Categories and
General Algebraic Structures
with Applications
Volume 13, Number 1, July 2020, 23-44.

The non-abelian tensor product of normal crossed submodules of groups

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Communicated by Themba Dube

Abstract. In this article, the notions of non-abelian tensor and exterior products of two normal crossed submodules of a given crossed module of groups are introduced and some of their basic properties are established. In particular, we investigate some common properties between normal crossed modules and their tensor products, and present some bounds on the nilpotency class and solvability length of the tensor product, provided such information is given at least on one of the normal crossed submodules.

1 Introduction

The notion of the non-abelian tensor product of groups was introduced by Brown and Loday [5,6] following ideas of Miller [14], Dennis [8], and has arisen from applications in homotopy theory of a generalized Van Kampen theorem. Group theorical aspects of this concept have been studied extensively by several authors (see [2,3,4,9,16,17,22]). Especially, one of the main themes of the group theoretical part of research on non-abelian ten-

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Keywords: Crossed module, tensor product, exterior product.
Mathematics Subject Classification [2010]: 18A99, 18G99, 20F19.
Received: 25 March 2019, Accepted: 8 July 2019.
ISSN: Print 2345-5853, Online 2345-5861.

sor products has been to determine which group theoretical properties are closed with respect to forming the tensor product. For instance, Ellis [10] proved that the non-abelian tensor product of two finite groups is again a finite group. Visscher [22] and Nakaoka [17] had investigated nilpotency and solvability of the non-abelian tensor product. Donadze, et al. [9] showed that the classes of the nilpotent-by-finite, solvable-by-finite, and supersolvable groups are each closed under the formation of the non-abelian tensor product. These results are excellent tools for studying groups.

The algebraic study of the category of crossed modules was initiated by Norrie [18] and has led to a substantial algebraic theory contained essentially in the following papers: [1,7,11,12,13,15,20,21]. In particular, Pirashvili [19] presented the concept of the tensor product of two abelian crossed modules and investigated its relation to the low-dimensional homology of crossed modules. He also generalized Whitehead's universal quadratic functor of abelian groups to abelian crossed modules.

In this article, we introduce the notions of non-abelian tensor and exterior products of two normal crossed submodules of some crossed module, which are a generalization of the work of Brown and Loday. We give some of their important properties and study the connection of nilpotency and solvability between the normal crossed modules and their tensor products.

2 Preliminaries on crossed modules

This section is devoted to recalling some basic definitions in the category of crossed modules and giving some results related to non-abelian tensor products of groups which will be needed in the sequel.

A *crossed module* (T, G, ∂) is a group homomorphism $\partial : T \longrightarrow G$ together with an action of G on T, written ${}^{g}t$ for $t \in T$ and $g \in G$, satisfying $\partial({}^{g}t) = g\partial t g^{-1}$ and ${}^{\partial(t)}t' = tt't^{-1}$, for all $t, t' \in T$, $g \in G$. It is worth noting that for any crossed module (T, G, ∂), $\mathrm{Im}\partial$ is a normal subgroup of G and $\ker\partial$ is a G-invariant subgroup in the centre of T. Evidently, for any normal subgroup N of a group G, (N, G, i) is a crossed module, where i is the inclusion and G acts on N by conjugation. In this way, every group G can be seen as a crossed module in two obvious ways: $(1, G, i)$ or (G, G, id).

A *morphism of crossed modules* $(\gamma_1, \gamma_2) : (T, G, \partial) \longrightarrow (T', G', \partial')$ is a pair of homomorphisms $\gamma_1 : T \longrightarrow T'$ and $\gamma_2 : G \longrightarrow G'$ such that

$\partial'\gamma_1 = \gamma_2\partial$ and $\gamma_1({}^g t) = {}^{\gamma_2(g)}\gamma_1(t)$ for all $g \in G$, $t \in T$.

Taking objects and morphisms as defined above, we obtain the category of crossed modules. In this category one can find the familiar notions of injection, surjection, (normal) subobject, kernel, cokernel, exact sequence, etc.; most of them can be found in detail in [11,18].

Let (T, G, ∂) be a crossed module with normal crossed submodules (S, H, ∂) and (L, K, ∂). The following is a list of notations which will be used:

- $Z(T, G, \partial) = (T^G, Z(G) \cap st_G(T), \partial)$ is the *center* of (T, G, ∂), where $Z(G)$ is the centre of G, $T^G = \{t \in T \mid {}^g t = t \text{ for all } g \in G\}$, and $st_G(T) = \{g \in G \mid {}^g t = t \text{ for all } t \in T\}$.
- $(T, G, \partial)' = ([G, T], G', \partial)$ is the *commutator crossed submodule* of (T, G, ∂), where $G' = [G, G]$ and $[G, T] = \langle\, {}^g t t^{-1} \mid t \in T, g \in G \rangle$ is the displacement subgroup of T relative to G.
- $[(S, H, \partial), (L, K, \partial)]$ is the normal crossed submodule $([K, S][H, L], [H, K], \partial)$ of (T, G, ∂).
- $\gamma_n(T, G, \partial)$ denotes the nth term of lower central series of (T, G, ∂) defined inductively by

$$\gamma_1(T, G, \partial) = (T, G, \partial) \text{ and } \gamma_{n+1}(T, G, \partial) = [\gamma_n(T, G, \partial), (T, G, \partial)],$$

for $n \geq 1$.
- $(T, G, \partial)^{(n)}$ denotes the nth term of derived series of (T, G, ∂) defined inductively by $(T, G, \partial)^{(0)} = (T, G, \partial)$ and $(T, G, \partial)^{(n+1)} = ((T, G, \partial)^{(n)})'$, for $n \geq 0$.
- $(T, G, \partial)_{ab} = (T/[G, T], G_{ab}, \overline{\partial})$ denotes the *abelianization* of (T, G, ∂), where $G_{ab} = G/G'$ and $\overline{\partial}$ is induced by ∂.

We say a crossed module (T, G, ∂) is *finite* if the groups T and G are both finite. Also, the crossed module (T, G, ∂) is *nilpotent* (respectively, solvable) when there is $n \geq 1$ such that $\gamma_{n+1}(T, G, \partial) = 1$ (respectively, $(T, G, \partial)^{(n)} = 1$). The smallest n with this property is called the nilpotency class (respectively, solvability length) of (T, G, ∂). Especially, the nilpotent crossed module of class 1 is *abelian* and the solvable crossed module of length 2 is *metabelian*.

Let (S, G, ∂) and (L, G, σ) be two crossed modules. There are actions of S on L and of L on S given by ${}^s l = {}^{\partial(s)} l$ and ${}^l s = {}^{\sigma(l)} s$. We take S (and L) to act on itself by conjugation. The *non-abelian tensor product* $S \otimes L$

is defined in [6] as the group generated by symbols $s \otimes l$ for $s \in S, l \in L$, subject to the relations

$$ss' \otimes l = ({}^{s}s' \otimes {}^{s}l)(s \otimes l), \tag{2.1}$$

$$s \otimes ll' = (s \otimes l)({}^{l}s \otimes {}^{l}l'), \tag{2.2}$$

for all $s, s' \in S$ and $l, l' \in L$. Note that the identity homomorphism $id_G : G \longrightarrow G$ is a crossed module with G acting on itself by conjugation, so we can always form the tensor products $S \otimes G$, $L \otimes G$ and $G \otimes G$.

Let S and L be as above, and let T be a group. A function $\phi : S \times L \longrightarrow T$ is called a *crossed pairing* if $\phi(ss', l) = \phi({}^{s}s', {}^{s}l)\phi(s, l)$ and $\phi(s, ll') = \phi(s, l)\phi({}^{l}s, {}^{l}l')$ for all $s, s' \in S$, $l, l' \in L$. It is apparent that the function $S \times L \longrightarrow S \otimes L$, $(s, l) \longmapsto s \otimes l$, is the universal crossed paring in the sense that any crossed pairing $\phi : S \times L \longrightarrow T$ determines a unique homomorphism $\phi^* : S \otimes L \longrightarrow T$ such that $\phi^*(s \otimes l) = \phi(s, l)$.

Let $S \square L$ denote the subgroup of $S \otimes L$ generated by the elements $s \otimes l$ with $\partial(s) = \sigma(l)$. This is a normal subgroup of $S \otimes L$ and, following [6], the *non-abelian exterior product* $S \wedge L$ is defined to be the quotient $S \otimes L / S \square L$.

The following proposition summarizes the rather elementary properties of the non-abelian tensor product, the proof of which is left to the reader (see also [4,6]).

Proposition 2.1. *With the above assumptions and notations, we have*

(i) *If* ${}^{s}l = l$, ${}^{l}s = s$ *for all* $s \in S$, $l \in L$, *then* $S \otimes L \cong S_{ab} \otimes L_{ab}$, *where the right-hand side of the isomorphism denotes the usual tensor product of abelian groups.*

(ii) *There is an isomorphism* $S \otimes L \xrightarrow{\cong} L \otimes S$, $s \otimes l \longmapsto (l \otimes s)^{-1}$.

