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Some aspects of cosheaves on diffeological spaces

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Abstract. We define a notion of cosheaves on diffeological spaces by cosheaves on the site of plots. This provides a framework to describe diffeological objects such as internal tangent bundles, the Poincaré groupoids, and furthermore, homology theories such as cubic homology in diffeology by the language of cosheaves. We show that every cosheaf on a diffeological space induces a cosheaf in terms of the D-topological structure. We also study quasi-cosheaves, defined by pre-cosheaves which respect the colimit over covering generating families, and prove that cosheaves are quasi-cosheaves. Finally, a so-called quasi-Čech homology with values in pre-cosheaves is established for diffeological spaces.

1 Introduction

Diffeology was introduced by J.M. Souriau [19] in the 1980's, one of the set-based generalizations of smooth manifolds (see [20]) by focusing on smooth maps from open subsets of Euclidean spaces, so-called plots. Diffeology systematizes geometric spaces such as orbifolds and even infinite-dimensional

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spaces. The category of diffeological spaces and smooth maps between them is complete, cocomplete, and cartesian closed. As it is shown in [2], this category forms a quasitopos. It is also closed under constructions such as subspaces and quotient spaces. The main reference for this theory is the book [13] by P. Iglesias-Zemmour.

Sheaves and cosheaves are robust tools to study local information on sites (categories with Grothendieck topologies), given by functors that preserve (co)limits over coverings. Sheaves and quasi-sheaves on diffeological spaces were introduced by the authors [9] with respect to the *site of plots* and *covering generating families*, respectively, to study relations between data on spaces and those on plots. In this paper, we investigate cosheaves on diffeological spaces, defined as cosheaves on the site of plots (Definition 3.2). The purpose is to exhibit the applications of cosheaves in diffeology and to show how naturally diffeological objects and structures appear in this framework by the use of the cosections of cosheaves. For example, internal tangent bundles and path-connected components are cosections of cosheaves (see Examples 3.6 and 3.7). In addition, we describe the Poincaré groupoids in the context of cosheaves (Proposition 4.1) in Section 4. This result may be considered as the counterpart of the van Kampen's Theorem in diffeology.

As another application, in Section 5 we explain the cubic homology by a chain complex of pre-cosheaves (Proposition 5.2). In this manner, one can suggest a version of other homology theories defined on manifolds for diffeological spaces. These facts demonstrate that cosheaf tools unify and simplify the nature of such contravariant objects in diffeology.

In Section 6, the relationship between cosheaves and the D-topological structure of a diffeological space are studied. While a cosheaf on a diffeological space, is in essence, nothing more than an assignment of ordinary cosheaves to plots, we prove that every cosheaf on a diffeological space gives rise to a cosheaf with respect to the D-topology of the space (Theorem 6.2).

Covering generating families play a central role in diffeology, families of plots in a diffeological space that generate the whole diffeology, and data on a diffeological space are given over its covering generating families. In Section 7, we define quasi-cosheaves on diffeological spaces, a dual notion to quasi-sheaves, by pre-cosheaves respecting the colimit over covering generating families (Definition 7.3). We prove that every cosheaf is a quasi-cosheaf (Theorem 7.4). In other words, cosections of a cosheaf are recognizable by the data over covering generating families. However, not every quasi-cosheaf is a cosheaf on diffeological spaces (see Example 7.6).

When we deal with pre-cosheaves of abelian groups, it is natural to talk about homology. Čech homology in topology is an extension of the (global) cosection functor. In the context of diffeology, a (pre)-cosheaf only consider local data over each plot. We also need data about the whole space. In Subsection 7.1, a so-called quasi-Čech homology is associated with pre-cosheaves of abelian groups, which extends the cosection functor of quasi-cosheaves (Proposition 7.15). Quasi-Čech homology can be regarded as the diffeological counterpart of Čech homology. In fact, this provides a combinatorial approach to determine further data on a diffeological space form given data over covering generating families.

2 Preliminaries

In this section we give some definition we need in the sequel (see [13] for more details).

Definition 2.1. An *n-domain*, for a nonnegative integer n, is an open subset of Euclidean space $\mathbb{R}^n$ with the standard topology. All n-domains, n ranges over nonnegative integers, together with smooth maps between them define a category denoted by $\mathsf{Domains}$. Objects in $\mathsf{Domains}$ are called domains.

Definition 2.2. Any map from a domain to a set X is said to be a *parametrization* in X. If the domain of definition of a parametrization P, denoted by $\mathrm{dom}(P)$, is an n-domain, P is an n-parametrization. The only 0-parametrization with the value $x \in X$ is denoted by the bold letter $\mathbf{x}$. A family $\{P_i : U_i \to X\}_{i \in J}$ of n-parameterizations is *compatible* if $P_i|_{U_i \cap U_j} = P_j|_{U_i \cap U_j}$, for all $i, j \in J$. Given such a family, the parametrization $P : \bigcup_{i \in J} U_i \to X$ defined by $P(r) = P_i(r)$ for $r \in U_i$, is called the *supremum* of this family. By convention, the supremum of the empty family is the empty parametrization $\varnothing \to X$.

Definition 2.3. A *diffeology* $\mathcal{D}$ on a set X is a set of parameterizations in X with the following axioms:

D1. The union of the images of the elements of $\mathcal{D}$ covers X.

D2. For every element $P : U \to X$ of $\mathcal{D}$ and every smooth map $F : V \to U$ between domains, the parametrization $P \circ F$ belongs to $\mathcal{D}$.

D3. The supremum of any compatible family of elements of $\mathcal{D}$ is also belongs to $\mathcal{D}$.

Definition 2.4. A *prediffeology* on a set X is a set $\mathcal{P}$ of parameterizations in X satisfying D1 and D2. A *parametrized cover* of X is a set $\mathcal{C}$ of parameterizations in X satisfying D1.

Definition 2.5. A *diffeological space* $(X, \mathcal{D})$ is an underlying set X equipped with a diffeology $\mathcal{D}$, whose elements are called the *plots* in X. A diffeological space is just denoted by the underlying set, when the diffeology is understood.

The axioms of diffeology all together imply that in any diffeological space, the locally constant parametrizations are plots.

Definition 2.6. Let X and Y be two diffeological spaces. A map $f : X \to Y$ is *smooth* if for every plot P in X, the composition $f \circ P$ is a plot in the space Y. The set of all smooth maps from X to Y is denoted by $\mathrm{C}^\infty(X, Y)$. We denote by Diff the category of diffeological spaces and smooth maps. The isomorphisms in the category Diff are called *diffeomorphisms*.

Example 2.7. Any smooth manifold has a *standard diffeology* by thinking of smooth parameterizations as plots. A map between manifolds is smooth in the usual sense if and only if it is smooth in diffeological sense. In other words, the category of smooth manifolds is a full subcategory of diffeological spaces. In particular, $\mathsf{Domains}$ is a full subcategory of Diff.

Definition 2.8. Let X be a diffeological space. A *diffeological subspace* of X is a subset $X' \subseteq X$ equipped with the *subspace diffeology*, which is the set of all plots in X with values in X'. In this situation, the inclusion map $X' \hookrightarrow X$ is smooth.

Definition 2.9. The *functional diffeology* on the set of all smooth maps from X to Y, $\mathrm{C}^\infty(X, Y)$, is given by the following condition: A parametrization $Q : V \to \mathrm{C}^\infty(X, Y)$ is a plot for the functional diffeology if and only if for every plot $P : U \to X$, the parametrization $Q.P : V \times U \to Y$ with $(Q.P)(r, s) = Q(r)\big(P(s)\big)$ is a plot in Y.

Definition 2.10. A subset U of a diffeological space X is D-*open* if $P^{-1}(U)$ is open for all plots P in X. D-open subsets of X constitute a topology, which is called the D-*topology* on X. In this situation, every smooth map is continuous.

3 Cosheaves on diffeological spaces

Cosheaves on Grothendieck sites are standard and well known (see, e.g. [18]). In this section, we work with cosheaves on the site of plots and give some examples.

