$G_3(x)$	0	b	a	d	c	1
$G_4(x)$	0	a	b	c	d	1

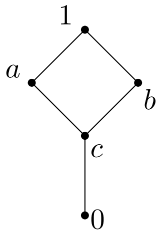
Then, it is clear that G_i $(1 \leq i \leq 4)$ is an involutive operator on X.

(iv) Let $X = [0, 1]$ and $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ be an equality algebra, where $x \sim y = 1 - |x - y|$ and $x \wedge y = \min\{x, y\}$, for any $x, y \in X$. It is easy to see that $\mathcal{E}$ is an ivolutive equality algebra. Define $G : X \to X$ by $G(x) = |1 - |1 - 3x||$, for any $x \in X$. Then G is an involutive and decreasing operator.

Let $\mathcal{E} = \langle X, \sim, \wedge, 0, 1\rangle$ be a bounded involutive equality algebra and $G : X \to X$ be an involutive operator on X, we define the kernel of G as

$$\ker G = \{x \in X : \ G(x) = 1\}$$

Example 3.3. [18] Let $X = \{0, c, a, b, 1\}$ be a lattice with the diagram as given below. Define the operations $\sim$ and $\to$ on X by



$\sim$	0	c	a	b	1
0	1	0	0	0	0
c	0	1	b	a	c
a	0	b	1	c	a
b	0	a	c	1	b
1	0	c	a	b	1

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	b	1
1	0	c	a	b	1

Then $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ is an equality algebra. Now, define $G_i : X \to X$, $i = 1, 2, 3$, by

x	0	c	a	b	1
$G_1(x)$	0	1	c	1	1
$G_2(x)$	0	1	a	1	1
$G_3(x)$	0	1	b	1	1

Then G_i's are involutive operators on X.

Remark 1. L*et* $\mathcal{E} = \langle X, \sim, \wedge, 0, 1\rangle$ *be a bounded involutive equality algebra and* $G : X \to X$ *be an involutive operator. Then* $\ker G$ *is not equality subalgebra of* X*, in general. For example in the above example,* $c, b \in \ker G_1$ *but* $G_1(c \sim b) \neq 1$*. Therefore,* $c \sim b \notin \ker G_1$*.*

But if we assume that $G(x \sim y) = G(x) \sim G(y)$ *for all* $x, y \in X$*, then* $x \sim y \in \ker G$*, whenever* $x, y \in \ker G$*. Since* $1 \in \ker G$*,* $\ker G$ *is an equality subalgebra of* $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$*.*

Definition 3.4. Let $\mathcal{E} = \langle X, \sim, \wedge, 0, 1\rangle$ be a bounded involutive equality algebra. A *tense like equality algebra* is a tripled $(\mathcal{E}, G, H)$, where $G, H : X \to X$ are unary operators on X and G is an involutive operator on X and for any $x, y \in X$, the following hold:
(T0) $G(1) = 1$ and $H(1) = 1$,
(T1) $G(x \to y) \geq G(x) \to G(y)$ and $H(x \to y) \geq H(x) \to H(y)$,
(T2) $G(x \sim y) \leq G(x) \sim G(y)$ and $H(x \sim y) \leq H(x) \sim H(y)$,
(T3) $G(x) \wedge G(y) = G(x \wedge y)$ and $H(x) \wedge H(y) = H(x \wedge y)$,
(T4) $x \leq H(G(x))$.

Example 3.5. (i) Let $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ be a linearly ordered equality algebra and $G = H$ be identity maps on X. Then $(\mathcal{E}, G, H)$ is a tense like equality algebra.
(ii) Let $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ be an equality algebra and $G_1 : X \to X$ be an involutive and increasing operator on X as Example 3.2(ii). If we define the map $H : X \to X$ by

x	0	a	b	1
$H(x)$	a	a	1	1

then $(\mathcal{E}, G_1, H)$ is a tense like equality algebra. Also, $\ker G_1$ and $\ker H$ are subalgebras of $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$, where

$$\ker G_1 = \{b, 1\} = \ker H.$$

(iii) Let $X = \{0, c, a, b, 1\}$ be an equality algebra as Example 3.3. Now, define the map $G : X \to X$ by

x	0	c	a	b	1
$G(x)$	0	c	b	a	1

Then by routine calculations, we can see that G is an involutive operator and satisfies in the axioms (T0)-(T3). If we take $H = G$, then, clearly the axiom (T4) holds. Hence, $(\mathcal{E}, G, H)$ is a tense like equality algebra.

Now, we give some properties of tense like equality algebras.

Proposition 3.6. *Let $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ be an involutive equality algebra. Then, for any $x, y \in X$, the following statements hold:*

(i) *If $G(x) \leq G(y)$, then $G(x \to y) = 1$. Moreover, if G is an injective, then $x \leq y$ if and only if $G(x) \leq G(y)$.*

(ii) *$H((x \sim y)') = H(x \sim y) \to H(0)$.*

(iii) *if the negation "$'$" satisfies condition (N), then $H(G(x'))' \leq x$.*

(iv) *$H(G(x) \sim G(y)) \leq x \to HG(y)$ and $H(G(x) \sim G(y)) \leq y \to HG(x)$.*

(v) *$H(G(x) \to G(y)) \geq HG(x) \sim HG(y)$ and $H(G(x) \to G(y)) \geq HG(y)$.*

(vi) *If $x \in \ker G$, then*

$$G(x \to y) \geq G(y), G(y \to x) \geq G(y), G(x \sim y) \leq G(y), G(x \wedge y) = G(y).$$

Moreover, if $x \leq y$, then $G(x \sim y) = G(y)$.

