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The notions of closedness and D-connectedness in quantale-valued approach spaces

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Abstract. In this paper, we characterize local T_0 and T_1 quantale-valued gauge spaces, show how these concepts are related to each other and apply them to $\mathcal{L}$-approach distance spaces and $\mathcal{L}$-approach system spaces. Furthermore, we give the characterization of a closed point and D-connectedness in quantale-valued gauge spaces. Finally, we compare all these concepts to each other.

1 Introduction

Approach spaces have been introduced by Lowen [30, 31] to generalize metric and topological concepts, have many applications in almost all areas of mathematics including probability theory [15], convergence theory [16], domain theory [17], and fixed point theory [18]. Due to its huge importance, several generalizations of approach spaces appeared recently such as probabilistic approach spaces [22], quantale-valued gauge spaces [23], quantale-

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valued approach system [24], and quantale-valued approach with respect to (w.r.t.) closure operators [29]. Recently, some quantale-valued approach spaces [25] are also characterized by using quantale-valued bounded interior spaces and bounded strong topological spaces which are commonly used by fuzzy mathematicians.

In 1991, Baran [2] introduced local separation axioms and the notion of closedness in set-based topological categories which are used to define several distinct Hausdorff objects [4], T_3 and T_4 objects [6], regular, completely regular, normal objects [7], the notion of compactness and minimality, perfectness [9]. He also showed that the notion of closedness induces closure operators in the sense of Guili and Dikranjan [19] in some well-known topological categories **Conv** (the category of convergence spaces and filter convergence maps) [8, 32, 33], **Lim** (the category of limit spaces and filter convergence maps) [9, 32, 33], **Prord** (the category of preordered sets and monotone maps) [10, 32] and **SUConv** (the category of semiuniform convergence spaces and uniformly continuous maps) [11, 33].

The main objective of this paper is:

• to characterize local T_0 and local T_1 quantale-valued gauge spaces, quantale-valued distance approach and quantale-valued approach systems, and to show their relationship with each other;

• to provide the characterization of the notion of closedness and D-connectedness in quantale-valued approach spaces, and to show how they are linked to each other;

• to give a comparison between local T_0 and T_1 quantale-valued approach spaces, and between the notion of closedness and D-connectedness, and to examine their relationships.

2 Preliminaries

Recall [23, 24], that for every non-empty set L, a relation $\leq$ on L is called a partial order if it satisfies reflexivity ($\forall a \in L$, $a \leq a$), anti-symmetry ($\forall a, b \in L$, $a \leq b \wedge b \leq a \Rightarrow a = b$), and transitivity ($\forall a, b, c \in L$, $a \leq b \wedge b \leq c \Rightarrow a \leq c$). If $\leq$ is a partial order on L, then $(L, \leq)$ is called a partially ordered set or a poset. A poset $(L, \leq)$ is called a complete lattice if all subsets of L have both supremum ($\bigvee$) and infimum ($\bigwedge$). For any complete lattice, the top element and the bottom element are denoted by $\top$ and $\bot$, respectively.

In any complete lattice $(L, \leq)$, we define the *well-below relation*, $\alpha \lhd \beta$ if for all subsets $A \subseteq L$ such that $\beta \leq \bigvee A$ there is $\delta \in A$ such that $\alpha \leq \delta$. Similarly, we define the *well-above relation*, $\alpha \prec \beta$ if for all subsets $A \subseteq L$ such that $\bigwedge A \leq \alpha$ there exists $\delta \in A$ such that $\delta \leq \beta$. Furthermore, a complete lattice $(L, \leq)$ is called a *completely distributive lattice* if and only if we have $\alpha = \bigvee\{\beta : \beta \lhd \alpha\}$ for any $\alpha \in L$.

The triple $(L, \leq, *)$ is called a *quantale* if $(L, *)$ is a semigroup, and the operation $*$ satisfies: for all $\alpha_i, \beta \in L$, $(\bigvee_{i\in I} \alpha_i) * \beta = \bigvee_{i\in I}(\alpha_i * \beta)$ and $\beta * (\bigvee_{i\in I} \alpha_i) = \bigvee_{i\in I}(\beta * \alpha_i)$ and $(L, \leq)$ is a complete lattice.

A quantale $(L, \leq, *)$ is called *commutative* if $(L, *)$ is a commutative semigroup and it is called *integral* if $\alpha * \top = \top * \alpha = \alpha$ for all $\alpha \in L$.

Note that we denote a quantale by $\mathcal{L} = (L, \leq, *)$ if it is commutative and integral where $(L, \leq)$ is completely distributive.

In a quantale $\mathcal{L} = (L, \leq, *)$, we define the implication map $\rightarrow: L \times L \longrightarrow L$ by $\alpha \rightarrow \beta = \bigvee\{\gamma \in L : \alpha * \gamma \leq \beta\}$ for all $\alpha, \beta \in L$. Then $\alpha * \beta \leq \gamma$ if and only if $\alpha \leq \beta \rightarrow \gamma$ for all $\alpha, \beta, \gamma \in L$. In addition, a quantale $\mathcal{L} = (L, \leq, *)$ satisfies the *strong De Morgan law* if and only if $(\bigwedge_{i\in I} \alpha_i) \rightarrow \beta = \bigvee_{i\in I}(\alpha_i \rightarrow \beta)$ for all $\alpha_i, \beta \in L, i \in I$, where $I \neq \emptyset$.

A quantale $\mathcal{L} = (L, \leq, *)$ is called a *value quantale* if $(L, \leq)$ is a completely distributive lattice such that for all $\alpha, \beta \lhd \top$, $\alpha \vee \beta \lhd \top$ [20]. Furthermore, a quantale $(L, \leq, *)$ is called a *linearly ordered quantale* if for all $\alpha, \beta \in L$ either $\alpha \leq \beta$ or $\beta \leq \alpha$.

Example 2.1. (i) **Lawvere′s quantale**. $(L = [0, \infty], \geq, +)$ with $\gamma + \infty = \infty + \gamma = \infty$ for all $\gamma \in L$, is a linearly ordered value quantale [20]. Moreover, it has the distributive property w.r.t. the quantale operation (that is, $+$) distributes over arbitrary meets and satisfies the strong De Morgan law.

(ii) A commutative and integral quantale $(L, \leq, *)$, which satisfies $(\alpha \rightarrow \beta) \rightarrow \beta = \alpha \vee \beta$ for all $\alpha, \beta \in L$, is a complete *MV-algebra* [21]. Furthermore, a complete *MV-algebra* satisfies distribution over meets w.r.t. the quantale operation and holds the strong De Morgan law.

(iii) **Distance distribution functions quantale**. A function $\varphi : [0, \infty] \rightarrow [0, 1]$ which satisfies $\varphi(x) = \sup\limits_{z<x} \varphi(z)$ is called a *distance distribution function* [34]. We note that a distance distribution function is non-decreasing and satisfies $\varphi(0) = 0$. The set of all distance distribution functions is

denoted by $\triangle^+$; for example, for all $a \in [0, \infty]$,

$$\epsilon_a(x) = \begin{cases} 0, & x \in [0, a] \\ 1, & x \in (a, \infty] \end{cases}$$

is in $\triangle^+$. The set $\triangle^+$ is ordered pointwise, and the top and bottom elements are ϵ_0 and ϵ_∞, respectively, and the set $\triangle^+$ with pointwise order then becomes a complete lattice ([20]). We note that $\bigwedge_{i \in I} \varphi_i$ is not the pointwise infimum in general.

A binary operation $* : \triangle^+ \times \triangle^+ \to \triangle^+$ which is commutative, associative, non-decreasing, and $\varphi * \epsilon_0 = \varphi$ for all $\varphi \in \triangle^+$, is called a *triangle function* [34]. A triangle function is called *sup-continuous* if $(\bigvee_{i \in I} \varphi_i) * \Psi = \bigvee_{i \in I} (\varphi_i * \Psi)$ for all $\varphi_i, \Psi \in \triangle^+$ [34]. Then $(\triangle^+, \leq, *)$ is a commutative and integral quantale.

Definition 2.2. [23] A quantale $\mathcal{L} = (L, \leq, *)$ satisfies the condition (I) if for all $\beta, \gamma \in L$ with $\perp \prec \beta$ and $\gamma \lhd \top$ we have $\beta \nleq \gamma * \beta$.

