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From torsion theories to closure operators and factorization systems

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Abstract. Torsion theories are here extended to categories equipped with an ideal of 'null morphisms', or equivalently a full subcategory of 'null objects'. Instances of this extension include closure operators viewed as generalised torsion theories in a 'category of pairs', and factorization systems viewed as torsion theories in a category of morphisms. The first point has essentially been treated in [15].

Introduction

Classically, a *torsion theory* (T, F) in an abelian category A (see Borceux [3], Section 1.2) is a pair (T, F) of full, replete subcategories of A such that:

(i) every morphism $T \to F$ from an object of T to an object of F is null,

(ii) for every object A in A there exists a short exact sequence $T \rightarrowtail A \twoheadrightarrow F$, with T in T and F in F.

The objects of T and F are called, respectively, the *torsion objects* and the *torsion-free objects* of the theory. One easily proves that:

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(a) every object A in A has a short exact sequence, determined up to isomorphism

$$T(A) \rightarrowtail A \twoheadrightarrow F(A) \qquad \text{(with } T(A) \text{ in } \mathsf{T} \text{ and } F(A) \text{ in } \mathsf{F}\text{)}, \tag{0.1}$$

and this determines a subfunctor T and a quotient functor F of the identity of A,

(b) T and F correspond to each other in the Galois connection determined by the *orthogonality relation* $A \perp B$ in $\mathrm{Ob}\mathsf{A}$, meaning that every morphism $A \to B$ is null.

All this can be extended, with the same words, to any pointed category E with kernels and cokernels, a short exact sequence (m, p) being any pair of consecutive maps where m is a kernel of p and the latter a cokernel of m (see e.g. [16]).

In fact, we are interested in wider extensions, to *non-pointed* categories where we still have some form of exactness with respect to an ideal of 'null morphisms', as in [11]– [13]; these extensions, which we call *multi-pointed categories* and *pre-pointed categories*, are briefly analysed in Sections 1 and 4.

Section 2 studies torsion theories in a multi-pointed category E, essentially defined as above. More generally, a *torsion operator* (τ, φ) consists of two endofunctors $T, F \colon \mathsf{E} \to \mathsf{E}$ and two natural transformations $\tau \colon T \to 1$ and $\varphi \colon 1 \to F$ giving a short exact sequence (0.1) for every object A. This operator is a torsion theory if and only if τ and φ give invertible transformations $T\tau = \tau T$ and $F\varphi = \varphi F$, which means that T and F can be (uniquely) extended to an idempotent comonad and monad, respectively.

In Section 3 we show that a closure operator for topological spaces amounts to a torsion operator on the category Top_2 of *pairs of topological spaces* (in the sense of Algebraic Topology, see 1.2), which is multi-pointed with respect to a natural ideal of null morphisms. More generally, this works for any category C_2 'of pairs', constructed over a category C equipped with a suitable choice of 'distinguished monomorphisms'. The stronger case of a torsion theory corresponds to a closure operator which is weakly hereditary and idempotent. All this is outlined in the first table below

1	in a multi-pointed category	in C_2
2	torsion operator	closure operator in C
3	torsion theory	closure operator weakly hereditary and idempotent

1	in a pre-pointed category	in $\mathsf{C}^{\mathbf{2}}$
2	torsion operator	functorial factorization in C
$3a$	algebraic torsion operator	algebraic factorization
$3b$	torsion theory	factorization system

In Sections 4 and 5 we develop the second table. Torsion operators and torsion theories are generalised to pre-pointed categories. Any category of morphisms $\mathsf{C}^{\mathbf{2}}$ comes equipped with a subcategory of null objects, the isomorphisms of C; in this sense $\mathsf{C}^{\mathbf{2}}$ is pre-pointed and has a notion of *short pre-exact sequence*, which amounts to a factorization $x = x''.x'$ of a morphism in C. Now a torsion operator in $\mathsf{C}^{\mathbf{2}}$ amounts to a functorial factorization in C; the former is a torsion theory if and only if the latter is an (orthogonal) factorization system.

This correspondence can be refined: we already remarked that a torsion theory is a torsion operator (τ, φ) where (T, τ) 'is' an idempotent comonad and (F, φ) an idempotent monad. This suggests to consider an intermediate case: an *algebraic torsion operator*, where the endofunctors T and F are respectively enriched to a comonad and a monad. In the previous case of a multi-pointed category, the additional structure is automatically idempotent and we just get a torsion theory. For a pre-pointed category E this is not the case: in fact, for a category of morphisms $\mathsf{E} = \mathsf{C}^{\mathbf{2}}$, we get an *algebraic factorization* on C, introduced in [14] under the name of a 'natural weak factorization system'; the more explicit term 'algebraic', instead of 'natural', was used in [10].

1 Short exact sequences in categories with an ideal of null morphisms

Short exact sequences are a basic ingredient of torsion theories. We explore some fairly general situations where they can be defined.

Categories with an assigned ideal of null morphisms and (co)kernels with respect to this ideal have been considered by various authors, including C. Ehresmann [7] and R. Lavendhomme [17]. The present results have been developed in [11]– [13]; we generally refer to the recent book [13], following its terminology and notation. The symbol $\subset$ denotes *weak* inclusion.

After recalling the notion of semiexact category, we briefly explore two extensions: a *multi-pointed category*, and – more generally – a *pre-pointed category.*

1.1 Semiexact categories An *ex0-category*, as defined in [13], is a category E equipped with a set of *null morphisms* $\mathcal{N}$, where:

(ex.0) $\mathcal{N}$ is a *closed ideal* of E.

The condition that $\mathcal{N}$ be an ideal means that every composite with a null morphism is null, while the 'closedness' of $\mathcal{N}$ means that every null morphism factorizes through a null object, that is, an object whose identity belongs to $\mathcal{N}$. Equivalently, one can assign a full subcategory E_0 closed under retracts, called the subcategory of *null objects.* (The equivalence comes out of an obvious covariant Galois connection between subsets of morphisms and subsets of objects, in any category.)