(iii) *There are group homomorphisms*

$$\begin{aligned} \lambda_G &: S \otimes L \longrightarrow G, & s \otimes l &\longmapsto [\partial(s), \sigma(l)] \\ \lambda_S &: S \otimes L \longrightarrow S, & s \otimes l &\longmapsto s({}^{l}s^{-1}) \\ \lambda_L &: S \otimes L \longrightarrow L, & s \otimes l &\longmapsto {}^{s}ll^{-1} \end{aligned}$$

(iv) *These homomorphisms are crossed modules in which the action of* G *on* $S \otimes L$ *is given by* ${}^{g}(s \otimes l) = {}^{g}s \otimes {}^{g}l$, *and* S *and* L *act on* $S \otimes L$ *via* ∂ *and* σ.

(v) $\lambda_S(x) \otimes l = x\,{}^{l}x^{-1}$, $s \otimes \lambda_L(x) = {}^{s}xx^{-1}$ *for all* $x \in S \otimes L$, $s \in S$, $l \in L$ *and thus the actions of* S *on* $\ker \lambda_L$, L *on* $\ker \lambda_S$ *are trivial.*

(vi) $S\square L \subseteq \ker\lambda_S \cap \ker\lambda_L$, *whence* $S\square L \subseteq Z(S\otimes L)$, *and* S, L *act trivially on* $S\square L$. *In particular, for any* $s\otimes l \in S\square L$, $s\otimes l = s^{-1}\otimes l^{-1}$.

(vii) *There is a natural exact sequence* $\Gamma(S_{ab}) \xrightarrow{\psi} S\otimes S \overset{\pi}{\twoheadrightarrow} S\wedge S$, *where* $\psi(\gamma(\bar{s})) = s\otimes s$ *and* $\pi(s_1\otimes s_2) = (s_1\otimes s_2)S\square S$.

Here $\Gamma(S_{ab})$ denotes J.H.C. Whitehead's universal quadratic functor [23], defined for each abelian group $A = S_{ab}$ as the abelian group generated by symbols $\gamma(a)$ for $a\in A$, subject to the relations

$$\begin{aligned}\gamma(a^{-1}) &= \gamma(a),\\ \gamma(abc)\gamma(a)\gamma(b)\gamma(c) &= \gamma(ab)\gamma(ac)\gamma(bc),\end{aligned}$$

for all $a,b,c\in A$. Note that the last condition yields that the map $\Delta\gamma : A\times A \longrightarrow \Gamma(A)$, $(a,b)\longmapsto \gamma(ab)\gamma(a)^{-1}\gamma(b)^{-1}$, is bilinear. Therefore one has a natural homomorphism $\Delta : A\otimes A \longrightarrow \Gamma(A)$ given by $\Delta(a\otimes b) = \Delta\gamma(a,b)$.

Lemma 2.2. *With the above assumptions and notations, we have*

(i) *For all* $s,s'\in S$, $l,l'\in L$, $x,y\in S\otimes L$, *the following identities hold in* $S\otimes L$*:*

$$\begin{aligned}{}^{s}(s^{-1}\otimes l) = (s\otimes l)^{-1} &= {}^{l}(s\otimes l^{-1}), &(2.3)\\ s'\otimes {}^{s}l l^{-1} &= {}^{s'}(s\otimes l)(s\otimes l)^{-1}, &(2.4)\\ s\,{}^{l}s^{-1}\otimes l' &= (s\otimes l)\,{}^{l'}(s\otimes l)^{-1}, &(2.5)\\ {}^{s\otimes l}(s'\otimes l') &= (s\,{}^{l}s^{-1}\otimes {}^{s'}l'l'^{-1})(s'\otimes l'), &(2.6)\\ {}^{\lambda_S(x)}y &= {}^{\lambda_L(x)}y. &(2.7)\end{aligned}$$

(ii) *For any* $s\otimes l$, $s'\otimes l'\in S\square L$, $(s'\otimes l)(s\otimes l')\in S\square L$ *and* $ss'\otimes ll' = (s'\otimes l)(s\otimes l')(s\otimes l)(s'\otimes l')$.

(iii) *In the group* $G\otimes S$, $\partial(y)\otimes y = 1$ *for all* $y\in[G,S]$.

Proof. Parts (i) and (ii) are found in [4; Proposition 3] and in the proof of [6; Theorem 2.12], respectively.

(iii) Taking into account that any element of $[G,S]$ is written as a finite product of the elements of the form $({}^{g}ss^{-1})^{\pm 1}$, where $g\in G$, $s\in S$, and using Proposition 2.1(vi), we need only verify the result for the case $y = {}^{g}ss^{-1}$. Applying (2.4), we have

$$\begin{aligned}\partial(y)\otimes y = \partial({}^{g}ss^{-1})\otimes {}^{g}ss^{-1} &= {}^{\partial({}^{g}ss^{-1})}(g\otimes s)(g\otimes s)^{-1}\\ &= {}^{{}^{g}ss^{-1}}(g\otimes s)(g\otimes s)^{-1} = {}^{g\otimes s}(g\otimes s)(g\otimes s)^{-1} = 1,\end{aligned}$$

which gives the result. $\square$

3 The tensor and exterior products of crossed submodules

Let (S, H, ∂) and (L, K, ∂) be two normal crossed submodules of a crossed module (T, G, ∂). We can form the non-abelian tensor products $S \otimes K$, $H \otimes L$, and $H \otimes K$. By Proposition 2.1(iv), there are crossed modules $\lambda_H : H \otimes L \longrightarrow H$, $\lambda'_H : H \otimes K \longrightarrow H$ and $\lambda_S : S \otimes K \longrightarrow S$ from which we find the actions of $H \otimes L$ on $S \otimes K$, $H \otimes K$, of $H \otimes K$ on $S \otimes K$, $H \otimes L$, and of $S \otimes K$ on $H \otimes L$, $H \otimes K$. We now form the semidirect product $(S \otimes K) \rtimes (H \otimes L)$ and consider the maps

$$\begin{array}{rlcrl} \alpha : S \otimes L & \longrightarrow (S \otimes K) \rtimes (H \otimes L) & , & \beta : (S \otimes K) \rtimes (H \otimes L) & \longrightarrow \; H \otimes K \\ x & \longmapsto (id_S \otimes \partial(x), (\partial \otimes id_L(x))^{-1}) & & (y, z) & \longmapsto (\partial \otimes id_K(y))(id_H \otimes \partial(z)) \end{array}$$

in which $id_S \otimes \partial : S \otimes L \longrightarrow S \otimes K$, $\partial \otimes id_L : S \otimes L \longrightarrow H \otimes L$, $\partial \otimes id_K : S \otimes K \longrightarrow H \otimes K$ and $id_H \otimes \partial : H \otimes L \longrightarrow H \otimes K$ are the functorial homomorphisms.

The following lemmas play a crucial role in our investigation.

Lemma 3.1. *With the above assumptions and notations, we have*

(i) *for any $x \in S \otimes K$, $y \in H \otimes L$ and $z \in (S \otimes K) \cup (H \otimes L)$, ${}^{\partial \otimes id_K(x)}z = {}^{\lambda_S(x)}z$ and ${}^{id_H \otimes \partial(y)}z = {}^{\lambda_H(y)}z$.*

(ii) *For any $x \in S \otimes L$ and $y \in S \otimes K$, ${}^{\partial \otimes id_L(x)}y = {}^{id_S \otimes \partial(x)}y$.*

(iii) *For any $h \in H$ and $x \in S \otimes L$, ${}^{h}(id_S \otimes \partial(x)) = id_S \otimes \partial({}^{h}x)$ and ${}^{h}(\partial \otimes id_L(x)) = \partial \otimes id_L({}^{h}x)$.*

(iv) *For any $s \in S$, $l \in L$ and $x \in S \otimes K$, ${}^{(\partial(s) \otimes l)^{-1}}x = {}^{l \otimes \partial(s)}x$.*

(v) *For any $l \in L$, $h \in H$ and $x \in S \cap L$, $(\partial(x) \otimes {}^{h}ll^{-1})^{-1} = \partial({}^{h}ll^{-1}) \otimes x$ and $(x \otimes \partial({}^{h}ll^{-1}))^{-1} = {}^{h}ll^{-1} \otimes \partial(x)$.*

(vi) *There are actions of $H \otimes L$ and $H \otimes K$ on $(S \otimes K) \rtimes (H \otimes L)$ defined by ${}^{x}(y, z) = ({}^{\lambda_H(x)}y, {}^{\lambda_H(x)}z)$ and ${}^{x'}(y, z) = ({}^{\lambda'_H(x')}y, {}^{\lambda'_H(x')}z)$, for all $x \in H \otimes L$, $x' \in H \otimes K$ and $(y, z) \in (S \otimes K) \rtimes (H \otimes L)$.*

Proof. Parts (i)-(iv) immediately follow from the definitions of the actions and the properties of crossed module ∂.