Lemma 2.3. *[23] A quantale $\mathcal{L} = (L, \leq, *)$ satisfies strong cancellation property if for all $\gamma, \alpha \in L$, $\perp \prec \beta$: $\gamma * \beta \leq \alpha * \beta$ implies $\gamma \leq \alpha$. Moreover, if $\top \ntriangleleft \top$, then the condition* (I) *is satisfied.*

Example 2.4. (i) The Lawvere's quantale, that is, $([0, \infty], \geq, +)$ satisfies the condition (I).

(ii) A linearly ordered *MV-algebra* quantale $(L, \leq, *)$ satisfies the condition (I) but it is not true in general for every *MV-algebra* (see [24]).

Definition 2.5. A quantale $\mathcal{L} = (L, \leq, *)$ is said to be a *linear DM-I value quantale* if it is a linearly ordered value quantale for which the condition (I), distributivity property over arbitrary meets w.r.t. the quantale operation, and the strong De Morgan law hold.

Example 2.6. (i) The Lawvere's quantale $([0, \infty], \geq, +)$ is a linearly ordered value quantale which enjoys all these three conditions and thus, $([0, \infty], \geq, +)$ is a linear DM-I value quantale.

(ii) Let $\mathcal{L} = ([0, 1], \leq, *)$ be a triangular norm with a binary operation $*$ defined as for all $\alpha, \beta \in [0, 1], \alpha * \beta = \alpha \cdot \beta$ and named as a *product triangular norm* [26]. The triple $\mathcal{L} = ([0, 1], \leq, \cdot)$ is a commutative and integral quantale which satisfies the strong cancellation property and thus, the condition (I) holds. Furthermore, it satisfies the strong De Morgan law and enjoys distributivity property over arbitrary meets w.r.t. the quantale operation. Hence, it is a linear DM-I value quantale.

(iii) Let $\mathcal{L} = ([0,1], \leq, *)$, where for all $\alpha, \beta \in [0,1]$, $\alpha * \beta = (\alpha - 1 + \beta) \vee 0$ (Lukasiewicz t-norm) [26]. Then, $\mathcal{L}$ is clearly a linearly ordered value quantale which satisfies the condition (I), but the strong cancellation law is not valid. Moreover, it satisfies the strong De Morgan law and enjoys the distributivity property over arbitrary meets. Furthermore, $\mathcal{L}$ is a commutative and integral quantale. Thus, it is a linear DM-I value quantale.

(iv) Let $\mathcal{L} = (\triangle^+, \leq, *)$ (a probabilistic quantale), where $\varphi * \psi = \varphi \cdot \psi$ for all $\varphi, \psi \in \triangle^+$. Then $\mathcal{L}$ satisfies the condition (I), but it is not linearly ordered [23].

(v) Let $\mathcal{L} = ([0,1] \cup \{\bot = -1, \top = \infty\}, \leq, \cdot)$. Clearly, $\mathcal{L}$ is a linearly ordered quantale but it does not satisfy the condition (I) as $\top \lhd \top$.

(vi) If $\mathcal{L} = (\{0,1\}, \leq, \wedge)$, then $\mathcal{L}$ does not satisfy the condition (I) as $1 \lhd 1$ [23].

Definition 2.7. [23] Let X be a nonempty set. A map $d : X \times X \longrightarrow \mathcal{L} = (L, \leq, *)$ is called an *$\mathcal{L}$-metric* on X if it satisfies for all $x \in X$, $d(x,x) = \top$, and for all $x, y, z \in X$, $d(x,y) * d(y,z) \leq d(x,z)$. The pair (X, d) is called an *$\mathcal{L}$-metric space.*

A map $f : (X, d_X) \longrightarrow (Y, d_Y)$ is called an *$\mathcal{L}$-metric morphism* if $d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.

The category whose objects are $\mathcal{L}$-metric spaces and morphisms are $\mathcal{L}$-metric morphisms is denoted by **L-MET**. Furthermore, we define **L-MET(X)** as the set of all $\mathcal{L}$-metrics on X.

Example 2.8. (i) If $\mathcal{L} = (\{0,1\}, \leq, \wedge)$, then an $\mathcal{L}$-metric space is a preordered set.

(ii) If $\mathcal{L}$ is a Lawvere's quantale, that is, $\mathcal{L} = ([0,\infty], \geq, +)$, then an $\mathcal{L}$-metric space is an extended pseudo-quasi metric space.

(iii) If $\mathcal{L} = (\triangle^+, \leq, *)$, then an $\mathcal{L}$-metric space is a probabilistic quasi metric space [20].

Definition 2.9. [23] Let $\mathcal{H} \subseteq$ **L-MET(X)** and $d \in$ **L-MET(X)**.

(i) d is called *locally supported* by $\mathcal{H}$ if for all $x \in X$, $\alpha \lhd \top$, $\bot \prec \omega$, there is $e \in \mathcal{H}$ such that $e(x, .) * \alpha \leq d(x, .) \vee \omega$.

(ii) $\mathcal{H}$ is called *locally directed* if for all finite subsets $\mathcal{H}_0 \subseteq \mathcal{H}$, $\bigwedge_{d\in\mathcal{H}_0} d$ is locally supported by $\mathcal{H}$.

(iii) $\mathcal{H}$ is called *locally saturated* if for all $d \in$ **L-MET(X)**, we have $d \in \mathcal{H}$ whenever d is locally supported by $\mathcal{H}$.

(iv) The set $\tilde{\mathcal{H}} = \{d \in$ **L-MET(X)** $: d$ is locally supported by $\mathcal{H}\}$ is called the *local saturation* of $\mathcal{H}$.

Definition 2.10. [23] Let X be a set. Then, $\mathcal{G} \subseteq$ **L-MET(X)** is called an $\mathcal{L}$-*gauge* if $\mathcal{G}$ satisfies the following:

(i) $\mathcal{G} \neq \emptyset$.

(ii) $d \in \mathcal{G}$ and $d \leq e$ implies $e \in \mathcal{G}$.

(iii) $d, e \in \mathcal{G}$ implies $d \wedge e \in \mathcal{G}$.

(iv) $\mathcal{G}$ is locally saturated.

The pair $(X, \mathcal{G})$ is called an $\mathcal{L}$-*gauge space.*

A map $f : (X, \mathcal{G}) \longrightarrow (X', \mathcal{G}')$ is called an $\mathcal{L}$-*gauge morphism* if $d' \circ (f \times f) \in \mathcal{G}$ whenever $d' \in \mathcal{G}'$.

The category whose objects are $\mathcal{L}$-gauge spaces and morphisms are $\mathcal{L}$-gauge morphisms is denoted by **L-GS** (cf. [23]).

Definition 2.11. [23] Let $(X, \mathcal{G})$ be an $\mathcal{L}$-gauge space and let $\mathcal{H} \subseteq$ **L-MET(X)**. If $\tilde{\mathcal{H}} = \mathcal{G}$, then $\mathcal{H}$ is called a *basis* for the gauge $\mathcal{G}$.

Proposition 2.12. *[23] Let $\mathcal{L} = (L, \leq, *)$ be a value quantale. If $\emptyset \neq \mathcal{H} \subseteq$ **L-MET(X)** is locally directed, then $\mathcal{G} = \tilde{\mathcal{H}}$ is a gauge with $\mathcal{H}$ as a basis.*

Proposition 2.13. *Let X be a nonempty set. The discrete $\mathcal{L}$-gauge structure on X is given by $\mathcal{G}_{dis} =$ **L-MET(X)**.*

Proof. Note that for all $x, y \in X$,

$$d_{dis}(x, y) = \begin{cases} \top, & x = y \\ \bot, & x \neq y \end{cases}$$

is the smallest $\mathcal{L}$-metric structure on X. To show $\mathcal{H} = \{d_{dis}\}$ is an $\mathcal{L}$-gauge basis, by Proposition 2.12, it suffices to show that $\mathcal{H}$ is locally directed. Since a basis with one element is always locally directed, then $\mathcal{H} = \{d_{dis}\}$ is an $\mathcal{L}$-gauge basis. In addition, the associated $\mathcal{L}$-gauge is just the principal filters of d_{dis}, that is, $\mathcal{G}_{dis} = \{e \in$ **L-MET(X)** $: e \geq d_{dis}\}$. Therefore, all the

$\mathcal{L}$-metrics are in $\mathcal{L}$-gauge. Thus, $\mathcal{G}_{dis} = \mathbf{L\text{-}MET(X)}$. Furthermore, every $f : (X, \mathcal{G}_{dis} = \mathbf{L\text{-}MET(X)}) \to (X', \mathcal{G}')$ is an $\mathcal{L}$-gauge morphism for any $(X', \mathcal{G}')$ $\mathcal{L}$-gauge space. □

Definition 2.14. [23] A map $\delta : X \times P(X) \longrightarrow \mathcal{L} = (L, \leq, *)$ is called an *$\mathcal{L}$-approach distance* if δ satisfies the following:

(i) $\forall x \in X$, $\delta(x, \{x\}) = \top$.