A *semiexact category* is an ex0-category E where:

(ex.1) every morphism of E has a *kernel* and a *cokernel*, with respect to $\mathcal{N}$, written as:

$$\mathrm{ker} f\colon \mathrm{Ker} f \rightarrowtail A, \qquad \mathrm{cok} f\colon B \twoheadrightarrow \mathrm{Cok} f. \tag{1.1}$$

The kernel is characterised up to isomorphism by the 'usual' universal property, written with respect to the ideal of null morphisms (see the left diagram below):

- $f\cdot \mathrm{ker} f$ is null,

- for every map u in E such that fu is null, there is a unique map v such that $u = (\mathrm{ker} f)v$

$$
\begin{tikzcd}
\mathrm{Ker}f \arrow[r, tail] & A \arrow[r, "f"] & B \\
& \bullet \arrow[ul, dashed, "v"] \arrow[u, "u"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, two heads] \arrow[d, "u"'] & \mathrm{Cok}f \arrow[dl, dashed, "v"] \\
& \bullet &
\end{tikzcd}
\tag{1.2}
$$

The cokernel is defined by the dual property, represented in the right diagram above. It follows easily that $\mathrm{ker}f$ is mono and $\mathrm{cok}f$ is epi. A *normal mono* is, by definition, a kernel (of some morphism), and a *normal epi* is a cokernel; the arrows $\rightarrowtail$, $\twoheadrightarrow$ *are reserved here for such morphisms.*

One proves that the morphism f has a unique *normal factorization* $f = mgp$ ([13], Section 1.1.5) through its *normal coimage* p (the cokernel of the kernel morphism) and its *normal image* m (the kernel of the cokernel morphism)

$$
\begin{tikzcd}
\mathrm{Ker}f \arrow[r, tail, "\mathrm{ker}f"] & A \arrow[r, "f"] \arrow[d, two heads, "p"'] & B \arrow[r, two heads, "\mathrm{cok}f"] & \mathrm{Cok}f \\
& \mathrm{Ncm}f \arrow[r, "g"'] & \mathrm{Nim}f \arrow[u, tail, "m"'] &
\end{tikzcd}
\tag{1.3}
$$

$$p = \mathrm{ncm}f = \mathrm{cok}(\mathrm{ker}f), \qquad m = \mathrm{nim}f = \mathrm{ker}(\mathrm{cok}f).$$

This factorization is natural; f is said to be an *exact* morphism if this g is an isomorphism.

A sequence

$$M \xrightarrow{m} A \xrightarrow{p} B \tag{1.4}$$

in E will be said to be *short exact* if m is a kernel of p, and p a cokernel of m.

The morphism $f\colon A \to B$ is null if and only if $\mathrm{ker}f = 1_A$, if and only if $\mathrm{cok}f = 1_B$.

The semiexact category E is said to be *pointed* (or p-semiexact) if it has a zero object 0 *and* its null morphisms are precisely the zero morphisms (those which factorize through 0). In this case kernels and cokernels acquire the usual meaning. Incidentally, we recall that a p-semiexact category where each morphism is exact is the same as an exact category in the sense of

Puppe and Mitchell [19, 20], called a *p-exact category* in [13]; such a category is abelian if and only if it has all finite products, if and only if it has all finite sums.

The ideal $\mathcal{N}$ cannot be empty unless E is. In the trivial case $\mathcal{N} = \mathsf{E}$ all kernels and cokernels are identities. The opposite category E^{op} is equipped with the ideal $\mathcal{N}^{\mathrm{op}}$, and is semiexact.

A functor between semiexact categories is said to be *exact* if it preserves kernels and cokernels.

1.2 A basic example The general behaviour of semiexact categories can be very different from that of the pointed ones, as one can see analysing the classical category Top_2 of *pairs of topological spaces*, in the sense of Algebraic Topology (cf. [8]).

An object (X, A) is a space X equipped with a subspace A; a *morphism* $f\colon (X, A) \to (Y, B)$ is a continuous mapping $f\colon X \to Y$ such that $f(A) \subset B$; the composition is plain. Top is fully embedded in Top_2, identifying the space X with the pair $(X, \varnothing)$.

The object (X, A) is usually read as X *modulo* A, and viewed as a sort of 'formal quotient'. It is, therefore, quite natural to *define* the morphism $f\colon (X, A) \to (Y, B)$ to be null whenever $f(X) \subset B$. The null objects are thus the 'diagonal' pairs (T, T) and f is null if and only if it factorizes through one of them, for instance (X, X) (or (B, B)).

Kernels and cokernels exist and the normal factorization of f can be presented as

$$\begin{tikzcd}
(f^{-1}(B), A) \arrow[r, tail] & (X, A) \arrow[r, "f"] \arrow[d, "p"'] & (Y, B) \arrow[r, two heads] & (Y, B \cup fX) \\
& (X, f^{-1}(B)) \arrow[r, "g"'] & (B \cup fX, B) \arrow[u, "m"'] &
\end{tikzcd} \tag{1.5}$$

The morphism f is exact if and only if it is injective and $f(X) \supset B$.

Every short exact sequence in Top_2 is (up to isomorphism) of the form

$$(B, A) \rightarrowtail (X, A) \twoheadrightarrow (X, B) \qquad (A \subset B \subset X), \tag{1.6}$$

where both maps carry all elements to themselves. This also determines the normal subobjects and the normal quotients of the object (X, A).

Now *the pair* (X, A) *is indeed the normal quotient*

$$X/A = (X, \varnothing)/(A, \varnothing)$$

in Top_2. Every homology theory for pairs of spaces, in the sense of Eilenberg and Steenrod [8], carries the sequence (1.6) to a *long* exact sequence of abelian groups, called the *homology sequence of the triple* (X, B, A).

Note the following facts, in contrast with the behaviour of abelian (or just pointed) categories. A null morphism $(X, A) \to (Y, B)$ between two given objects need neither exist (take $X \neq \varnothing = B$) nor be unique. A monomorphism (given by an injective map) need not have a null kernel. A null morphism need not be exact, a null monomorphism need not be normal. An exact monomorphism need not be a normal mono: for example, the normal quotient $(X, A) \twoheadrightarrow (X, B)$ is mono but it is not a normal mono (unless $A = B$). The initial object $(\varnothing, \varnothing)$ and the terminal object $(\{*\}, \{*\})$ are distinct and both null, but do not determine the null morphisms in any useful way.