Site of plots. ([9]) The *category of the plots* in a diffeological space X, which we denote it by $\mathsf{Plots}(X)$, has the plots in X for objects and a morphism $Q \xrightarrow{F} P$ between two plots $P : U \to X$ and $Q : V \to X$ is a commutative triangle

$$\begin{tikzcd} & X & \\ V \arrow[ur, "Q"] \arrow[rr, "F"'] & & U \arrow[ul, "P"'] \end{tikzcd}$$

where F is a smooth map between domains (see [6]). If $P' : U' \to X$ is a restriction of $P : U \to X$, the inclusion $\imath : U' \hookrightarrow U$ gives the inclusion morphism $P' \overset{\imath}{\hookrightarrow} P$. In particular, for a compatible family $\{P_i\}_{i\in J}$ of plots with the supremum P, one has the inclusion morphisms $P_i \hookrightarrow P$. For such a family, let $\mathsf{E}_J = \{P_i \times_P P_j\}_{(i,j)\in J\times J}$, which is a compatible family with the supremum P. Note that every $P_i = P_i \times_P P_i$ belongs to E_J. Consider E_J as a subcategory of $\mathsf{Plots}(X)$ with $P_i \times_P P_j$ for objects and the inclusions $P_i \hookleftarrow P_i \times_P P_j \hookrightarrow P_j$ for morphisms, and let $e_J : \mathsf{E}_J \to \mathsf{Plots}(X)$ be the canonical functor.

The category of the plots in a diffeological space X is endowed with a Grothendieck pretopology in which a covering for a plot P is a compatible family of plots with the supremum P. This site is called the *site of plots* in X and denoted by X_{Plots}.

Definition 3.1. A *pre-cosheaf* S on a diffeological space X with values in a cocomplete category D is a functor $S : \mathsf{Plots}(X) \to \mathsf{D}$. We denote the corresponding morphism to $Q \xrightarrow{F} P$ by $F_* : S(Q) \to S(P)$ and call it the *pushforward* by F. We denote by $s \downharpoonleft P$ the pushforward of s by an inclusion $P' \overset{\imath}{\hookrightarrow} P$, for $s \in S(P')$.

Definition 3.2. A *cosheaf* S on a diffeological space X is a cosheaf on the site X_{Plots}, meaning that S is a pre-cosheaf on X such that the sequence

$$\coprod_{(i,j)\in J\times J} S(P_i \times_P P_j) \rightrightarrows \coprod_{i\in J} S(P_i) \longrightarrow S(P)$$

is a coequalizer, for every plot P in X and every compatible family $\{P_i\}_{i\in J}$ of plots with the supremum P, or equivalently, the canonical morphism $\eta_J : \varinjlim S \circ e_J \to S(P)$ is an isomorphism.

Remark 3.3. The definition implies that every cosheaf S on a diffeological space X assigns to the empty plot $\varnothing \to X$ the initial object.

Definition 3.4. A *morphism* $\phi : S \to S'$ of (pre-)cosheaves on a diffeological space X is a natural transformation of functors.

Denote the category of pre-cosheaves and cosheaves on a diffeological space X by $\mathsf{PreCoshv}(X)$ and $\mathsf{Coshv}(X)$, respectively.

Definition 3.5. We denote the colimit of a pre-cosheaf S on a diffeological space X by $\Gamma S(X)$ and call it the *cosections* of S.

For every plot P in X, let $P_* : S(P) \to \Gamma S(X)$ denote the morphism in the definition of the colimit of S. Hence we can write $Q_* = P_* \circ F_*$, for morphisms $Q \xrightarrow{F} P$ of plots. By the universal property of pre-cosheaves $\phi : S \to S'$ on a diffeological space X induces a unique morphism $\Gamma\phi : \Gamma S(X) \to \Gamma S'(X)$ between cosections with the property that $\Gamma\phi \circ P_* = P_{*'} \circ \phi_P$, where $P_{*'} : S'(P) \to \Gamma S'(X)$ is the morphism in the definition of the colimit of S'.

Example 3.6. Let DVS denote the category of diffeological vector spaces over diffeological spaces [7, Definition 4.5] and let VSD denote the category of vector spaces with diffeology over diffeological spaces. The category DVS is a full subcategory VSD (see [7, Subsection 4.2]).

For a diffeological space X, one can see that the pre-cosheaf $T : \mathsf{Plots}(X) \to \mathsf{DVS}$ defined by

$$\begin{tikzcd} & X & \\ V \arrow[ur, "Q"] \arrow[rr, "F"'] & & U \arrow[ul, "P"'] \end{tikzcd} \longmapsto \begin{tikzcd} TV \arrow[r, "TF"] \arrow[d, "\pi_V"'] & TU \arrow[d, "\pi_U"] \\ V \arrow[r, "F"'] & U \end{tikzcd}$$

is a cosheaf on X. Moreover, by [7, Theorem 4.17], the cosections of T is exactly the internal tangent bundle $\pi_X : T^{dvs}(X) \to X$.

If one considers the functor T into the category VSD, another example of cosheaves is obtained and again by [7, Theorem 4.17], the cosections of T is the Hector's tangent bundle $\pi_X : T^H(X) \to X$.

Example 3.7. The assignment to each diffeological space X, the set $\pi_0(X)$ of its components and to each smooth map $f : X \to X'$ the induced map $f_* : \pi_0(X) \to \pi_0(X')$ with $f_* \circ \mathsf{comp}_X = \mathsf{comp}_{X'} \circ f$ defines a functor π_0 from Diff to Set (see [13, art. 5.9]). The pre-cosheaf on a diffeological space X, which associates to each plot P in X, the connected components of $\mathrm{dom}(P)$, $\pi_0(\mathrm{dom}(P))$ and to each $Q \xrightarrow{F} P$, the induced map $F_* : \pi_0(\mathrm{dom}(Q)) \to \pi_0(\mathrm{dom}(P))$ is a cosheaf, and the set of its cosections is the same as $\pi_0(X)$.

4 The Poincaré groupoids as cosheaves

We now intend to describe the Poincaré groupoids as an interesting example of cosheaves on diffeological spaces. We begin with smooth paths.

A path in a diffeological space X is any smooth map from $\mathbb{R}$ to X. Let $\mathsf{Paths}(X)$ denote the set of all paths in X equipped with the functional diffeology.

The *smashing function* [13, art. 5.5] is given by an increasing smooth function $\lambda : \mathbb{R} \to \mathbb{R}$ with $\lambda|_{(-\infty,\epsilon)} = 0$ and $\lambda|_{(1-\epsilon,\infty)} = 1$, where $0 < \epsilon < 1$ is a fixed real number. One can construct a concrete example of such a function (see [10, p. 31]): Consider the bump function

$$f(t) = \begin{cases} e^{\frac{-1}{t}}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

and let

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}, \quad t \in \mathbb{R}.$$

For some $0 < \epsilon < \frac{1}{2}$, let

$$\mu(t) = \frac{1}{1-2\epsilon}(t-\epsilon), \quad t \in \mathbb{R}.$$

Now $\lambda = g \circ \mu$ is a smashing function.

λ is fixed-ends homotopic to the identity $1_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$. For every path γ in X, $\gamma^\star = \gamma \circ \lambda$ is a stationary path fixed-ends homotopic to γ. If γ and γ' are juxtaposable paths in X, the *smashed concatenation* is defined by $\gamma \star \gamma' = \gamma^\star \vee \gamma'^\star$, where $\vee$ denotes the usual concatenation of paths.

Recall from [13, art. 5.15] that the Poincaré groupoid $\mathbf{X}$ of a diffeological space X has points of X for objects and fixed-ends homotopy classes of paths for morphisms. The composition in the Poincaré groupoids is the projection of the smashed concatenation of paths, and the inverse of a class of paths is the class of the reverse of one of paths. This gives rise to a functor from Diff to Gpd taking any diffeological space X to its corresponding Poincaré groupoid $\mathbf{X}$ and any smooth map $f : X \to Y$ to the functor $\mathbf{f}_* : \mathbf{X} \to \mathbf{Y}$ with $\mathbf{f}_*(x) = f(x)$ on objects and $\mathbf{f}_*(\mathsf{class}(\gamma)) = \mathsf{class}(f \circ \gamma)$ on morphisms.

