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Witt rings of quadratically presentable fields

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Abstract. This paper introduces an approach to the axiomatic theory of quadratic forms based on *presentable* partially ordered sets, that is partially ordered sets subject to additional conditions which amount to a strong form of local presentability. It turns out that the classical notion of the Witt ring of symmetric bilinear forms over a field makes sense in the context of *quadratically presentable fields*, that is, fields equipped with a presentable partial order inequationaly compatible with the algebraic operations. In particular, Witt rings of symmetric bilinear forms over fields of arbitrary characteristics are isomorphic to Witt rings of suitably built quadratically presentable fields.

1 Introduction

In this work we approach the axiomatic theory of quadratic forms by generalising the underlying principles of hyperrings [16] to certain partial orders we call *presentable*. Roughly speaking, presentable posets generalise the behaviour of *pierced powersets*, that is powersets excluding the empty set with order given by inclusion. The most salient order-theoretic feature of pierced

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powersets is that they exhibit a generating set of minimal elements, since a non-empty set is a union of singletons. It is precisely this feature which is captured in the definition of presentable posets. The objective here is to build an axiomatic theory of quadratic forms by describing the behaviour of their value sets.

In Section 2 we formally introduce presentable posets and provide some examples including the set of integers greater or equal 1 augmented with a point at infinity and ordered by division, as well as the set of proper ideals of a Noetherian ring reversely ordered by inclusion.

In Sections 3 and 4 we introduce presentable monoids, groups, rings and fields. In particular, we exhibit presentable groups, rings and fields arising in a natural way from hypergroups, hyperrings and hyperfields, respectively. This provides the main link between our theory and already existing axiomatic theories of quadratic forms. A word of caution might be in order here as far as the terminology is concerned: a presentable group is not a group, not even a cancellative monoid. We just choose to stick to established terminology. A similar comment can be made about the notion of hypergroup underlying the notion of hyperring [16]. On the other hand, the Witt rings we construct are rings without further ado.

In Section 5 we define pre-quadratically and quadratically presentable fields which share certain similarities with groups of square classes of fields, endowed with partial order and addition. We then exhibit a Witt ring structure naturally occuring in quadratically presentable fields. As an application, for every field one can form a hyperfield by defining on the multiplicative group of its square classes multivalued addition that corresponds to value sets of binary forms. The presentable field induced by this hyperfield is quadratically presentable, and its Witt ring in our sense is isomorphic to its standard Witt ring. What makes this construction of interest is the fact that it uniformely works for fields of both characteristic 2 and $\neq 2$. It is technically reminiscent of the one used by Dickmann and Miraglia to build Witt rings of special groups [6].

In Sections 6 and 7 we explain how a pre-quadratically presentable field can be obtained from any presentable field. For that purpose we introduce in Section 6 quotients of presentable fields, the quotienting being performed with respect to the multiplicative structure. In Section 7 we use these quotients in order to build pre-quadratically presentable fields from presentable fields. The techniques here heavily rely on the connection between presentable algebras and hyperalgebras.

2 Presentable posets

Recall that a partially-ordered set or *poset* is a set equipped with a reflexive, transitive, and anti-symmetric relation. Let A be a poset. We shall write $\bigsqcup X$ for the supremum of $X \subseteq A$ and $x \sqcup y$ for $\bigsqcup\{x, y\}$. An element $a \in A$ is *minimal* if $a' \leqslant a$ implies $a' = a$. Let A be a poset and $\mathcal{S}_A$ be the set of A's minimal elements. We shall write $\mathcal{S}_a :=\downarrow a \cap \mathcal{S}_A$ for the set of all minimal elements below $a \in A$, and $\mathcal{S}_X := \bigcup_{x \in X} \mathcal{S}_x$ for the set of minimal elements below $X \subseteq A$.

Definition 2.1. A poset $(A, \leqslant)$ is *presentable* if

(i) every non-empty subset $R \subseteq A$ admits a supremum;

(ii) $\mathcal{S}_a$ is non-empty and $a = \bigsqcup \mathcal{S}_a$ for all $a \in A$;

(iii) every minimal element $s \in \mathcal{S}_A$ is *compact* in the following sense: if $Y \subseteq A$ is a nonempty subset and $s \leqslant \bigsqcup Y$, then there is an element $y \in Y$ such that $s \leq y$.

We shall call the minimal elements of a presentable poset *supercompact*.

Proposition 2.2. *Let* $(A, \leqslant)$ *be a poset satisfying conditions* (i) *and* (ii) *of* Definition 2.1. *Then the following are equivalent:*

(i) *Every minimal element is compact;*

(ii) *given* $x \in A$, *if* $x = \bigsqcup S$ *for some* $S \subseteq \mathcal{S}_A$ *then* $S = \mathcal{S}_x$.

Proof. (i) $\Rightarrow$ (ii) Let $x \in A$ with $x = \bigsqcup S$, for some $S \subseteq \mathcal{S}_A$. Clearly, $S \subseteq \mathcal{S}_x$. Suppose that, for some $a \in \mathcal{S}_x$, $a \notin S$. But $a \leq x = \bigsqcup S$, so, by compactness of a, $a \leq b$, for some $b \in S$. As b is a minimal element, this yields $a = b$, which leads to a contradiction.

(ii) $\Rightarrow$ (i) Let $a \in \mathcal{S}_A$ and assume that $a \leq \bigsqcup Y$, for some $Y \subseteq A$. Thus $a \leq \bigsqcup \bigcup_{y \in Y} \mathcal{S}_y$ and hence $a \in \bigcup_{y \in Y} \mathcal{S}_y$, say $a \leq y$ for some $y \in Y$. □

Remark 2.3. Let A be a presentable poset and $x, y \in A$. The following are equivalent:

(i) $x \leqslant y$;

(ii) $\mathcal{S}_x \subseteq \mathcal{S}_y$.

Example 2.4. The *pierced powerset* $\mathcal{P}^*(X)$ of a set X (that is, its set of nonempty subsets) is presentable with respect to the ordering by inclusion. The singletons are the supercompacts.

Example 2.5. The set $\mathcal{Z}^{\infty}_{\geq 2}$ of greater and equal than 2 and square-free integers with a point at infinity added is a presentable poset with respect to the ordering by division. Prime numbers are the supercompacts. Clearly, $\mathcal{Z}^{\infty}_{\geq 2}$ here can be replaced with conjugation classes of square-free non-units of any unique factorization domain with classes of nonzero irreducibles as supercompacts.

Example 2.6. The set $(\mathcal{I}^*(R), \supseteq)$ of all ideals of an absolutely flat (that is, von Neumann regular) Noetherian ring R is a presentable poset with respect to the ordering by reverse inclusion. The primary ideals in absolutely flat rings are maximal, and so is the trivial ideal R, hence they constitute the supercompacts. Indeed, every element of $\mathcal{I}^*(R)$ is either contained in a maximal ideal (which, in particular, is primary), or is equal to R. Every proper ideal is an intersetion of some primary ideals due to the Noether-Lasker theorem, and, clearly, an intersection of any family of proper ideals is a proper ideal. Clearly, this example can be also phrased in the language of affine algebraic sets that also satisfy some extra conditions.