Every object (X, A) has a least normal subobject, the kernel of its identity, and a least normal quotient, which are null objects, generally non-isomorphic

$$(X, A)_0 = \operatorname{Ker} 1_{(X,A)} = (A, A), \quad (X, A)^0 = \operatorname{Cok} 1_{(X,A)} = (X, X). \qquad (1.7)$$

Further properties show that Top_2 is a 'homological category', in the sense of [11]– [13]. Similarly, the homological category Gp_2 of 'pairs of groups' is the domain of relative (co)homology of groups, and a source of modified categories where exact sequences and spectral sequences coming out of homotopy theory can be studied.

General 'categories of pairs' C_2 on a category C 'with distinguished subobjects' have been studied in [13], Section 2.5. This topic will be reviewed in Section 3 under more general assumptions on C, still sufficient to characterise the short exact sequences of C_2 as in diagram (1.6): here we need not have a semiexact (or homological) category, but just a 'multi-pointed' one, as defined below.

1.3 Multi-pointed categories A first extension of semiexact categories will allow us to view closure operators as torsion operators, in full generality.

By a *multi-pointed category* we mean a category E equipped with a closed ideal $\mathcal{N}$ (or equivalently with a full subcategory E_0 closed under retracts in E) where we assume the existence of the kernel and cokernel of any identity (and a choice of them). The latter are written as

$$0_A = \ker 1_A \colon A_0 \rightarrowtail A, \quad 0^A = \operatorname{cok} 1_A \colon A \twoheadrightarrow A^0. \tag{1.8}$$

This can be viewed as a 'multi-object extension' of the notion of pointed category: in fact the latter amounts to a multi-pointed category where the category E_0 is equivalent to the singleton category.

Kernels and cokernels are defined as in 1.1, but we do not assume their existence, in general. We only note that the kernel of a morphism $f \colon A \to B$ amounts to the pullback of $0_B \colon B_0 \rightarrowtail B$ along f while the cokernel of f is the same as the pushout of $0^A \colon A \twoheadrightarrow A^0$ along f. Moreover,

$$\operatorname{cok} 0_A = 1_A = \ker 0^A.$$

A short exact sequence (m, p) is defined as in 1.1 (see (1.4)); the morphisms $m \colon M \rightarrowtail A$ and $p \colon A \twoheadrightarrow P$ appearing there will be said to be a *normal monomorphism* and a *normal epimorphism*, respectively. Note that *a kernel morphism that does not have a cokernel is not considered as a normal mono.*

Incidentally we note that, in a category equipped with a closed ideal, *(co)kernels can only be defined as (co)limits* under some additional assumption, as in (1.8). To wit, the category Set of sets, equipped with the closed ideal of constant mappings (that factorize through a singleton), has all cokernels, which are pushouts, but lacks kernels – even though all limits exist.

1.4 Normal subobjects and quotients Extending topics studied in [13] for semiexact categories, in the multi-pointed category E every object A has a (possibly large) ordered set $\mathrm{Nsb}(A)$ of normal subobjects and an ordered set $\mathrm{Nqt}(A)$ of normal quotients. They are linked by an anti-isomorphism produced by cokernels and kernels

$$\mathrm{Nsb}(A) \rightleftarrows \mathrm{Nqt}(A), \tag{1.9}$$

which will be called *kernel-cokernel duality*, or *kernel duality* for short.

Let us remark that kernel duality only operates on normal subobjects and normal quotients of E and *reverses* their order relation. On the other hand, *categorical duality*, which operates on all items, takes a normal mono of E to a normal epi of the opposite multi-pointed category E^{op} and *preserves* their order. When speaking of 'duality' without specification we shall mean the categorical one.

These ordered sets are bounded: the object A has a least normal subobject $0_A \colon A_0 \rightarrowtail A$ and a greatest one $1_A \colon A \rightarrowtail A$. Similarly, it has a least normal quotient $0^A \colon A \twoheadrightarrow A^0$ and a greatest one $1_A \colon A \twoheadrightarrow A$. (If E is semiexact these ordered sets are lattices; if it is p-exact they are modular lattices [13].)

A is a null object if and only if $0_A = 1_A$, if and only if $0^A = 1_A$, and then $A_0 = A = A^0$. An object is null if and only if it has precisely one normal subobject (the null one), if and only if it has precisely one normal quotient.

1.5 The subcategory of null objects A multi-pointed category E can be simply described by properties of its full subcategory E_0 of null objects, in the same line as in [15] for semiexact categories.

In fact $0_A \colon A_0 \rightarrowtail A$ is a universal arrow from the embedding $\mathsf{E}_0 \to \mathsf{E}$ to the object A, while $0^A \colon A \twoheadrightarrow A^0$ is a universal arrow the other way round. Therefore the subcategory E_0 satisfies the following condition:

(ex.0a) E_0 is a full, replete, reflective and coreflective subcategory, with a componentwise-mono counit and a componentwise-epi unit (for the coreflector D and the reflector C, respectively)

$$\begin{gathered} U \colon \mathsf{E}_0 \to \mathsf{E} \qquad\qquad C \dashv U \dashv D, \\ 0_A \colon D(A) = A_0 \rightarrowtail A, \qquad\qquad 0^A \colon A \twoheadrightarrow A^0 = C(A), \\ (DU = \mathrm{id}\mathsf{E}_0 = CU). \end{gathered} \tag{1.10}$$

This is indeed an equivalent definition of multi-pointed categories, as we shall see in Proposition 1.3. The pointed case is characterised by E_0 being equivalent to the singleton category, that is, being an indiscrete category.

We now introduce a further extension by dropping the cancellation conditions inside axiom (ex.0a). This will allow us to include factorization structures in the reach of torsion structures.

Definition 1.1 (Pre-pointed categories)**.** By a *pre-pointed category* we mean a pair $(\mathsf{E}, \mathsf{E}_0)$ satisfying the following axiom:

(ex.0b) E is a category and E_0 is a full replete reflective and coreflective subcategory.

The adjunctions of the inclusion $U\colon \mathsf{E}_0 \to \mathsf{E}$ will be written as

$$\begin{aligned} &C \dashv U \dashv D \qquad && DU = \mathrm{id}\mathsf{E}_0 = CU, \\ &\kappa A\colon D(A) = A_0 \to A, \qquad && \gamma A\colon A \to A^0 = C(A). \end{aligned} \tag{1.11}$$

E_0 is closed in E under retracts (as any reflective subcategory), and therefore determined by the ideal $\mathcal{N}$ of the morphisms which factorize through its objects.