With a similar argument to the van Kampen's Theorem for fundamental groupoids of topological spaces (see, e.g., [17]), the pre-cosheaf $\prod_1 : \mathsf{Plots}(X) \to \mathsf{Gpd}$ assigning to each plot P, the Poincaré groupoid $\prod_1(P)$ of $\mathrm{dom}(P)$ and to each $Q \xrightarrow{F} P$ the functor $\mathbf{F}_* : \prod_1(Q) \to \prod_1(P)$ is a cosheaf on X.

Proposition 4.1. *The groupoid of cosections of the cosheaf $\prod_1$ is the Poincaré groupoid $\boldsymbol{X}$ of a diffeological space X.*

Proof. First of all,

$$\begin{tikzcd} & \mathbf{X} & \\ \prod_1(Q) \arrow[ur, "\mathbf{Q}_*"] \arrow[rr, "\mathbf{F}_*"'] & & \prod_1(P) \arrow[ul, "\mathbf{P}_*"'] \end{tikzcd}$$

is a cocone. To verify the universal property, let $\varphi : \prod_1 \Rightarrow \mathbf{Y}$ be another cocone. Define the functor $u : \mathbf{X} \to \mathbf{Y}$ with $u(x) = \varphi_{\mathbf{x}}(0)$ on objects, $\mathbf{x}$ is the 0-plot corresponding to x. Note that $u(x) = \varphi_P(r)$ for any plot P with $P(r) = x$, for some $r \in \mathrm{dom}(P)$, by the naturality of φ. On morphisms, define $u\big(\mathsf{class}(\gamma)\big) = \varphi_\gamma\big(\mathsf{class}(\lambda)\big)$, where λ is the smashing function. To see that the definition is independent of the choice of paths, assume that γ' is fixed-ends homotopic to γ, $\gamma(0) = x = \gamma'(0)$ and $\gamma(1) = x' = \gamma'(1)$, through a path $H : \mathbb{R} \to \mathsf{Paths}(X)$, equivalently, a smooth map $\overline{H} : \mathbb{R}^2 \to X$. Let $a, b, c, d : \mathbb{R} \to \mathbb{R}^2$ be the paths defined by $a(t) = (0, t), b(t) = (1, t), c(t) = (t, 0), d(t) = (1 - t, 1)$, respectively, so that $\overline{H} \circ a = \gamma$, $\overline{H} \circ b = \gamma'$, $\overline{H} \circ c = \widetilde{\mathbf{x}}$ and $\overline{H} \circ d = \widetilde{\mathbf{x}'}$, where $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{x}'}$ denote the paths with the constant values x and x'. Let $\alpha := c \star b \star d$, the smashed concatenation of c, b, d. Then α is fixed-ends homotopic to a and we can write

$$
\begin{aligned}
\varphi_\gamma(\mathsf{class}(\lambda)) &= \varphi_{\overline{H}\circ a}(\mathsf{class}(1_\mathbb{R})) \\
&= \varphi_{\overline{H}}(\mathsf{class}(a)) \\
&= \varphi_{\overline{H}}(\mathsf{class}(\alpha)) \\
&= \varphi_{\overline{H}}(\mathsf{class}(c \star b \star d)) \\
&= \varphi_{\overline{H}}(\mathsf{class}(c)) \circ \varphi_{\overline{H}}(\mathsf{class}(b)) \circ \varphi_{\overline{H}}(\mathsf{class}(d)) \\
&= \varphi_{\overline{H}\circ c}(\mathsf{class}(1_\mathbb{R})) \circ \varphi_{\overline{H}\circ b}(\mathsf{class}(1_\mathbb{R})) \circ \varphi_{\overline{H}\circ d}(\mathsf{class}(1_\mathbb{R})) \\
&= \varphi_{\widetilde{\mathbf{x}}}(\mathsf{class}(1_\mathbb{R})) \circ \varphi_{\gamma'}(\mathsf{class}(1_\mathbb{R})) \circ \varphi_{\widetilde{\mathbf{x}'}}(\mathsf{class}(1_\mathbb{R})) \\
&= \varphi_{\gamma'}(\mathsf{class}(\lambda)).
\end{aligned}
$$

In the last equality, we used the fact that

$$u(1_x) = u(\mathsf{class}(\widetilde{\mathbf{x}})) = \varphi_{\widetilde{\mathbf{x}}}(\mathsf{class}(1_\mathbb{R})) = \varphi_{\mathbf{x}}(\mathsf{class}(\widetilde{\mathbf{0}})) = 1_{u(x)},$$

by the naturality of φ for the morphism $\widetilde{\mathbf{x}} \to \mathbf{x}$, where $\widetilde{\mathbf{0}}$ is the only path in $\mathbb{R}^0$. We now prove that u preserves the compositions. Consider the functions $v(t) = \frac{1}{2}t$ and $w(t) = \frac{1}{2}(t + 1)$ on $\mathbb{R}$. Then $v \star w$ is equal to λ. One can observe that $(\gamma \star \gamma') \circ v$ and $(\gamma \star \gamma') \circ w$ are fixed-ends homotopic to γ and γ', respectively. So we have

$$
\begin{aligned}
u(\mathsf{class}(\gamma) \circ \mathsf{class}(\gamma')) &= u(\mathsf{class}(\gamma \star \gamma') \\
&= \varphi_{\gamma\star\gamma'}(\mathsf{class}(\lambda)) \\
&= \varphi_{\gamma\star\gamma'}(\mathsf{class}(v \star w)) \\
&= \varphi_{\gamma\star\gamma'}(\mathsf{class}(v)) \circ \varphi_{\gamma\star\gamma'}(\mathsf{class}(w)) \\
&= \varphi_{(\gamma\star\gamma')\circ v}(\mathsf{class}(1_\mathbb{R})) \circ \varphi_{(\gamma\star\gamma')\circ w}(\mathsf{class}(1_\mathbb{R})) \\
&= \varphi_\gamma(\mathsf{class}(\lambda)) \circ \varphi_{\gamma'}(\mathsf{class}(\lambda)) \\
&= u(\mathsf{class}(\gamma)) \circ u(\mathsf{class}(\gamma')).
\end{aligned}
$$

It is easy to check that $u \circ \mathbf{P}_* = \varphi_P$ for every plot P in X and that u is unique with this property. $\square$

5 Homology theories in (pre-)cosheaf framework

In this section, some classical homology theories are exhibited in the context of cosheaves of abelian groups.

By Definition 3.2, a cosheaf of abelian groups on a diffeological space X is a pre-cosheaf A for which the sequence

$$\bigoplus_{(i,j)\in J\times J} A(P_i \times_P P_j) \xrightarrow{g} \bigoplus_{i\in J} A(P_i) \xrightarrow{f} A(P) \longrightarrow 0$$

is exact, for any plot P and any compatible family $\{P_i\}_{i\in J}$ of plots with the supremum P, where

$$g\Big(\sum_{(i,j)\in J\times J} s_{ij}\Big) = \sum_{(i,j)\in J\times J} s_{ij} \downharpoonleft P_i - s_{ij} \downharpoonleft P_j, \qquad \text{and}$$
$$f\Big(\sum_{i\in J} s_i\Big) = \sum_{i\in J} s_i \downharpoonleft P.$$

A description of the group $\Gamma A(X)$ of cosections of a pre-cosheaf A of abelian groups on X is as the quotient group $\bigoplus_{P\in\mathcal{D}} A(P)/\Lambda_X$, where Λ_X is the subgroup generated by the elements in the form $F_*(s)-G_*(s)$, for morphisms $R \xleftarrow{G} Q \xrightarrow{F} P$ of plots in X and for $s \in A(Q)$.

Definition 5.1. Let X be a diffeological space. A *chain complex* $(A_\bullet, \partial)$ of pre-cosheaves of abelian groups on X is a sequence of pre-cosheaves and morphisms

$$\cdots \longrightarrow A_{k+1} \xrightarrow{\partial} A_k \xrightarrow{\partial} A_{k-1} \longrightarrow \cdots,$$

with $\partial \circ \partial = 0$.