3 Presentable groups

Recall that a *pointed poset* is a poset with a distinguished element called a *point*. In what follows, points of presentable posets will always be supercompacts.

Definition 3.1. A *presentable monoid* $(M, \leq, 0, +)$ is a pointed presentable poset $(M, \leq, 0)$ with a distinguished supercompact 0 and a suprema-preserving binary addition $+ : M \times M \to M$ such that

(i) $a + (b + c) = (a + b) + c$ for all $a, b, c \in M$;

(ii) $a + 0 = 0 + a = a$ for all $a \in M$;

(iii) $a + b = b + a$ for all $a, b \in M$.

Remark 3.2. (i) The addition is in particular monotone, that is,

$$(a \leq b) \wedge (c \leq d) \Rightarrow (a + c \leq b + d)$$

for all $a, b, c, d \in M$.

(ii) Suppose $a \leq b + c$. Let $s \in \mathcal{S}_a$. We have

$$\begin{aligned} s &\leqslant a \\ &\leqslant b + c \\ &= \bigsqcup \mathcal{S}_b + \bigsqcup \mathcal{S}_c \\ &= \bigsqcup \{t + u \mid t \in \mathcal{S}_b, u \in \mathcal{S}_c\}. \end{aligned}$$

Hence, by Proposition 2.2, there are $t \in \mathcal{S}_b$ and $u \in \mathcal{S}_c$ such that

$$s \leqslant t + u.$$

Example 3.3. Let $(M, 0, \cdot)$ be a commutative monoid. The pointed pierced powerset

$$(\mathcal{P}^*(M), \subseteq, \{0\})$$

(cf. Example 2.4) is a presentable monoid with addition given by

$$\begin{aligned} + : \mathcal{P}^*(M) \times \mathcal{P}^*(M) &\longrightarrow \mathcal{P}^*(M) \\ (A, B) &\mapsto \{a \cdot b \mid a \in A, b \in B\} \end{aligned}$$

The singleton $\{0\}$ is the neutral element. The addition preserves suprema:

$$\left(\bigcup_{i \in I} A_i\right) + \left(\bigcup_{j \in J} B_j\right) = \bigcup_{i \in I, j \in J} (A_i + B_j).$$

Definition 3.4. A *hypermonoid* is a pointed set $(M, 0, \oplus)$ equipped with a multivalued addition

$$\oplus : M \times M \to \mathcal{P}^*(M)$$

such that

(i) $a \oplus 0 = a = 0 \oplus a$ for all $a \in M$;
(ii) $a \oplus b = b \oplus a$ for all $a, b \in M$;
(iii) $(a \oplus b) \oplus c = \bigcup \{a \oplus x \mid x \in b \oplus c\} = a \oplus (b \oplus c)$ for all $a, b, c \in M$.

Remark 3.5. Let $(M, 0, \oplus)$ be a hypermonoid. The pointed pierced powerset $(\mathcal{P}^*(M), \subseteq, \{0\})$ is a presentable monoid with addition given by

$$\begin{aligned} + : \mathcal{P}^*(M) \times \mathcal{P}^*(M) &\longrightarrow \mathcal{P}^*(M) \\ (A, B) &\mapsto \bigcup \{a \oplus b | a \in A, b \in B\} \end{aligned}$$

Definition 3.6. A *presentable group* G is a presentable monoid equipped with a suprema preserving involutive homomorphism $- : G \to G$ called *inversion*, verifying

$$(s \leq t + u) \Rightarrow (t \leq s + (-u))$$

for all $s, t, u \in \mathcal{S}_G$.

Remark 3.7. Consider a presentable group $(G, \leqslant, 0, +, -)$.

(i) Notice that the inversion is, in particular, monotone, so we have quite counterintuitively

$$(a \leq b) \Rightarrow (-a \leq -b)$$

for all $a, b \in G$.

(ii) The above, in particular, implies that $-$ maps $\mathcal{S}_G$ bijectively onto $\mathcal{S}_G$: indeed, suppose that $s \in \mathcal{S}_G$ and $-s \notin \mathcal{S}_G$. Then $t \leqslant -s$, for some $t \in \mathcal{S}_G$, which implies $-t \leqslant s$, so that, by the minimality of s, $-t = s$ and, consequently, $t = -s$. Hence, as a well-defined involution on $\mathcal{S}_G$, it is necessarily bijective.

(iii) We have $0 \leqslant s + (-s)$, for all $s \in \mathcal{S}_G$, since $s \leqslant 0 + s$ implies that $0 \leqslant s + (-s)$. This entails that, in fact

$$0 \leqslant a + (-a)$$

for *any* $a \in G$. Since $\mathcal{S}_a \neq \varnothing$ there is a supercompact $s \in \mathcal{S}_a$ such that $s \leqslant a$, hence

$$\begin{aligned} 0 &\leqslant s - s \\ &\leqslant a - a. \end{aligned}$$

It is in general not true that $a \leq b+c$ implies $b \leq a-c$ for arbitrary $a, b, c \in G$. Take the presentable group $(\mathcal{P}^*(\mathbb{Z}), \subseteq, \{0\}, +)$, where $\mathbb{Z}$ is endowed with the usual addition. Then

$$\{1, 3\} \subseteq \{0, 1\} + \{0, 2\} = \{0, 1, 2, 3\}$$

but

$$\{0, 1\} \nsubseteq \{1, 3\} - \{0, 2\} = \{1, -1, 3\}.$$

Example 3.8. Let $(G, 0, \boxplus)$ be an abelian group and denote by $\multimap a$ the opposite element of a with respect to $\boxplus$. The presentable monoid $\mathcal{P}^*(G)$ as defined in Example 3.3 is a presentable group with inversion given by

$$\begin{array}{rcl} - : \mathcal{P}^*(G) & \longrightarrow & \mathcal{P}^*(G) \\ A & \mapsto & \{\multimap a \mid a \in A\} \end{array}$$

Definition 3.9. A *hypergroup* G is a hypermonoid together with a map $\ominus : G \to G$ such that
(i) $0 \in a \oplus (\ominus a)$ for all $a \in G$;
(ii) $(a \in b \oplus c) \Rightarrow (c \in a \oplus (\ominus b))$ for all $a, b, c \in G$.

Example 3.10. Let $(G, 0, \oplus, \ominus)$ be a hypergroup. The presentable monoid $(\mathcal{P}^*(G), \subseteq, \{0\}, +)$ is a presentable group with inversion given by

$$\begin{array}{rcl} - : \mathcal{P}^*(G) & \longrightarrow & \mathcal{P}^*(G) \\ A & \mapsto & \{\ominus a \mid a \in A\} \end{array}$$

4 Presentable rings and fields

We shall now extend the theory developed so far to ring- and field-like objects that will be called presentable rings and presentable fields.