(v) Applying the relations (2.4), (2.5) and using the action of T on G, we have

$$\begin{aligned} (\partial(x) \otimes {}^{h}ll^{-1})^{-1} &= ({}^{\partial(x)}(h \otimes l)(h \otimes l)^{-1})^{-1} = (h \otimes l)\, {}^{x}(h \otimes l)^{-1} \\ &= h\, {}^{l}h^{-1} \otimes x = h\, {}^{\partial(l)}h^{-1} \otimes x \\ &= {}^{h}\partial(l)\partial(l)^{-1} \otimes x = \partial({}^{h}ll^{-1}) \otimes x. \end{aligned}$$

For the proof of the second formula, assuming $l_0 = {}^h l l^{-1}$ and using again the relations (2.3)-(2.5) we see that

$$\begin{aligned}(l_0 \otimes \partial(x)) &= ({}^{l_0}(l_0^{-1} \otimes \partial(x)))^{-1} = {}^{l_0}(l\, {}^h l^{-1} \otimes \partial(x))^{-1} \\ &= {}^{l_0}((l \otimes h)\, {}^{\partial(x)}(l \otimes h)^{-1})^{-1} = {}^{l_0}({}^x(l \otimes h)(l \otimes h)^{-1}) \\ &= {}^{l_0}(x \otimes {}^l h h^{-1}) = {}^{l_0}(x \otimes \partial(l)\, {}^h\partial(l)^{-1}) \\ &= {}^{l_0}(x \otimes \partial(l_0^{-1})) = {}^{\partial(l_0)}(x \otimes \partial(l_0^{-1})),\end{aligned}$$

and consequently $(x \otimes \partial(l_0))(l_0 \otimes \partial(x)) = 1$.

(vi) We only verify the condition ${}^x((y_1, z_1)(y_2, z_2)) = {}^x(y_1, z_1)\, {}^x(y_2, z_2)$, for all $x \in H \otimes L$ and $(y_1, z_1), (y_2, z_2) \in (S \otimes K) \rtimes (H \otimes L)$; the rest is easily shown.

$$\begin{aligned}{}^x(y_1, z_1)\, {}^x(y_2, z_2) &= ({}^{\lambda_H(x)}y_1, {}^{\lambda_H(x)}z_1)({}^{\lambda_H(x)}y_2, {}^{\lambda_H(x)}z_2) \\ &= ({}^{\lambda_H(x)}y_1\, {}^{{}^{\lambda_H(x)}z_1}({}^{\lambda_H(x)}y_2), {}^{\lambda_H(x)}z_1\, {}^{\lambda_H(x)}z_2) \\ &= ({}^{\lambda_H(x)}y_1\, {}^{\lambda'_H({}^{\lambda_H(x)}z_1)}({}^{\lambda_H(x)}y_2), {}^{\lambda_H(x)}z_1\, {}^{\lambda_H(x)}z_2) \\ &= ({}^{\lambda_H(x)}y_1\, {}^{\lambda_H(x)\lambda'_H(z_1)\lambda_H^{-1}(x)}({}^{\lambda_H(x)}y_2), {}^{\lambda_H(x)}z_1\, {}^{\lambda_H(x)}z_2) \\ &= ({}^{\lambda_H(x)}y_1\, {}^{\lambda_H(x)\lambda'_H(z_1)}y_2, {}^{\lambda_H(x)}z_1\, {}^{\lambda_H(x)}z_2) \\ &= {}^x(y_1\, {}^{z_1}y_2, z_1 z_2) = {}^x((y_1, z_1)(y_2, z_2)).\end{aligned}$$

The proof is complete. □

Lemma 3.2. *With the above assumptions and notations, we have*

(i) β *is a homomorphism.*

(ii) *$Im\alpha$ is a normal subgroup of $(S \otimes K) \rtimes (H \otimes L)$.*

Proof. (i) Straightforward.

(ii) By virtue of Lemma 3.1(ii), for any $x_1, x_2 \in S \otimes L$, we have

$$\begin{aligned}\alpha(x_1)\alpha(x_2) &= (id_S \otimes \partial(x_1)^{(\partial \otimes id_L(x_1))^{-1}}(id_S \otimes \partial(x_2)), \\ &\qquad (\partial \otimes id_L(x_1))^{-1}(\partial \otimes id_L(x_2))^{-1}) \\ &= (id_S \otimes \partial(x_1)^{(id_S \otimes \partial(x_1))^{-1}}(id_S \otimes \partial(x_2)), \\ &\qquad (\partial \otimes id_L(x_2)\partial \otimes id_L(x_1))^{-1}) \\ &= (id_S \otimes \partial(x_2 x_1), (\partial \otimes id_L(x_2 x_1))^{-1}) \in \mathrm{Im}\alpha.\end{aligned}$$

In particular, $\alpha(x_1)^{-1} = (id_S \otimes \partial(x_1^{-1}), (\partial \otimes id_L(x_1^{-1}))^{-1}) \in \mathrm{Im}\alpha$. We now prove that for all elements $x \in S \otimes L$ and $(y, z) \in (S \otimes K) \rtimes (H \otimes L)$, $A := {}^{(y,z)}(id_S \otimes \partial(x), (\partial \otimes id_L(x))^{-1}) \in \mathrm{Im}\alpha$. A simple computation indicates that $A = (y\, {}^{z}(id_S \otimes \partial(x))^{\,z(\partial \otimes id_L(x))^{-1}} y^{-1}, {}^{z}(\partial \otimes id_L(x))^{-1})$, from which we infer from Lemma 3.1(ii),(vi) that

$$\begin{aligned} {}^{z^{-1}}A &= ({}^{z^{-1}}y(id_S \otimes \partial(x))^{(\partial \otimes id_L(x))^{-1}}({}^{z^{-1}}y^{-1}), (\partial \otimes id_L(x))^{-1}) \\ &= ({}^{z^{-1}}y(id_S \otimes \partial(x))^{(id_S \otimes \partial(x))^{-1}}({}^{z^{-1}}y)^{-1}, (\partial \otimes id_L(x))^{-1}) \\ &= (id_S \otimes \partial(x), (\partial \otimes id_L(x))^{-1}) \in \mathrm{Im}\alpha. \end{aligned}$$

Hence,

$$\begin{aligned} A &= {}^{z}(id_S \otimes \partial(x), (\partial \otimes id_L(x))^{-1}) \\ &= (id_S \otimes \partial({}^{\lambda_H(z)}x), (\partial \otimes id_L({}^{\lambda_H(z)}x))^{-1}) \in Im\alpha, \end{aligned}$$

thanks to Lemma 3.1(iii). □

In the above lemma, it is easy to see that $\mathrm{Im}\alpha$ lies in the kernel of β. So, if we denote by $Coker\alpha$ the quotient group $(S \otimes K) \rtimes (H \otimes L)/\mathrm{Im}\alpha$, then β induces a homomorphism $\delta : Coker\alpha \longrightarrow H \otimes K$. On the other hand, the parts (iii) and (vi) of Lemma 3.1 show that $\mathrm{Im}\alpha$ is invariant under the action of $H \otimes K$ and so, we have an action of $H \otimes K$ on $Coker\alpha$. We can hence get the following

Proposition 3.3. (i) *$(Coker\alpha, H \otimes K, \delta)$ is a crossed module.*

(ii) *If I is a subgroup of $Coker\alpha$ generated by the elements $(x \otimes y, (y \otimes x)(\partial(z) \otimes z))Im\alpha$ for all $x, z \in S \cap L$, $y \in H \cap K$, then $(I, H\square K, \delta)$ is a normal crossed submodule of $(Coker\alpha, H \otimes K, \delta)$.*

Proof. (i) It is straightforward to check that $\delta({}^{u}v) = {}^{u}\delta(v)$ for all $u \in H \otimes K$, $v \in Coker\alpha$. We now establish that ${}^{\delta(v_1)}v = {}^{v_1}v$ for all $v, v_1 \in Coker\alpha$. Without loss of generality, we may assume that $v = (s \otimes k, h \otimes l)\mathrm{Im}\alpha$ and $v_1 = (s_1 \otimes k_1, h_1 \otimes l_1)\mathrm{Im}\alpha$. We must prove that

$$\begin{aligned} &({}^{(\partial(s_1) \otimes k_1)(h_1 \otimes \partial(l_1))}(s \otimes k)\ ,\ {}^{(\partial(s_1) \otimes k_1)(h_1 \otimes \partial(l_1))}(h \otimes l))\mathrm{Im}\alpha = \\ &\qquad ((s_1 \otimes k_1)^{h_1 \otimes l_1}(s \otimes k)^{(h_1 \otimes l_1)(h \otimes l)}(s_1 \otimes k_1)^{-1}\ ,\ {}^{h_1 \otimes l_1}(h \otimes l))\mathrm{Im}\alpha, \end{aligned}$$

or equivalently, letting

$$\begin{aligned}
x_1 &= {}^{(\partial(s_1)\otimes k_1)(h_1\otimes\partial(l_1))}(s\otimes k)\ ,\\
x_2 &= {}^{(\partial(s_1)\otimes k_1)(h_1\otimes\partial(l_1))}(h\otimes l)\ ,\\
y_1 &= (s_1\otimes k_1)^{h_1\otimes l_1}(s\otimes k)^{(h_1\otimes l_1)(h\otimes l)}(s_1\otimes k_1)^{-1}\ ,\\
y_2 &= {}^{h_1\otimes l_1}(h\otimes l),
\end{aligned}$$

we show that

$$(x_1,x_2)^{-1}(y_1,y_2) = {}^{x_2^{-1}}(x_1^{-1}y_1, y_2x_2^{-1}) \in \mathrm{Im}\alpha.$$