In this situation, we have chain complexes $(A_\bullet(P), \partial_p)$ for all plots P in X. The morphism $\partial_k : A_k \to A_{k-1}$ is called the *kth boundary operator*. Because the boundary operators are natural transformation, the assignment

$$H_k(A_\bullet) : P \longmapsto H_k(A_\bullet(P)),$$

is a pre-cosheaf, which we call it the *kth homology pre-cosheaf* of the chain complex $(A_\bullet, \partial)$. Moreover, a chain complex $(A_\bullet, \partial)$ induces an associated chain complex $(\Gamma A_\bullet, \Gamma\partial)$ of groups of cosections

$$\cdots \longrightarrow \Gamma A_{k+1}(X) \xrightarrow{\Gamma\partial} \Gamma A_k(X) \xrightarrow{\Gamma\partial} \Gamma A_{k-1}(X) \longrightarrow \cdots,$$

where $\Gamma\partial(s_P + \Lambda_k) = \partial_P\big(s_P\big) + \Lambda_{k-1}$ and Λ_k is described as above, for $s_P \in A_k(P)$. Let $H_k(\Gamma A_\bullet)$ denote the kth homology group of the chain complex $(\Gamma A_\bullet, \Gamma\partial)$. Since the homomorphisms $P_\# : H_k(A_\bullet(P)) \to H_k(\Gamma A_\bullet)$ induced by the chain maps $P_* : A_k(P) \to \Gamma A_k(X)$ construct a cocone, universal property gives us a unique homomorphism

$$\Theta : \Gamma H_k(A_\bullet)(X) \to H_k(\Gamma A_\bullet)$$

such that $\Theta \circ \varphi_P = P_\#$ for every plot P in X, where $\varphi_P : H_k(A_\bullet)(P) \to \Gamma H_k(A_\bullet)(X)$ is the morphism given by the colimit of $H_k(A_\bullet)$.

5.1 The cubic homology Now, we describe the cubic homology in this framework. Let us first review the cubic homology of diffeological spaces from [13].

Let X be a diffeological space. A (smooth) *k-cube* in X is any smooth map from $\mathbb{R}^k$ to X, denoted by $\mathrm{Cub}_k(X)$. Denoted by $\mathbf{C}_k(X)$ the free abelian group generated by $\mathrm{Cub}_k(X)$ and call the elements of $\mathbf{C}_k(X)$ *cubic k-chains* in X with coefficients in $\mathbb{Z}$. A *reduction* from $\mathbb{R}^k$ to $\mathbb{R}^l$ is any projection $\mathrm{Pr} : \mathbb{R}^k \to \mathbb{R}^l$ with $\mathrm{Pr}(r_1, \ldots, t_k) = (r_{i_1}, \ldots, r_{i_l})$, where $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, k\}$ is a subset of indices, $i_1 < \cdots < i_l$. A k-cube σ is *degenerate* if $\sigma = \sigma' \circ \mathrm{Pr}$, for some l-cube σ' and a reduction Pr from $\mathbb{R}^k$ to $\mathbb{R}^l$, for some integer l. The set of degenerate k-cubes in X is denoted by $\mathrm{Cub}_k^\bullet(X)$ and the free abelian group generated by $\mathrm{Cub}_k^\bullet(X)$ is denoted by $\mathbf{C}_k^\bullet(X)$. The quotient $\mathsf{C}_k(X) = \mathbf{C}_k(X)/\mathbf{C}_k^\bullet(X)$ of the group of cubic k-chains of X by the subgroup of degenerate k- chains is called the *reduced group of cubic k-chains* of X.

Any smooth map $f : X \to Y$ induces a homomorphism

$$f_\# : \mathbf{C}_k(X) \longrightarrow \mathbf{C}_k(Y) \qquad \text{with} \qquad f_\#(\textstyle\sum_\sigma n_\sigma \sigma) = \sum_\sigma n_\sigma f \circ \sigma$$

between groups of cubic k-chains. Since $f_\#$ preserves degenerate cubic k-chains, a homomorphism $f_* : \mathsf{C}_k(X) \longrightarrow \mathsf{C}_k(Y)$ between reduced groups of cubic k-chains is obtained. This defines a functor from Diff to the category

Ab of abelian groups. There exists also a boundary operator $\partial_X : \mathsf{C}_k(X) \longrightarrow \mathsf{C}_{k-1}(X)$ satisfying the homological condition $\partial_X \circ \partial_X = 0$ (see [13, art. 6.60]) and that $\partial_Y \circ f_* = f_* \circ \partial_X$. This gives rise to a chain complex and consequently, *the cubic homology* $H_\bullet(X)$ of the space X.

Now for every nonnegative integer k, consider the pre-cosheaf $\mathsf{C}_k : \mathsf{Plots}(X) \to \mathsf{Ab}$ assigning to every plot P, $\mathsf{C}_k(P) := \mathsf{C}_k(\mathrm{dom}(P))$ the reduced group of cubic k-chains on the domain of P, and to every $Q \xrightarrow{F} P$, the homomorphism $F_* : \mathsf{C}_k(\mathrm{dom}(Q)) \to \mathsf{C}_k(\mathrm{dom}(P))$ between reduced groups of cubic k-chains. Let $\partial : \mathsf{C}_k \to \mathsf{C}_{k-1}$ be the morphism of pre-cosheaves consists of the boundary operators $\partial_P : \mathsf{C}_k(P) \to \mathsf{C}_{k-1}(P)$ on domains of plots. Then $(\mathsf{C}_\bullet, \partial)$ is a chain complex of pre-cosheaves.

Proposition 5.2. *The associated chain complex* $(\Gamma\mathsf{C}_\bullet, \Gamma\partial)$ *of groups of co-sections is the same as the chain complex* $(\mathsf{C}_\bullet(X), \partial_X)$ *of cubics on* X*, and hence the associated homology of* $(\Gamma\mathsf{C}_\bullet, \Gamma\partial)$ *coincides with the cubic homology of diffeological space* X*.*

Proof. Let us show that $\mathsf{C}_k(X)$ is the colimit of the functor C_k, for every nonnegative integer k. It is clear that

$$\begin{tikzcd} & \mathsf{C}_k(X) & \\ \mathsf{C}_k(Q) \arrow[ur, "Q_*"] \arrow[rr, "F_*"'] & & \mathsf{C}_k(P) \arrow[ul, "P_*"'] \end{tikzcd}$$

is a cocone. Suppose that $\varphi : \mathsf{C}_k \Rightarrow C$ is another cocone. Define

$$h : \mathsf{C}_k(X) \to C \qquad \text{by}$$
$$h\big(\mathsf{coset}(\textstyle\sum_\sigma n_\sigma \sigma)\big) = \textstyle\sum_\sigma n_\sigma \varphi_\sigma\big(\mathsf{coset}(1_{\mathbb{R}^k})\big).$$

Notice that $1_{\mathbb{R}^k}$ is a k-cube in $\mathrm{dom}(\sigma)$. If σ is a degenerate k-cube, $\sigma = \sigma' \circ \mathrm{Pr}$ as above, then

$$\varphi_\sigma\big(\mathsf{coset}(1_{\mathbb{R}^k})\big) = \varphi_{\sigma'} \circ \mathrm{Pr}_*\big(\mathsf{coset}(1_{\mathbb{R}^k})\big) = \varphi_{\sigma'}\big(\mathsf{coset}(\mathrm{Pr})\big) = 0.$$

So h is well-defined. It is easy to see that h is a unique homomorphism with $h \circ P_* = \varphi_p$ for plots P in X. Thus, $\mathsf{C}_k(X)$ is the colimit of C_k.