Proof. (i) If $G(x) \leq G(y)$ then, by Proposition 2.2(iv), we have $G(x) \to G(y) = 1$. Thus, by (T1), $G(x \to y) = 1$. If G is injective, then $x \to y = 1$, and so $x \leq y$. The proof of the converse is clear. Now, assume that $x \leq y$. Then, by Proposition 2.2(viii), we have $x \sim y = x \to y$. Hence, by (T2) and Proposition 2.2(iii), the proof is clear.

(ii) Let $x, y \in X$. Then

$$\begin{aligned} H((x \sim y)') &= H(x \sim y \to 0)\ , \text{ by (T1)} \\ &\geq H(x \sim y) \to H(0)\ , \text{ by Proposition 2.2(i)} \\ &\geq H(x \sim y) \sim H(0)\ , \text{ by (T2)} \\ &\geq H((x \sim y) \sim 0)) \\ &= H((x \sim y)'). \end{aligned}$$

Hence, $H((x \sim y)') = H(x \sim y) \to H(0)$.

(iii) From (T4), we have $x' \leq H(G(x'))$, for every $x' \in X$. Since condition (N) holds, the proof is clear.

(iv) By Proposition 2.2(i), $x \sim y \leq x \to y$ and, by Proposition 2.3(v) and (vi), for any $x, y, z \in X$,

$$x \to (x \wedge y) = x \to y\ ,\ x \to y \leq (x \wedge z) \to (y \wedge z).$$

By these facts and (T2), we have

$$\begin{aligned} H(G(x) \sim G(y)) &\leq HG(x) \sim HG(y), \text{ by Proposition 2.2(i)} \\ &\leq HG(x) \to HG(y), \text{ by Proposition 2.3(vi)} \\ &\leq (HG(x) \wedge x) \to (x \wedge HG(y)), \qquad \text{by (T4)} \\ &= x \to (x \wedge HG(y)), \text{ by Proposition 2.3(v)} \\ &= x \to HG(y). \end{aligned}$$

(v) By (T1) and Proposition 2.2(i), for any $x, y \in X$ we have,

$$H(G(x) \to G(y)) \geq HG(x) \sim HG(y).$$

Again, by the axiom (T1) and Proposition 2.2(vii), for any $x, y \in X$ we have $H(G(x) \to G(y)) \geq HG(y)$.

(vi) Let $x \in \ker G$. Then by (T1), (T2) and Proposition 2.2(vi), for all $y \in X$,

$$G(x \to y) \geq G(x) \to G(y) = 1 \to G(y) = G(y). \qquad (3.1)$$

Again, by (T1), (T2), and Proposition 2.2(vi), for all $y \in X$,

$$G(y \to x) \geq G(y) \to G(x) = G(y) \to 1 = 1.$$

Then $G(y \to x) = 1$, for all $y \in X$. Hence, $G(x \to y) \geq G(y)$, for any $y \in X$. By (T2) and (E4) we have,

$$G(x \sim y) \leq G(x) \sim G(y) = 1 \sim G(y) = G(y) \tag{3.2}$$

for all $y \in X$. By (T3), we have,

$$G(x \wedge y) = G(x) \wedge G(y) = 1 \wedge G(y) = G(y)$$

for any $y \in X$. If $x \leq y$, then $x \to y = x \sim y$. Thus, by (3.2) and (3.1), for any $y \in X$, we get that,

$$G(x \sim y) \leq G(y) \leq G(x \to y) = G(x \sim y).$$

Hence, $G(x \sim y) = G(y)$, for any $x, y \in X$. □

Remark 2. T*he obtained results in the implications (i) and (vi) of Proposition 3.6, hold for H, too.*

Proposition 3.7. *[17]Let $\mathcal{E} = \langle X, \sim, \wedge, 0, 1\rangle$ be a bounded lattice equality algebra with the negation "$'$". Then the following properties hold, for all $x, y \in X$,*
(i) $(x \vee y)' = x' \wedge y'$,
(ii) $x \leq (x')'$,
(iii) $x \to y \leq y' \to x'$, *and if X is involutive, then* $x \to y = y' \to x'$.

Proposition 3.8. *Let $(X, \sim, \wedge, 0, 1)$ be a tense like equality algebra, where X is an involutive lattice. Then, for any $x, y \in X$,*

$$H(x \to y) \geq H(x')' \to H(y')'.$$

Proof. For any $x, y \in X$, by Proposition 3.7(iii) and (T1), we have

$$H(x \to y) = H(y' \to x') \geq H(y') \to H(x') = H(x')' \to H(y')'.$$

□

Definition 3.9. [11] Let $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ be an equality algebra. A subset $D \subseteq X$ is called a *deductive system* of X if, for any $x, y \in X$:
(DS$_1$) $1 \in D$,
(DS$_2$) if $x \in D$ and $x \leq y$, then $y \in D$,
(DS$_3$) if $x, x \sim y \in D$, then $y \in D$.
The set of all deductive systems of X is denoted by $\mathcal{DS}(X)$.

Proposition 3.10. *[11]Every deductive system of an equality algebra $\mathcal{E}$ is a subalgebra of $\mathcal{E}$.*

Let $(X_A, \sim_A, \wedge_A, 1_A)$ and $(X_B, \sim_B, \wedge_B, 1_B)$ be two equality algebras. A map $f : X_A \to X_B$ is called an *equality-homomorphism*, if for any $x, y \in X_A$,

$$f(x \sim_A y) = f(x) \sim_B f(y) \quad \text{and} \quad f(x \wedge_A y) = f(x) \wedge_B f(y)$$

A homomorphism between two equality algebras is defined in [18], where the authors are investigated its relation with state morphisms. If the above equality algebras are bounded, then $f(0_A) = 0_B$.