The pair $(\mathsf{E}, \mathsf{E}_0)$ will often be denoted as E, provided no ambiguity arises.

We shall see in Section 4 that every category of morphisms $\mathsf{C}^{\mathbf{2}}$ is pre-pointed, in a natural way.

Lemma 1.2 (Annihilation properties)**.** *Let* $(\mathsf{E}, \mathsf{E}_0)$ *be a pre-pointed category.*

(a) *For every object A, the objects A_0 and A^0 are null. The counit component $\kappa A\colon A_0 \to A$ is cancellable with respect to pairs of morphisms $Z \rightrightarrows A_0$ defined on a null object. Dually for γA.*

(b) *An object A is null if and only if κA is an isomorphism, if and only if γA is an isomorphism.*

(c) *A morphism $f\colon A \to B$ is null if and only if it factorizes through $\kappa B\colon B_0 \to B$, if and only if it factorizes through $\gamma A\colon A \to A^0$.*

Proof. (a) is obvious. This implies the non-trivial part of (b): if A is null then 1_A factorizes as $\kappa A.v$, whence κA is a split epi in E_0, and therefore invertible. For (c), if $f\colon A \to B$ is null it factorizes through some $g\colon Z \to B$ where Z is null, and the latter factorizes through κB. □

Proposition 1.3 (Characterising multi-pointed categories)**.** *A category* E *is multi-pointed with respect to a subcategory* E_0 (*of null objects*) *if and only if the latter satisfies axiom* (ex.0a) *of* 1.5.

When this holds, the kernel of a morphism $f\colon A \to B$ amounts to the pullback of $0_B\colon B_0 \rightarrowtail B$ along f, and the cokernel of f amounts to the pushout of $0^A\colon A \twoheadrightarrow A^0$ along f. Therefore E *is semiexact if and only if all* these *pullbacks and pushouts exist.*

Proof. Follows easily from the previous lemma. □

2 Torsion theories and torsion operators in multi-pointed categories

Torsion theories in abelian categories are dealt with in the book by Borceux [3], Section 1.2. We show here that the basic results can be easily extended to multi-pointed categories.

Throughout this section E is a multi-pointed category. For a category C, writing $X \in \mathsf{C}$ we mean $X \in \mathrm{Ob}\mathsf{C}$.

Definition 2.1 (Torsion theories)**.** By a *torsion theory* in the multi-pointed category E we mean a pair (T, F) of full, replete subcategories of E such that

(tor.1) every morphism $T \to F$, with $T \in \mathsf{T}$ and $F \in \mathsf{F}$, is null,

(tor.2) for every A in E there exists a short exact sequence $T \rightarrowtail A \twoheadrightarrow F$ with $T \in \mathsf{T}$ and $F \in \mathsf{F}$.

The objects of T are called the *torsion objects* of the theory, those of F the *torsion-free objects.* Categorical duality gives a torsion theory $(\mathsf{F}^{\mathrm{op}}, \mathsf{T}^{\mathrm{op}})$ in the opposite category E^{op}.

Condition (tor.1) means that $\mathsf{T} \perp \mathsf{F}$ with respect to the *orthogonality relation* in $\mathrm{Ob}\mathsf{E}$, defined by:

(i) $A \perp B$ if and only if every morphism $A \to B$ is null.

A stronger result, in the presence of (tor.2), will be given in Corollary 2.3(b).

We shall see that the torsion objects are always closed under normal quotients. A torsion theory is said to be *hereditary* if the torsion objects are also closed under normal subobjects.

Proposition 2.2 (The torsion exact sequence)**.** *Let* (T, F) *be a torsion theory in the multi-pointed category* E*.*

(a) *Every object A has a short exact sequence, determined up to isomorphism*

$$T(A) \rightarrowtail A \twoheadrightarrow F(A) \qquad (T(A) \in \mathsf{T} \text{ and } F(A) \in \mathsf{F}), \tag{2.1}$$

and called the torsion exact sequence *of A. (Implicitly, we are making a choice of a* distinguished *sequence for every A, in its isomorphism class.)*

(b) *The normal subobject in this sequence*

$$\tau A\colon T(A) \rightarrowtail A, \tag{2.2}$$

is the greatest normal subobject of A with domain in T. This defines the torsion function $\tau\colon \mathrm{Ob}\mathsf{E} \to \mathrm{Mor}\mathsf{E}$ *of the theory.*

(b*) *The normal quotient*

$$\varphi A\colon A \twoheadrightarrow F(A), \tag{2.3}$$

is the greatest normal quotient of A with codomain in F. This defines the torsion-free function $\varphi\colon \mathrm{Ob}\mathsf{E} \to \mathrm{Mor}\mathsf{E}$ *of the theory.*

(c) *All this determines a subfunctor T and a quotient-functor F of the identity of E.*

Proof. The points (a), (b) and (b*) can be proved by a single argument. By hypothesis there exists a short exact sequence (m,p) as in the diagram below, with $T \in \mathsf{T}$ and $F \in \mathsf{F}$

$$\begin{tikzcd}
T \arrow[r, tail, "m"] & A \arrow[r, two heads, "p"] \arrow[d, equal] & F \\
T' \arrow[u, dashed] \arrow[r, tail, "m'"'] & A &
\end{tikzcd} \tag{2.4}$$

Given any normal monomorphism $m'\colon T' \rightarrowtail A$ with domain in T, the morphism pm' is null, whence $m' \prec m$. Therefore, in our sequence, m is determined as *the* greatest normal subobject of A with domain in T. By duality p is determined as in (b*), and the sequence itself is determined (up to isomorphism).