(ii) $\forall x \in X$, $\delta(x, \emptyset) = \bot$.

(iii) $\forall x \in X$ and for all $A, B \subseteq X$, $\delta(x, A \cup B) = \delta(x, A) \vee \delta(x, B)$.

(iv) $\forall x \in X$, for all $A \subseteq X$ and for all $\alpha \in \mathcal{L}$, $\delta(x, A) \geq \delta(x, \overline{A}^{\alpha}) * \alpha$, where $\overline{A}^{\alpha} = \{x \in X : \delta(x, A) \geq \alpha\}$.

The pair (X, δ) is called an *$\mathcal{L}$-approach distance space.*

A map $f : (X, \delta) \longrightarrow (X', \delta')$ is called an *$\mathcal{L}$-approach morphism* if $\delta(x, A) \leq \delta'(f(x), f(A))$ for all $x \in X$ and $A \subseteq X$.

The category whose objects are $\mathcal{L}$-approach distance spaces and morphisms are $\mathcal{L}$-approach morphisms is denoted by **L-AP**.

Definition 2.15. [24] Let $\mathcal{A} \subseteq L^X$ and $\varphi \in L^X$.

(i) φ is *supported* by $\mathcal{A}$ if for all $\alpha \lhd \top$, $\bot \prec \omega$ there exists $\varphi_{\alpha}^{\omega} \in \mathcal{A}$ such that $\varphi_{\alpha}^{\omega} * \alpha \leq \varphi \vee \omega$.

(ii) $\mathcal{A}$ is *saturated* if $\varphi \in \mathcal{A}$ whenever φ is supported by $\mathcal{A}$.

(iii) For $\mathcal{B} \subseteq L^X$, $\tilde{\mathcal{B}} = \{\varphi \in \mathcal{L}^X : \varphi$ is supported by $\mathcal{B}\}$ is called the *saturation of* $\mathcal{B}$.

Definition 2.16. [24] Let $\mathcal{A}(x) \subseteq L^X$ for all $x \in X$. Then $\mathcal{A} = (\mathcal{A}(x))_{x \in X}$ is called an *$\mathcal{L}$-approach system* if for all $x \in X$,

(i) $\mathcal{A}(x)$ is a filter in $\mathcal{L}^X$, that is, $\varphi \in \mathcal{A}(x)$ and $\varphi \leq \varphi'$ implies $\varphi' \in \mathcal{A}(x)$, and $\varphi, \varphi' \in \mathcal{A}(x)$ implies $\varphi \wedge \varphi' \in \mathcal{A}(x)$.

(ii) $\varphi(x) = \top$ whenever $\varphi \in \mathcal{A}(x)$.

(iii) $\mathcal{A}(x)$ is saturated.

(iv) For all $\varphi \in \mathcal{A}(x)$, $\alpha \lhd \top$, $\bot \prec \omega$ there exists a family $(\varphi_z)_{z \in X} \in \prod_{z \in X} \mathcal{A}(z)$ such that $\varphi_x(z) * \varphi_z(y) * \alpha \leq \varphi(y) \vee \omega$, $\forall y, z \in X$.

The pair $(X, \mathcal{A})$ is called an *$\mathcal{L}$-approach system space.*

The map $f : (X, \mathcal{A}) \longrightarrow (X', \mathcal{A}')$ is called an *$\mathcal{L}$-approach system morphism* if for all $x \in X$, $\varphi' \circ f \in \mathcal{A}(x)$ whenever $\varphi' \in \mathcal{A}'(f(x))$.

The category whose objects are $\mathcal{L}$-approach system spaces and morphisms are $\mathcal{L}$-approach system morphisms is denoted by **L-AS**.

Definition 2.17. [24] Let $\mathcal{B}(x) \subset L^X$ for all $x \in X$. Then $(\mathcal{B}(x))_{x\in X}$ is called an L-*approach system basis* if for all $x \in X$,

(i) $\mathcal{B}(x)$ is a filter basis in $\mathcal{L}^X$.

(ii) $\varphi(x) = \top$ whenever $\varphi \in \mathcal{B}(x)$.

(iii) For all $\varphi \in \mathcal{B}(x)$, $\alpha \lhd \top$, $\bot \prec \omega$ there exists a family $(\varphi_z)_{z\in X} \in \prod_{z\in X} \mathcal{B}(z)$ such that $\varphi_x(z) * \varphi_z(y) * \alpha \leq \varphi(y) \vee \omega$, $\forall y, z \in X$.

Definition 2.18. [24] Let $(\mathcal{A}(x))_{x\in X}$ be an $\mathcal{L}$-approach system and $(\mathcal{B}(x))_{x\in X}$ be the collection of filter bases on L^X. Then, $(\mathcal{B}(x))_{x\in X}$ is called a *basis* for $\mathcal{L}$-approach system if $\tilde{\mathcal{B}}(x) = \{\varphi \in L^X : \varphi \text{ is supported by } \mathcal{B}(x)\} = \mathcal{A}(x)$.

A functor $\mathcal{U} : \mathcal{E} \longrightarrow$ **Set** (the category of sets and functions) is called *topological* if $\mathcal{U}$ is concrete, consists of small fibers, and each $\mathcal{U}$-source has an initial lift [1, 32, 33].

A topological functor which has a left adjoint is called a *discrete functor.*

Lemma 2.19. *[23, 24] Let $\mathcal{L} = (L, \leq, *)$ be a value quantale, $(X_i, \mathfrak{B}_i)$ be the collection of $\mathcal{L}$-approach spaces, and let $f_i : X \longrightarrow (X_i, \mathfrak{B}_i)$ be a source and $x \in X$.*

(i) *A basis for the initial $\mathcal{L}$-gauge on X is given by*

$$\mathcal{H} = \{\bigwedge_{i\in K} d_i \circ (f_i \times f_i) : K \subseteq I \ finite, d_i \in \mathcal{G}_i, \forall i \in I\}.$$

(ii) *A basis for the initial $\mathcal{L}$-approach system is provided by*

$$\mathcal{B}(x) = \{\bigwedge_{i\in K} \varphi_i \circ f_i : K \subseteq I \ finite, \varphi_i \in \mathcal{A}_i(f_i(x)), \forall i \in I\}.$$

Note that for a value quantale $\mathcal{L}$, the categories **L-GS**, **L-AP**, and **L-AS** are topological categories over **Set** [23, 24] and we will denote any $\mathcal{L}$-approach space by $(X, \mathfrak{B})$.

Remark 2.20. [23, 24] Let $\mathcal{L}$ be a value quantale and let $(X, \mathcal{G})$ (respectively, (X, δ) and $(X, \mathcal{A})$) be an $\mathcal{L}$-gauge space (respectively, $\mathcal{L}$-approach space and $\mathcal{L}$-approach system space). For all $A \subseteq X$ and for all $x \in X$,

(i) The transition from an $\mathcal{L}$-gauge to an $\mathcal{L}$-approach distance is determined by

$$\delta(x, A) = \bigwedge_{d\in\mathcal{G}} \bigvee_{a\in A} d(x, a).$$

Conversely, if $\delta : X \times P(X) \to \mathcal{L}$ is an $\mathcal{L}$-approach distance, then an associated $\mathcal{L}$-gauge is defined by

$$\mathcal{G} = \{d \in \mathbf{L\text{-}MET(X)} : \delta(x, A) \leq \bigvee_{a \in A} d(x, a)\}.$$

(ii) If $\delta : X \times P(X) \to \mathcal{L}$ is an $\mathcal{L}$-approach distance, then an associated $\mathcal{L}$-approach system is provided by

$$\mathcal{A}(x) = \{\varphi \in L^X : \delta(x, A) \leq \bigvee_{a \in A} \varphi(a)\}.$$

Conversely, the transition from an $\mathcal{L}$-approach system to an $\mathcal{L}$-approach distance is determined by

$$\delta(x, A) = \bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{a \in A} \varphi(a).$$

(iii) The transition from an $\mathcal{L}$-approach system to an $\mathcal{L}$-gauge is given by

$$\mathcal{G}^{\mathcal{A}} = \{d \in \mathbf{L\text{-}MET(X)} : d(x, .) \in \mathcal{A}(x), \ \forall x \in X\}.$$

Conversely, the transition from an $\mathcal{L}$-gauge to an $\mathcal{L}$-approach system is determined by the following $\mathcal{L}$-approach system basis

$$\mathcal{B}(x) = \{d(x, .) : d \in \mathcal{G}\}.$$

Remark 2.21. (i) If the quantale $\mathcal{L}$ is a linear DM-I value quantale, then the above transition formulas provide isomorphisms functors among the categories, that is, **L-GS**, **L-AP**, and **L-AS** are isomorphic. It was shown in [23, 24] that these assumptions on the quantale are necessary.