Definition 4.1. A *presentable ring* R is a presentable group $(R, \leq, 0, +, -)$ consisting of at least two elements as well as a commutative monoid $(R, \cdot, 1)$, such that the element $1 \in R$ is a supercompact, $\cdot$ is compatible with $\leq$ (that is, $a \leq$b implies $a \cdot c \leq b \cdot c$, for all $a, b, c \in R$) and $-$(that is, $a \cdot (-b) = -(a \cdot b)$, for all $a, b \in R$), is distributative with respect to $+$, $0 \cdot a = 0$, for all $a \in R$, and $\cdot$ satisfies

$$\mathcal{S}_{a \cdot b} \quad = \quad \{s \cdot t \mid s \in \mathcal{S}_a, t \in \mathcal{S}_b\}$$

for all $a, b \in R$. A presentable ring R such that $\mathcal{S}_R^* = \mathcal{S}_R \setminus \{0\}$ is a multiplicative group will be called a *presentable field*.

Example 4.2. Let $(R, 0, +, \cdot, 1)$ be a ring. The presentable group $\mathcal{P}^*(R)$ (cf. Example 3.8) is a presentable ring with identity $\{1\}$ and multiplication given by

$$A \cdot B := \{a \cdot b \mid a \in A, b \in B\}.$$

If R is a field, then $\mathcal{P}^*(R)$ becomes a presentable field.

Remark 4.3. Consider a presentable ring $(R, \leq, 0, +, -, \cdot, 1)$.

(i) The element $1 \in R$ is the unique element $i \in R$ such that $i \cdot a = a$, for all $a \in R$.

(ii) $1 \neq 0$.

(iii) $-1 \in \mathcal{S}_R$.

The proof of (i) is no different than the proof of the corresponding results for rings. (ii) follows from the fact that $0 \cdot a = 0$, for all $a \in R$: suppose that $1 = 0$ and that $b \in R$ is an element such that $b \neq 0$ (as R has at least 2 elements, it always exists). Then $0 \neq b = 1 \cdot b = 0 \cdot b = 0$, which is a contradiction. For (iii) fix $s \in S_{-1}$, then $s \leq -1$ and, consequently, $-s \leq 1$, but 1 is a supercompact, so $-s = 1$ and hence $-1 = s \in \mathcal{S}_R$.

Definition 4.4. A *hyperring* $(R, 0, \oplus, \ominus, 1, \odot)$ is a hypergroup $(R, 0, \oplus, \ominus)$ such that $(R, 1, \odot)$ is a commutative monoid and

(i) $0 \odot a = 0$ for all $a \in R$;

(ii) $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ for all $a, b, c \in R$;

(iii) $0 \neq 1$.

If, in addition, every non-zero element has a multiplicative inverse, then R is called a *hyperfield.*

Example 4.5. Let $(R, 0, \oplus, \ominus, 1, \odot)$ be a hyperring (or a hyperfield). The presentable group $\mathcal{P}^*(R)$ is a presentable ring (or a presentable field, respectively) with multiplication given by

$$A \cdot B = \{a \odot b \mid a \in A, b \in B\}$$

for $A, B \in \mathcal{P}^*(R)$. The identity is $\{1\}$.

Remark 4.6. Let $(\boldsymbol{F}, 0, \oplus, \ominus, 1, \odot)$ be a hyperfield and T be a subgroup of the multiplicative group $(\boldsymbol{F}, 1, \odot)$. The relation $\sim$ on $\boldsymbol{F}$ given by

$$x \sim y \overset{\text{def.}}{\Longleftrightarrow} x \odot s = y \odot t \quad \text{for some } s, t \in T$$

is an equivalence. Let $\boldsymbol{F}/_m T$ be its set of equivalence classes and $\bar{x}$ the class of $x \in F$. The induced operations

- $\bar{x} \in \bar{y} \bar{\oplus} \bar{z} \overset{\text{def.}}{\Longleftrightarrow} x \odot s \in (y \odot t) \oplus (z \odot u) \quad$ for some $s, t, u \in T$,
- $\bar{x} \bar{\odot} \bar{y} = \overline{x \odot y}$,

and

- $\bar{\ominus}\bar{x} = \overline{\ominus x}$

are well-defined and $(\boldsymbol{F}/_m T, \bar{0}, \bar{\oplus}, \bar{\ominus}, \bar{1}, \bar{\odot})$ is a hyperfield which we shall call the *quotient hyperfield* of $\boldsymbol{F}$ modulo T [16].

Example 4.7. Let k be a field with $\mathrm{char}(k) \neq 2$ and $k \neq \mathbb{F}_3, \mathbb{F}_5$. This yields an example of a hyperfield with $a \oplus b = \{a+b\}$. Let $T = k^{*2}$ be k's *multiplicative group of squares* and $x, y, z \in k$. It is easy to see that the following are equivalent:

(i) $x = s^2 y + t^2 z$ for some $s, t \in k$;

(ii) $\bar{x} \in \bar{y}\bar{\oplus}\bar{z}$ in $k/_m k^{*2}$.

We thus have $\bar{y}\bar{\oplus}\bar{z} = D(y,z) \cup \{\bar{0}\}$, where $D(y,z)$ is the value set of the binary quadratic form (y,z).

Assume now $\mathrm{char}(k) = 2$ or $k \in \{\mathbb{F}_3, \mathbb{F}_5\}$. In this case the above assertions are not equivalent in general. However, $k/_m k^{*2}$ with a modified addition given by

$$\bar{y}\bar{\oplus}'\bar{z} = \begin{cases} \bar{y}\bar{\oplus}\bar{z} & \text{if } \bar{y} = \bar{0} \text{ or } \bar{z} = \bar{0} \\ \bar{y}\bar{\oplus}\bar{z} \cup \{\bar{y}, \bar{z}\} & \text{if } \bar{y} \neq \bar{0}, \bar{z} \neq \bar{0}, \bar{y} \neq -\bar{z} \\ k/_m k^{*2} & \text{if } \bar{y} \neq \bar{0}, \bar{z} \neq \bar{0}, \bar{y} = -\bar{z} \end{cases}$$

is again a hyperfield. Observe that this addition is well-defined for any hyperfield [9, Proposition 2.1], we just have $\bar{\oplus}' = \oplus$ whenever $\mathrm{char}(k) \neq 2$ and $k \neq \mathbb{F}_3, \mathbb{F}_5$.

Definition 4.8. Let k be a field. Then $Q(k) := (k/_m k^{*2}, \bar{\oplus}')$ is called *k's quadratic hyperfield*.