This gives another description of $\mathsf{C}_k(X)$. That is, $\mathsf{C}_k(X)$ is isomorphic to $\Gamma\mathsf{C}_k(X) = \bigoplus_{P\in\mathcal{D}} \mathsf{C}_k(P)/\Lambda_k$ by the isomorphism $h : \mathsf{C}_k(X) \to \Gamma\mathsf{C}_k(X)$, given by $h\big(\mathsf{coset}(\sum_\sigma n_\sigma \sigma)\big) = \sum_\sigma n_\sigma \big(\mathsf{coset}(1_{\mathbb{R}^k})\big) + \Lambda_k$ according to the discussion above, where Λ_k is the subgroup generated by the elements in the form $F_*(c) - G_*(c)$, for morphisms $R \xleftarrow{G} Q \xrightarrow{F} P$ of plots in X and $c \in \mathsf{C}_k(Q)$. The following diagram is commutative.

$$\begin{tikzcd}
\mathsf{C}_k(X) \arrow[r, "h"] \arrow[d, "\partial_X"'] & \Gamma\mathsf{C}_k(X) \arrow[d, "\Gamma\partial"] \\
\mathsf{C}_{k-1}(X) \arrow[r, "h"'] & \Gamma\mathsf{C}_{k-1}(X)
\end{tikzcd}$$

Therefore, the chain complexes $(\mathsf{C}_\bullet(X), \partial_X)$ and $(\Gamma\mathsf{C}_\bullet, \Gamma\partial)$ are the same. □

5.2 Čech homology One approach toward Čech homology on diffeological spaces can be considering them as D-topological spaces and the use of open coverings. However, here we intend to see this homology theory by pre-cosheaves. Let X be a diffeological space, A be a pre-cosheaf on X, and $\mathcal{U}$ be a D-open covering of X. Then A induces an ordinary pre-cosheaf A_P on dom(P) and $\mathcal{U}_P = P^{-1}\mathcal{U}$ is an open covering of the domain of definition of any plot P in X. Then the assignment $P \mapsto \check{C}_k(\mathcal{U}_P; A_P)$ to every plot P is a pre-cosheaf on diffeological space X, where $\check{C}_k(\mathcal{U}_P; A_P)$ is the Čech chain complex subordinate to $\mathcal{U}_P$ on dom(P). If A is a cosheaf, then $\check{C}_k$ is a cosheaf also by [4, Lemma VI.4.3].

Let $\delta : \check{C}_k \to \check{C}_{k-1}$ be the morphism of pre-cosheaves consists of the Čech boundary operators $\delta_P : \check{C}_k(P) \to \check{C}_{k-1}(P)$ on domains of plots. Then $(\check{C}_\bullet, \delta)$ is a chain complex of pre-cosheaves. In this manner, one obtains the homology groups $\Gamma H_k(\check{C}_\bullet)(X)$ and $H_k(\Gamma\check{C}_\bullet)$, where $H_k(\check{C}_\bullet)$ is the pre-cosheaf assigning to each plot P, the Čech homology subordinate to $\mathcal{U}_P$ on dom(P). There is a natural transformation $\phi : H_0(\check{C}_\bullet) \to A$ (see [4, VI.4.]). As a result, if A is a cosheaf then ϕ is a natural isomorphism.

6 Cosheaves and D-topology

Here we show how a cosheaf on diffeological spaces induces an ordinary cosheaf.

Definition 6.1. Let $f : X \to Y$ be a smooth map and S be a pre-cosheaf on diffeological space Y. The *pullback* f^*S of the pre-cosheaf S by f on X

is given by the assignment

$$\begin{aligned} f^*S\ (P) &:= \quad S(f \circ P), \\ f^*S(Q \xrightarrow{F} P) &:= \quad F_* : S(f \circ Q) \longrightarrow S(f \circ P), \end{aligned}$$

for plots P in X and morphisms $Q \xrightarrow{F} P$.

If S is a cosheaf on Y, then the pullback f^*S is a cosheaf on X.

By universal property, there is a unique morphism $\Gamma f : \Gamma f^*S(X) \to \Gamma S(Y)$ with $(\Gamma f) \circ P_* = (f \circ P)_*$ for every plot P in X, where $P_* : f^*S(P) \to \Gamma f^*S(X)$ and $(f \circ P)_* : S(f \circ P) \to \Gamma S(Y)$ are the morphisms in the definition of $\Gamma f^*S(X)$ and $\Gamma S(Y)$, respectively. If $g : Y \to Z$ is another smooth map and S is a pre-cosheaf on Z, then

$$(g \circ f)^*S = f^*(g^*S), \qquad \text{but} \qquad \Gamma(g \circ f) = (\Gamma g) \circ (\Gamma f).$$

When f is an inclusion $X \hookrightarrow Y$, we denote by $S|_X$ the pullback of a pre-cosheaf S on Y, we also denote by $\Gamma S_{X,Y}$ the induced morphism between cosections and call it the extension of cosections of X to Y.

Theorem 6.2. Any cosheaf S on a diffeological space X induces a cosheaf ΓS on the D-topological space X by assigning $\Gamma S(U) := \Gamma S|_U(U)$ to every D-open subspace U of X, and the extensions $\Gamma S_{U,V} : \Gamma S(U) \to \Gamma S(V)$ to inclusions $U \hookrightarrow V$ of D-open subspaces of X.

Proof. From the discussion above, it is clear that ΓS is a pre-cosheaf on the D-topological space X. Let U be any D-open subspace of X. Assume that $\mathcal{U} = \{U_i\}_{i \in J}$ is any D-open cover of U and let $U_{ij} = U_i \cap U_j$, for every $i, j \in J$. Consider the D-open cover $\mathcal{U}_J = \{U_{ij}\}_{(i,j) \in J \times J}$ as a full subcategory of $\mathsf{Open}(X)$ and the canonical functor $\varepsilon_J : \mathcal{U}_J \to \mathsf{Open}(X)$. Notice that $U_i = U_i \cap U_i = U_{ii}$, so $\mathcal{U}_J$ contains $\mathcal{U}$. Obviously,

$$\begin{tikzcd} & \Gamma S(U) & \\ \Gamma S(U_{ij}) \arrow[ur, "\Gamma S_{U_{ij},U}"] \arrow[rr, "\Gamma S_{U_{ij},U_i}"'] & & \Gamma S(U_i) \arrow[ul, "\Gamma S_{U_i,U}"'] \end{tikzcd}$$

is a cocone on $\Gamma S \circ \varepsilon_J$. To show that ΓS is a cosheaf, we must prove that the canonical morphism $\varinjlim \Gamma S \circ \varepsilon_J \to \Gamma S(U)$ is an isomorphism.

Let $\varphi : \Gamma S \circ \varepsilon_J \Rightarrow C$ be an arbitrary cocone. Every plot P in U can be written as the supremum of a compatible family $\{P_i\}_{i\in J}$ of plots such that P_i is a plot in U_i. This induces a cocone

$$\begin{tikzcd} & C & \\ S(P_{ij}) \arrow[ur, "\psi_{P_{ij}}"] \arrow[rr, "(\imath_{P_{ij},P_i})_*"'] & & S(P_i) \arrow[ul, "\psi_{P_i}"'] \end{tikzcd}$$

on $S \circ e_J$, where $\psi_{P_{ij}} = \varphi_{U_{ij}} \circ (P_{ij})_*$, P_{ij} denotes $P_i \times_P P_j$ and $\imath_{P_{ij},P_i}$ is the inclusion morphism from P_{ij} to P_i. Since S is a cosheaf on diffeological space X, there exists a unique morphism $\psi_P : S(P) \to C$ with $\psi_P \circ (\imath_{P_{ij},P})_* = \psi_{P_{ij}}$ for every plot P in U. To show that ψ is a cocone on $S|_U$, let $Q \xrightarrow{F} P$ be a morphism of plots in U and let $Q_{ij} = Q \times_P P_{ij}$. We can write

$$\begin{aligned} \psi_P \circ F_* \circ (\imath_{Q_{ij},Q})_* &= \psi_P \circ (\imath_{P_{ij},P})_* \circ (F_{ij})_* \\ &= \varphi_{ij} \circ (P_{ij})_* \circ (F_{ij})_* \\ &= \varphi_{ij} \circ (Q_{ij})_* = \psi_{Q_{ij}}, \end{aligned}$$

where $Q_{ij} \xrightarrow{F_{ij}} P_{ij}$ is the restriction of F to Q_{ij}, and by uniqueness, we obtain $\psi_P \circ F_* = \psi_Q$. Hence, there is a unique morphism $u : \Gamma S(U) \to C$ such that $u \circ P_* = \psi_P$ for plots P in U.