(ii) For any arbitrary quantale, for example, in $\mathcal{L} = (\triangle^+, \leq, *)$ (probabilistic case), by Example 5.11 of [23, 24], **L-GS**, **L-AP** and **L-AS** are not isomorphic.

3 Local T_0 and T_1 quantale-valued approach spaces

Let X be a set and p be a point in X. Let $X \vee_p X$ be the *wedge product* of X at p ([2], p. 334), that is, two disjoint copies of X identified at p, or in other words, the pushout of $p : 1 \to X$ along itself (where 1 is the terminal object in **Set**). More precisely, if i_1 and $i_2 : X \to X \vee_p X$ denote the inclusion of X as the first and second factor, respectively, then $i_1 p = i_2 p$ is the pushout diagram [9].

A point x in $X \vee_p X$ is denoted as x_1 if it lies in the first component and as x_2 if it lies in the second component.

Let X^2 be the cartesian product of X.

Definition 3.1. [2] The *principal p-axis map*, $A_p : X \vee_p X \longrightarrow X^2$ is defined by

$$A_p(x_i) = \begin{cases} (x, p), & i = 1 \\ (p, x), & i = 2 \end{cases}$$

Definition 3.2. [2] The *skewed p-axis map*, $S_p : X \vee_p X \longrightarrow X^2$ is defined by

$$S_p(x_i) = \begin{cases} (x, x), & i = 1 \\ (p, x), & i = 2 \end{cases}$$

Definition 3.3. [2] The *fold map at p*, $\nabla_p : X \vee_p X \longrightarrow X$ is defined by $\nabla_p(x_i) = x$ for $i = 1, 2$.

Definition 3.4. [2] Let $U : \mathcal{E} \longrightarrow$ **Set** be topological, $X \in Ob(\mathcal{E})$ with $U(X) = B$, and $p \in B$.

(i) X is $\overline{T_0}$ at p if and only if the initial lift of the U-source $\{A_p : B \vee_p B \longrightarrow U(X^2) = B^2$ and $\nabla_p : B \vee_p B \longrightarrow UD(B) = B\}$ is discrete, where D is the discrete functor.

(ii) X is T_1 at p if and only if the initial lift of the U-source $\{S_p : B \vee_p B \longrightarrow U(X^2) = B^2$ and $\nabla_p : B \vee_p B \longrightarrow UD(B) = B\}$ is discrete.

Remark 3.5. In **Top** (the category of topological spaces and continuous maps), the condition that a topological space (X, τ) is $\overline{T_0}$ at p (respectively, T_1 at p) is reduced to the condition that for each $x \in X$ with $x \neq p$, there exists a neighborhood of x does not contain p or (respectively, and) there exists a neighborhood of p does not contain x. Moreover, (X, τ) is T_0 (respectively, T_1) if and only if (X, τ) is T_0 at p (respectively, T_1 at p) for all $p \in X$ [5].

Theorem 3.6. *Let $(X,\mathcal{G})$ be an $\mathcal{L}$-gauge space and $p \in X$. Then, $(X,\mathcal{G})$ is $\overline{T_0}$ at p if and only if for all $x \in X$ with $x \neq p$, there exists $d \in \mathcal{G}$ such that $d(x,p) \wedge d(p,x) = \perp$.*

Proof. Suppose that $(X,\mathcal{G})$ is $\overline{T_0}$ at p, $x \in X$, and $x \neq p$. Let $\overline{\mathcal{G}}$ be the initial $\mathcal{L}$-gauge on $X \vee_p X$ induced by $A_p : X \vee_p X \to U(X^2,\mathcal{G}^2) = X^2$ and $\nabla_p : X \vee_p X \to U(X,\mathcal{G}_{dis}) = X$, where $\mathcal{G}_{dis}$ is the discrete structure on X and $\mathcal{G}^2$ is the product structure on X^2 induced by $\pi_i : X^2 \to X$, projection maps for $i = 1,2$. Suppose that $\mathcal{H}_{dis} = \{d_{dis}\}$ is a basis for the discrete $\mathcal{L}$-gauge, where d_{dis} is the discrete $\mathcal{L}$-metric on X. Let $\mathcal{H}$ be an $\mathcal{L}$-gauge basis of $\mathcal{G}$ and $d \in \mathcal{H}$, and $\overline{\mathcal{H}} = \{\overline{d}_{dis}\}$ be the initial $\mathcal{L}$-gauge basis of $\overline{\mathcal{G}}$, where $\overline{d}_{dis}$ is the discrete $\mathcal{L}$-metric on $X \vee_p X$. For $x_1, x_2 \in X \vee_p X$ with $x_1 \neq x_2$, note that

$$d_{dis}(\nabla_p(x_1), \nabla_p(x_2)) = d_{dis}(x,x) = \top,$$

$$d(\pi_1 A_p(x_1), \pi_1 A_p(x_2)) = d(x,p),$$

$$d(\pi_2 A_p(x_1), \pi_2 A_p(x_2)) = d(p,x).$$

Since $x_1 \neq x_2$, $\overline{d}_{dis}$ is the discrete $\mathcal{L}$-metric on $X \vee_p X$ and $(X,\mathcal{G})$ is $\overline{T_0}$ at p, by Lemma 2.19,

$$\begin{aligned}
\perp &= \overline{d}_{dis}(x_1,x_2) \\
&= \bigwedge \{d_{dis}(\nabla_p(x_1), \nabla_p(x_2)), d(\pi_1 A_p(x_1), \pi_1 A_p(x_2)), \\
&\qquad d(\pi_2 A_p(x_1), \pi_2 A_p(x_2))\} \\
&= \bigwedge \{\top, d(x,p), d(p,x)\} \\
&= d(x,p) \wedge d(p,x).
\end{aligned}$$

Conversely, let $\overline{\mathcal{H}}$ be the initial $\mathcal{L}$-gauge basis on $X \vee_p X$ induced by $A_p : X \vee_p X \to U(X^2,\mathcal{G}^2) = X^2$ and $\nabla_p : X \vee_p X \to U(X,\mathcal{G}_{dis}) = X$ where, by Proposition 2.13, $\mathcal{G}_{dis} = \mathbf{L\text{-}MET(X)}$ discrete $\mathcal{L}$-gauge on X and $\mathcal{G}^2$ be the product structure on X^2 induced by $\pi_i : X^2 \to X$ the projection maps for $i = 1,2$.

Suppose that for all $x \in X$ with $x \neq p$, there exists $d \in \mathcal{G}$ such that $d(x,p) \wedge d(p,x) = \perp$. Let $\overline{d} \in \overline{\mathcal{H}}$ and $u, v \in X \vee_p X$.