Example 4.9. Let k be a field with two square classes, for instance when k is *real closed*. The two square classes are represented by $1, -1$, so that k is Euclidean (for example, $k = \mathbb{R}$, or the field of real algebraic numbers, or the field of real constructible numbers, etc.), and $Q(k) = \{\bar{0}, \bar{1}, \overline{-1}\}$ with multivalued addition given by

$\bar{\oplus}$	$\bar{0}$	$\bar{1}$	$\overline{-1}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\overline{-1}$
$\bar{1}$	$\bar{1}$	$\bar{1}$	$\{\bar{0}, \bar{1}, \overline{-1}\}$
$\overline{-1}$	$\overline{-1}$	$\{\bar{0}, \bar{1}, \overline{-1}\}$	$\overline{-1}$

along with the obvious multiplication. The presentable ring $\mathcal{P}^*(Q(k))$ with identity I consists of 7 elements

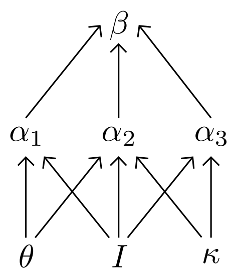
$$\begin{array}{llll} \theta = \{\bar{0}\} & I = \{\bar{1}\} & \kappa = \{-1\} & \beta = \{\bar{0}, \bar{1}, \overline{-1}\} \\ \alpha_1 = \{\bar{0}, \bar{1}\} & \alpha_2 = \{\bar{0}, \overline{-1}\} & \alpha_3 = \{\bar{1}, \overline{-1}\} & \end{array}$$

with arithmetic given by

$+$	θ	I	κ	α_1	α_2	α_3	β
θ	θ	I	κ	α_1	α_2	α_3	β
I	I	I	β	I	β	β	β
κ	κ	β	κ	β	κ	β	β
α_1	α_1	I	β	α_1	β	β	β
α_2	α_2	β	κ	β	α_2	β	β
α_3	α_3	β	β	β	β	β	β
β	β	β	β	β	β	β	β

$\cdot$	θ	I	κ	α_1	α_2	α_3	β
θ	θ	θ	θ	θ	θ	θ	θ
I	θ	I	κ	α_1	α_2	α_3	β
κ	θ	κ	I	α_2	α_1	α_3	β
α_1	θ	α_1	α_2	α_1	α_2	β	β
α_2	θ	α_2	α_1	α_2	α_1	β	β
α_3	θ	α_3	α_3	β	β	α_3	β
β	θ	β	β	β	β	β	β

The partial order in $\mathcal{P}^*(Q(k))$ is generated by



5 Witt rings of quadratically presentable fields

We are now in position to define the central object of interest in this paper, namely the Witt ring. In view of the foreseeable applications that will become apparent in subsequent sections, we shall restrict our considerations to a rather special class of presentable fields that we will call quadratically presentable. We start with explaining these concepts in some detail.

Definition 5.1. Let $(R, \leq, 0, +, -, \cdot, 1)$ be a presentable field. We shall call R *pre-quadratically presentable*, if the following conditions hold:

(i) $a \leq a + b$ for all $a \in \mathcal{S}_R^*, b \in \mathcal{S}_R$;
(ii) $(a \leq 1 - b) \wedge (a \leq 1 - c) \Rightarrow (a \leq 1 - bc)$ for all $a, b, c \in \mathcal{S}_R$;
(iii) $a^2 = 1$ for all $a \in \mathcal{S}_R \setminus \{0\}$.

Remark 5.2. Note that in the axiom (i) the assumption that $a \in \mathcal{S}_R^*$ is crucial if $a = 0$ then $a \leq a + b$ is just $0 \leq 0 + b = b$, which means $b = 0$ for all $b \in \mathcal{S}_R$. This, in turn, entails $R = \{0\}$.

Proposition 5.3. *Let k be a field, $Q(k)$ be its quadratic hyperfield and $\mathcal{P}^*(Q(k))$ be the induced presentable field (cf. Examples 4.7 and 4.9). Then $\mathcal{P}^*(Q(k))$ is a pre-quadratically presentable field.*

Proof. It suffices to verify the conditions (i)–(iii) of Definition 5.1. Item (iii) is immediate. For (i), fix $a \in \mathcal{S}_R^*$ and $b \in \mathcal{S}_R$. If $b = 0$, then, since, by Example 4.7, $a + b = D_k(a, b) \cup \{0\}$, $a \leq a + b$. If $b \neq 0$ then either $a \neq -b$, in which case $a + b = D_k(a, b) \cup \{0\} \cup \{a, b\}$, or $a = -b$ and $a + b = k/_m k^{*2}$, in both cases leading to $a \leq a + b$.

For (ii), fix $a, b, c \in \mathcal{S}_R$ and assume $a \leq 1 - b$ and $a \leq 1 - c$. If either $b = 0$ or $c = 0$, then $a = 1$ and the conclusion of the argument follows obviously, so assume $b \neq 0$ and $c \neq 0$. If either $b = 1$ or $c = 1$, then $1 - bc = 1 - c$ or $1 - b$, respectively, and there is nothing to prove as well, so assume further that $b \neq 1$ and $c \neq 1$. Thus $1 - b = D_k(1, -b) \cup \{0\} \cup \{1, -b\}$, $1 - c = D_k(1, -c) \cup \{0\} \cup \{1, -c\}$ and $1 - bc$ is either $k/_m k^{*2}$ or $D_k(1, -bc) \cup \{0\} \cup \{1, -bc\}$ depending on whether $bc = 1$ or not. The case when a is either 0 or 1 is obvious. If $a = -b$ and $a = -c$, then $bc = a^2 = 1$ and $1 - bc = k/_m k^{*2}$, so this is clear as well. If $a = -b$ and $a \in D_k(1, -c)$, then $a = s^2 - ct^2$, for some $s, t \in k^*$; but then $ab = bs^2 - bct^2$, or, equivalently, $-b = \frac{1}{s^2}(-ab) - bc\frac{t^2}{s^2}$ – as $a = -b$ this yields $a = \frac{a^2}{s^2} - bc\frac{t^2}{s^2}$, that is $a \in D_k(1, -bc)$. The case when $a = -c$ and $a \in D_k(1, -b)$ is symmetric. This leaves us with the case when $a \in D_k(1, -b)$ and $a \in D_k(1, -c)$, say $a = s^2 - bt^2 = s'^2 - ct'^2$, for some $s, s', t, t' \in k^*$. Thus $bc(tt')^2 = (s^2 - a)(s'^2 - a) = (ss' - a)^2 - a(s - s')^2$: as $s - s' = 0$ leads to the cases $bc = 1$ or $bc = 0$, which is either easy or has been already ruled out, this yields $a = s''^2 - bct''^2$, for some $s'', t'' \in k^*$, that is, $a \in D_k(1, -bc)$. □

Example 5.4. The presentable field $\mathcal{P}^*(R)$ constructed from a field $(R, 0, +, \cdot, 1)$ (cf. Example 4.2) is usually not pre-quadratically presentable, since it is, in general, not true that $\{a\} \subset \{a\} + \{b\} = \{a + b\}$.

Definition 5.5. A *form* on a pre-quadratically presentable field R is an n-tuple $\langle a_1,\dots,a_n\rangle$ of elements of $\mathcal{S}_R^*$. The relation $\cong$ of *isometry* of forms of the same dimension is given by induction:

$\langle a\rangle \cong \langle b\rangle \iff a=b$;

$\langle a_1,a_2\rangle \cong \langle b_1,b_2\rangle \iff a_1a_2=b_1b_2$ and $b_1\leq a_1+a_2$;

$\langle a_1,\dots,a_n\rangle \cong \langle b_1,\dots,b_n\rangle \iff$ there exist $x,y,c_3,\dots,c_n\in\mathcal{S}_R^*$ such that

(i) $\langle a_1,x\rangle\cong\langle b_1,y\rangle$;

(ii) $\langle a_2,\dots,a_n\rangle\cong\langle x,c_3,\dots,c_n\rangle$;

(iii) $\langle b_2,\dots,b_n\rangle\cong\langle y,c_3\dots,c_n\rangle$.