(c) Given a morphism $f\colon A \to B$, the composite $\varphi B \cdot f\tau A\colon T(A) \to F(B)$ is null. This gives a (unique) commutative diagram with distinguished short exact rows

$$\begin{tikzcd}
TA \arrow[r, tail, "\tau A"] \arrow[d, "Tf"'] & A \arrow[r, two heads, "\varphi A"] \arrow[d, "f"] & FA \arrow[d, "Ff"] \\
TB \arrow[r, tail, "\tau B"'] & B \arrow[r, two heads, "\varphi B"'] & FB
\end{tikzcd} \tag{2.5}$$

and defines the functors T and F on the morphisms of E. □

Corollary 2.3. *Let* (T, F) *be a torsion theory in the multi-pointed category* E*, with torsion function* $\tau A\colon T(A) \rightarrowtail A$ *and torsion-free function* $\varphi A\colon A \twoheadrightarrow F(A)$*.*

(a) *An object* A *belongs to* T *if and only if* τA *is invertible, if and only if* φA *is null.*

(a*) *An object* A *belongs to* F *if and only if* φA *is invertible, if and only if* τA *is null.*

(b) $\mathrm{Ob}\mathsf{T}$ *and* $\mathrm{Ob}\mathsf{F}$ *correspond to each other in the Galois connection determined by the orthogonality relation* $A \perp B$ *in* $\mathrm{Ob}\mathsf{E}$*, of* 2.1(i)*. In other words:*

$$\begin{aligned} \mathrm{Ob}\mathsf{T} &= \{A \in \mathrm{Ob}\mathsf{E} \mid \textit{every map } A \to B \textit{ with } B \in \mathsf{F} \textit{ is null}\}, \\ \mathrm{Ob}\mathsf{F} &= \{B \in \mathrm{Ob}\mathsf{E} \mid \textit{every map } A \to B \textit{ with } A \in \mathsf{T} \textit{ is null}\}. \end{aligned} \tag{2.6}$$

(c) *If* $A \in \mathsf{T}$ *and there is a morphism* $h\colon A \to A'$ *with a null cokernel, then* $A' \in \mathsf{T}$*. In particular,* T *is closed under normal quotients.*

(c*) *If* $B \in \mathsf{F}$ *and there is a morphism* $h\colon B' \to B$ *with a null kernel, then* $B' \in \mathsf{F}$*. In particular,* F *is closed under normal subobjects.*

(d) *An object of* E *belongs to* T *and* F *if and only if it is null.*

Proof. (a) and (a*) follow trivially from the previous proposition.

(b) By (tor.1) every morphism $T \to F$, with $T \in \mathsf{T}$ and $F \in \mathsf{F}$, is null. Let an object A be given, such that every morphism $A \to F$ with $F \in \mathsf{F}$ is null. Then $\varphi A\colon A \twoheadrightarrow F(A)$ is the null normal quotient of A, and the latter belongs to T by (a).

(c) In the given hypotheses, take a morphism $f\colon A' \to B$ with $B \in \mathsf{F}$. Then fh is null, and f is also; therefore $A' \in \mathsf{T}$, by (b).

(d) If A belongs to T and F then 1_A is null. The converse follows from (b). $\square$

Theorem 2.4 (The torsion function)**.** *In the multi-pointed category* E *it is equivalent to assign:*

(i) *a torsion theory* (T, F)*,*

(ii) *a* torsion function $\tau\colon \mathsf{ObE} \to \mathrm{Mor}\mathsf{E}$ *that satisfies the following three axioms:*

(tf.1) *for every A in E, $\tau A\colon T(A) \rightarrowtail A$ is a normal subobject producing a subfunctor T of the identity (that is, every map $f\colon A \to B$ in E has a restriction $T(f)\colon T(A) \to T(B)$,*

(tf.2) *for every A in E, $\tau(TA)$ is invertible (and the functor T can be made idempotent),*

(tf.3) *for every A in E, $T(A/T(A))$ is a null object,*

Given the torsion theory (T, F), the corresponding torsion function is defined as in 2.2(b). *Given the torsion function τ the corresponding torsion theory (T, F) is defined as:*

$$\mathsf{T} = \{A \in \mathsf{E} \mid \tau A \text{ is invertible}\}, \quad \mathsf{F} = \{A \in \mathsf{E} \mid \tau A \text{ is null}\}. \tag{2.7}$$

Proof. (i) $\Rightarrow$ (ii). If (T, F) is a torsion theory and τ the associated function, the properties (tf.1–3) are proved in 2.2 and 2.3.

(ii) $\Rightarrow$ (i). Define T and F as in (2.7). For (tor.1) we take a morphism $f\colon A \to B$ with $A \in \mathsf{T}$ and $B \in \mathsf{F}$; then we have a commutative diagram

$$\begin{tikzcd} TA \arrow[r, tail, "\tau A"] \arrow[d, "Tf"'] & A \arrow[d, "f"] \\ TB \arrow[r, tail, "\tau B"'] & B \end{tikzcd} \tag{2.8}$$

with τA invertible and τB null, so that f is null. For (tor.2) we take the short exact sequence

$$TA \xrightarrow{\tau A} A \xrightarrow{p} A/T(A) \tag{2.9}$$

where TA is in T by (tf.2) and $A/T(A)$ is in F, by (tf.3). $\square$

2.1 The torsion-free function Dually, one can give axioms for a torsion-free function on the multi-pointed category E. It is thus equivalent to assign:

(i) a torsion theory (T, F),

(ii) a *torsion-free function* $\varphi\colon \mathrm{Ob}\mathsf{E} \to \mathrm{Mor}\mathsf{E}$ that satisfies the following three axioms:

(ff.1) for every A in E, $\varphi A\colon A \twoheadrightarrow F(A)$ is a normal quotient producing a quotient functor F of the identity (that is, every map $f\colon A \to B$ in E induces $F(f)\colon F(A) \to F(B)$),

(ff.2) for every A in E, $\varphi(F(A))$ is invertible (and the functor F can be made idempotent),

(ff.3) for every A in E, $F(\mathrm{Ker}\,\varphi A)$ is a null object.

Given a torsion theory (T, F), the corresponding torsion-free function is defined as in (2.3). Given a torsion-free function φ, the corresponding torsion theory (T, F) is defined as:

$$\mathsf{T} = \{A \in \mathsf{E} \,|\, \varphi A \text{ is null}\}, \quad \mathsf{F} = \{A \in \mathsf{E} \,|\, \varphi A \text{ is invertible}\}. \tag{2.10}$$

Definition 2.5 (Torsion operators)**.** More generally a *torsion operator* (τ, φ) on the multi-pointed category E will consist of assigning:

(top.1) for every object A, a natural short exact sequence $(\tau A, \varphi A)$ with central object A.