If $u = v$, then

$$\begin{aligned}\overline{d}(u,v) &= \bigwedge \{d_{dis}(\nabla_p(u), \nabla_p(u)), d(\pi_1 A_p(u), \pi_1 A_p(u)),\\ &\qquad d(\pi_2 A_p(u), \pi_2 A_p(u))\}\\ &= \top.\end{aligned}$$

If $u \neq v$ and $\nabla_p(u) \neq \nabla_p(v)$, then $d_{dis}(\nabla_p(u), \nabla_p(v)) = \bot$, since d_{dis} is discrete. By Lemma 2.19,

$$\begin{aligned}\overline{d}(u,v) &= \bigwedge \{d_{dis}(\nabla_p(u), \nabla_p(v)), d(\pi_1 A_p(u), \pi_1 A_p(v)),\\ &\qquad d(\pi_2 A_p(u), \pi_2 A_p(v))\}\\ &= \bigwedge \{\bot, d(\pi_1 A_p(u), \pi_1 A_p(v)), d(\pi_2 A_p(u), \pi_2 A_p(v))\}\\ &= \bot.\end{aligned}$$

Suppose that $u \neq v$ and $\nabla_p(u) = \nabla_p(v)$. If $\nabla_p(u) = x = \nabla_p(v)$ for some $x \in X$ with $x \neq p$, then $u = x_1$ and $v = x_2$ or $u = x_2$ and $v = x_1$, since $u \neq v$. Let $u = x_1$ and $v = x_2$. Then

$$\begin{aligned}d_{dis}(\nabla_p(u), \nabla_p(v)) &= d_{dis}(\nabla_p(x_1), \nabla_p(x_2))\\ &= d_{dis}(x, x)\\ &= \top,\end{aligned}$$

$$\begin{aligned}d(\pi_1 A_p(u), \pi_1 A_p(v)) &= d(\pi_1 A_p(x_1), \pi_1 A_p(x_2))\\ &= d(x, p),\end{aligned}$$

and

$$\begin{aligned}d(\pi_2 A_p(u), \pi_2 A_p(v)) &= d(\pi_2 A_p(x_1), \pi_2 A_p(x_2))\\ &= d(p, x).\end{aligned}$$

It follows that

$$\begin{aligned}\overline{d}(u,v) &= \overline{d}(x_1, x_2)\\ &= \bigwedge \{d_{dis}(\nabla_p(x_1), \nabla_p(x_2)), d(\pi_1 A_p(x_1), \pi_1 A_p(x_2)),\\ &\qquad d(\pi_2 A_p(x_1), \pi_2 A_p(x_2))\}\\ &= \bigwedge \{\top, d(x,p), d(p,x)\}\\ &= \bigwedge \{d(x,p), d(p,x)\}\\ &= d(x,p) \wedge d(p,x).\end{aligned}$$

By the assumption, $d(x,p) \wedge d(p,x) = \bot$ and we have $\overline{d}(u,v) = \bot$.
Let $u = x_2$ and $v = x_1$. Similarly,

$$\begin{aligned} d_{dis}(\nabla_p(u), \nabla_p(v)) &= d_{dis}(\nabla_p(x_2), \nabla_p(x_1)) \\ &= d_{dis}(x,x) \\ &= \top, \end{aligned}$$

$$\begin{aligned} d(\pi_1 A_p(u), \pi_1 A_p(v)) &= d(\pi_1 A_p(x_2), \pi_1 A_p(x_1)) \\ &= d(p,x), \end{aligned}$$

and

$$\begin{aligned} d(\pi_2 A_p(u), \pi_2 A_p(v)) &= d(\pi_2 A_p(x_2), \pi_2 A_p(x_1)) \\ &= d(x,p). \end{aligned}$$

It follows that

$$\begin{aligned} \overline{d}(u,v) &= \overline{d}(x_2,x_1) \\ &= \bigwedge \{d_{dis}(\nabla_p(x_2), \nabla_p(x_1)), d(\pi_1 A_p(x_2), \pi_1 A_p(x_1)), \\ &\qquad d(\pi_2 A_p(x_2), \pi_2 A_p(x_1))\} \\ &= \bigwedge \{\top, d(p,x), d(x,p)\} \\ &= \bigwedge \{d(x,p), d(p,x)\} \\ &= d(x,p) \wedge d(p,x). \end{aligned}$$

By the assumption, we get $\overline{d}(u,v) = \bot$. Therefore, for all $u, v \in X \vee_p X$, we have

$$\overline{d}(u,v) = \begin{cases} \top, & u = v \\ \bot, & u \neq v \end{cases}$$

and by Proposition 2.13, $\overline{d}$ is the discrete $\mathcal{L}$-metric on $X \vee_p X$, that is, $\overline{\mathcal{H}} = \{\overline{d}\}$, which means $\mathcal{G}_{dis} = \textbf{L-MET(X)}$. By Definition 3.4 (i), $(X, \mathcal{G})$ is $\overline{T_0}$ at p. $\square$

In a quantale $(L, \leq, *)$, if $a \in L$ and $a \neq \top$, then a is called a *prime element* if and only if $\alpha \wedge \beta \leq a$ implies $\alpha \leq a$ or $\beta \leq a$ for all $\alpha, \beta \in L$.

Corollary 3.7. *Let $(X, \mathcal{G})$ be an $\mathcal{L}$-gauge space, where $\mathcal{L}$ has a prime bottom element and $p \in X$. Then $(X, \mathcal{G})$ is $\overline{T_0}$ at p if and only if for all $x \in X$ with $x \neq p$, there exists $d \in \mathcal{G}$ such that $d(x,p) = \bot$ or $d(p,x) = \bot$.*

Proof. It follows from Theorem 3.6 and the definition of the prime bottom element. □

Theorem 3.8. *Let $\mathcal{L}$ be a value quantale which has a prime bottom element, and let $(X, \mathfrak{B})$ be an $\mathcal{L}$-approach space, and $p \in X$. Then, the following are equivalent:*

(i) *$(X, \mathfrak{B})$ is $\overline{T_0}$ at p.*

(ii) *$\forall\ x \in X$ with $x \neq p$, there exists $d \in \mathcal{G}$ such that $d(x,p) = \bot$ or $d(p,x) = \bot$.*

(iii) *$\forall\ x \in X$ with $x \neq p$, $\delta(x, \{p\}) = \bot$ or $\delta(p, \{x\}) = \bot$.*

(iv) *$\forall\ x \in X$ with $x \neq p$, there exists $\varphi \in \mathcal{A}(p)$ such that $\varphi(x) = \bot$ or there exists $\varphi \in \mathcal{A}(x)$ such that $\varphi(p) = \bot$.*

Proof. (i) $\Leftrightarrow$ (ii) follows from Corollary 3.7.

(ii) $\Rightarrow$ (iii) Suppose that for all $x \in X$ with $x \neq p$, there exists $d \in \mathcal{G}$ such that $d(x,p) = \bot$ or $d(p,x) = \bot$. By Remark 2.20 (i), $\delta(x, \{p\}) = \bigwedge\limits_{d' \in \mathcal{G}} d'(x,p) = \bot$, and consequently, $\delta(x, \{p\}) = \bot$. Similarly, $\delta(p, \{x\}) = \bigwedge\limits_{d' \in \mathcal{G}} d'(p,x) = \bot$ implies $\delta(p, \{x\}) = \bot$.

(iii) $\Rightarrow$ (iv) Suppose that for all $x \in X$ with $x \neq p$, $\delta(x, \{p\}) = \bot$ or $\delta(p, \{x\}) = \bot$. Let $A = \{p\}$, then, by Remark 2.20(ii), for all $\varphi' \in L^X$, $\delta(x, \{p\}) \leq \varphi'(p)$. In particular, there exists $\varphi \in \mathcal{A}(x)$ such that $\varphi(p) = \bot$. Similarly, if $A = \{x\}$, then, by Remark 2.20(ii), for all $\varphi'' \in L^X$, $\delta(p, \{x\}) \leq \varphi''(x)$ and, particularly, there exists $\varphi \in \mathcal{A}(p)$ such that $\varphi(x) = \bot$.

(iv) $\Rightarrow$ (ii) Suppose that the condition holds. By Remark 2.20(iii), $d'(x,p) \in \mathcal{A}(x)$ for all $x \in X$ and, by Definition 2.18, for all $x \in X$, $\alpha \lhd \top$, $\bot \prec \omega$, there exists $\varphi \in \mathcal{B}(x)$ such that $\varphi(p) * \alpha \leq d'(x,p) \vee \omega$. Since $\varphi(p) = \bot$ and $\bot * \alpha = \bot$, $\bot \leq d'(x,p) \vee \omega$, and, in particular, there exists $d \in \mathcal{G}$ such that $d(x,p) = \bot$. In a similar way, there exists $d \in \mathcal{G}$ such that $d(p,x) = \bot$. □

Theorem 3.9. *Let $(X, \mathcal{G})$ be an $\mathcal{L}$-gauge space and $p \in X$. Then $(X, \mathcal{G})$ is T_1 at p if and only if for all $x \in X$ with $x \neq p$, there exists $d \in \mathcal{G}$ such that $d(x,p) = \bot = d(p,x)$.*