Proposition 5.6. *The relation $\cong$ is an equivalence on the sets of all unary and binary forms of a pre-quadratically presentable field R.*

Proof. The statement is clear for unary forms. For binary forms, reflexivity follows from the axiom (i). For symmetry assume that $\langle a,b\rangle\cong\langle c,d\rangle$, for $a,b,c,d\in\mathcal{S}_R^*$. Thus $ab=cd$ and $a\leq c+d$. But then $a=bcd$, so that $bcd\leq c+d$. Thus $b\leq cd(c+d)=c+d$. For transitivity assume $\langle a,b\rangle\cong\langle c,d\rangle$ and $\langle c,d\rangle\cong\langle e,f\rangle$, for $a,b,c,d,e,f\in\mathcal{S}_R^*$. This means $ab=cd$, $cd=ef$, $a\leq c+d$ and, by symmetry, $e\leq c+d$. Therefore, $c\leq a-d$ and $c\leq e-d$, which gives $-cd\leq 1-ad$ and $-cd\leq 1-ed$. This implies $-cd\leq 1-ae$. Since $cd=ef$, this is just $-ef\leq 1-ae$, or, equivalently, $ef\leq ae-1$. But this is the same as $ae\leq 1+ef$, so $a\leq e+f$. □

Definition 5.7. A pre-quadratically presentable field $(R,\leq,0,+,-,\cdot,1)$ will be called *quadratically presentable* if the isometry relation is an equivalence on the set of all forms of the same dimension.

Example 5.8. The pre-quadratically presentable field $\mathcal{P}^*(Q(k))$, for a field k, is quadratically presentable. That $\cong$ is an equivalence relation on the set of all forms of the same dimension follows from the well-known inductive description of the isometry relation of quadratic forms (see, for example, [15, Theorem 1.13]).

Definition 5.9. Let R be a pre-quadratically presentable field and $\phi=\langle a_1,\dots,a_n\rangle$, $\psi=\langle b_1,\dots,b_m\rangle$ be two forms. The *orthogonal sum* $\phi\oplus\psi$ is defined as the form

$$\langle a_1,\dots,a_n,b_1,\dots,b_m\rangle$$

and the *tensor product* $\phi \otimes \psi$ as

$$\langle a_1b_1, \dots, a_1b_m, a_2b_1, \dots, a_2b_m, \dots, a_nb_1, \dots, a_nb_m \rangle.$$

We will write $k \times \phi$ for the form $\underbrace{\phi \oplus \ldots \oplus \phi}_{k \text{ times}}$.

Proposition 5.10. (i) *Let R be a pre-quadratically presentable field. The direct sum and the tensor product of isometric forms are isometric.*

(ii) *(Witt cancellation) Let R be a quadratically presentable field. If $\phi_1 \oplus \psi \cong \phi_2 \oplus \psi$, then $\phi \cong \psi$.*

Proof. Proofs of both assertions are by induction. We shall outline the proof of (ii) to see where the assumption on transitivity of $\cong$ is used. Let $n = \dim(\phi_1) = \dim(\phi_2)$ and first suppose that $\psi = \langle a \rangle$, $a \in \mathcal{S}_R^*$. Thus there are $x, y, c_3, \dots, c_n \in \mathcal{S}_R^*$ such that $\langle a, x \rangle \cong \langle a, y \rangle$, $\phi_1 \cong \langle x, c_3, \dots, c_n \rangle$ and $\phi_2 \cong \langle y, c_3, \dots, c_n \rangle$. By the first isometry, $ax = ay$ and so $x = y$, but then the transitivity of $\cong$ ensures that $\phi_1 \cong \phi_2$. For the general case one uses induction on $\dim(\psi)$ noting that $\phi_1 \oplus \psi \cong \phi_2 \oplus \psi$ can be written as $\phi_1' \oplus \langle a \rangle \cong \phi_2' \oplus \langle a \rangle$, for suitably chosen ϕ_1' and ϕ_2'. □

Definition 5.11. Let R be a quadratically presentable field. Two forms ϕ and ψ will be called *Witt equivalent*, denoted as $\phi \sim \psi$, if, for some integers $m, n \geq 0$:

$$\phi \oplus m \times \langle 1, -1 \rangle \cong \psi \oplus n \times \langle 1, -1 \rangle$$

Remark 5.12. It is easily verified that $\sim$ is an equivalence relation on forms over R, compatible with (and, clearly, coarser than) the isometry. One also easily checks that Witt equivalence is a congruence with respect to orthogonal sum and tensor product of forms. Denote by $W(R)$ the set of equivalence classes of forms over R under Witt equivalence, and by $\bar{\phi}$ the equivalence class of ϕ. With the operations

$$\bar{\phi} + \bar{\psi} = \overline{\phi \oplus \psi} \qquad \bar{\phi} \cdot \bar{\psi} = \overline{\phi \otimes \psi}$$

$W(R)$ is a commutative ring, having as zero the class $\overline{\langle 1, -1 \rangle}$, and $\overline{\langle 1 \rangle}$ as multiplicative identity.

Definition 5.13. Let R be a quadratically presentable field. Then $W(R)$ with binary operations as defined above is called the *Witt ring of R*.

As one might expect, the main example of a Witt ring of a quadratically presentable field, is the Witt ring of the quadratically presentable field induced by the quadratic hyperfield of a field.

Theorem 5.14. *For a field k, $W(\mathcal{P}^*(Q(k)))$ is just the usual Witt ring $W(k)$ of non-degenerate symmetric bilinear forms of k.*

Proof. Consider the map $\omega : W(k) \to W(\mathcal{P}^*(Q(k)))$ defined by

$$\omega(\langle a_1, \dots, a_n\rangle) = \overline{\langle\{a_1\}, \dots \{a_n\}\rangle},$$

where $\langle a_1, \dots, a_n\rangle$ denotes the similarity class of a nonsingular quadratic form whose entries in its diagonalized form are $a_1, \dots, a_n \in k^*/k^{*2}$. By design, this map is well-defined: if $\langle\{a_1\}, \dots, \{a_n\}\rangle \sim \langle\{b_1\}, \dots, \{b_m\}\rangle$ then $\langle\{a_1\}, \dots, \{a_n\}\rangle \oplus k \times \langle 1, -1\rangle \cong \langle\{b_1\}, \dots, \{b_m\}\rangle \oplus l \times \langle 1, -1\rangle$, which translates to $\langle a_1, \dots, a_n\rangle \oplus k \times \langle 1, -1\rangle \cong \langle b_1, \dots, b_m\rangle \oplus l \times \langle 1, -1\rangle$ as forms over fields, meaning $\langle a_1, \dots, a_n\rangle$ and $\langle b_1, \dots, b_m\rangle$ are the same similarity class of a nonsingular quadratic form.