Now we have

$$u \circ \Gamma S_{U_{ij},U} \circ (P_{ij})_* = u \circ (\iota \circ P_{ij})_* = \psi_{P_{ij}} = \varphi_{U_{ij}} \circ (P_{ij})_*,$$

for every plot P_{ij} in U_{ij} considered as a plot in U by the inclusion $\iota : U_{ij} \to U$. Therefore $u \circ \Gamma S_{U_{ij},U} = \varphi_{U_{ij}}$, by the universal property of the colimit of the functor $S|_{U_{ij}}$. This completes the proof. $\square$

7 Quasi-cosheaves and quasi-Čech homology

We shall define and study quasi-cosheaves, a notion associated with covering generating families. Followed by that, quasi-Čech homology for diffeological spaces is established. First, we recall covering generating families from [13].

Definition 7.1. Let $\mathcal{C}$ be a parametrized cover of X. The *prediffeology generated* by $\mathcal{C}$ denoted by $\lfloor \mathcal{C} \rfloor$, consists of parametrizations $P \circ F$, where P is an element of $\mathcal{C}$ and F is a smooth map between domains. The *diffeology generated* by $\mathcal{C}$, denoted by $\langle \mathcal{C} \rangle$, is the set of parametrizations P which are as the supremum of a compatible family $\{P_i\}_{i \in J}$ of parametrizations in X with $P_i \in \lfloor \mathcal{C} \rfloor$. A *covering generating family* of a diffeological space $(X, \mathcal{D})$ is a parametrized cover $\mathcal{C}$ of X generating the diffeology of the space, that is $\langle \mathcal{C} \rangle = \mathcal{D}$. Let $\mathsf{CGF}(X)$ denote the collection of all covering generating families of the space X. Note that the diffeology $\mathcal{D}$ of the space X is itself a covering generating family.

Example 7.2. For any diffeological space X, the collection of plots whose domains are open balls, the collection of global plots $\mathbb{R}^n \to X$ (n ranges over nonnegative integers), the collection of centered plots, i.e., plots $U \to X$ with $0 \in U$, are all covering generating families. For smooth manifolds or orbifolds, any atlas is a covering generating family. If U is a domain, the singleton $\{1_U : U \to U\}$ is a covering generating family of U.

Let X be a diffeological space and $\mathcal{C} \in \mathsf{CGF}(X)$. Consider the prediffeology $\lfloor \mathcal{C} \rfloor$ as a full subcategory of $\mathsf{Plots}(X)$. Denote by $\Gamma S(\mathcal{C})$ the colimit of the restriction of a pre-cosheaf S to $\lfloor \mathcal{C} \rfloor$. By universal property, there is a canonical morphism $\rho : \Gamma S(\mathcal{C}) \to \Gamma S(X)$ with $\rho \circ \varphi_P = P_*$ for every plot $P \in \lfloor \mathcal{C} \rfloor$, where $\varphi_p : S(P) \to \Gamma S(\mathcal{C})$ is the morphism in the definition of the colimit of the restriction of S to $\lfloor \mathcal{C} \rfloor$.

Definition 7.3. A pre-cosheaf S on a diffeological space X is a *quasi-cosheaf* if the canonical morphism $\rho : \Gamma S(\mathcal{C}) \to \Gamma S(X)$ is an isomorphism, for every $\mathcal{C} \in \mathsf{CGF}(X)$. We denote the category of quasi-cosheaves on X by $\mathsf{QuasiCoshv}(X)$ as a full subcategory of $\mathsf{PreCoshv}(X)$.

Obviously, if S is a quasi-cosheaf, $\Gamma S(\mathcal{C})$ and $\Gamma S(\mathcal{C}')$ are isomorphic for every $\mathcal{C}, \mathcal{C}' \in \mathsf{CGF}(X)$.

Theorem 7.4. Every cosheaf S on a diffeological space X is a quasi-cosheaf.

Proof. Let $\mathcal{C} \in \mathsf{CGF}(X)$ and P be an arbitrary plot in X. Then P is as the supremum of compatible family $\{P_i\}_{i \in J}$ with $P_i \in \lfloor \mathcal{C} \rfloor$. Since $\mathsf{E}_J = \{P_i \times_P P_j\}_{(i,j) \in J \times J}$ is a subcategory of $\lfloor \mathcal{C} \rfloor$, there is a unique morphism $\alpha_J : \varinjlim S \circ e_J \to \Gamma S(\mathcal{C})$. On the other hand, because S is a cosheaf, the morphism $\eta_J : \varinjlim S \circ e_J \to S(P)$ is an isomorphism. The composition of α_J with the inverse morphism $\eta_J^{-1} : S(P) \to \varinjlim S \circ e_J$ gives us a morphism $\varphi_J = \alpha_J \circ \eta_J^{-1} : S(P) \to \Gamma S(\mathcal{C})$. Let P be the supremum of another compatible family $\{P'_{i'}\}_{i' \in J'}$ with $P'_{i'} \in \lfloor \mathcal{C} \rfloor$. Let JJ' denote the disjoint union $J \amalg J'$. By universal property, the inner triangles in the following diagram are commutative:

$$\begin{tikzcd}
 & \varinjlim S \circ e_J \arrow[d] \arrow[dr, "\alpha_J"] & \\
S(P) \arrow[ur, "\eta_J^{-1}"] \arrow[r, "\eta_{JJ'}^{-1}"] \arrow[dr, "\eta_{J'}^{-1}"'] & \varinjlim S \circ e_{JJ'} \arrow[r, "\alpha_{JJ'}"] & \Gamma S(\mathcal{C}) \\
 & \varinjlim S \circ e_{J'} \arrow[u] \arrow[ur, "\alpha_{J'}"'] &
\end{tikzcd}$$

Thus, the entire diagram is commutative and $\varphi_J = \varphi_{J'}$. So we obtain a well-defined morphism $\varphi_P : S(P) \to \Gamma S(\mathcal{C})$. By definition, $\rho \circ \varphi_P = P_*$ for all $P \in \lfloor \mathcal{C} \rfloor$. But we have $\rho \circ \alpha_J = P_* \circ \eta_J$, which implies $\rho \circ \varphi_P = P_*$ for all plots P in X.

Now suppose $R \xrightarrow{F} P$ is a morphism of plots in X and P is the supremum of a compatible family $\{P_i\}_{i \in J}$ with $P_i \in \lfloor \mathcal{C} \rfloor$. Then R is the supremum of the compatible family $\{R_i = R \times_P P_i\}_{i \in J}$ and we have the restriction $R_i \xrightarrow{F_i} P_i$ of F to R_i. Again by universal property, the diagram

$$\begin{tikzcd}
S(P) \arrow[r, "\eta_J^{-1}"] & \varinjlim S \circ e_J \arrow[dr, "\alpha_J"] & \\
S(R) \arrow[u, "F_*"] \arrow[r, "\eta_J'^{-1}"'] & \varinjlim S \circ e'_J \arrow[u, "\varinjlim F_J"] \arrow[r, "\alpha'_J"'] & \Gamma S(\mathcal{C})
\end{tikzcd}$$

is commutative and $\varphi_P \circ F_* = \varphi_R$, where the lower morphisms are corresponding to R and $\{R_i\}_{i \in J}$. In other words, $\varphi : S \Rightarrow \Gamma S(\mathcal{C})$ is a cocone. So there exists a unique morphism $\xi : \Gamma S(X) \to \Gamma S(\mathcal{C})$ with $\xi \circ P_* = \varphi_P$ for every plot P in X. We have $\xi \circ \rho \circ \varphi_P = \xi \circ P_* = \varphi_P$ for $P \in \lfloor \mathcal{C} \rfloor$, and $\rho \circ \xi \circ P_* = \rho \circ \varphi_P = P_*$ for all plots P in X. By uniqueness, we conclude that $\xi \circ \rho = 1_{\Gamma S(\mathcal{C})}$ and $\rho \circ \xi = 1_{\Gamma S(X)}$. Therefore, ρ is an isomorphism and S is a quasi-cosheaf on X. $\square$

Example 7.5. Let X be a diffeological space. The domain functor *dom* : $\mathsf{Plots}(X) \to \mathsf{Diff}$ given by

$$(Q \xrightarrow{F} P) \quad \longmapsto \quad \big(F : \mathrm{dom}(Q) \longrightarrow \mathrm{dom}(P)\big),$$

is a cosheaf and the space of its cosections is X by [6, Proposition 2.7]. As a result, the domain functor is a quasi-cosheaf (compare with [13, art. 1.76]).