Proof. Suppose that $(X, \mathcal{G})$ is T_1 at p, $x \in X$ and $x \neq p$. Let $u = x_1$, $v = x_2 \in X \vee_p X$. Note that

$$d_{dis}(\nabla_p(u), \nabla_p(v)) = d_{dis}(\nabla_p(x_1), \nabla_p(x_2)) = d_{dis}(x, x) = \top,$$

$$d(\pi_1 S_p(u), \pi_1 S_p(v)) = d(\pi_1 S_p(x_1), \pi_1 S_p(x_2)) = d(x, p),$$

$$d(\pi_2 S_p(u), \pi_2 S_p(v)) = d(\pi_2 S_p(x_1), \pi_2 S_p(x_2)) = d(x, x) = \top,$$

where d_{dis} is the discrete $\mathcal{L}$-metric on $X \vee_p X$ and $\pi_i : X^2 \to X$ are the projection maps for $i = 1, 2$. Since $u \neq v$ and $(X, \mathcal{G})$ is T_1 at p, by Lemma 2.19,

$$\begin{aligned} \perp &= \bigwedge \{d_{dis}(\nabla_p(u), \nabla_p(v)), d(\pi_1 S_p(u), \pi_1 S_p(v)), d(\pi_2 S_p(u), \pi_2 S_p(v))\} \\ &= \bigwedge \{\top, d(x, p)\} \\ &= d(x, p). \end{aligned}$$

Let $u = x_2$, $v = x_1 \in X \vee_p X$. Similarly,

$$d_{dis}(\nabla_p(u), \nabla_p(v)) = d_{dis}(\nabla_p(x_2), \nabla_p(x_1)) = d_{dis}(x, x) = \top$$

$$d(\pi_1 S_p(u), \pi_1 S_p(v)) = d(\pi_1 S_p(x_2), \pi_1 S_p(x_1)) = d(p, x)$$

$$d(\pi_2 S_p(u), \pi_2 S_p(v)) = d(\pi_2 S_p(x_2), \pi_2 S_p(x_1)) = d(x, x) = \top$$

It follows that

$$\begin{aligned} \perp &= \bigwedge \{d_{dis}(\nabla_p(u), \nabla_p(v)), d(\pi_1 S_p(u), \pi_1 S_p(v)), d(\pi_2 S_p(u), \pi_2 S_p(v))\} \\ &= \bigwedge \{\top, d(p, x)\} \\ &= d(p, x). \end{aligned}$$

Conversely, let $\overline{\mathcal{H}}$ be the initial $\mathcal{L}$-gauge basis on $X \vee_p X$ induced by $S_p : X \vee_p X \to U(X^2, \mathcal{G}^2) = X^2$ and $\nabla_p : X \vee_p X \to U(X, \mathcal{G}_{dis}) = X$ where, by Proposition 2.13, $\mathcal{G}_{dis} = \mathbf{L\text{-}MET(X)}$ is the discrete $\mathcal{L}$-gauge on X and $\mathcal{G}^2$ is the product $\mathcal{L}$-gauge structure on X^2 induced by $\pi_i : X^2 \to X$, the projection maps for $i = 1, 2$.

Suppose that for all $x \in X$ with $x \neq p$, there exists $d \in \mathcal{G}$ such that $d(x,p) = \bot = d(p,x)$. Let $\overline{d} \in \overline{\mathcal{H}}$ and $u, v \in X \vee_p X$. If $u = v$, then

$$\begin{aligned}
\overline{d}(u,v) &= \bigwedge \{d_{dis}(\nabla_p(u), \nabla_p(u)), d(\pi_1 S_p(u), \pi_1 S_p(u)), \\
&\qquad d(\pi_2 S_p(u), \pi_2 S_p(u))\} \\
&= \top.
\end{aligned}$$

If $u \neq v$ and $\nabla_p(u) \neq \nabla_p(v)$, then $d_{dis}(\nabla_p(u), \nabla_p(v)) = \bot$, since d_{dis} is discrete. By Lemma 2.19

$$\begin{aligned}
\overline{d}(u,v) &= \bigwedge \{d_{dis}(\nabla_p(u), \nabla_p(v)), d(\pi_1 S_p(u), \pi_1 S_p(v)), \\
&\qquad d(\pi_2 S_p(u), \pi_2 S_p(v))\} \\
&= \bigwedge \{\bot, d(\pi_1 S_p(u), \pi_1 S_p(v)), d(\pi_2 S_p(u), \pi_2 S_p(v))\} \\
&= \bot.
\end{aligned}$$

Suppose that $u \neq v$ and $\nabla_p(u) = \nabla_p(v)$. If $\nabla_p(u) = x = \nabla_p(v)$ for some $x \in X$ with $x \neq p$, then $u = x_1$ and $v = x_2$ or $u = x_2$ and $v = x_1$, since $u \neq v$. If $u = x_1$ and $v = x_2$, then, by Lemma 2.19,

$$\begin{aligned}
\overline{d}(u,v) &= \overline{d}(x_1, x_2) \\
&= \bigwedge \{d_{dis}(\nabla_p(x_1), \nabla_p(x_2)), d(\pi_1 S_p(x_1), \pi_1 S_p(x_2)), \\
&\qquad d(\pi_2 S_p(x_1), \pi_2 S_p(x_2))\} \\
&= \bigwedge \{\top, d(x,p)\} \\
&= d(x,p) \\
&= \bot,
\end{aligned}$$

since $x \neq p$ and $d(x,p) = \bot$. Similarly, if $u = x_2$ and $v = x_1$, then

$$\begin{aligned}
\overline{d}(u,v) &= \overline{d}(x_2, x_1) \\
&= \bigwedge \{d_{dis}(\nabla_p(x_2), \nabla_p(x_1)), d(\pi_1 S_p(x_2), \pi_1 S_p(x_1)), \\
&\qquad d(\pi_2 S_p(x_2), \pi_2 S_p(x_1))\} \\
&= \bigwedge \{\top, d(p,x)\} \\
&= d(p,x) \\
&= \bot,
\end{aligned}$$

since $x \neq p$ and $d(p,x) = \perp$. Hence, for all $u, v \in X \vee_p X$, we get

$$\overline{d}(u,v) = \begin{cases} \top, & u = v \\ \perp, & u \neq v \end{cases}$$

and it follows that $\overline{d}$ is the discrete $\mathcal{L}$-metric on $X \vee_p X$, that is, $\overline{\mathcal{H}} = \{\overline{d}\}$, which means $\mathcal{G}_{dis} = \mathbf{L\text{-}MET(X)}$. By Definition 3.4(ii), $(X, \mathcal{G})$ is T_1 at p. $\square$

Theorem 3.10. *Let $\mathcal{L}$ be a value quantale, and let $(X, \mathfrak{B})$ be an $\mathcal{L}$-approach space and $p \in X$. Then, the following are equivalent:*

(i) *$(X, \mathfrak{B})$ is T_1 at p.*

(ii) *for all $x \in X$ with $x \neq p$, there exists $d \in \mathcal{G}$ such that $d(x,p) = \perp = d(p,x)$.*

(iii) *for all $x \in X$ with $x \neq p$, $\delta(x, \{p\}) = \perp = \delta(p, \{x\})$.*

(iv) *for all $x \in X$ with $x \neq p$, there exists $\varphi \in \mathcal{A}(p)$ such that $\varphi(x) = \perp$ and there exists $\varphi \in \mathcal{A}(x)$ such that $\varphi(p) = \perp$.*

Proof. It is analogous to the proof of Theorem 3.8. $\square$

Remark 3.11. If $\mathcal{L} = ([0, \infty], \geq, +)$ (Lawvere's quantale), then local $\overline{T_0}$ (respectively, local T_1) $\mathcal{L}$-approach spaces are reduced to classical local $\overline{T_0}$ (respectively, local T_1) approach spaces defined in [12, 13].

Example 3.12. Let $\mathcal{L} = ([0,1], \leq, \cdot)$ be a triangular product norm. Let $X = \{a, b, c\}$, $A \subset X$ and $\delta : X \times 2^X \to ([0,1], \leq, \cdot)$ be a map defined by $\forall x \in X$, $\delta(x, \emptyset) = 0$, $\delta(x, A) = 1$ if $x \in A$, $\delta(b, \{a\}) = 0 = \delta(c, \{a\})$, $\delta(a, \{b\}) = 1/2 = \delta(a, \{b, c\})$, $\delta(c, \{b\}) = 1/3 = \delta(c, \{a, b\})$, $\delta(a, \{c\}) = 1/4$ and $\delta(b, \{c\}) = 1/5 = \delta(b, \{a, c\})$. Clearly, $\delta(x, A)$ is an $\mathcal{L}$-approach distance space. By Theorem 3.8, (X, δ) is $\overline{T_0}$ at $p = a$ but neither $\overline{T_0}$ at $p = b$ nor $p = c$. Similarly, by Theorem 3.10, (X, δ) is not T_1 at p, for all $p \in X$.