Naturality, of course, means that for every morphism $f\colon A \to B$ there is a (unique) commutative diagram with distinguished short exact rows

$$\begin{tikzcd} TA \arrow[r, tail, "\tau A"] \arrow[d, "Tf"'] & A \arrow[r, two heads, "\varphi A"] \arrow[d, "f"] & FA \arrow[d, "Ff"] \\ TB \arrow[r, tail, "\tau B"'] & B \arrow[r, two heads, "\varphi B"'] & FB \end{tikzcd} \tag{2.11}$$

This defines the endofunctors T and F of E, as, respectively, a subfunctor and a quotient-functor of the identity.

Equivalently, we can assign *one* of the following functions:

(top.1′) a function $\tau\colon \mathrm{Ob}\mathsf{E} \to \mathrm{Mor}\mathsf{E}$ that satisfies axiom (tf.1) of Theorem 2.4,

(top.1″) a function $\varphi\colon \mathrm{Ob}\mathsf{E} \to \mathrm{Mor}\mathsf{E}$ that satisfies axiom (ff.1) of Section 2.1.

A torsion operator is a torsion theory if and only if the additional axioms of τ in 2.4 (or equivalently those of φ in 2.1) are satisfied. Plainly, this is equivalent to *each* of the following conditions:

(top.2) for every A in E, $\tau(TA)$ and $\varphi(FA)$ are invertible,

(top.2′) for every A in E, $F(TA)$ and $T(FA)$ are null objects.

Let us note that $T(\tau A) = \tau(TA)$ (cancelling the monomorphism τA from the naturality equation $\tau A \cdot T(\tau A) = \tau A \cdot \tau(TA)$); similarly $F\varphi = \varphi F$. Therefore (F, φ) 'is' an idempotent monad, with multiplication $\mu = (F\varphi)^{-1} = (\varphi F)^{-1}$, and dually for (T, τ). Finally, condition (top.2) also amounts to:

(top.2″) (T, τ) is an idempotent comonad and (F, φ) an idempotent monad.

3 Closure operators as torsion operators in categories of pairs

Given a category C with assigned 'distinguished subobjects', we construct a 'category of pairs' C_2 which is multi-pointed with respect to a natural ideal of null morphisms. Torsion operators in C_2 amount to closure operators on the distinguished subobjects of C, while a torsion theory corresponds to the idempotence and weak hereditariness of the associated closure operator.

For closure operators we follow the terminology of Dikranjan and Tholen [6].

3.1 Categories with distinguished subobjects We have recalled the classical category Top_2 of pairs of topological spaces, based on a choice of distinguished subobjects in Top, namely the subspaces. This construction has a natural extension to a *semiexact* category of pairs C_2 over a 'ds-category' C ([13], Section 2.5). Forsaking the existence of all kernels and cokernels, this procedure can be further generalised as follows.

A *ds1-category* is a category C equipped with a set d of *distinguished* subobjects, called *d-subobjects* for short. We assume that:

(ds.0) for each object X, 1_X represents a d-subobject,

(ds.1) if $a \colon A \to X$ and $b \colon B \to X$ represent d-subobjects and $a \leq b$, the monomorphism $u \colon A \to B$ such that $a = bu$ is equivalent to a d-subobject.

We speak of a *ds2-category* if, moreover:

(ds.2) every morphism factorizes through a smallest d-subobject of its codomain.

We shall write d_X for the ordered set of d-subobjects of X, and d'_X for the preordered set of the *d-monomorphisms* with values in X, equivalent to the previous ones. We show below that conditions (ds.0) and (ds.1) are sufficient to construct a multi-pointed category of pairs C_2 and to characterise its short exact sequences, while the additional condition (ds.2) will be used to introduce closure operators in C.

Remark. If in (ds.1) we assume that the monomorphism $u\colon A \to B$ is indeed a d-subobject, our construction below can be simplified (up to equivalence of categories) taking as objects of C_2 the d-subobjects instead of all the d-monomorphisms. This stronger assumption is 'sound' in the usual concrete categories, and gives a category of pairs Top_2 as classically constructed.

3.2 General categories of pairs

Given a ds1-category C, we construct the multi-pointed *category of pairs* $\mathsf{C}_2 = \mathrm{Pair}(\mathsf{C}, \mathsf{d})$ as a full subcategory of the category of morphisms $\mathsf{C}^{\mathbf{2}}$.

Its objects are the distinguished monomorphisms in d' (the union of all d'_X), its morphisms $f = (f, f')\colon a \to b$ are the commutative squares as in the left diagram below, with obvious composition

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed, "h"] & Y \\ A \arrow[u, "a"] \arrow[r, "f'"'] & B \arrow[u, "b"'] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "h"] & B \arrow[r, "b"] \arrow[d, equal] & Y \\ A \arrow[u, "a"] \arrow[r, "f'"'] & B \arrow[r, equal] & B \arrow[u, "b"'] \end{tikzcd} \tag{3.1}$$

Such a morphism $f\colon a \to b$ is assumed to be null if one can insert a map h (necessarily unique), forming a commutative diagram. The null objects are the isomorphisms of C and $f\colon a \to b$ is null if and only if it factorizes through $1\colon B \to B$ as in the right diagram above (or equivalently through $1\colon X \to X$).

The kernel $0_a\colon a_0 \to a$ of 1_a and its cokernel $0^a\colon a \to a^0$ are shown in the left diagram below, proving that C_2 is multi-pointed

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal] & X \arrow[r, equal] & X \arrow[d, equal] \\
A \arrow[r, equal] & A \arrow[u, "a"'] \arrow[r, "a"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "b"] & X \arrow[r, equal] & X \\
A \arrow[u, "u"] \arrow[r, equal] & A \arrow[u, "a"'] \arrow[r, "u"'] & B \arrow[u, "b"']
\end{tikzcd}
\tag{3.2}
$$

The *short exact sequences* of C_2, the *normal subobjects* and the *normal quotients* of the object a are – up to isomorphism – of the form shown in the right diagram above, determined by an inequality $a \leq b$ in d_X (since the monomorphism u is in d', by (ds.1).)