Example 7.6. Let X be a diffeological space and D be an non-initial object in a category D. The constant pre-cosheaf D assigning to any plot P the object D, and to any morphism $Q \xrightarrow{F} P$ the identity morphism 1_D on D is not a cosheaf by Remark 3.3. However, it is not hard to see that D is a quasi-cosheaf. This example shows that the converse to Theorem 7.4 does not hold. Also, Theorem 6.2 is not true for quasi-cosheaves.

To reach a characterization of quasi-cosheaves we need the notion of simplices on covering generating families.

Definition 7.7. Let X be a diffeological space and $\mathcal{C} \in \mathsf{CGF}(X)$. We define n-simplices on $\mathcal{C}$ inductively:

(i) A 0-*simplex* is just an element P_0 of $\mathcal{C}$. The nerve plot of a 0-simplex P_0 is the plot P_0 itself by convention.

(ii) A 1-*simplex* is any diagram

$$P_0 \xleftarrow{F_1} Q \xrightarrow{F_0} P_1$$

with $P_0, P_1 \in \mathcal{C}$ and a nonempty plot Q. In this situation, Q is called the nerve plot. Notice that Q is the nerve plot of the diagram not that of P_0, P_1.

(iii) For integers $n \geqslant 2$, an n-*simplex* $(P_0, \ldots, P_n)$ consists of $n+1$ plots $P_0, \ldots, P_n$ belonging to $\mathcal{C}$ and a nonempty *nerve plot* Q in X such that any n plots $P_0, \ldots, \widehat{P_i}, \ldots, P_n$ (the hat indicates the omission of P_i) form an $(n-1)$-simplex with the nerve plot Q_i. In addition, for each $i = 0, \ldots, n$, there exist a morphism $Q \xrightarrow{F_i} Q_i$ commuting with the morphisms $Q_i \xrightarrow{F_{i,j}} Q_{i,j}$, for $(n-2)$-simplices $P_0, \ldots, \widehat{P_i}, \ldots, \widehat{P_j}, \ldots, P_n$ with the nerve plots $Q_{i,j}$; that is, $F_{i,j} \circ F_i = F_{j,i} \circ F_j$.

For instance, a 2-simplex is as the following commutative diagram.

$$\begin{tikzcd}
& & P_0 & & \\
& Q_2 \arrow[ur, "F_{2,1}"] \arrow[ddl, "F_{2,0}"'] & & Q_1 \arrow[ul, "F_{1,2}"'] \arrow[ddr, "F_{1,0}"] & \\
& & Q \arrow[ul, "F_2"'] \arrow[ur, "F_1"] \arrow[d, "F_0"] & & \\
P_1 & & Q_0 \arrow[ll, "F_{0,2}"] \arrow[rr, "F_{0,1}"'] & & P_2
\end{tikzcd}$$

Denote by $n-\mathsf{simplex}(\mathcal{C})$ the set of n-simplices on $\mathcal{C}$. Let S be a pre-cosheaf on X. For an n-simplex $(P_0, \ldots, P_n)$ with the nerve plot Q, let $S(P_0, \ldots, P_n) := S(Q)$. Note that the nerve plots are elements of $\lfloor \mathcal{C} \rfloor$.

Proposition 7.8. *A pre-cosheaf S on a diffeological space X is a quasi-cosheaf if and only if for every $\mathcal{C} \in \mathsf{CGF}(X)$, the sequence*

$$\coprod_{1-\mathsf{simplex}(\mathcal{C})} S(P_0 \xleftarrow{F_1} Q \xrightarrow{F_0} P_1) \quad \overset{\beta_0}{\underset{\beta_1}{\rightrightarrows}} \quad \coprod_{P \in \mathcal{C}} S(P) \xrightarrow{\ \alpha\ } \Gamma S(X)$$

is a coequalizer, where the arrows β_0 and β_1 are induced by F_{0*} and F_{1*}.

Proof. There exists a morphism $\gamma : \coprod_{P \in \mathcal{C}} S(P) \to \Gamma S(\mathcal{C})$ with $\gamma \circ \beta_0 = \gamma \circ \beta_1$, which is in fact the coequalizer of β_0 and β_1. Moreover, $\rho : \Gamma S(\mathcal{C}) \to \Gamma S(X)$ is commutative with α and γ. Therefore, the above diagram is a coequalizer if and only if S is a quasi-cosheaf. $\square$

7.1 Quasi-Čech homology

In the sequel, let A be a pre-cosheaf of abelian groups on a diffeological space X and $\mathcal{C} \in \mathsf{CGF}(X)$. We define the group of n-chains with coefficients in the cosheaf A subordinated to the covering generating family $\mathcal{C}$ to be

$$C_n(X, \mathcal{C}, A) = \bigoplus_{n-\mathsf{simplex}(\mathcal{C})} A(P_0, \ldots, P_n),$$

that is, finite formal sums $\sum_\sigma c_\sigma \sigma$, where the sum ranges over all n-simplices $\sigma = (P_0, \ldots, P_n)$. The operators

$$\delta_n : C_n(X, \mathcal{C}, A) \longrightarrow C_{n-1}(X, \mathcal{C}, A)$$

for integers $n \geq 1$, are defined $\mathbb{Z}$-linearity by

$$\delta_n(c_\sigma(P_0, \ldots, P_n)) = \sum_{i=0}^{n} (-1)^i (F_i)_* c_\sigma(P_0, \ldots, \widehat{P}_i, \ldots, P_n).$$

Proposition 7.9. *The sequence*

$$\begin{aligned} \cdots \longrightarrow C_n(X, \mathcal{C}, A) &\xrightarrow{\delta_n} C_{n-1}(X, \mathcal{C}, A) \longrightarrow \cdots \\ \longrightarrow C_1(X, \mathcal{C}, A) &\xrightarrow{\delta_1} C_0(X, \mathcal{C}, A) \longrightarrow 0 \end{aligned}$$

is a chain complex.

Proof. We must show that $\delta_{n-1} \circ \delta_n = 0$, for every integer $n \geq 1$.

$$\begin{aligned} \delta_{n-1} \circ \delta_n(c_\sigma(P_0, \ldots, P_n)) &= \delta_{n-1}\big(\sum_{i=0}^{n} (-1)^i (F_i)_* c_\sigma(P_0, \ldots, \widehat{P}_i, \ldots, P_n)\big) \\ &= \sum_{i=0}^{n} (-1)^i \delta_{n-1}\big((F_i)_* c_\sigma(P_0, \ldots, \widehat{P}_i, \ldots, P_n)\big) \\ &= \sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^{i+j} (F_{i,j} \circ F_i)_* c_\sigma(P_0, \ldots, \widehat{P_j}, \ldots, \widehat{P}_i, \ldots, P_n) \\ &+ \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} (-1)^{i+j-1} (F_{i,j} \circ F_i)_* c_\sigma(P_0, \ldots, \widehat{P}_i, \ldots, \widehat{P_j}, \ldots, P_n) \\ &= \sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^{i+j} (F_{i,j} \circ F_i)_* c_\sigma(P_0, \ldots, \widehat{P_j}, \ldots, \widehat{P}_i, \ldots, P_n) \\ &- \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j} (F_{j,i} \circ F_j)_* c_\sigma(P_0, \ldots, \widehat{P_j}, \ldots, \widehat{P}_i, \ldots, P_n) \\ &= 0, \end{aligned}$$

where $\sigma = (P_0, \ldots, P_n)$ is an n-simplex. $\square$

Denote the nth homology group of the chain complex $C_\bullet(X, \mathcal{C}, A)$ by $H_n(X, \mathcal{C}; A)$.