Similarly, by design, ω preserves the orthogonal sum and tensor product of similarity classes of nonsingular quadratic forms, so it is a ring homomorphism. It is clearly a surjection, and the class $\langle a_1, \dots, a_n\rangle$ is in the kernel of ω if and only if $\langle\{a_1\}, \dots, \{a_n\}\rangle \oplus k \times \langle 1, -1\rangle \cong l \times \langle 1, -1\rangle$, that is $\langle a_1, \dots, a_n\rangle \oplus k \times \langle 1, -1\rangle \cong l \times \langle 1, -1\rangle$ as forms over fields, or, equivalently, if $\langle a_1, \dots, a_n\rangle$ is zero in $W(k)$. □

Remark 5.15. Notice that Theorem 5.14 provides a uniform construction of the Witt ring for all charateristics as well as for $\mathbb{F}_3$ and $\mathbb{F}_5$.

Definition 5.16 (Dickmann-Miraglia)**.** A *pre-special group [6, Definition 1.2]* is a group G of exponent 2 together with a distinguished element -1 and a binary operation $\cong$ on $G \times G$ such that, for all $a, b, c, d \in G$:

(i) $\cong$ is an equivalence relation;
(ii) $(a, b) \cong (b, a)$;
(iii) $(a, -1 \cdot a) \cong (1, -1)$;
(iv) $[(a, b) \cong (c, d)] \Rightarrow [ab = cd]$;
(v) $[(a, b) \cong (c, d)] \Rightarrow [(a, -1 \cdot c) \cong (-1 \cdot b, d)]$;
(vi) $[(a, b) \cong (c, d)] \Rightarrow \forall x \in G, [(xa, xb) \cong (xc, xd)]$.

Remark 5.17. Let $(G, \cong, -1)$ be a pre-special group. The relation $\cong$ can be extended to the set $\underbrace{G \times \ldots \times G}_{n}$ as follows: for $n = 1$, $(a_1) \cong_1 (b_1)$ is just $a_1 = b_1$, and for $n \geq 3$

$$(a_1, \ldots, a_n) \cong_n (b_1, \ldots, b_n)$$

holds provided that there exist $x, y, c_3, \ldots, c_n \in G$ such that

(i) $(a_1, x) \cong (b_1, y)$;

(ii) $(a_2, \ldots, a_n) \cong_{n-1} (x, c_3, \ldots, c_n)$;

(iii) $(b_2, \ldots, b_n) \cong_{n-1} (y, c_3 \ldots, c_n)$.

A *special group* [6, Definition 1.2] is a pre-special group $(G, \cong, -1)$ such that $\cong_n$ is an equivalence relation for all $n \in \mathbb{N}$.

Remark 5.18. Let $(R, \leq, 0, +, -, \cdot, 1)$ be a (pre-) quadratically presentable field. Then $(\mathcal{S}_R^*, \cong, -1)$ is a (pre-) special group. The only non-trivial parts to check are that $\langle a, -a\rangle \cong \langle 1, -1\rangle$ and that $\langle a, b\rangle \cong \langle c, d\rangle$ implies $\langle a, -c\rangle \cong \langle -b, d\rangle$, for $a, b, c, d \in \mathcal{S}_R^*$. The first statement follows from the fact that $a \leq a + 1$ implies $1 \leq a - a$. For the second, assume $ab = cd$ and $a \leq c + d$. Thus $d \leq a - c$, by the exchange law, so that $\langle d, -b\rangle \cong \langle a, -c\rangle$.

Now, given suitable notions of morphisms, it can be shown that the assignment

$$S : (R, \leq, 0, +, -, \cdot, 1) \mapsto (\mathcal{S}_R^*, \cong, -1)$$

is functorial and that S is in fact an equivalence of categories. The construction of Witt rings of quadratically presentable fields that we provide (cf. Theorem 5.14 and Remark 5.15) corresponds to the construction of Witt rings of special groups. Conversely, the relevant constructions could be carried out directly in the category of special groups. However, the formalism of presentable algebras is of independent interest, this since the category of presentable groups as well as the category of *presentable modules* over a presentable ring (not formally introduced here) exhibit quite good properties. This circle of ideas will be addressed in a forthcoming paper.

6 Quotients in presentable fields

In order to investigate Witt rings of presentable fields, one needs to know how to pass from presentable fields to quadratically presentable fields. We are "almost" able to do that, and will show how one can build a pre-quadratically presentable field from arbitrary presentable field – it is, however, an open question when the resulting presentable field is quadratically presentable. The main tool to be used are quotients of presentable fields. Before we proceed to general quotients, we focus on a rather special case of quotients "modulo" multiplicative subsets of supercompacts. These are, in fact, the only quotients that we need in the sequel, which explains why we choose to present our exposition in this particular manner.

Theorem 6.1. *Let $(R, \leq, 0, +, -, \cdot, 1)$ be a presentable field. Let $T \subseteq \mathcal{S}_R^*$ be a multiplicative set, that is, for all $s, t \in T$, $st \in T$. Define the relation $\sim$ on $\mathcal{S}_R$ by*

$$a \sim b \Longleftrightarrow \exists s, t \in T, \quad as = bt.$$

This is an equivalence relation, whose equivalence classes will be denoted by $\bar{a}$, $a \in \mathcal{S}_R$. Let

$$\bar{a} \odot \bar{b} = \overline{ab} \qquad \ominus \bar{a} = \overline{-a}$$

and let

$$\bar{a} \in \bar{b} \oplus \bar{c} \Longleftrightarrow \exists s, t, u \in T, \quad as \leq bt + cu.$$

Then $(\mathcal{S}_R/\sim, \bar{0}, \oplus, \ominus, \bar{1}, \odot)$ is a hyperfield.

Proof. The relation $\sim$ is clearly reflexive and symmetric, and for transitivity assume $as = bt$ and $bu = cv$, for some $a, b, c \in \mathcal{S}_R$, $s, t, u, v \in T$. Then $asu = btu$ and $btu = cvt$ with $su, tu, vt \in T$ thanks to the multiplicativity of T.

Next, the operation $\odot$ is clearly well-defined, and to see that so is $\oplus$, assume $\bar{b} = \overline{b'}$ and $\bar{c} = \overline{c'}$, say, $vb = v'b'$ and $wc = w'c'$, for some $v, v', w, w' \in T$. Then

$$\begin{aligned} \bar{a} \in \bar{b} \oplus \bar{c} & \Leftrightarrow \exists s, t, u \in T, \quad as \leq bt + cu \\ & \Rightarrow \exists s, t, u, \quad asvw \leq bvwt + cwvu \\ & \Leftrightarrow \exists s, t, u, \quad asvw \leq b'v'wt + c'w'vu \\ & \Leftrightarrow \bar{a} \in \overline{b'} \oplus \overline{c'}. \end{aligned}$$

In order to show that $\mathcal{S}_R/\sim$ with operations defined as above is, indeed, a hyperring, we note that both the commutativity of $\oplus$ and the fact that $(\mathcal{S}_R/\sim \setminus \{\bar{0}\}, \bar{1}, \odot)$ forms a commutative group are obvious, that $\bar{0} \in \bar{a} \ominus \bar{a}$, for all $\bar{a} \in \mathcal{S}_R/\sim$, follows immediately from $0 \leq a - a$ for all $a \in \mathcal{S}_R$, that $\bar{0} \odot \bar{a} = \bar{0}$ is clear in view of $0 \cdot 1 = 0$, and that $\bar{0} \neq \bar{1}$ is apparent, as $1 \cdot t = 0$, for some $t \in T$, leads to $0 = 1$. It remains to show the neutrality of $\bar{0}$, associativity of $\oplus$, cancellation and distributativity of $\oplus$ and $\odot$.