The simple calculations, together with the results of Lemmas 2.2(i) and 3.1(i), allow us to get

$$\begin{aligned}
x_1 &= {}^{s_1\otimes k_1}({}^{h_1\otimes l_1}(s\otimes k));\\
x_2 &= {}^{s_1\otimes k_1}({}^{h_1\otimes l_1}(h\otimes l)) = {}^{s_1\otimes k_1}((h_1{}^{l_1}h_1^{-1}\otimes {}^{h}ll^{-1})(h\otimes l))\\
&= {}^{s_1k_1k_1^{-1}}(h_1{}^{l_1}h_1^{-1}\otimes {}^{h}ll^{-1})({}^{s_1}k_1k_1^{-1}\otimes {}^{h}ll^{-1})(h\otimes l);\\
y_1 &= x_1(s_1\otimes k_1)^{y_2}(s_1\otimes k_1)^{-1};\\
y_2 &= (h_1{}^{l_1}h_1^{-1}\otimes {}^{h}ll^{-1})(h\otimes l).
\end{aligned}$$

Now, assuming $l_2 = {}^{h}ll^{-1}$, $l_3 = {}^{h_1}l_1l_1^{-1}$ and $s_2 = s_1{}^{k_1}s_1^{-1}$, one easily sees that $l_2, l_3, s_2 \in S\cap L$, $\partial(l_3) = h_1{}^{l_1}h_1^{-1}$ and $\partial(s_2) = {}^{s_1}k_1k_1^{-1}$, implying that

$$\begin{aligned}
x_2 &= {}^{\partial(s_2)}(\partial(l_3)\otimes l_2)(\partial(s_2)\otimes l_2)(h\otimes l) = (\partial(s_2l_3)\otimes l_2)(h\otimes l);\\
y_2 &= (\partial(l_3)\otimes l_2)(h\otimes l).
\end{aligned}$$

Putting $a = \partial(s_2l_3)\otimes l_2$ and $b = \partial(l_3)\otimes l_2$, we hence deduce that

$$\begin{aligned}
{}^{x_2^{-1}}(x_1^{-1}y_1, y_2x_2^{-1}) &= {}^{(h\otimes l)^{-1}}({}^{a^{-1}}((s_1\otimes k_1)^{b(h\otimes l)}(s_1\otimes k_1)^{-1}), a^{-1}b)\\
&= {}^{(h\otimes l)^{-1}}({}^{a^{-1}b}({}^{b^{-1}}(s_1\otimes k_1)^{h\otimes l}(s_1\otimes k_1)^{-1}), a^{-1}b).
\end{aligned}$$

Therefore, if we set $c = {}^{b^{-1}}(s_1\otimes k_1)^{h\otimes l}(s_1\otimes k_1)^{-1}$, it is enough to prove that $({}^{a^{-1}b}c, a^{-1}b)^{-1} = (c^{-1}, b^{-1}a) \in \mathrm{Im}\alpha$. But

$$
\begin{aligned}
c^{-1} &= {}^{h\otimes l}(s_1 \otimes k_1)({}^{\partial(l_2)\otimes l_3}(s_1 \otimes k_1))^{-1} && \text{(by Lemma 3.1(v))}\\
&= {}^{hll^{-1}}(s_1 \otimes k_1)({}^{l_2 l_3 l_3^{-1}}(s_1 \otimes k_1))^{-1} && \text{(by Lemma 3.1(i))}\\
&= (l_2 \otimes \partial(s_2))({}^{l_2}l_3 l_3^{-1} \otimes \partial(s_2))^{-1} && \text{(by (2.4))}\\
&= (l_2 \otimes \partial(s_2))({l_2}^{\partial(l_3)} l_2^{-1} \otimes \partial(s_2))^{-1} && \text{(by the properties of } \partial)\\
&= (l_2 \otimes \partial(s_2))\,{}^{\partial(s_2)}(l_2 \otimes \partial(l_3))(l_2 \otimes \partial(l_3))^{-1} && \text{(by (2.5))}\\
&= (l_2 \otimes \partial(s_2 l_3))(l_2 \otimes \partial(l_3))^{-1} && \text{(by (2.2))}\\
&= (l_2 \otimes \partial(l_3))^{-1}\,{}^{l_2\otimes\partial(l_3)}(l_2 \otimes \partial(s_2 l_3)) && \\
&= (l_3 \otimes \partial(l_2))\,{}^{(l_3\otimes\partial(l_2))^{-1}}(l_2 \otimes \partial(s_2 l_3)) && \text{(by Lemma 3.1(v))}\\
&= (l_3 \otimes \partial(l_2))\,{}^{(\partial(l_3)\otimes l_2)^{-1}}(l_2 \otimes \partial(s_2 l_3)) && \text{(by Lemma 3.1(ii)).}
\end{aligned}
$$

So, $(c^{-1}, b^{-1}a) = (l_3 \otimes \partial(l_2), (\partial(l_3) \otimes l_2)^{-1})(l_2 \otimes \partial(s_2 l_3), (\partial(l_2) \otimes s_2 l_3)^{-1}) \in \mathrm{Im}\alpha$.

(ii) Recall that $H\Box K = \langle x \otimes x \mid x \in H \cap K\rangle$. Taking into account Lemma 2.2(ii), we have

$$
\begin{aligned}
\delta((x \otimes y, (y \otimes x)(\partial(z) \otimes z))\mathrm{Im}\alpha) &= (\partial(x) \otimes y)(y \otimes \partial(x))(\partial(z) \otimes \partial(z))\\
&= (y\partial(x) \otimes y\partial(x))(\partial(x) \otimes \partial(x))^{-1}(y \otimes y)^{-1}(\partial(z) \otimes \partial(z))\\
&\in H\Box K
\end{aligned}
$$

for all $x, z \in S \cap L$, $y \in H \cap K$ and then $\delta(I) \subseteq H\Box K$. The other conditions are easily verified. $\Box$

Setting $(S, H, \partial)\Box(L, K, \partial) = (I, H\Box K, \delta)$, we are now ready to give the following main definition.

Definition 3.4. The *non-abelian tensor and exterior products* of normal crossed submodules (S, H, ∂) and (L, K, ∂) are defined, respectively, as

$$
\begin{aligned}
(S, H, \partial) \otimes (L, K, \partial) &= (Coker\alpha, H \otimes K, \delta),\\
(S, H, \partial) \wedge (L, K, \partial) &= \frac{(Coker\alpha, H \otimes K, \delta)}{(S, H, \partial)\Box(L, K, \partial)} = (\frac{Coker\alpha}{I}, H \wedge K, \bar{\delta}).
\end{aligned}
$$

Note that if $[(S, H, \partial), (L, K, \partial)] = 1$, then

$$(S, H, \partial) \otimes (L, K, \partial) \cong (S, H, \partial)_{ab} \otimes (L, K, \partial)_{ab},$$

where the right-hand side of the isomorphism denotes the tensor product of abelian crossed modules introduced in [19]. Also, one easily sees that the tensor and exterior products are symmetric.