4 Closedness and D-connectedness

Let X be a set and p be a point in X. The *infinite wedge product* $\bigvee_p^\infty X$ is formed by taking countably many disjoint copies of X and identifying them at the point p.

A point x in $\bigvee_p^\infty X$ is denoted as x_i if it lies in the i-th component.

Definition 4.1. [3] Let $X^\infty = X \times X \times \cdots$ be the countable cartesian product of X.

(i) The *infinite principle axis map at* p, $A_p^\infty : \bigvee_p^\infty X \longrightarrow X^\infty$ is defined by $A_p^\infty(x_i) = (p, p, \ldots, p, x, p, \ldots)$.

(ii) The *infinite fold map at* p, $\nabla_p^\infty : \bigvee_p^\infty X \longrightarrow X^\infty$ is defined by $\nabla_p^\infty(x_i) = x$ for all $i \in I$.

Note that the map A_p^∞ is the unique map arising from the multiple pushout of $p : 1 \to X$ for which $A_p^\infty i_j = (p, p, \ldots, p, id, p, \ldots) : X \to X^\infty$, where the identity map, id, is in the j-th place [9].

Definition 4.2. Let $U : \mathcal{E} \longrightarrow$ **Set** be topological, $X \in Ob(\mathcal{E})$ with $U(X) = B$, and $p \in B$.

(i) $\{p\}$ is *closed* if and only if the initial lift of the U-source $\{A_p^\infty : \bigvee_p^\infty B \longrightarrow B^\infty$ and $\nabla_p^\infty : \bigvee_p^\infty B \longrightarrow UD(B^\infty) = B^\infty\}$ is discrete, where D is the discrete functor [3].

(ii) X is *D-connected* if and only if any morphism from X to any discrete object is constant [28].

Theorem 4.3. *Let $(X, \mathcal{G})$ be an $\mathcal{L}$-gauge space and $p \in X$. Then, $\{p\}$ is closed in X if and only if for all $x \in X$, with $x \neq p$, there exists $d \in \mathcal{G}$ such that $d(x,p) \wedge d(p,x) = \perp$.*

Proof. Let $(X, \mathcal{G})$ be an $\mathcal{L}$-gauge space, $p \in X$, and $\{p\}$ be closed in X. Let $\overline{\mathcal{G}}$ be the initial $\mathcal{L}$-gauge on $\bigvee_p^\infty X$ induced by $A_p^\infty : \bigvee_p^\infty X \to U(X^\infty, \mathcal{G}^*) = X^\infty$ and $\nabla_p^\infty : \bigvee_p^\infty X \to U(X, \mathcal{G}_{dis}) = X$, where $\mathcal{G}_{dis}$ is the discrete structure on X, and $\mathcal{G}^*$ be the product structure on X^∞ induced by $\pi_i : X^\infty \to X$ ($i \in I$) projection maps. Suppose that $\mathcal{H}_{dis} = \{d_{dis}\}$ is a basis for the discrete $\mathcal{L}$-gauge where d_{dis} is the discrete $\mathcal{L}$-metric on X. Let $\mathcal{H}$ be an $\mathcal{L}$-gauge basis of $\mathcal{G}$ and $d \in \mathcal{H}$, and $\overline{\mathcal{H}} = \{\overline{d}_{dis}\}$ be the initial $\mathcal{L}$-gauge basis of $\overline{\mathcal{G}}$, where $\overline{d}_{dis}$ is the discrete $\mathcal{L}$-metric on $\bigvee_p^\infty X$.

We will show that for all $x \in X$ with $x \neq p$, there exists $d \in \mathcal{G}$ such that $d(x,p) \wedge d(p,x) = \perp$. Suppose that $d(x,p) \wedge d(p,x) > \perp$ for all $d \in \mathcal{G}$ and $x \in X$ with $x \neq p$. For $i, j, k \in I$ with $i \neq j$ and $i \neq k \neq j$, note that

$$d_{dis}(\nabla_p^\infty(x_i), \nabla_p^\infty(x_j)) = d_{dis}(x,x) = \top$$

$$d(\pi_i A_p^\infty(x_i), \pi_i A_p^\infty(x_j)) = d(x,p)$$

$$d(\pi_j A_p^\infty(x_i), \pi_j A_p^\infty(x_j)) = d(p, x)$$
$$d(\pi_k A_p^\infty(x_i), \pi_k A_p^\infty(x_j)) = d(p, p) = \top.$$

Since $x_i \neq x_j$ $(i \neq j)$ and p is closed in X, by Lemma 2.19,

$$\begin{aligned} \overline{d}_{dis}(x_i, x_j) &= \bigwedge \{d_{dis}(\nabla_p^\infty(x_i), \nabla_p^\infty(x_j)), d(\pi_i A_p^\infty(x_i), \pi_i A_p^\infty(x_j)), \\ &\quad d(\pi_j A_p^\infty(x_i), \pi_j A_p^\infty(x_j)), d(\pi_k A_p^\infty(x_i), \pi_k A_p^\infty(x_j))\} \\ &= \bigwedge \{\top, d(x, p), d(p, x)\} \\ &= d(x, p) \wedge d(p, x) \\ &> \bot, \end{aligned}$$

which is a contradiction to the fact that $\overline{d}_{dis}$ is the discrete $\mathcal{L}$-metric on $\bigvee_p^\infty X$. Hence, $d(x, p) \wedge d(p, x) = \bot$.

Conversely, let $\overline{\mathcal{H}}$ be the initial $\mathcal{L}$-gauge basis on $\bigvee_p^\infty X$ induced by $A_p^\infty : \bigvee_p^\infty X \to U(X^\infty, \mathcal{G}^*) = X^\infty$ and $\nabla_p^\infty : \bigvee_p^\infty X \to U(X, \mathcal{G}_{dis}) = X$, where, by Proposition 2.13, $\mathcal{G}_{dis} = \mathbf{L\text{-}MET(X)}$ discrete $\mathcal{L}$-gauge on X and $\mathcal{G}^*$ be the product structure on X^∞ induced by $\pi_i : X^\infty \to X$ $(i \in I)$ projection maps. Suppose that for all $x \in X$ with $x \neq p$, there exists $d \in \mathcal{G}$ such that $d(x, p) \wedge d(p, x) = \bot$. Let $\overline{d} \in \overline{\mathcal{H}}$ and $u, v \in \bigvee_p^\infty X$. If $u = v$, then for $i \in I$,

$$\begin{aligned} \overline{d}(u, v) &= \bigwedge \{d_{dis}(\nabla_p^\infty(u), \nabla_p^\infty(u)), d(\pi_i A_p^\infty(u), \pi_i A_p^\infty(u))\} \\ &= \top. \end{aligned}$$

If $u \neq v$ and $\nabla_p^\infty(u) \neq \nabla_p^\infty(v)$, then $d_{dis}(\nabla_p^\infty(u), \nabla_p^\infty(v)) = \bot$ since d_{dis} is a discrete structure. By Lemma 2.19 for $i \in I$,

$$\begin{aligned} \overline{d}(u, v) &= \bigwedge \{d_{dis}(\nabla_p^\infty(u), \nabla_p^\infty(v)), d(\pi_i A_p^\infty(u), \pi_i A_p^\infty(v))\} \\ &= \bigwedge \{\bot, d(\pi_i A_p^\infty(u), \pi_i A_p^\infty(v))\} \\ &= \bot. \end{aligned}$$