Assume $\bar{b} \in \bar{a} \oplus \bar{0}$, so $bs \leq at + 0 = at$, for some $s, t \in T$. But then $bst^{-1} \leq a$, and, since a is a supercompact, this yields $bst^{-1} = a$ and, consequently, $\bar{b} = \bar{a}$.

Assume $\bar{d} \in \bar{a} \oplus (\bar{b} \oplus \bar{c})$, so that $\bar{d} \in \bar{a} \oplus \bar{e}$ with $\bar{e} \in \bar{b} \oplus \bar{c}$. Hence, $ds \leq at + eu$ and $es' \leq bt' + cu'$, for some $s, t, u, s', t', u' \in T$. Thus $dss' \leq ats' + eus'$ and $eus' \leq but' + cuu'$, so that $dss' \leq ats' + (but' + cuu') = (ats' + but') + cuu'$. By Remark 3.2 (ii), it follows that there exist supercompacts $d', f, c' \in \mathcal{S}_R$ with $d' \leq dss'$, $f \leq ats' + but'$ and $c' \leq cuu'$ with $d' \leq f + c'$. Using the same argument as in the proof of neutrality of $\bar{0}$, we easily check that $d' = dss'$ and $c' = cuu'$. Therefore, $\bar{d} = \overline{d'}$, $\bar{c} = \overline{c'}$, $d \leq f + c$ and $f \leq ats' + but'$. This yields $\bar{d} \in \bar{f} \oplus \bar{c}$ with $\bar{f} \in \bar{a} \oplus \bar{b}$, so that $\bar{d} \in (\bar{a} \oplus \bar{b}) \oplus \bar{c}$.

Assume $\bar{a} \in \bar{b} \oplus \bar{c}$, so that $at \leq bs + cu$, for some $s, t, u \in T$. Then there are supercompacts $a' \leq at$, $b' \leq bs$ and $c' \leq cu$ such that $a' \leq b' + c'$. Using the same trick as before we conclude $\bar{a} = \overline{a'}$, $\bar{b} = \overline{b'}$, $\bar{c} = \overline{c'}$ and, since $a' \leq b' + c'$, thus $b' \leq a' - c'$, which implies $\bar{b} \in \bar{a} \ominus \bar{c}$.

Finally, if $\bar{d} \in \bar{a} \odot (\bar{b} \oplus \bar{c})$, then $\bar{d} = \overline{ae}$ with $\bar{e} \in \bar{b} \oplus \bar{c}$, and thus $es \leq bt + cu$, for some $s, t, u \in T$. But then $aes \leq abt + acu$, so $\overline{ae} \in \overline{ab} \oplus \overline{ac}$. This shows that $\bar{a} \odot \left(\bar{b} \oplus \bar{c}\right) \subseteq \bar{a} \odot \bar{b} \oplus \bar{a} \odot \bar{c}$, and the other inclusion holds if $a, b, c \in \mathcal{S}_R$ (or, to be more precise, if $a, b, c \in U \cup \{0\}$, where U is the group of invertible elements of the multiplicative monoid $(\mathcal{S}_R, \cdot, 1)$, which is just $\mathcal{S}_R^*$ when R is a presentable field). □

Remark 6.2. Observe that the above works in fact for any presentable ring $(R, \leq, 0, +, -, \cdot, 1)$ and a subgroup $T \subseteq \mathcal{S}_R^*$ of the multiplicative monoid $\mathcal{S}_R^*$. That is, we only need to be able to invert the elements of T for the argument to go through.

Definition 6.3. The *quotient* of $(R, \leq, 0, +, -, \cdot, 1)$ *modulo the multiplicative set* T is the presentable field $(\mathcal{P}^*(\mathcal{S}_R/_\sim), \subseteq, \{\bar{0}\})$ with the hyperfield $(\mathcal{S}_R/_\sim, \bar{0}, \oplus, \ominus, \bar{1}, \odot)$ defined in Theorem 6.1 and will be denoted by $R/_m T$.

Theorem 6.1, as remarked before, is a special case of the following, more general result.

Theorem 6.4. *Let* $(R, \leq, 0, +, -, \cdot, 1)$ *be a presentable field. Let* $\sim$ *be a nontrivial congruence on the set* $\mathcal{S}_R^*$ *of supercompacts of* R*, that is, an equivalence relation such that* $0 \nsim 1$*, and, for all* $a, a', b, b' \in \mathcal{S}_R^*$*, if* $a \sim a'$*, and* $b \sim b'$ *then*

$$ab \sim a'b' \qquad a + b \sim a' + b' \qquad -a \sim -a'$$

Denote by $\bar{a}$ *the equivalence class of* $a \in \mathcal{S}_R$*. Let*

$$\bar{a} \odot \bar{b} = \overline{ab} \qquad \ominus\, \bar{a} = \overline{-a}$$

and let

$$\bar{a} \in \bar{b} \oplus \bar{c} \overset{def.}{\iff} \exists a' \in \bar{a}, b' \in \bar{b}, c' \in \bar{c}, \quad [a' \leq b' + c'].$$

Then $(\mathcal{S}_R/\sim, \bar{0}, \oplus, \ominus, \bar{1}, \odot)$ *is a hyperfield.*

The proof mimics the one of Theorem 6.1. That $\bar{0} \neq \bar{1}$ follows from the fact that $0 \nsim 1$.

7 From presentable fields to pre-quadratically presentable fields

In this section we shall explain how one can construct a pre-quadratically presentable field from any given presentable field. This almost allows for a construction of a Witt ring for arbitrary presentable fields: in particular, if the presentable field at the start of the construction is induced by the quadratic hyperfield of a field, then the resulting pre-quadratically presentable field is, in fact, quadratically presentable, and its Witt ring coincides with the usual Witt ring of the underlying field.

Remark 7.1. Let $(R, \leq, 0, +, -, \cdot, 1)$ be a presentable field and define the following operations on the set $\mathcal{S}_R$ of supercompacts of R:

$$a \odot b = a \cdot b \qquad \ominus\, a = -a$$

and

$$a \in b \oplus c \overset{def.}{\iff} a \leq b + c$$

Then $(\mathcal{S}_R, 0, \oplus, \ominus, 1, \odot)$ is a hyperfield. Further, define the prime addition on $\mathcal{S}_R$ as

$$a \oplus' b = \begin{cases} a \oplus b, & \text{if } a = 0 \text{ or } \ b = 0 \\ a \oplus b \cup \{a, b\}, & \text{if } a \neq 0, b \neq 0, a \neq -b \\ \mathcal{S}_R, & \text{if } a \neq 0, b \neq 0, a = -b. \end{cases}$$

Then $(\mathcal{S}_R, 0, \oplus', \ominus, 1, \odot)$ is again a hyperfield [9, Proposition 2.1], called the *prime hyperfield* of $(R, \leq, 0, +, -, \cdot, 1)$. The induced presentable field $(\mathcal{P}^*(\mathcal{S}_R), \subseteq, \{0\}, +', -, \cdot, \{1\})$, which will be called the *prime presentable field*, satisfies the condition:

$$\{a\} \subseteq \{a\} +' \{b\} \text{ for all } \{a\}, \{b\} \in \mathcal{S}^*_{\mathcal{P}^*(\mathcal{S}_R)}.$$

Theorem 7.2. *Let $(R, \leq, 0, +, -, \cdot, 1)$ be a presentable field such that*

$$a \leq a + b \textit{ for all } a \in \mathcal{S}^*_R, b \in \mathcal{S}_R.$$

Then $T := \{s \in \mathcal{S}^*_R \mid s \leq a^2$ for some $a \in R\}$ is a multiplicative set and the quotient $R/_mT$ of R modulo T is a pre-quadratically presentable field.