We give the basic properties of bounded equality homomorphisms as follows.

Proposition 3.11. *[18]Let $\mathcal{E}_A = (X_A, \sim_A, \wedge_A, 0_A, 1_A)$, $\mathcal{E}_B = (X_B, \sim_B, \wedge_B, 0_B, 1_B)$ be two bounded equality algebras and $f : X_A \to X_B$ be a bounded equality homomorphism. Then the following statements hold:*
(i) $f(1_A) = 1_B$.
(ii) *f is monotone.*
(iii) $f(x \sim_A 0) = f(x) \sim_B 0_B$.
(iv) $\ker f$ *is a proper deductive system of X_A.*
(v) *$f(X_A)$ is a subalgebra of $\mathcal{E}_B$.*
(vi) *f is injective if and only if* $\ker f = \{1\}$.
(vii) *if $D_B \in \mathcal{DS}(X_B)$, then $f^{-1}(D_B) \in \mathcal{DS}(X_A)$.*
(viii) *if f is surjective and* $\ker f \subseteq D \in \mathcal{DS}(X_A)$, *then $f(D) \in \mathcal{DS}(X_B)$.*

By the following theorem, we show that if there exists an equality homomorphism between two equality algebras, it can transfer tense like equality algebras.

Theorem 3.12. *Let $\mathcal{E}_A = (X_A, \sim_A, \wedge_A, 0_A, 1_A)$ and $\mathcal{E}_B = (X_B, \sim_B, \wedge_B, 0_B, 1_B)$ be two involutive bounded equality algebras and $f : X_A \to X_B$ be an equality* *homomorphism. If $(\mathcal{E}_A, G_A, H_A)$ is a tense like equality algebra, then there are mappings $G_B, H_B : f(X_A) \to f(X_A)$ such that $(f(X_A), G_B, H_B)$ is a tense like equality algebra.*

Proof. Let $Y = f(X_A)$. By Proposition 3.11(v), Y is a subalgebra of $\mathcal{E}_B$. Define $G : Y \to Y$ and $H : Y \to Y$, for any $x \in X_A$, by

$$G(f(x)) = f(G_A(x)) \quad \text{and} \quad H(f(x)) = f(H_A(x)) \tag{3.3}$$

Since, $f(1_A) = 1_B \in Y$, $G(1_B) = H(1_B) = 1$. Thus, the axiom (T0) holds. Moreover, for any $a, b \in X_A$, we have

$$f(a \to b) = f(a) \to f(b) \tag{3.4}$$

and if $a \leq_A b$, then

$$f(a) \leq_B f(b) \tag{3.5}$$

Thus, for any $x, y \in Y$, there are $a, b \in X_A$ such that $f(a) = x$, $f(b) = y$ and we have

$$\begin{array}{rcl} G(x \to y) & = & G(f(a) \to f(b)) \ , \text{ by } (3.4) \\ & = & G(f(a \to b)) \ , \text{ by } (3.3) \\ & = & f(G_A(a \to b)) \ , \text{ by } (3.5) \\ & \geq_B & f(G_A(a)) \to f(G_A(b)) \ , \quad \text{by } (3.3) \\ & = & G(f(a)) \to G(f(b)) \\ & = & G(x) \to G(y). \end{array}$$

By the similar way, for any $x, y \in Y$, $H(x \to y) \geq_B H(x) \to H(y)$. Hence, (T1) holds. By similar arguments, (T2)-(T4) hold. Now, it is enough to prove (T5). Suppose $x \in Y$. Then there is an element $a \in X_A$ such that $f(a) = x$ and $a \leq H_A G_A(a)$. Thus $f(a) \leq f(H_A G_A(a))$. Hence, for any $x \in Y$, we have

$$x = f(a) \leq_B f(H_A(G_A(a))) = H(f(G_A(a))) = HG(f(a)) = HG(x).$$

Now, we show that G is an involutive operator. Let " $'$" be a negation on X_A and "$^-$" be a negation on X_B. Suppose $x \in Y$ such that $f(a) = x$, for some $a \in X_A$. Then

$$\begin{aligned}
G(x^-) &= G(f(a)^-) = G(f(a) \sim_B 0_B) \\
&= G(f(a) \sim_B f(0_A)) \ , \text{ by Proposition 3.11(iii)} \\
&= G(f(a \sim_A 0_A)) \\
&= G(f(a')) \ , \text{ by (3.3)} \\
&= f(G_A(a')) = f(G_A(a)') \\
&= f(G_A(a) \sim_A 0_A) \ , \text{ by Proposition 3.11(iii)} \\
&= f(G_A(a)) \sim_B f(0_A) \ , \text{ by (3.3)} \\
&= G(f(a)) \sim_B 0_B = G(x) \sim_B 0_B \\
&= G(x)^-.
\end{aligned}$$

Therefore, $(f(X_A), G, H)$ is a tense like equality algebra. □

Corollary 3.13. *Let $\mathcal{E}_A = (X_A, \sim_A, \wedge_A, 0_A, 1_A)$ and $\mathcal{E}_B = (X_B, \sim_B, \wedge_B, 0_B, 1_B)$ be two involutive bounded equality algebras and $f : X_A \to X_B$ be a surjective equality homomorphism. If $(\mathcal{E}_A, G_A, H_A)$ is a tense like equality algebra, then there are mappings $G, H : X_B \to X_B$ such that (X_B, G, H) is a tense like equality algebra.*