Example 3.5. (i) If H and K are normal subgroups of a group G, then $(H,H,id)\otimes(K,K,id)\cong(H\otimes K,H\otimes K,id)$. For, assume that $(H,H,id)\otimes(K,K,id)=(Coker\alpha,H\otimes K,\delta)$, where the homomorphism δ is defined by $\delta((x,y)\mathrm{Im}\alpha)=xy$. Then, it is an easy verification that δ is an isomorphism and $(\delta,id):(Coker\alpha,H\otimes K,\delta)\longrightarrow(H\otimes K,H\otimes K,id)$ is a morphism of crossed modules. Analogously, one can see that

$$\begin{aligned}(H,H,id)\wedge(K,K,id)&\cong(H\wedge K,H\wedge K,id),\\(1,H,i)\otimes(1,K,i)&\cong(1,H\otimes K,i),\\(1,H,i)\wedge(1,K,i)&\cong(1,H\wedge K,i).\end{aligned}$$

(ii) For any crossed module (T,G,∂), $(T,G,\partial)\wedge(T,G,\partial)\cong(G\wedge T,G\wedge G,id\wedge\partial)$. Because, by the definition, we have $(T,G,\partial)\wedge(T,G,\partial)=(Coker\alpha/I,G\wedge G,\bar{\delta})$, in which $\bar{\delta}$ is defined by $\bar{\delta}(\overline{(x,y)}I)=\partial\wedge id_G(x)id_G\wedge\partial(y)$ for all $\overline{(x,y)}\in Coker\alpha$. According to Proposition 2.1(ii), the map $\theta:T\otimes G\longrightarrow G\otimes T$, $t\otimes g\longmapsto(g\otimes t)^{-1}$ is an isomorphism. So, we get the epimorphism $\mu:(T\otimes G)\rtimes(G\otimes T)\longrightarrow G\otimes T$ given by $\mu(x,y)=\theta(x)y$. Since, by Lemma 2.2(ii),

$$\mu(t_1\otimes\partial(t_2),(\partial(t_1)\otimes t_2)^{-1})=((\partial(t_1)\otimes t_2)(\partial(t_2)\otimes t_1))^{-1}\in G\square T,$$

for all $t_1,t_2\in T$, μ induces an epimorphism $\bar{\mu}:Coker\alpha\longrightarrow G\wedge T$. We claim that $\ker\bar{\mu}=I$. Plainly, $I\subseteq\ker\bar{\mu}$. So, suppose that $\bar{\mu}((x,y)\mathrm{Im}\alpha)=\theta(x)yG\square T=1_{G\wedge T}$. Then $y=\theta(x)^{-1}z$ for some $z\in G\square T$, implying that $(x,y)\mathrm{Im}\alpha=(x,\theta(x)^{-1}z)\mathrm{Im}\alpha\in I$. Thus, $\bar{\mu}$ gives rise to an isomorphism $\widetilde{\mu}:Coker\alpha/I\longrightarrow G\wedge T$. Now, an easy verification shows that the pair $(\widetilde{\mu},id_{G\wedge G})$ is a crossed module morphism.

The following extends [6; Proposition 2.3] and is important in obtaining the next results.

Proposition 3.6. (i) *There is a morphism* $(\tau_1,\tau_2):(S,H,\partial)\otimes(L,K,\partial)\longrightarrow(T,G,\partial)$ *defined by* $\tau_1((x,y)Im\alpha)=\lambda_S(x)\lambda_L(y)$ *and* $\tau_2(z)=\lambda_H(z)$, *for all* $x\in S\otimes K$, $y\in H\otimes L$ *and* $z\in H\otimes K$.

(ii) $ker(\tau_1,\tau_2)$ *is an abelian crossed module.*

(iii) *If* (L,K,∂) *is simply connected or* H *acts trivially on* L, *then*

$$\ker(\tau_1,\tau_2)\subseteq Z((S,H,\partial)\otimes(L,K,\partial)).$$

Proof. (i) The only non-trivial part is to verify that τ_1 is a homomorphism. But this follows from the following observations. (1) τ_1 is well-defined, because it is induced by the homomorphism $(S\otimes K)\rtimes(H\otimes L)\longrightarrow T$, $(x,y)\longmapsto\lambda_S(x)\lambda_L(y)$. (2) For any $x,x'\in S\otimes K$, $y,y'\in H\otimes L$,

$$\begin{aligned}\lambda_S(x\,{}^{y}x')\lambda_L(yy')&=\lambda_S(x\,{}^{\lambda_L(y)}x')\lambda_L(y)\lambda_L(y')\\&=\lambda_S(x)\,{}^{\lambda_L(y)}\lambda_S(x')\lambda_L(y)\lambda_L(y')\\&=\lambda_S(x)\lambda_L(y)\lambda_S(x')\lambda_L(y').\end{aligned}$$

Note that the second equality follows from the first property of the crossed module λ_S and the fact that $\lambda_L(y)\in L\cap S$.

(ii) It is sufficient to note that $\ker\tau_2=H\Box K$ is a central subgroup of $H\otimes K$ and acts trivially on $Coker\alpha$.

(iii) We first assume that (L,K,∂) is simply connected. Owing to part (ii), we only need to prove that $\ker\tau_1$ is contained in $Coker\alpha^{H\otimes K}$, or equivalently, ${}^{x}y=y$ for all $x\in H\otimes K$ and $y\in\ker\tau_1$. Let $x=h'\otimes k'$ and $y=(s\otimes k,h\otimes l)\mathrm{Im}\alpha$ for $h,h'\in H$, $k,k'\in K$, $s\in S$, $l\in L$. Then ${}^{k}ss^{-1}={}^{h}ll^{-1}$and also, $k'=\partial(l')$ for some $l'\in L$, forcing that ${}^{h'}k'k'^{-1}=\partial({}^{h'}l'l'^{-1})$. Hence, we have

$$\begin{aligned}{}^{h'\otimes k'}(s\otimes k)&={}^{{}^{h'}k'k'^{-1}}(s\otimes k)\\&=(s\,{}^{k}s^{-1}\otimes{}^{h'}k'k'^{-1})^{-1}(s\otimes k)&&\text{(by (2.5))}\\&={}^{s\,{}^{k}s^{-1}}({}^{h}ll^{-1}\otimes{}^{h'}k'k'^{-1})(s\otimes k)&&\text{(by (2.3))}\\&={}^{s\otimes k}({}^{h}ll^{-1}\otimes{}^{h'}k'k'^{-1})(s\otimes k)\\&=(s\otimes k)({}^{h}ll^{-1}\otimes{}^{h'}k'k'^{-1})\\&=({}^{h}ll^{-1}\otimes{}^{h'}k'k'^{-1})\,{}^{({}^{h}ll^{-1}\otimes{}^{h'}k'k'^{-1})^{-1}}(s\otimes k)\\&=({}^{h}ll^{-1}\otimes{}^{h'}k'k'^{-1})\,{}^{({}^{h'}k'k'^{-1}\otimes{}^{h}ll^{-1})}(s\otimes k)\\&&&\text{(by Lemma 3.1(i), (iv))}\\&=({}^{h}ll^{-1}\otimes\partial({}^{h'}l'l'^{-1}))\,{}^{(\partial({}^{h'}l'l'^{-1})\otimes{}^{h}ll^{-1})}(s\otimes k),\end{aligned}$$

and using Lemma 3.1(i),

$$\begin{aligned}{}^{h'\otimes k'}(h\otimes l)={}^{h'\otimes\partial(l')}(h\otimes l)={}^{h'\otimes l'}(h\otimes l)&={}^{{}^{h'}l'l'^{-1}}(h\otimes l)\\&={}^{\partial({}^{h'}l'l'^{-1})}(h\otimes l)=(\partial({}^{h'}l'l'^{-1})\otimes{}^{h}ll^{-1})(h\otimes l).\end{aligned}$$

It therefore follows that

$$
\begin{aligned}
{}^{x}y &= ({}^{h'\otimes k'}(s \otimes k),\ {}^{h'\otimes k'}(h \otimes l))\mathrm{Im}\alpha \\
&= ({}^{h}ll^{-1} \otimes \partial({}^{h'}l'l'^{-1}), \partial({}^{h'}l'l'^{-1}) \otimes {}^{h}ll^{-1})(s \otimes k, h \otimes l)\mathrm{Im}\alpha \\
&= ({}^{h}ll^{-1} \otimes \partial({}^{h'}l'l'^{-1}), (\partial({}^{h}ll^{-1}) \otimes {}^{h'}l'l'^{-1})^{-1})(s \otimes k, h \otimes l)\mathrm{Im}\alpha \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 3.1(v))} \\
&= (s \otimes k, h \otimes l)\mathrm{Im}\alpha = y.
\end{aligned}
$$

The proof for the other case is analogous. □

The following corollary is an immediate consequence of the above proposition.

Corollary 3.7. *Let (T, G, ∂) be a crossed module such that ∂ is onto or G acts trivially on T. Then $(T, G, \partial)\square(T, G, \partial)$ is a central crossed submodule of $(T, G, \partial) \otimes (T, G, \partial)$.*

In order to study the relation between the exterior product and tensor product of crossed modules, under some conditions, we use a generalized version of Whitehead's universal quadratic functor, the generalization being due to [19].

Definition 3.8. ([19]). Let (A, B, ∂) be an abelian crossed module, and $B\underline{\otimes}A$ be the quotient of $B \otimes A$ by the subgroup generated by the elements $(\partial(a_1) \otimes a_2)(\partial(a_2) \otimes a_1)^{-1}$ with $a_1, a_2 \in A$. Then we define $\Gamma(A, B, \partial)$ to be the abelian crossed module $(\bar{\Gamma}(A, B, \partial), \Gamma(B), \partial_\Gamma)$, in which $\bar{\Gamma}(A, B, \partial)$ is the cokernel of the group homomorphism $f : A \otimes A \longrightarrow \overline{(B\underline{\otimes}A) \times \Gamma(A)}$ given by $f(a_1 \otimes a_2) = (\overline{\partial(a_1) \otimes a_2}, \Delta(a_1 \otimes a_2)^{-1})$, and $\partial_\Gamma(\overline{\overline{b \otimes a}}, \gamma(a_1)) = \Delta(b \otimes \partial(a))\gamma(\partial(a_1))$.