In fact, it is easy to check that $(b, 1_A)\colon c \to a$ is a kernel of $(1_X, c)\colon a \to b$, and the latter a cokernel of the former. Conversely, the diagonal embedding $\mathsf{C} \to \mathsf{C}_2$ (that sends X to 1_X) has for reflector the codomain-projection

$$C\colon \mathsf{C}_2 \to \mathsf{C}, \qquad\qquad C\colon a \mapsto X$$

(with unit $0^a\colon a \to 1_X$); therefore the functor C preserves the existing colimits. Now, if a morphism $(f, f')\colon t \to a$ has a cokernel, the latter is the pushout of $0^t = (1_Y, t)\colon t \to 1_Y$ along (f, f'), and therefore must be of the form $(1_X, u)\colon a \to b$, for some $b \geq a$ in d_X

$$
\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "1"'] & & X \arrow[dr, "1"] & \\
& Y \arrow[rr] & & X \\
T \arrow[uu, "t"] \arrow[rr, dashed] \arrow[dr, "t"'] & & A \arrow[uu, dashed, "a"] \arrow[dr, dashed, "u"] & \\
& Y \arrow[uu, "1"'] \arrow[rr] & & \bullet \arrow[uu, "b"']
\end{tikzcd}
\tag{3.3}
$$

3.3 Closure operators on distinguished subobjects Now we take C to be a ds2-category. We shall simplify notation, denoting distinguished subobjects $a\colon A \to X$, $b\colon B \to X, \ldots$ by their domain $A, B, \ldots$ and writing $A \subset B$ for $a \leq b$; which is appropriate for the usual concrete categories (see the last remark in 3.1).

Following (and extending) the definition of [6] in Chapter 2, by a *closure operator* on the ds2-category C we mean a family $c = (c_X)$ of operators on the ordered sets d_X (for $X \in \mathrm{Ob}\mathsf{C}$)

$$c_X\colon d_X \to d_X, \qquad A \mapsto c_X(A) = \overline{A}, \tag{3.4}$$

that satisfies the following axioms, for every A, B in d_X and every morphism $f\colon X \to Y$

$$\begin{array}{lll} \text{(cl.1)} & A \subset \overline{A} & (\textit{extension}), \\ \text{(cl.2)} & A \subset B \;\;\Rightarrow\;\; \overline{A} \subset \overline{B} & (\textit{monotonicity}), \\ \text{(cl.3)} & f_*(c_X A) \subset c_Y(f_* A) & (\textit{continuity}). \end{array}$$

Here f_*A is the smallest d-subobject of Y through which the composite $A \subset X \to Y$ factorizes. As usual, a subobject A is said to be *closed* in X if $\overline{A} = A$, and *dense* in X if $\overline{A} = X$.

One says that the closure operator c is *idempotent* if it satisfies the following condition (cl.4), and *weakly hereditary* if it satisfies (cl.5):

$$\begin{array}{lll} \text{(cl.4)} & \overline{\overline{A}} \subset \overline{A} & (\textit{idempotence}), \\ \text{(cl.5)} & c_{\overline{A}}(A) = \overline{A} & (\textit{weak hereditariness}), \end{array}$$

Theorem 3.1 (Closure operators as torsion operators)**.** *Let* C *be a ds2-category and* $\mathsf{C}_2 = \mathrm{Pair}(\mathsf{C}, \mathsf{d})$ *its multi-pointed category of pairs. To assign a closure operator* c *on* (*the distinguished subobjects of*) C *is equivalent to assigning a torsion operator* (τ, φ) *in* C_2, *as defined in Section 2. Moreover,* c *is idempotent and weakly hereditary if and only if* (τ, φ) *is a torsion theory.*

Proof. (a) Given a closure operator, we define the associated torsion operator (τ, φ) in C_2 by letting

$$T(X, A) = (\overline{A}, A), \qquad F(X, A) = (X, \overline{A}). \tag{3.5}$$

$$\tau(X, A)\colon (\overline{A}, A) \rightarrowtail (X, A), \quad \varphi(X, A)\colon (X, A) \twoheadrightarrow (X, \overline{A}), \tag{3.6}$$

$$\mathsf{T} = \{(X, A) \in \mathsf{C}_2 \mid A \text{ is dense in } X\} = \{(X, A) \in \mathsf{C}_2 \mid \overline{A} = X\}, \tag{3.7}$$

$$\mathsf{F} = \{(X, A) \in \mathsf{C}_2 \mid A \text{ is closed in } X\} = \{(X, A) \in \mathsf{C}_2 \mid \overline{A} = A\}. \tag{3.8}$$

Then T is indeed a subfunctor of the identity.

(b) Conversely, let (τ, φ) be a torsion operator in C_2. We define the associated closure operator on distinguished subobjects of C by noting that

$T(X, A)$, as a normal subobject of (X, A), determines a d-subobject $c_X(A)$ of X

$$T(X, A) = (c_X(A), A), \qquad\qquad A \subset c_X(A) \subset X. \tag{3.9}$$

This is indeed a closure operator on (C, d), in the sense of 3.3.

- Monotonicity: if $A \subset B$ in d_X, the normal quotient $(X, A) \twoheadrightarrow (X, B)$ induces a normal epimorphism $F(X, A) \to F(X, B)$ (by 2.2(c)), that is, $(X, c_X(A)) \twoheadrightarrow (X, c_X(B)$, which shows that $c_X(A) \subset c_X(B)$.

- Continuity: every morphism $f\colon (X, A) \to (Y, B)$ restricts to a morphism $T(X, A) \to T(Y, B)$, which amounts to saying that $f_*(c_X A) \subset c_Y(f_* A)$.

(c) The two procedures are inverse to each other. Starting from the operator c, the associated torsion operator (τ, φ) defined in (3.5)–(3.8) gives back the original transformation $A \mapsto c_X(A)$. Starting from a torsion operator (τ, φ), the associated operator c defined in (3.9) gives back, in (3.6), the original torsion function $\tau(X, A)\colon T(X, A) \rightarrowtail (X, A)$, whence the original torsion operator.

(d) Finally the following computations, for a closure operator c and the associated torsion operator (τ, φ) defined in (3.5)–(3.8),

$$\begin{gathered} F(T(X, A)) = F(\overline{A}, A) = (\overline{A}, c_{\overline{A}}(A)), \\ T(F(X, A)) = T(X, \overline{A}) = (\overline{\overline{A}}, \overline{A}), \end{gathered} \tag{3.10}$$

show that the weak hereditariness and idempotence of c amount to (τ, φ) being a torsion theory (by (top.2′) in 2.5). □

4 Pre-pointed categories and categories of morphisms

After a brief analysis of pre-pointed categories, defined in 1.1, we show that any category of morphisms $\mathsf{E} = \mathsf{C}^{\mathbf{2}}$ is pre-pointed, and that a 'short pre-exact sequence' in $\mathsf{C}^{\mathbf{2}}$ amounts to a factorization $x = x''.x'$ of a morphism in C.