Lemma 3.14. *Let $\mathcal{E}_A = (X_A, \sim_A, \wedge_A, 0_A, 1_A)$ and $\mathcal{E}_B = (X_B, \sim_B, \wedge_B, 0_B, 1_B)$ be two involutive bounded equality algebras and $f : X_A \to X_B$ be an injective equality homomorphism. Then, for any $x, y \in f(X_A)$, the left inverse of f satisfies*

(i) $f^{-1}(1_B) = 1_A$,
(ii) $f^{-1}(x \sim_B y) = f^{-1}(x) \sim_A f^{-1}(y)$,
(iii) $f^{-1}(x \to y) = f^{-1}(x) \to_A f^{-1}(y)$,
(iv) *if $x \leq_B y$, then $f^{-1}(x) \leq_A f^{-1}(y)$.*

Proof. The proof is a straightforward. □

Theorem 3.15. *Let $\mathcal{E}_A = (X_A, \sim_A, \wedge_A, 0_A, 1_A)$ and $\mathcal{E}_B = (X_B, \sim_B, \wedge_B, 0_B, 1_B)$ be two involutive bounded equality algebras and $f : X_A \to X_B$ be an injective equality homomorphism. If $(\mathcal{E}_B, G_B, H_B)$ is a tense like equality algebra, then there are mappings $G, H : X_A \to X_A$ such that $(\mathcal{E}_A, G, H)$ is a tense like equality algebra.*

Proof. It is clear that $f(X_A)$ is a subalgebra of X_B. If $(\mathcal{E}_B, G_B, H_B)$ is a tense like equality algebra, then $(f(X_A), G_B, H_B)$ becomes a tense like equality algebra. Now, for any $x \in X_A$, define $G, H : X_A \to X_A$ by

$$G(a) = f^{-1}(G_B(f(a)))\ ,\ H(a) = f^{-1}(H_B(f(a))).$$

By Lemma 3.14(i), we have $G(1_A) = H(1_A) = 1$. Thus, (T0) holds. Let $a, b \in X_A$ such that $f(a) = x$, $f(b) = y$, $a = f^{-1}(x)$ and $b = f^{-1}(y)$. Then

$$\begin{aligned} G(a \to b) &= f^{-1}(G_B(f(a \to b)))\ ,\ \text{by (3.4)} \\ &= f^{-1}(G_B(f(a) \to f(b))),\ \text{by (T1) and Lemma 3.14(iv)} \\ &\geq_A f^{-1}(G_B(f(a)) \to G_B(f(b))),\ \text{by Lemma 3.14(iii)} \\ &= f^{-1}(G_B(f(a))) \to f^{-1}(G_B(f(b))),\ \text{by (3.4)} \\ &= G(a) \to G(b). \end{aligned}$$

Similarly, we can show that for any $a, b \in X_A$, $H(a \to b) \geq_A H(a) \to H(b)$. This follows that (T1) holds. By the similar arguments and Lemma 3.14, (T2)-(T4) hold. Now, suppose $a \in X_A$. Then $f(a) \in f(X_A)$. Since, $(\mathcal{E}_B, G_B, H_B)$ is a tense like equality algebra, by (T5), we get that $f(a) \leq H_B G_B(f(a))$. According to definition of H, for any $a \in f(X_A)$, we have $H(f^{-1}(x)) = f^{-1}(H_B(x))$ such that $a = f^{-1}(x)$. Thus, for any $a \in X_A$;

$$a \leq_A f^{-1}(H_B(G_B(x))) = H(f^{-1}(G_A(f(a)))) = HG(a).$$

Now, we show that G is an involutive operator. Let " $'$" be negation on X_A and "$^-$" be negation on X_B. Then by Lemma 3.14(ii), we have

$$\begin{aligned} G(a') &= f^{-1}(G_B(f(a'))) \\ &= f^{-1}(G_B(f(a \sim_A 0_A)) \\ &= f^{-1}(G_B(f(a) \sim_B 0_B)),\ \text{by Proposition 3.11(iii)} \\ &= f^{-1}(G_B(f(a)^-)) \\ &= f^{-1}(G_B(f(a))^-) \\ &= f^{-1}(G_B(f(a)) \sim_B 0_B),\ \text{by Lemma 3.14(ii)} \\ &= f^{-1}(G_B(f(a))) \sim_A 0_A \\ &= G(a) \sim_A 0_A \\ &= G(a)'. \end{aligned}$$

Therefore, (X_A, G, H) is a tense like equality algebra. □

If an equality homomorphism between two equality algebras is injective and surjective, then it is called an *equality isomorphism*.

An immediate result of Corollary 3.13 and Theorem 3.15, is as follows:

Corollary 3.16. *Let* $\mathcal{E}_A = (X_A, \sim_A, \wedge_A, 0_A, 1_A)$ *and* $\mathcal{E}_B = (X_B, \sim_B, \wedge_B, 0_B, 1_B)$ *be two involutive bounded equality algebras and* $f : X_A \to X_B$ *be an equality isomorphism. Then* $(\mathcal{E}_A, G_A, H_A)$ *is a tense like equality algebra if and only if* (X_B, G_B, H_B) *is a tense like equality algebra.*

Proposition 3.17. *Let* $(\mathcal{E}, G, H)$ *be a tense like equality algebra and* $G(H)$ *is increasing. Then* $\ker G \in \mathcal{DS}(X)$ $(\ker H \in \mathcal{DS}(X))$.