Theorem 3.9. *Let (T, G, ∂) be a crossed module such that ∂ is onto or G acts trivially on T. Then there is a natural exact sequence*

$$\Gamma((T, G, \partial)_{ab}) \longrightarrow (T, G, \partial) \otimes (T, G, \partial) \twoheadrightarrow (T, G, \partial) \wedge (T, G, \partial).$$

Proof. We only prove the result for the case of ∂ is onto. The proof for the other case is identical. It is sufficient to define a surjective morphism $(\eta_1, \eta_2) : (\bar{\Gamma}((T, G, \partial)_{ab}), \Gamma(G_{ab}), \partial_\Gamma) \longrightarrow (I, G\square G, \delta)$. We take η_2 to be the epimorphism given in Proposition 2.1(vii). Putting $\bar{T} = T/[G, T]$ and $\bar{G} = G_{ab}$, we construct η_1 in the following three steps.

Step 1. Here we show the existence of a homomorphism $\widetilde{\phi}_1 : \bar{G}\underline{\otimes}\bar{T} \longrightarrow I$.

Let $\phi_1 : \bar{G} \times \bar{T} \longrightarrow I$ be defined by $\phi_1(\bar{g}, \bar{t}) = (t \otimes g, g \otimes t)\mathrm{Im}\alpha$. We first show that ϕ_1 is well-defined. Suppose that $g_1 = g_2 x$ and $t_1 = t_2 y$ for some $x \in G'$ and $y \in [G, T]$. Then

$$\begin{aligned}\phi_1(\bar{g}_1, \bar{t}_1) &= (t_2 y \otimes g_2 x, g_2 x \otimes t_2 y)\mathrm{Im}\alpha \\ &= ({}^{t_2}(y \otimes g_2)(t_2 \otimes g_2)\, {}^{g_2 t_2}(y \otimes x)\, {}^{g_2}(t_2 \otimes x), \\ &\qquad {}^{g_2}(x \otimes t_2)\, {}^{g_2 t_2}(x \otimes y)(g_2 \otimes t_2)\, {}^{t_2}(g_2 \otimes y))\mathrm{Im}\alpha \\ &= ({}^{t_2}(y \otimes g_2)(t_2 \otimes g_2)\, {}^{g_2 t_2}(y \otimes x), 1)\, {}^{g_2}(t_2 \otimes x, x \otimes t_2) \\ &\qquad (1, {}^{g_2 t_2}(x \otimes y)(g_2 \otimes t_2)\, {}^{t_2}(g_2 \otimes y))\mathrm{Im}\alpha.\end{aligned}$$

The surjectivity of ∂ yields that $\partial([G, T]) = G'$ and then there exists $t' \in [G, T]$ such that $\partial(t') = x$. Since, by Lemma 3.1(v),

$$\begin{aligned}{}^{g_2}(t_2 \otimes x, x \otimes t_2) &= {}^{g_2}(t_2 \otimes \partial(t'), \partial(t') \otimes t_2) \\ &= {}^{g_2}(t_2 \otimes \partial(t'), (\partial(t_2) \otimes t')^{-1}) \in \mathrm{Im}\alpha,\end{aligned}$$

it follows that

$$\begin{aligned}\phi_1(\bar{g}_1, \bar{t}_1) &= ({}^{t_2}(y \otimes g_2)(t_2 \otimes g_2)\, {}^{g_2 t_2}(y \otimes x), 1) \\ &\qquad (1, {}^{g_2 t_2}(x \otimes y)(g_2 \otimes t_2)\, {}^{t_2}(g_2 \otimes y))\mathrm{Im}\alpha \\ &= ({}^{t_2}(y \otimes g_2)(t_2 \otimes g_2), 1)\, {}^{g_2 t_2}(y \otimes x, x \otimes y) \\ &\qquad (1, (g_2 \otimes t_2)\, {}^{t_2}(g_2 \otimes y))\mathrm{Im}\alpha.\end{aligned}$$

Using the same arguments as above, one sees that ${}^{g_2 t_2}(y \otimes x, x \otimes y) \in \mathrm{Im}\alpha$, whence

$$\begin{aligned}\phi_1(\bar{g}_1, \bar{t}_1) &= ({}^{t_2}(y \otimes g_2)(t_2 \otimes g_2), 1)(1, (g_2 \otimes t_2)\, {}^{t_2}(g_2 \otimes y))\mathrm{Im}\alpha \\ &= {}^{t_2}(y \otimes g_2, 1)(t_2 \otimes g_2, g_2 \otimes t_2)\, {}^{t_2}(1, g_2 \otimes y)\mathrm{Im}\alpha \\ &= (t_2 \otimes g_2, g_2 \otimes t_2)\, {}^{t_2}(y \otimes g_2, g_2 \otimes y)\mathrm{Im}\alpha,\end{aligned}$$

since $(t_2 \otimes g_2, g_2 \otimes t_2) \in I \subseteq Z(Coker\alpha)$, thanks to Corollary 3.6. Noting that $y \in [G, T]$ and $g_2 = \partial(t_3)$ for some $t_3 \in T$, we can again conclude that ${}^{t_2}(y \otimes g_2, g_2 \otimes y) \in \mathrm{Im}\alpha$ and therefore $\phi_1(\bar{g}_1, \bar{t}_1) = (t_2 \otimes g_2, g_2 \otimes t_2)\mathrm{Im}\alpha = \phi_1(\bar{g}_2, \bar{t}_2)$. It is easily verified that ϕ_1 is a crossed paring, and the universal property of the tensor product then implies the homomorphism $\bar{\phi}_1 : \bar{G} \otimes \bar{T} \longrightarrow I$. Now, we claim that $\bar{\phi}_1$ annihilates the subgroup generated by the elements $(\overline{\partial(t_1)} \otimes \bar{t}_2)(\overline{\partial(t_2)} \otimes \bar{t}_1)^{-1}$, $t_1, t_2 \in T$; because, we have

$$
\begin{aligned}
&(t_2 \otimes \partial(t_1), \partial(t_1) \otimes t_2)(t_1 \otimes \partial(t_2), \partial(t_2) \otimes t_1)^{-1} \\
&= (t_2 \otimes \partial(t_1), \partial(t_1) \otimes t_2)({}^{(\partial(t_2)\otimes t_1)^{-1}}(t_1 \otimes \partial(t_2))^{-1}, (\partial(t_2) \otimes t_1)^{-1}) \\
&= ((t_2 \otimes \partial(t_1))\ {}^{(\partial(t_1)\otimes t_2)(\partial(t_2)\otimes t_1)^{-1}}(t_1 \otimes \partial(t_2))^{-1}, (\partial(t_1) \otimes t_2)(\partial(t_2) \otimes t_1)^{-1}) \\
&= ((t_2 \otimes \partial(t_1))(t_1 \otimes \partial(t_2))^{-1}, (\partial(t_1) \otimes t_2)(\partial(t_2) \otimes t_1)^{-1}) \\
&= ((t_1 \otimes \partial(t_2))^{-1}\ {}^{\partial(t_1)\otimes t_2}(t_2 \otimes \partial(t_1)), (\partial(t_1) \otimes t_2)(\partial(t_2) \otimes t_1)^{-1}) \\
&= ((t_1 \otimes \partial(t_2))^{-1}, \partial(t_1) \otimes t_2)(t_2 \otimes \partial(t_1), (\partial(t_2) \otimes t_1)^{-1}) \\
&= ({}^{\partial(t_1)\otimes t_2}(t_1 \otimes \partial(t_2))^{-1}, \partial(t_1) \otimes t_2)(t_2 \otimes \partial(t_1), (\partial(t_2) \otimes t_1)^{-1}) \\
&= (t_1 \otimes \partial(t_2), (\partial(t_1) \otimes t_2)^{-1})^{-1}(t_2 \otimes \partial(t_1), (\partial(t_2) \otimes t_1)^{-1}) \in \mathrm{Im}\alpha.
\end{aligned}
$$

Note that the third and sixth equalities follow from the parts (ii) and (iv) of Lemma 3.1. Consequently, $\bar{\phi}_1((\overline{\partial(t_1)} \otimes \bar{t}_2)(\overline{\partial(t_2)} \otimes \bar{t}_1)^{-1}) = 1$ and $\bar{\phi}_1$ induces a homomorphism $\widetilde{\phi}_1 : \bar{G}\underline{\otimes}\bar{T} \longrightarrow I$.

Step 2. Here we show the existence of a homomorphism $\widetilde{\phi}_2 : \Gamma(\bar{T}) \longrightarrow I$.