Suppose that $u \neq v$ and $\nabla_p^\infty(u) = \nabla_p^\infty(v)$. If $\nabla_p^\infty(u) = x = \nabla_p^\infty(v)$ for some $x \in X$ with $x \neq p$, then $u = x_i$ and $v = x_j$ for $i, j \in I$ with $i \neq j$, since $u \neq v$. Let $u = x_i$, $v = x_j$ and $i, j, k \in I$ with $i \neq j$ and $i \neq k \neq j$. Then

$$\begin{aligned} d_{dis}(\nabla_p^\infty(u), \nabla_p^\infty(v)) &= d_{dis}(\nabla_p^\infty(x_i), \nabla_p^\infty(x_j)) \\ &= d_{dis}(x, x) \\ &= \top, \end{aligned}$$

$$\begin{aligned} d(\pi_i A_p^\infty(u), \pi_i A_p^\infty(v)) &= d(\pi_i A_p^\infty(x_i), \pi_i A_p^\infty(x_j)) \\ &= d(x, p), \end{aligned}$$

$$\begin{aligned} d(\pi_j A_p^\infty(u), \pi_j A_p^\infty(v)) &= d(\pi_j A_p^\infty(x_i), \pi_j A_p^\infty(x_j)) \\ &= d(p, x), \end{aligned}$$

and

$$\begin{aligned} d(\pi_k A_p^\infty(u), \pi_k A_p^\infty(v)) &= d(\pi_k A_p^\infty(x_i), \pi_k A_p^\infty(x_j)) \\ &= d(p, p) \\ &= \top. \end{aligned}$$

It follows that

$$\begin{aligned} \overline{d}(u, v) &= \bigwedge \{d_{dis}(\nabla_p^\infty(x_i), \nabla_p^\infty(x_j)), d(\pi_i A_p^\infty(x_i), \pi_i A_p^\infty(x_j)), \\ &\qquad d(\pi_j A_p^\infty(x_i), \pi_j A_p^\infty(x_j)), d(\pi_k A_p^\infty(x_i), \pi_k A_p^\infty(x_j))\} \\ &= \bigwedge \{\top, d(x, p), d(p, x)\} \\ &= \bigwedge \{d(x, p), d(p, x)\} \\ &= d(x, p) \wedge d(p, x). \end{aligned}$$

By the assumption, $d(x, p) \wedge d(p, x) = \perp$ and we have $\overline{d}(u, v) = \perp$. Therefore, $\forall\ u, v \in \bigvee_p^\infty X$, we get

$$\overline{d}(u, v) = \begin{cases} \top, & u = v \\ \perp, & u \neq v \end{cases}$$

and, by Proposition 2.13, $\overline{d}$ is the discrete $\mathcal{L}$-metric on $\bigvee_p^\infty X$, that is, $\overline{\mathcal{H}} = \{\overline{d}\}$, which means $\mathcal{G}_{dis} = \mathbf{L\text{-}MET(X)}$. By Definition 4.2 (i), $\{p\}$ is closed in X. □

Corollary 4.4. *Let $(X, \mathcal{G})$ be an $\mathcal{L}$-gauge space where $\mathcal{L}$ has a prime bottom element and $p \in X$. Then $\{p\}$ is closed in X if and only if for all $x \in X$, with $x \neq p$, there exists $d \in \mathcal{G}$ such that $d(x, p) = \perp$ or $d(p, x) = \perp$.*

Proof. It follows from Theorem 4.3 and the definition of the prime bottom element. □

Theorem 4.5. *Let $(X, \mathcal{G})$ be an $\mathcal{L}$-gauge space and $p \in X$. Then, $(X, \mathcal{G})$ is $\overline{T_0}$ at p if and only if $\{p\}$ is closed in X.*

Proof. It follows from Theorems 3.6 and 4.3. $\square$

Theorem 4.6. *An $\mathcal{L}$-gauge space $(X, \mathcal{G})$ is D-connected if and only if for each distinct points x and y in X, there exists $d \in \mathcal{G}$ such that $d(x, y) = \top = d(y, x)$.*

Proof. Let $(X, \mathcal{G})$ be an $\mathcal{L}$-gauge space, $(Y, \mathcal{G}_{dis})$ be a discrete $\mathcal{L}$-gauge space with $CardY > 1$ and $f\colon (X, \mathcal{G}) \to (Y, \mathcal{G}_{dis})$ be a contraction map. Suppose that $(X, \mathcal{G})$ is D-connected. Since f is a contraction map and by definition, for all $d_{dis} \in \mathcal{G}_{dis}$, $d_{dis} \circ (f \times f) \in \mathcal{G}$, that is, there exists $d \in \mathcal{G}$ such that $d = d_{dis}(f, f)$. It follows that for each distinct points x and y in X, $d(x, y) = d_{dis}(f(x), f(y))$. Since $(X, \mathcal{G})$ is D-connected, f is a constant map. Therefore, for $x, y \in X$, $f(x) = f(y) = \alpha \in Y$, and consequently,

$$d(x, y) = d_{dis}(f(x), f(y)) = d_{dis}(\alpha, \alpha) = \top,$$

and

$$d(y, x) = d_{dis}(f(y), f(x)) = d_{dis}(\alpha, \alpha) = \top.$$

Thus, if $(X, \mathcal{G})$ is D-connected, then there exists $d \in \mathcal{G}$ such that $d(x, y) = \top = d(y, x)$.

Conversely, suppose that the condition holds, that is, for each distinct points x and y in X, there exists $d \in \mathcal{G}$ such that $d(x, y) = \top = d(y, x)$. We show that $(X, \mathcal{G})$ is D-connected. Let $f\colon (X, \mathcal{G}) \to (Y, \mathcal{G}_{dis})$ be a contraction map and $(Y, \mathcal{G}_{dis})$ be a discrete $\mathcal{L}$-gauge space. If $CardY = 1$, then $(X, \mathcal{G})$ is D-connected, since f is a constant map. Suppose that $CardY > 1$ and f is not a constant map. Then, there exist distinct points x and y in X such that $f(x) \neq f(y)$ and consequently,

$$d(x, y) = d_{dis}(f(x), f(y)) = \perp,$$

and

$$d(y, x) = d_{dis}(f(y), f(x)) = \perp,$$

which is a contradiction, since $d(x, y) = \top = d(y, x)$ for $x, y \in X$ with $x \neq y$. Hence, f is a constant map. By Definition 4.2(ii), $(X, \mathcal{G})$ is D-connected. $\square$

Theorem 4.7. *Let $\mathcal{L}$ be a value quantale and let $(X, \mathfrak{B})$ be an $\mathcal{L}$-approach space. Then, the following are equivalent:*

(i) *$(X, \mathfrak{B})$ is D-connected.*

(ii) *for all $x, y \in X$ with $x \neq y$, there exists $d \in \mathcal{G}$ such that $d(x, y) = \top = d(y, x)$.*

(iii) *for all $x, y \in X$ with $x \neq y$, $\delta(x, \{y\}) = \top = \delta(y, \{x\})$.*

(iv) *for all $x, y \in X$ with $x \neq y$, there exists $\varphi \in \mathcal{A}(x)$ such that $\varphi(y) = \top$ and there exists $\varphi \in \mathcal{A}(x)$ such that $\varphi(y) = \top$.*

Proof. (i) $\Leftrightarrow$ (ii) follows from Theorem 4.6. Using Remark 2.20, the proof of the other implications are straightforward. □

Example 4.8. Suppose that $\mathcal{L} = ([0, 1], \leq, *)$ is a Lukasiewicz t-norm. Let X be a non-empty set, and let $\delta : X \times 2^X \to \mathcal{L} = (L, \leq, *)$ be a map defined by for all $x \in X$ and $A \subseteq X$,

$$\delta(x, A) = \begin{cases} 1, & A \neq \emptyset \\ 0, & A = \emptyset \end{cases}$$

By Theorem 4.7, (X, δ) is D-connected.

Remark 4.9. (i) In **Top** as well as in **CHY** (the category of Cauchy spaces and Cauchy continuous maps) T_1 at p (that is, local T_1) and the notion of closedness are equivalent [5, 27], and T_1 at p implies $\overline{T_0}$ at p. However, in **L-AP**, by Theorem 4.5, $\overline{T_0}$ at p and the notion of closedness are equivalent, and by Example 3.12, T_1 at p implies $\overline{T_0}$ at p, but the converse is not true in general.

(ii) In **CP** (the category of pairs and pair preserving maps) and in **Prox** (the category of proximity spaces and proximity maps), $\overline{T_0}$ at p, T_1 at p and the notion of closedness are equivalent [3, 28]. Moreover, in ∞**psMet** (the category of extended pseudo semi metric spaces and non-expansive maps) (see [30]) $\overline{T_0}$ at p and T_1 at p are the only discrete objects at p, that is, for all $x \in X$ with $x \neq p$, $d(x, p) = \infty$ [14].

(iii) By Theorems 3.4, 4.5 and 4.14 of [28], and by Examples 3.12 and 4.8, there is no relation between the notion of closedness and D-connected objects, T_1 at p and D-connected objects.

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