Proof. By (T0), $1 \in \ker G$. Then (DS_1) holds. If $x \in \ker G$ and $x \leq y$, since G is increasing, then $1 = G(x) \leq G(y)$. Thus, $G(y) = 1$, and so $y \in \ker G$. Hence, (DS_2) holds. Now, suppose $x, x \sim y \in \ker G$. Then, $G(x) = G(x \sim y) = 1$. Since G is increasing, by Proposition 2.3(iii), we have,

$$1 = G(x \sim y) \leq G(x \sim (x \wedge y)).$$

Thus, $G(x \sim (x \wedge y)) = 1$. By Proposition 2.3(iv), $x \leq (x \sim (x \wedge y)) \sim y$. Then

$$\begin{aligned} 1 &= G(x) && \text{since } G \text{ is increasing} \\ &\leq G((x \sim (x \wedge y)) \sim y)\ , && \text{by Proposition 2.3(iv) and (T2)} \\ &\leq G(x \sim (x \wedge y)) \sim G(y) && \\ &= 1 \sim G(y) && \\ &= G(y). && \end{aligned}$$

So, $G(y) = 1$ and $y \in \ker G$. Hence, (DS_3) holds. Therefore, $\ker G \in \mathcal{DS}(X)$. □

Corollary 3.18. *Let* $(\mathcal{E}, G, H)$ *be a tense like equality algebra and* G (H) *is increasing. Then* $\ker G$ $(\ker H)$ *is a subalgebra of* $\mathcal{E}$.

Proof. By Propositions 3.10 and 3.17, the proof is clear. □

Definition 3.19. [11] Let $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ be an equality algebra. A subset $\Theta \subseteq X \times X$ is called a *congruence* of X if it is an equivalence relation on X and, for any $x_1, y_1, x_2, y_2 \in X$ such that $(x_1, y_1), (x_2, y_2) \in X \times X$, the

following statements hold:
(CG$_1$) $(x_1 \wedge x_2, y_1 \wedge y_2) \in \Theta$,
(CG$_2$) $(x_1 \sim x_2, y_1 \sim y_2) \in \Theta$.
The set of all congruences of X is denoted by $\mathcal{C}on(X)$.

According to the aforementioned arguments, if G and H are (a): linear respect to $\sim$ or (b): are increasing, then $\ker G, \ker H \in \mathcal{DS}(X)$, and so $\ker G \cap \ker H \in \mathcal{DS}(X)$. Let $D_{G,H} = \ker G \cap \ker H$. Then, for any $x, y \in X$, define $\Theta_{D_{G,H}}$ by

$$x\Theta_{D_{G,H}}y \quad \text{if and only if} \quad x \sim y \in D_{G,H}$$

Proposition 3.20. *[11]If $D \in \mathcal{DS}(X)$, then $D \in \mathcal{C}on(X)$.*

Let $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ be an equality algebra and $D \in \mathcal{DS}(X)$. Denote

$$X/\Theta_D = \{x/\Theta_D : x \in X\},$$

where $x/\Theta_D = \{y \in X : (x, y) \in \Theta_D\}$. We define the operations $\sim$, $\to$ and $\wedge$ on X/Θ_D as

$$x/\Theta_D \sim y/\Theta_D = (x \sim y)/\Theta_D \ , \ x/\Theta_D \wedge y/\Theta_D = (x \wedge y)/\Theta_D$$

$$x/\Theta_D \to y/\Theta_D = (x \to y)/\Theta_D.$$

Theorem 3.21. *Let $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ be an equality algebra and $D \in \mathcal{DS}(X)$. Then*

$$\mathcal{E}/\Theta_D = \langle X/\Theta_D, \sim, \wedge, 1/\Theta_D\rangle$$

is an equality algebra.

Corollary 3.22. *Let $\mathcal{E} = \langle X, \sim, \wedge, 1, 0\rangle$ be a bounded equality algebra, $D \in \mathcal{DS}(X)$ and $X/\Theta_D = \{x/\Theta_D : x \in X\}$, where $x/\Theta_D = \{y \in X : (x, y) \in \Theta_D\}$. Then $\mathcal{E}/\Theta_D = \langle X/\Theta_D, \sim, \wedge, 1/\Theta_D\rangle$ is a bounded equality algebra.*

Proof. By Theorem 3.21, $\mathcal{E}/\Theta_D = \langle X/\Theta_D, \sim, \wedge, 1/\Theta_D\rangle$ is an equality algebra. So it suffices to show that $X/\Theta_D = \{x/\Theta_D : x \in X\}$ is bounded. From $x/\Theta_D \wedge y/\Theta_D = (x \wedge y)/\Theta_D$, for all $x, y \in X$, we have $x/\Theta_D \wedge 0/\Theta_D = (x \wedge 0)/\Theta_D = 0/\Theta_D$. Thus, for all $x \in X$, $x/\Theta_D \geq 0/\Theta_D$. This means that $\mathcal{E}/\Theta_D = \langle X/\Theta_D, \sim, \wedge, 1/\Theta_D\rangle$ is a bounded equality algebra. $\square$

Theorem 3.23. *Let $\mathcal{E} = \langle X, \sim, \wedge, 1, 0\rangle$ be a bounded involutive equality algebra and $(\mathcal{E}, G, H)$ be a tense like equality algebra, where G and H satisfy (a) or (b). Then $\mathcal{E}/\Theta_{D_{G,H}} = \langle X/\Theta_{D_{G,H}}, \sim, \wedge, 1/\Theta_{D_{G,H}}\rangle$ is a bounded involutive equality algebra. Moreover, there are involutive operators $\mathbb{G}, \mathbb{H} : X/\Theta_{D_{G,H}} \to X/\Theta_{D_{G,H}}$ such that $(\mathcal{E}/\Theta_{D_{G,H}}, \mathbb{G}, \mathbb{H})$ is a tense like equality algebra.*