Proof. T is multiplicative, for if $s \leq a^2$ and $t \leq b^2$, for some $s, t \in \mathcal{S}^*_R$, $a, b \in R$, then $st \leq a^2b^2 = (ab)^2$ and $st \neq 0$, since $\mathcal{S}^*_R$ is a group. The condition

$$a \leq a + b \text{ for all } a \in \mathcal{S}^*_R, b \in \mathcal{S}_R$$

carries over to $R/_mT$, non-zero supercompacts of $R/_mT$ form a group, since in the process of taking a quotient modulo multiplicative set we end up with a presentable field, and, finally, squares of all non-zero supercompacts of $R/_mT$ are equal to identity, as they are just classes of squares of non-zero supercompacts in R, which are, by definition, equivalent to 1.

It remains to show that for all supercompacts $\{\bar{a}\}$, $\{\bar{b}\}$ and $\{\bar{c}\}$ in $R/_mT$, if $\{\bar{a}\} \subseteq \{\bar{1}\} - \{\bar{b}\}$ and $\{\bar{a}\} \subseteq \{\bar{1}\} - \{\bar{c}\}$, then $\{\bar{a}\} \subseteq \{\bar{1}\} - \{\overline{bc}\}$. Fix three supercompacts as above and assume the antedecent. This is equivalent to $\bar{a} \in \bar{1} \ominus \bar{b}$ and $\bar{a} \in \bar{1} \ominus \bar{c}$ in the hyperfield $\mathcal{S}_R/\sim$, which, in turn, is equivalent to

$$sa \leq t - ub \text{ and } s'a \leq t' - u'c,$$

for some non-zero supercompacts $s, s', t, t', u, u' \in R$ such that $s \leq x^2$, $s' \leq x'^2$, $t \leq y^2$, $t' \leq y'^2$, $u \leq z^2$, $u' \leq z'^2$, for some $x, x', y, y', z, z' \in R$. Since $\mathcal{S}^*_R$

is a group, the elements $sa, s'a, ub, u'c$ are also supercompacts, which allows switching terms between both sides of the above inequalities, and gives

$$ub \leq t - sa \text{ and } u'c \leq t' - s'a'$$

and, in turn

$$ub \leq y^2 - x^2a \text{ and } u'c \leq y'^2 - x'^2a.$$

Hence,

$$\begin{aligned} uu'bc &\leq (y^2 - x^2a)(y'^2 - x'^2a) \\ &= y^2y'^2 - y^2x'^2a - x^2y'^2a + x^2x'^2a^2 \\ &\leq y^2y'^2 - y^2x'^2a - x^2y'^2a + x^2x'^2a^2 + 2xx'yy'a - 2xx'yy'a \\ &= (yy' + axx')^2 - a(x'y + xy')^2, \end{aligned}$$

by Remark 3.7 (iii).

In view of Remark 3.2 (ii), let v and w be supercompacts with $uu'bc \leq v - w$ and $v \leq (yy' + axx')^2$ and $w \leq a(x'y + xy')^2$. If both v and w are equal to zero, then one of b or c is zero, so $\bar{a} \in \bar{1} \ominus \overline{bc}$ is just $\bar{a} \in \bar{1} \ominus \bar{b}$ or $\bar{a} \in \bar{1} \ominus \bar{c}$. If $v = 0$ and $w \neq 0$, then $w = w'w''$ for $w', w'' \in \mathcal{S}_R^*$ with $w' \leq a$ and $w'' \leq (x'y + xy')^2$. Thus $w' = a$, since a is a supercompact itself, and hence a minimal element, and $w'' \in T$, so that $uu'bc \leq -aw''$, $w'' \in T$. But $-aw''$ is again a supercompact, as $\mathcal{S}_R^*$ is a group, so $uu'bc = -aw''$. But $-aw'' \leq -aw'' + 1$, so $uu'bc = -aw'' \leq 1 - aw''$, yielding $\bar{a} \in \bar{1} \ominus \overline{bc}$. Similarly, if $v \neq 0$ and $w = 0$, then $\overline{bc} = \bar{1} \in \bar{1} - \bar{a}$.

This leaves us with the case $v \neq 0$ and $w \neq 0$. Then $v \in T$ and $w = w'w''$, for some $w', w'' \in \mathcal{S}_R^*$ with $w' \leq a$ and $w'' \leq (x'y + xy')^2$. But then $w' = a$, and $w'' \in T$. So, at the end we obtain

$$uu'bc \leq v - aw''$$

with $uu', v, w'' \in T$, or, equivalently

$$aw'' \leq v - uu'bc,$$

which is the same as $\{\bar{a}\} \subseteq \{\bar{1}\} - \{\overline{bc}\}$. □

Proposition 7.3. *Let k be a field, and let $(\mathcal{P}^*(k), \subseteq, \{0\})$ be the induced presentable field. Then*

$$W(\mathcal{P}^*(\mathcal{S}_{\mathcal{P}^*(k)})) \cong W(\mathcal{P}^*(Q(k))) \cong W(k).$$

Proof. Apply Remark 7.1 to obtain the presentable field

$$(\mathcal{P}^*(\mathcal{S}_R), \subseteq, \{0\}, +', -, \cdot, \{1\})$$

which satisfies the condition

$$\{a\} \subseteq \{a\} +' \{b\} \text{ for all } \{a\}, \{b\} \in \mathcal{S}^*_{\mathcal{P}^*(\mathcal{S}_R)}.$$

Now apply Theorem 7.1 with

$$T = \{\{s\} \in \mathcal{S}_{\mathcal{P}^*(\mathcal{S}_{\mathcal{P}^*(k)})} \mid \{s\} \subseteq \{a\}^2 \text{ for some } \{a\} \in \mathcal{P}^*(\mathcal{S}_{\mathcal{P}^*(k)})\}$$

to obtain the pre-quadratically presentable field $\mathcal{P}^*(\mathcal{S}_{\mathcal{P}^*(k)})/_m T$. One readily checks that the assignment of singletons of classes of elements a modulo T in $\mathcal{P}^*(\mathcal{S}_{\mathcal{P}^*(k)})/_m T$ to singletons of classes of elements of a modulo k^{*2} in $\mathcal{P}^*(Q(k))$ extends to an isomorphism of presentable fields, which implies the isomorphism of underlying Witt rings. □

Remark 7.4. It is an open question when the resulting pre-quadratically presentable field is quadratically presentable.

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