Proposition 7.10. *If A is a quasi-cosheaf, the groups $H_0(X, \mathcal{C}; A)$ and $\Gamma A(X)$ are isomorphic.*

Proof. $H_0(X, \mathcal{C}; A)$ is the same as $\operatorname{coker}(\delta_0)$, which is exactly $\Gamma A(\mathcal{C})$. Since A is a quasi-cosheaf, we deduce that $H_0(X, \mathcal{C}; A)$ is isomorphic to $\Gamma A(X)$. $\square$

Given a morphism $\phi : A' \to A$ of pre-cosheaves on X, define homomorphisms $\phi_* : C_n(X, \mathcal{C}, A) \to C_n(X, \mathcal{C}, A')$ $\mathbb{Z}$-linearity by

$$\phi_*(c_\sigma \sigma) = \phi_Q(c_\sigma)\sigma.$$

where σ is an n-simplex with the nerve plot Q.

Proposition 7.11. *If $0 \longrightarrow A' \xrightarrow{\phi} A \xrightarrow{\psi} A'' \longrightarrow 0$ is a short exact sequence of pre-cosheaves on a diffeological space X, then the sequence*

$$0 \longrightarrow C_n(X, \mathcal{C}, A') \xrightarrow{\phi_*} C_n(X, \mathcal{C}, A) \xrightarrow{\psi_*} C_n(X, \mathcal{C}, A'') \longrightarrow 0$$

is exact, for every integer n.

Proof. The proof is straightforward. $\square$

Now by zigzag lemma, for such a short exact sequence there exist the connecting homomorphisms

$$\partial : H_n(X, \mathcal{C}; A'') \longrightarrow H_{n-1}(X, \mathcal{C}; A')$$

such that the long sequence

$$\begin{aligned} \cdots \longrightarrow H_n(X, \mathcal{C}; A') \longrightarrow H_n(X, \mathcal{C}; A) \longrightarrow H_n(X, \mathcal{C}; A'') \xrightarrow{\partial} \\ H_{n-1}(X, \mathcal{C}; A') \longrightarrow \cdots \end{aligned}$$

is exact. Furthermore, for any morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd} H_n(X,\mathcal{C};A'') \arrow[r] \arrow[d] & H_{n-1}(X,\mathcal{C};A') \arrow[d] \\ H_n(X,\mathcal{C};B'') \arrow[r] & H_{n-1}(X,\mathcal{C};B') \end{tikzcd}$$

Proposition 7.12. *For a morphism $\phi : A' \to A$ of pre-cosheaves, $\delta_n \circ \phi_* = \phi_* \circ \delta_n$.*

Proof. For any $c_\sigma(P_0,\dots,P_n) \in C_n(X,\mathcal{C},A')$, we can write

$$\begin{aligned} \delta_n \circ \phi_*\big(c_\sigma(P_0,\dots,P_n)\big) &= \sum_{i=0}^{n}(-1)^i(F_i)_* \circ \phi_Q(c_\sigma)(P_0,\dots,\widehat{P}_i,\dots,P_n) \\ &= \sum_{i=0}^{n}(-1)^i\phi_{Q_i} \circ (F_i)_*(c_\sigma)(P_0,\dots,\widehat{P}_i,\dots,P_n) \\ &= \phi_*\Big(\sum_{i=0}^{n}(-1)^i(F_i)_*(c_\sigma)(P_0,\dots,\widehat{P}_i,\dots,P_n)\Big) \\ &= \phi_* \circ \delta_n\big(c_\sigma(P_0,\dots,P_n)\big), \end{aligned}$$

where $\sigma = (P_0,\dots,P_n)$ is an n-simplex with the nerve plot Q. □

Thus, ϕ induces a homomorphism $\phi_\# : H_n(X,\mathcal{C};A') \to H_n(X,\mathcal{C};A)$ between homology groups, for every nonnegative integer n, such that $\mathrm{id}_\# = \mathrm{id}$ for the identity morphism $\mathrm{id} : A \to A$, and $(\psi \circ \phi)_\# = \psi_\# \circ \phi_\#$ for morphisms $\phi : A' \to A$ and $\psi : A \to A''$ of pre-cosheaves on X. Hence for each $n \geq 0$, we obtain an exact ∂-functor $H_n(X,\mathcal{C};-) : \mathsf{Ab}(\mathsf{PreCoshv})(X) \to \mathsf{Ab}$.

Definition 7.13. Let X be a diffeological space and $\mathcal{C} = \{P_\alpha\}_{\alpha\in I}$ be a covering generating family of X. A refinement of $\mathcal{C}$ is a covering generating family $\mathcal{C}' = \{P'_\beta\}_{\beta\in J}$ together with a map $\lambda : J \to I$ and a family $\{f_\beta\}_{\beta\in J}$ of morphisms $P'_\beta \xrightarrow{f_\beta} P_{\lambda(\beta)}$. Denote such a refining by $\lambda : \mathcal{C}' \to \mathcal{C}$. In this situation, we have $\mathcal{C}' \subseteq \lfloor\mathcal{C}\rfloor$.

Refinements of covering generating families of X turn $\mathsf{CGF}(X)$ into a category. If $\lambda : \mathcal{C}' \to \mathcal{C}$ is a refinement and $\sigma' = (P'_{\beta_0},\dots,P'_{\beta_n})$ is an

n-simplex of plots belonging to $\mathcal{C}'$ with the nerve plot Q, then $\lambda(\sigma') = (P_{\lambda(\beta)_0}, \ldots, P_{\lambda(\beta)_n})$ constitutes an n-simplex of plots belonging to $\mathcal{C}$ with the nerve plot Q. For example, we have

$$\begin{tikzcd}
P'_{\beta_0} \arrow[d, "f_{\beta_0}"'] & Q \arrow[l, "F_1"'] \arrow[r, "F_0"] \arrow[dl, dotted] \arrow[dr, dotted] & P'_{\beta_1} \arrow[d, "f_{\beta_1}"] \\
P_{\beta_0} & & P_{\beta_1}
\end{tikzcd}$$

for a 2-simplex. Thus, a refinement $\lambda : \mathcal{C}' \to \mathcal{C}$ defines a chain homomorphism $\lambda_* : C_\bullet(X, \mathcal{C}', A) \to C_\bullet(X, \mathcal{C}, A)$, $\mathbb{Z}$-linearity by

$$\lambda^0_*(c_{P'_\beta} P'_\beta) = (f_{\beta *} c_{P'_\beta}) P_{\lambda(\beta)} \qquad \text{and}$$
$$\lambda^n_*(c_{\sigma'}\sigma') = c_{\sigma'}\lambda(\sigma'), \text{ for } n \geq 1.$$

A not so hard calculation shows that $\lambda_* \circ \delta = \delta \circ \lambda_*$ and hence a homomorphism $\lambda_\# : H_\bullet(X, \mathcal{C}'; A) \to H_\bullet(X, \mathcal{C}; A)$ is achieved. One can easily check that $\mathrm{id}_\# = \mathrm{id}$ for the identity refinement $\mathrm{id} : \mathcal{C} \to \mathcal{C}$, and $\lambda_\# \circ \mu_\# = (\lambda \circ \mu)_\#$ for refinements $\lambda : \mathcal{C}' \to \mathcal{C}$ and $\mu : \mathcal{C}'' \to \mathcal{C}'$. Therefore, we obtain a functor $H^n(X, -; A) : \mathsf{CGF}(X) \to \mathsf{Ab}$. Now we define the quasi-Čech homology of diffeological spaces as below:

Definition 7.14. The n-th *quasi-Čech homology group* $\check{H}_n(X; A)$ of a diffeological space X with coefficients in a pre-cosheaf A on X is

$$\check{H}_n(X; A) = \varprojlim_{\mathcal{C}} H_n(X, \mathcal{C}; A).$$

As a consequence of Proposition 7.10, one can state the following:

Proposition 7.15. *$\check{H}_0(X; A)$ is isomorphic to $\Gamma A(X)$ if A is a quasi-cosheaf.*

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