Proof. By Theorem 3.21, $\mathcal{E}/\Theta_{D_{G,H}} = \langle X/\Theta_{D_{G,H}}, \sim, \wedge, 1/\Theta_{D_{G,H}}\rangle$ is an equality algebra. Also, define $(x/\Theta_{D_{G,H}})^* = x/\Theta_{D_{G,H}} \sim 0/\Theta_{D_{G,H}}$, for all $x/\Theta_{D_{G,H}} \in X/\Theta_{D_{G,H}}$. Thus, for all $x/\Theta_{D_{G,H}} \in \langle X/\Theta_{D_{G,H}}, \sim, \wedge, 1/_{D_{G,H}}\rangle$,

$$\begin{aligned}
((x/\Theta_{D_{G,H}})^*)^* &= (x/\Theta_{D_{G,H}} \sim 0/\Theta_{D_{G,H}})^* \\
&= ((x \sim 0)/\Theta_{D_{G,H}})^* \\
&= (x^*/\Theta_{D_{G,H}})^* \\
&= x^*/\Theta_{D_{G,H}} \sim 0/\Theta_{D_{G,H}} \\
&= x^{**}/\Theta_{D_{G,H}} \\
&= x/\Theta_{D_{G,H}}.
\end{aligned}$$

Hence, $\mathcal{E}/\Theta_{D_{G,H}} = \langle X/\Theta_{D_{G,H}}, \sim, \wedge, 1/_{D_{G,H}}\rangle$ is involutive. Also, for all $x, y \in X$,

$$\begin{aligned}
x/\Theta_{D_{G,H}} \leq y/\Theta_{D_{G,H}} &\Leftrightarrow x/\Theta_{D_{G,H}} \wedge y/\Theta_{D_{G,H}} = x/\Theta_{D_{G,H}} \\
&\Leftrightarrow (x \wedge y)/\Theta_{D_{G,H}} = x/\Theta_{D_{G,H}} \\
&\Leftrightarrow x \wedge y = x \\
&\Leftrightarrow x \leq y. \qquad (3.6)
\end{aligned}$$

Define $\mathbb{G}, \mathbb{H} : X/\Theta_{D_{G,H}} \to X/\Theta_{D_{G,H}}$, for all $x/\Theta_{D_{G,H}} \in X/\Theta_{D_{G,H}}$ as

$$\mathbb{G}(x/\Theta_{D_{G,H}}) = G(x)/\Theta_{D_{G,H}} \quad \text{and} \quad \mathbb{H}(x/\Theta_{D_{G,H}}) = H(x)/\Theta_{D_{G,H}}.$$

We show that $(\mathcal{E}/\Theta_{D_{G,H}}, \mathbb{G}, \mathbb{H})$ is a tense like equality algebra. We investigate the axioms (T0)-(T3) only for $\mathbb{G}$.
(T0) According to the definition of $\mathbb{G}$, we have

$$\mathbb{G}(1/\Theta_{D_{G,H}}) = G(1)/\Theta_{D_{G,H}} = 1/\Theta_{D_{G,H}}.$$

and similarly $\mathbb{H}(1/\Theta_{D_{G,H}}) = 1/\Theta_{D_{G,H}}$.
(T1) For all $x/\Theta_{D_{G,H}}, y/\Theta_{D_{G,H}} \in X/\Theta_{D_{G,H}}$,

$$\begin{aligned}
\mathbb{G}(x/\Theta_{D_{G,H}} \to y/\Theta_{D_{G,H}}) &= \mathbb{G}((x \to y)/\Theta_{D_{G,H}}) \\
&= G(x \to y)/\Theta_{D_{G,H}}, \qquad \text{by (3.6) and (T1)} \\
&\geq (G(x) \to G(y))/\Theta_{D_{G,H}} \\
&= G(x)/\Theta_{D_{G,H}} \to G(y)/\Theta_{D_{G,H}} \\
&= \mathbb{G}(x/\Theta_{D_{G,H}}) \to \mathbb{G}(y/\Theta_{D_{G,H}}).
\end{aligned}$$

(T2) For all $x/\Theta_{D_{G,H}}, y/\Theta_{D_{G,H}} \in X/\Theta_{D_{G,H}}$,

$$\begin{aligned}
\mathbb{G}(x/\Theta_{D_{G,H}} \sim y/\Theta_{D_{G,H}}) &= \mathbb{G}((x \sim y)/\Theta_{D_{G,H}}) \\
&= G(x \sim y)/\Theta_{D_{G,H}} \qquad \text{by (3.6) and (T2)} \\
&\leq (G(x) \sim G(y))/\Theta_{D_{G,H}} \\
&= G(x)/\Theta_{D_{G,H}} \sim G(y)/\Theta_{D_{G,H}} \\
&= \mathbb{G}(x/\Theta_{D_{G,H}}) \sim \mathbb{G}(y/\Theta_{D_{G,H}}).
\end{aligned}$$

(T3) For all $x/\Theta_{D_{G,H}}, y/\Theta_{D_{G,H}} \in X/\Theta_{D_{G,H}}$,

$$\begin{aligned}
\mathbb{G}(x/\Theta_{D_{G,H}} \wedge y/\Theta_{D_{G,H}}) &= \mathbb{G}((x \wedge y)/\Theta_{D_{G,H}}) \\
&= G(x \wedge y)/\Theta_{D_{G,H}} \quad \text{by (T3)} \\
&= (G(x) \wedge G(y))/\Theta_{D_{G,H}} \\
&= G(x)/\Theta_{D_{G,H}} \wedge G(y)/\Theta_{D_{G,H}} \\
&= \mathbb{G}(x/\Theta_{D_{G,H}}) \wedge \mathbb{G}(y/\Theta_{D_{G,H}}).
\end{aligned}$$