Let $\phi_2 : \bar{T} \longrightarrow I$ be defined by $\phi_2(\bar{t}) = (1, \partial(t) \otimes t)\mathrm{Im}\alpha$. Then ϕ_2 is correctly defined; because if $t_1 = t_2 y$ for some $y \in [G, T]$, then

$$
\begin{aligned}
\partial(t_2 y) \otimes t_2 y &= (\partial(y) \otimes t_2)(\partial(t_2) \otimes y)(\partial(t_2) \otimes t_2)(\partial(y) \otimes y) \\
&\qquad\qquad \text{(by Lemma 2.2(ii))} \\
&= (\partial(t_2) \otimes y)^{-1}(\partial(t_2) \otimes y)(\partial(t_2) \otimes t_2) \\
&\qquad\qquad \text{(by Lemmas 2.2(iii), 3.1(v))} \\
&= \partial(t_2) \otimes t_2.
\end{aligned}
$$

So, ϕ_2 induces a map $\widetilde{\phi}_2 : \Gamma(\bar{T}) \longrightarrow I$. We prove that $\widetilde{\phi}_2$ is a homomorphism by showing that $\widetilde{\phi}_2$ preserves the defining relations for $\Gamma(-)$. This is deduced for the first from Proposition 2.1(vi) and for the second from the following equalities, which all follow from Proposition 2.1(vi) and Lemma 2.2(ii). For any $t_1, t_2, t_3 \in T$,

$$\begin{aligned}
\partial(t_1t_3)\otimes t_1t_3 &= (\partial(t_3)\otimes t_1)(\partial(t_1)\otimes t_3)(\partial(t_1)\otimes t_1)(\partial(t_3)\otimes t_3),\\
\partial(t_2t_3)\otimes t_2t_3 &= (\partial(t_3)\otimes t_2)(\partial(t_2)\otimes t_3)(\partial(t_2)\otimes t_2)(\partial(t_3)\otimes t_3),\\
\partial(t_1t_2t_3)\otimes t_1t_2t_3 &= (\partial(t_3)\otimes t_1t_2)(\partial(t_1t_2)\otimes t_3)(\partial(t_1t_2)\otimes t_1t_2)(\partial(t_3)\otimes t_3)\\
&= (\partial(t_1t_2)\otimes t_1t_2)(\partial(t_3)\otimes t_1)\,{}^{t_1}((\partial(t_3)\otimes t_2)\\
&\qquad (\partial(t_2)\otimes t_3))(\partial(t_1)\otimes t_3)(\partial(t_3)\otimes t_3)\\
&= (\partial(t_1t_2)\otimes t_1t_2)(\partial(t_3)\otimes t_1)(\partial(t_1)\otimes t_3)\\
&\qquad (\partial(t_3)\otimes t_2)(\partial(t_2)\otimes t_3)(\partial(t_3)\otimes t_3).
\end{aligned}$$

Step 3. Here we define the homomorphism $\eta_1 : \overline{\Gamma}((T,G,\partial)_{ab}) \longrightarrow I$.

Since the groups $\bar{G}\underline{\otimes}\bar{T}$, $\Gamma(\bar{T})$ and I are abelian, we get the induced homomorphism $\widetilde{\phi} = \langle \widetilde{\phi_1}, \widetilde{\phi_2}\rangle$ in the coproduct $\bar{G}\underline{\otimes}\bar{T} \times \Gamma(\bar{T}) \longrightarrow I$, which is clearly surjective. Using Proposition 2.1(vi) and Lemma 2.2(ii), it is readily verified that $\widetilde{\phi}(\mathrm{Im} f) = 1$, in which f is the homomorphism introduced in the above definition. We thus obtain the epimorphism η_1 induced by $\widetilde{\phi}$.

Finally, the pair (η_1, η_2) is a crossed module morphism; because, using the definition of the homomorphism ∂_Γ, Proposition 2.1(vi) and Lemma 2.2(ii), we have

$$\begin{aligned}
\eta_2\partial_\Gamma((\overline{\bar{g}\otimes\bar{t}},\gamma(\bar{t}_1))\mathrm{Im} f) &= \eta_2(\Delta(\bar{g}\otimes\bar{\partial}(\bar{t}))\gamma(\bar{\partial}(\bar{t}_1)))\\
&= \eta_2(\gamma(\overline{g\partial(t)})\gamma(\bar{g})^{-1}\gamma(\overline{\partial(t)})^{-1}\gamma(\overline{\partial(t_1)}))\\
&= (g\partial(t)\otimes g\partial(t))(g\otimes g)^{-1}(\partial(t)\otimes\partial(t))^{-1}(\partial(t_1)\otimes\partial(t_1))\\
&= (\partial(t)\otimes g)(g\otimes\partial(t))(\partial(t_1)\otimes\partial(t_1))\\
&= \delta\eta_1((\overline{\bar{g}\otimes\bar{t}},\gamma(\bar{t}_1))\mathrm{Im} f),
\end{aligned}$$

for all $\bar{t},\bar{t}_1 \in \bar{T}$, $\bar{g}\in\bar{G}$ (note that $\bar{\partial}:\bar{T}\longrightarrow\bar{G}$ is the crossed module induced by ∂). The proof of theorem is complete. $\square$

Combining the above theorem with Proposition 3.6 and [18; Theorem 2.68], we immediately deduce that if (T,G,∂) is a simply connected perfect crossed module, then the central extension $\ker(\tau_1,\tau_2) \rightarrowtail (T,G,\partial)\otimes(T,G,\partial) \twoheadrightarrow (T,G,\partial)$ is universal.

4 On nilpotency and solvability of tensor products

Throughout, we assume that (S,H,∂), (L,K,∂) are normal crossed submodules of a given crossed module (T,G,∂). The goal of this section is to give bounds on the nilpotency class and solvability length of $(S,H,\partial)\otimes(L,K,\partial)$, provided such information is given in context with (S,H,∂) and (L,K,∂). The following lemma shortens the proof of our main result.

Lemma 4.1. *For any $n \geq 0$, we have*

(i) $\gamma_{n+1}(\frac{(T,G,\partial)}{(S,H,\partial)}) = \frac{\gamma_{n+1}(T,G,\partial)(S,H,\partial)}{(S,H,\partial)}$ *and* $(\frac{(T,G,\partial)}{(S,H,\partial)})^{(n)} = \frac{(T,G,\partial)^{(n)}(S,H,\partial)}{(S,H,\partial)}$.

(ii) $\gamma_{n+1}(T,G,\partial) = ([_nG,T],\gamma_{n+1}(G),\partial)$, *in which* $[_0G,T] = T$ *and inductively* $[_nG,T] = [G,[_{n-1}G,T]]$.

Proof. (i) Follows by induction on n.

(ii) By virtue of [20; Lemma 2.1],

$$\gamma_{n+1}(T,G,\partial) = ([_nG,T]\prod_{i=2}^{n}[_{n-i}G,[\gamma_i(G),T]],\gamma_{n+1}(G),\partial).$$

So, it is enough to prove that for any $i \geq 1$, $[\gamma_i(G),T] \subseteq [_iG,T]$. But this follows by induction on i, and using the fact that

$$[\gamma_{i+1}(G),T] \subseteq [\gamma_i(G),[G,T]][G,[\gamma_i(G),T]].$$

□

Theorem 4.2. (i) *If the commutator submodule $[(S,H,\partial),(L,K,\partial)]$ is solvable of length m, then $(S,H,\partial)\otimes(L,K,\partial)$ is solvable of length m or $m+1$.*

(ii) *If the commutator submodule $[(S,H,\partial),(L,K,\partial)]$ is nilpotent of class c, then $(S,H,\partial)\otimes(L,K,\partial)$ is nilpotent of class c or $c+1$.*

(iii) *If the commutator submodule $[(S,H,\partial),(L,K,\partial)]$ is abelian, then $(S,H,\partial)\otimes(L,K,\partial)$ is metabelian.*

Proof. Proposition 3.6 yields the following abelian extension

$$ker(\tau_1,\tau_2) \rightarrowtail (S,H,\partial)\otimes(L,K,\partial) \overset{(\tau_1,\tau_2)}{\twoheadrightarrow} [(S,H,\partial),(L,K,\partial)]. \quad (8)$$

Now, the parts (i) and (iii) are directly obtained from this extension.