(T4) For all $x/\Theta_{D_{G,H}} \in X/\Theta_{D_{G,H}}$,

$$\begin{aligned}
\mathbb{H}(\mathbb{G}(x/\Theta_{D_{G,H}})) &= \mathbb{H}(G(x)/\Theta_{D_{G,H}}) \\
&= H(G(x))/\Theta_{D_{G,H}} \text{ , by (3.6) and (T4)} \\
&\geq x/\Theta_{D_{G,H}}.
\end{aligned}$$

Finally, we show that $\mathbb{G}$ is an involutive operator. For all $x/\Theta_{D_{G,H}} \in$

$X/\Theta_{D_{G,H}}$,

$$\begin{aligned}
\mathbb{G}((x/\Theta_{D_{G,H}})^*) &= \mathbb{G}(x^*/\Theta_{D_{G,H}})\\
&= G(x^*)/\Theta_{D_{G,H}}\\
&= G(x)^*/\Theta_{D_{G,H}}\\
&= (G(x)\sim 0)/\Theta_{D_{G,H}}\\
&= G(x)/\Theta_{D_{G,H}}\sim 0/\Theta_{D_{G,H}}\\
&= (G(x)/\Theta_{D_{G,H}})^*\\
&= \mathbb{G}(x/\Theta_{D_{G,H}})^*.
\end{aligned}$$

Hence, $(\mathcal{E}/\Theta_{D_{G,H}}, \mathbb{G}, \mathbb{H})$ is a tense like equality algebra. □

4 Relation between tense like equality algebras and tense MV-algebras

In this section, we show that there is a connection between tense MV-algebras and tense like equality algebras.

In [17], the authors showed that there is a connection between equality algebras and MV-algebras as follows.

Theorem 4.1. *[17]The following two statements hold:*

(i) *For any MV-algebra $\mathcal{B} = (B, \oplus, -, 0)$, $\Psi(\mathcal{B}) = (B, \leftrightarrow, \wedge, 0, 1)$ is a bounded commutative equality algebra, where $\rightarrow$ and the top element 1 are defined by, $x \rightarrow y = x^- \oplus y$ and $1 = 0^-$, for all $x, y \in X$. Moreover, the equivalence operation $\leftrightarrow$ is defined by $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$ and $x \rightarrow y = x \leftrightarrow (x \wedge y)$.*

(ii) *For any bounded commutative equality algebra $\mathcal{E} = \langle X, \sim, \wedge, 0, 1\rangle$, $\Phi(\mathcal{E}) = (X, \oplus,, 0)$ is an MV-algebra, where the operations $\oplus$ and $^-$ defined by, $x \oplus y = x' \rightarrow y, x^- = x'$ and $\rightarrow$ denotes the implication of $\mathcal{E}$, for all $x, y \in X$.*

Definition 4.2. [12] Let $(X, \oplus, -, 0, 1)$ be an MV-algebra and $G, H : X \rightarrow X$ two unary operations on A. The structure (X, G, H) is called *tense* MV-*algebra* if, for any $x, y \in X$, the following conditions are satisfied:
(A0) $G(1) = 1$ and $H(1) = 1$,
(A1) $G(x \rightarrow y) \leq G(x) \rightarrow G(y)$ and $H(x \rightarrow y) \leq H(x) \rightarrow H(y)$, where

$x \to y$ is defined by $x^- \oplus y$,
(A2) $G(x) \oplus G(y) \leq G(x \oplus y)$ and $H(x) \oplus H(y) \leq H(x \oplus y)$,
(A3) $G(x) \oplus G(x) = G(x \oplus x)$ and $H(x) \oplus H(x) = H(x \oplus x)$,
(A4) $x \leq GP(x)$ and $x \leq HF(x)$ where P and F are the unary operations of X defined by $Fx = (Gx^-)^-$ and $Px = (Hx^-)^-$,
(A5) $F(x) \oplus F(x) = F(x \oplus x)$ and $P(x) \oplus P(x) = P(x \oplus x)$.
In axioms (A1), (A2) and (A5), if we replace $=$ by $\leq$, we call (X, G, H) an *equality tense* MV*-algebra*.

Proposition 4.3. *[12, Proposition 5.1]The following statements hold in any tense* MV*-algebra* (X, G, H)*,*

(i) *if* $x \leq y$*, then* $G(x) \leq G(y)$*,* $H(x) \leq H(y)$*,* $F(x) \leq F(y)$ *and* $P(x) \leq P(y)$*,*
(ii) $G(x \to y) \leq F(x) \to F(y)$ *and* $H(x \to y) \leq P(x) \to P(y)$*,*
(iii) $G(x) \odot G(y) \leq G(x \odot y)$ *and* $H(x) \odot H(y) \leq H(x \odot y)$*,*
(iv) $F(x \oplus y) \leq F(x) \oplus F(y)$ *and* $P(x \oplus y) \leq P(x) \oplus P(y)$*,*
(v) $G(x \vee y) \leq F(x) \vee G(y)$ *and* $H(x \vee y) \leq P(x) \vee H(y)$*,*
(vi) $G(x \odot x) = G(x) \odot G(x)$ *and* $H(x \odot x) = H(x) \odot H(x)$*,*
(vii) $F(x \odot x) = F(x) \odot F(x)$ *and* $P(x \odot x) = P(x) \odot P(x)$*,*
(viii) $x \odot F(y) \leq F(P(x) \odot y)$*,*
(ix) $PG(x) \leq x$ *and* $FH(x) \leq x$*,*
(x) $PGP = P, GPG = G, HFH = H$ *and* $FHF = F$*,*
(xi) G *and* H *preserve the arbitrary infima, whenever they exist,*
(xii) F *and* P *preserve the arbitrary suprema, whenever they exist.*

Corollary 4.4. *Let* (X, G, H) *be a tense MV-algebra such that* G *and* H *are involutive operators. Then* H *is the inverse map of* G*.*

Proof. Suppose G and H are involutive operators, that is, $G(x^-) = G(x)^-$ and $H(x^-) = H(x)^-$, for any $x \in X$. Then by (A4), we have $G = F$, $H = P$, $x \leq GH(x)$ and $x \leq HG(x)$, for any $x \in X$. On the other hand, by Proposition 4.3(ix), $PG(x) \leq x$ and $FH(x) \leq x$, for any $x \in X$. Thus, $HG(x) = GH(x) = x$, for any $x \in X$. □

Theorem 4.5. *Let* $(\mathcal{E}, G, H)$ *be a commutative tense like equality algebra and* $\Phi(\mathcal{E}) = (X, \oplus, -, 0)$ *be its corresponding MV-algebra. If* H *is involutive and* $G((H(x))) \geq x$*, for any* $x \in X$*, then* $(\Phi(\mathcal{E}), G, H)$ *is a tense* MV*-algebra, where the operations* $\oplus$ *and* $-$ *defined by* $x \oplus y = x' \to y$*,* $x^- = x'$ *and* $\to$ *denotes the implication of* $\mathcal{E}$*.*

Proof. By Theorem 4.1, $(X, \oplus, -, 0, 1)$ is an MV-algebra. We investigate the axioms (A0)-(A5) for G. Clearly, the axiom (A0) holds. Then by (T1), for any $x, y \in X$ we have,

$$G(x \to y) = G(x^- \oplus y) = G(x' \oplus y) = G(x \sim y) \geq G(x) \to G(y).$$

On the other hand, for any $x, y \in X$ we have,

$$\begin{aligned} G(x \to y) &= G(x \sim (x \wedge y)) \text{ , by (T2)} \\ &\leq G(x) \sim G((x \wedge y)) \\ &= G(x) \sim (G(x) \wedge G(y)) \\ &= G(x) \to G(y). \end{aligned}$$

Thus, (A1) holds. Moreover, for any $x, y \in X$, we have

$$\begin{aligned} G(x \oplus y) &= G(x' \to y) \text{ , by (T1)} \\ &\geq G(x') \to G(y) \\ &= G(x')' \oplus G(y) \\ &= G(x) \oplus G(y). \end{aligned}$$

Thus, (A2) holds. Also, for any $x \in X$, we have

$$\begin{aligned} G(x \oplus x) &= G(x^- \to x) \\ &\leq G(x^-) \to G(x) \\ &= G(x)' \to G(x) \\ &= G(x) \oplus G(x). \end{aligned}$$

Hence, we have (A3). Since G is involutive, we can take $G = F$. This implies that the axioms (A4) and (A5) hold for G. Similarly, we can show that (A0)-(A5) hold for H, where we can suppose that $H = P$. □

Theorem 4.6. *Let $\mathcal{X} = (X, \oplus, -, 0, 1)$ be an MV-algebra and (X, G, H) be a tense* MV*-algebra such that G and H are involutive operators. Then $(\Psi(\mathcal{X}), G, H)$ is a bounded tense like equality algebra, where $\Psi(\mathcal{X}) = (X, \wedge, \leftrightarrow, 0, 1)$, and for any $x, y \in X$, define the operations $\leftrightarrow$ and $\to$ as*

$$x \leftrightarrow y = (x \to y) \wedge (y \to x), x \to y = x \leftrightarrow (x \wedge y) \textit{and } x \to y = x^- \oplus y.$$

Proof. According to Theorem 4.1, $\Psi(\mathcal{X}) = (X, \wedge, \leftrightarrow, 0, 1)$ is a bounded involutive equality algebra. So it is clear that (T0) holds. Also, for any $x, y \in X$, we have

$$\begin{aligned} G(x \to y) &= G(x^- \oplus y) \\ &\geq G(x^-) \oplus G(y) \ , \text{ by (A2)} \\ &= G(x)^- \oplus G(y) \\ &= G(x) \to G(y). \end{aligned}$$

Then (T1) holds. Moreover, for any $x, y \in X$, by Proposition 4.3(i), we have

$$\begin{aligned} G(x \leftrightarrow y) &= G((x \to y) \wedge (y \to x)) \\ &\leq G(x \to y) \\ &\leq G(x) \to G(y). \text{ by (A1)}, \end{aligned} \tag{4.1}$$

Similarly, for any $x, y \in X$, we have

$$G(x \leftrightarrow y) \leq G(y) \to G(x). \tag{4.2}$$

Thus, (T2) holds. Clearly, we have $G(x \wedge y) \leq G(x) \wedge G(y)$, for any $x, y \in X$. Then by Proposition 4.3(iv),

$$\begin{aligned} G(x \wedge y) &= G(x \odot (x^- \oplus y)) \\ &\geq G(x) \odot G(x^- \oplus y) \\ &\geq G(x) \odot (G(x^-) \oplus G(y)) \ , \text{ by (A2)} \\ &= G(x) \odot (G(x)^- \oplus G(y)) \\ &= G(x) \wedge G(y). \end{aligned}$$

Hence, (T3) holds. Similarly, we can see that (T1)-(T3) hold, for H and (T4) holds by Corollary 4.4. Therefore, $(\Psi(\mathcal{X}), G, H)$ is a bounded tense like equality algebra. $\square$

5 Conclusion

In this paper, we have considered equality algebras, introduced involutive operators and tense like equality algebras. We investigated relations between these new notions with the notion of tense MV-algebras. Defining involutive operators on other logical algebras such as hoop algebras and considering these algebras as tense like algebras could be topics for our next task.

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