To prove (ii), it suffices to indicate that $\gamma_{c+2}((S,H,\partial)\otimes(L,K,\partial)) = 1$, or equivalently, by Lemma 4.1(ii), that $([_{c+1}H\otimes K, Coker\alpha],\gamma_{c+2}(H\otimes K),\delta) = 1$. It follows from the assumption and the above extension that $\gamma_{c+1}(H\otimes K) \subseteq \ker\tau_2$ and then $\gamma_{c+2}(H\otimes K) = 1$. We now prove that $[_{c+1}H\otimes K, Coker\alpha] = 1$. To do this, we will first show by induction that for any $n \geq 1$, every generator of $[_nH\otimes K, Coker\alpha]$ can be expressed as

$(\prod_{i=1}^{m_1}(x_i \otimes y_i), \prod_{j=1}^{m_2}(y'_j \otimes x'_j))\mathrm{Im}\alpha$, where $x_i, x'_j \in [_{n-1}[H,K],[K,S][H,L]]$ and $y_i, y'_j \in [H,K]$. For $n=1$, let $x = {}^c(a,b)(a,b)^{-1}\mathrm{Im}\alpha$ be an arbitrary generator of $[H \otimes K, Coker\alpha]$, where $a \in S \otimes K$, $b \in H \otimes L$ and $c \in H \otimes K$. Considering the homomorphisms $\lambda_S : S \otimes K \longrightarrow S$, $\lambda_H : H \otimes K \longrightarrow H$, $\lambda_K : H \otimes K \longrightarrow K$, $\lambda_L : H \otimes L \longrightarrow L$ and using the relations (2.4), (2.5), we have

$$\begin{aligned} x &= ({}^{\lambda_K(c)}a, {}^{\lambda_H(c)}b)({}^{b^{-1}}a^{-1}, b^{-1})\mathrm{Im}\alpha \\ &= ((\lambda_S(a) \otimes \lambda_K(c))^{-1}a, (\lambda_H(c) \otimes \lambda_L(b))b)({}^{b^{-1}}a^{-1}, b^{-1})\mathrm{Im}\alpha \\ &= ((\lambda_S(a) \otimes \lambda_K(c))^{-1}a\,{}^{(\lambda_H(c)\otimes\lambda_L(b))}a^{-1}, \lambda_H(c) \otimes \lambda_L(b))\mathrm{Im}\alpha. \end{aligned}$$

On the other hand, by the relation (2.5),

$$\begin{aligned} {}^{(\lambda_H(c)\otimes\lambda_L(b))}a^{-1} &= {}^{({}^{\lambda_H(c)}\lambda_L(b)\lambda_L(b)^{-1})}a^{-1} = {}^{({}^{\lambda_H(c)}\partial(\lambda_L(b))\partial(\lambda_L(b))^{-1})}a^{-1} \\ &= {}^{[\lambda_H(c),\partial(\lambda_L(b))]}a^{-1} = a^{-1}(\lambda_S(a) \otimes [\lambda_H(c), \partial(\lambda_L(b))]). \end{aligned}$$

It thus follows that

$$x = ((\lambda_S(a) \otimes \lambda_K(c))^{-1}(\lambda_S(a) \otimes [\lambda_H(c), \partial(\lambda_L(b))]), \lambda_H(c) \otimes \lambda_L(b))\mathrm{Im}\alpha,$$

where

$$\begin{aligned} &\lambda_S(a), \lambda_L(b) \in [K,S][H,L], \\ &\lambda_H(c), \lambda_K(c), [\lambda_H(c), \partial(\lambda_L(b))] \in [H,K]. \end{aligned}$$

Now, assume that the result holds for $n \geq 1$. Then any generator of $[_{n+1}H \otimes K, Coker\alpha]$ can be written as

$$x = {}^c(\prod_{i=1}^{m_1}(x_i \otimes y_i), \prod_{j=1}^{m_2}(y'_j \otimes x'_j))(\prod_{i=1}^{m_1}(x_i \otimes y_i), \prod_{j=1}^{m_2}(y'_j \otimes x'_j))^{-1}\mathrm{Im}\alpha,$$

where $x_i, x'_j \in [_{n-1}[H,K],[K,S][H,L]]$, $y_i, y'_j \in [H,K]$and $c \in H \otimes K$. Setting $a = \prod_{i=1}^{m_1}(x_i \otimes y_i)$, $b = \prod_{j=1}^{m_2}(y'_j \otimes x'_j)$ and applying arguments similar to the above, one can easily see that

$$x = ((\lambda_S(a) \otimes \lambda_K(c))^{-1}(\lambda_S(a) \otimes [\lambda_H(c), \partial(\lambda_L(b))]), \lambda_H(c) \otimes \lambda_L(b))\mathrm{Im}\alpha,$$

where

$$\lambda_S(a), \lambda_L(b) \in [_n[H,K],[K,S][H,L]],$$
$$\lambda_H(c), \lambda_K(c), [\lambda_H(c), \partial(\lambda_L(b))] \in [H,K].$$

This completes the induction. Since, by hypothesis, $[_c[H,K],[K,S][H,L]] = 1$, we therefore infer that $[_{c+1}H \otimes K, Coker\alpha] = 1$, as desired. $\square$

As an immediate consequence of the above theorem, we have

Corollary 4.3. (i) *If the crossed module (T,G,∂) is solvable of length m, then $(T,G,\partial) \otimes (T,G,\partial)$ is solvable of length $m-1$ or m.*

(ii) *If the commutator crossed submodule $(T,G,\partial)'$ is nilpotent of class c, then $(T,G,\partial) \otimes (T,G,\partial)$ is nilpotent of class c or $c+1$.*

(iii) *If the crossed module (T,G,∂) is metabelian, then so is $(T,G,\partial) \otimes (T,G,\partial)$.*

In the following, we indicate a result similar to part (i) of the above corollary for nilpotent crossed modules.

Theorem 4.4. (i) *If the crossed module (T,G,∂) is nilpotent of class c, then $(T,G,\partial) \otimes (T,G,\partial)$ is nilpotent of class at most c.*

(ii) *If the crossed module (T,G,∂) is nilpotent of class 2, then $(T,G,\partial) \otimes (T,G,\partial)$ is abelian.*

Proof. (i) By the assumption and Lemma 4.1(ii), $[_cG,T] = 1_T$ and $\gamma_{c+1}(G) = 1_G$. Hence, $G \otimes G$ is a nilpotent group of class to be c or $c+1$, thanks to [22; Proposition 3.2]. By an argument analogous to that used in the proof of Theorem 4.2, one observes that any generator of the group $[_cG \otimes G, Coker\alpha]$ may be exhibited as $(\prod_{i=1}^{m_1}(x_i \otimes y_i), \prod_{j=1}^{m_2}(y'_j \otimes x'_j))\mathrm{Im}\alpha$, where $x_i, x'_j \in [_{c-1}G',[G,T]]$ and $y_i, y'_j \in G'$. Inasmuch as $[_{c-1}G',[G,T]] \subseteq [_cG,T] = 1_T$, we must therefore have $[_cG \otimes G, Coker\alpha] = 1$ and then $(T,G,\partial) \otimes (T,G,\partial)$ is nilpotent of class at most c.

(ii) It follows from hypothesis that $([G,T],G',\partial)$ lies in $(T^G, Z(G) \cap st_G(T), \partial)$. So, the group $G \otimes G$ is abelian, due to [2; Proposition 3.1], and acts trivially on $Coker\alpha$. $\square$

Theorem 4.2(iii), Theorem 4.4(ii), and the following example show that both outcomes obtained in the above for the nilpotency class and the solvability length of tensor products occur.

Example 4.5. Let $G = \langle a, b, c\rangle$ be the free two-Engel group of rank three. Let H and K be the smallest normal subgroups of G containing the sets $\{a, b\}$ and $\{c\}$, respectively. Then it is shown in [22; Example 4.5] that $[H, K]$ is an abelian subgroup of G and $H \otimes K$ is nilpotent of class 2. So, considering the normal crossed submodules (H, H, id) and (K, K, id) of the crossed module (G, G, id), the commutator submodule

$$[(H, H, id), (K, K, id)] = ([H, K], [H, K], id)$$

is abelian and the tensor product $(H, H, id)\otimes(K, K, id) = (H\otimes K, H\otimes K, id)$ is nilpotent of class 2.

A crossed module (T, G, ∂) is called *nilpotent-by-finite* (respectively, *solvable-by-finite*) if it has a nilpotent (respectively, solvable) normal crossed submodule (S, H, ∂) such that $(T/S, G/H, \bar{\partial})$ is finite.

In [9], it was established that if H and K are groups acting on each other compatibly, then $H \otimes K$ is nilpotent-by-finite or solvable-by-finite whenever H or K satisfy such information. In the final result of this section, we extend this result by showing the following

Theorem 4.6. *With the assumptions of Proposition* 3.6(iii)*, we have*

(i) *if* (S, H, ∂) *or* (L, K, ∂) *is nilpotent-by-finite, then so is* $(S, H, \partial) \otimes (L, K, \partial)$.

(ii) *if* (S, H, ∂) *or* (L, K, ∂) *is solvable-by-finite, then so is* $(S, H, \partial) \otimes (L, K, \partial)$.

Proof. It is a routine exercise to check that the properties of nilpotent-by-finite and solvable-by-finite are closed under taking normal crossed submodules and central extensions. The results now follow from the extension (8) that is central because of Proposition 3.6(iii). $\square$

Acknowledgement

The authors would like to thank the editor and the referee for his/her valuable suggestions, which have made the improvement of the paper.

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