4.1 Pre-kernels and pre-cokernels Let $(\mathsf{E}, \mathsf{E}_0)$ be a pre-pointed category.

For every morphism $f\colon A \to B$, (a choice of) the pullback of $\kappa B\colon\ B_0 \to B$ along f, as in the left diagram below

$$
\begin{tikzcd}
A \arrow[r, "f"] & B \\
Kf \arrow[u, "kf"] \arrow[r, "g"'] & B_0 \arrow[u, "\kappa B"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\gamma A"'] & B \arrow[d, "cf"] \\
A_0 \arrow[r, "g"'] & Cf
\end{tikzcd}
\tag{4.1}
$$

will be written as $kf\colon Kf \to A$ and called a *pre-kernel* of f. We only fix the choice $k(1_A) = \kappa A$. We show below that pre-kernels are weak kernels, in a natural way. But this property puts a weak constraint on the object Kf, while pullbacks are determined up to isomorphism, of course.

Dually, the pushout of $\gamma A\colon A \to A^0$ along f (see the right diagram above) will be written as $cf\colon B \to Cf$ and called the *pre-cokernel* of f. Again we fix $c(1_A) = \gamma A$.

A sequence

$$M \xrightarrow{m} A \xrightarrow{p} P \tag{4.2}$$

will be said to be *short pre-exact* if m is a pre-kernel of p and p is a pre-cokernel of m.

Proposition 4.1. *Let* $\mathsf{E} = (\mathsf{E}, \mathsf{E}_0)$ *be a pre-pointed category. For every morphism* $f\colon A \to B$, *if the pre-kernel* $kf\colon Kf \to A$ *exists then it is a* weak kernel *of* f *with respect to the ideal of null morphisms of* E*.This means that* $f.kf$ *is null, and for every map* u *in* E *such that* fu *is null, there is* some *map* v *such that* $u = kf.v$, *as in the left diagram below*

$$
\begin{tikzcd}
Kf \arrow[r, "kf"] & A \arrow[r, "f"] & B \\
& \bullet \arrow[ul, dashed, "v"] \arrow[u, "u"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "cf"] \arrow[d, "u"'] & Cf \arrow[dl, dashed, "v"] \\
& \bullet &
\end{tikzcd}
\tag{4.3}
$$

Dually $cf\colon B \to Cf$, *if it exists, is a* weak cokernel *of* f, *with a weak universal property as in the right diagram above.*

In particular, $\kappa A = k(1_A)\colon A_0 \to A$ *is a weak kernel of* 1_A *and* $\gamma A = c(1_A)\colon A \to A^0$ *a weak cokernel of the latter.*

Proof. If fu is null, then it factorizes through $\kappa B\colon B_0 \to B$ as $fu = \kappa B.g$, whence the pair (u, g) factorizes uniquely through the pullback Kf and u just factorizes through kf. $\square$

Proposition 4.2. *Let* $\mathsf{E} = (\mathsf{E}, \mathsf{E}_0)$ *be a pre-pointed category with all pre-kernels and all pre-cokernels.*

(a) *The universal property of pullbacks determines a functor* $K\colon \mathsf{E}^{\mathbf{2}} \to \mathsf{E}$ *defined on the category of morphisms of* E*, and a natural transformation* $k\colon K \to \mathrm{Dom}\colon \mathsf{E}^{\mathbf{2}} \to \mathsf{E}$ *given by the family* $(kf)_f$*.*

Equivalently, we can present the pair (K, k) *as a functor* $k\colon \mathsf{E}^{\mathbf{2}} \to \mathsf{E}^{\mathbf{2}}$ *that sends the morphism* $(u, v)\colon f \to g$ *of* $\mathsf{E}^{\mathbf{2}}$ *to the morphism* $k(u, v) = (K(u, v), u)\colon kf \to kg$*, as in the left square below*

$$\begin{tikzcd} Kf \arrow[r, "kf"] \arrow[d, "K(u,v)"'] & A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[r, "cf"] \arrow[d, "v"] & Cf \arrow[d, "C(u,v)"] \\ Kg \arrow[r, "kg"'] & C \arrow[r, "g"'] & D \arrow[r, "cg"'] & Cg \end{tikzcd} \tag{4.4}$$

(a*) *Dually, the universal property of pushouts determines a functor* $C\colon \mathsf{E}^{\mathbf{2}} \to \mathsf{E}$ *and a natural transformation* $c\colon \mathrm{Cod} \to C\colon \mathsf{E}^{\mathbf{2}} \to \mathsf{E}$ *given by the family* $(cf)_f$*. Equivalently, we have a functor* $c\colon \mathsf{E}^{\mathbf{2}} \to \mathsf{E}^{\mathbf{2}}$ *that sends* $(u, v)\colon f \to g$ *to* $c(u, v) = (v, C(u, v))\colon cf \to cg$*, as in the right square above.*

(b) E *is semiexact* (*with respect to the ideal of null morphisms of* E) *if and only if* E *is multi-pointed* (*with respect to* E_0)*: all the unit components* $\kappa A\colon A_0 \to A$ *are mono and all the counit components* $\gamma A\colon A \to A^0$ *are epi. Then the pre-kernels* kf *are kernels* (*and monomorphisms*)*, while the pre-cokernels* cf *are cokernels* (*and epimorphisms*)*.*

(c) E *is p-semiexact if and only if* E *is pointed.*

Proof. Obvious. □

Remark 4.3. Therefore, in a pre-pointed category E with all pre-kernels, the family $(kf)_f$ can be extended to a *natural* assignment of weak kernels, that is, a natural transformation $k\colon K \to \mathrm{Dom}\colon \mathsf{E}^{\mathbf{2}} \to \mathsf{E}$ such that, for every map f in $\mathsf{E}^{\mathbf{2}}$, $kf\colon Kf \to \mathrm{Dom} f$ is a weak kernel of f.

Again, let us note that the weak universal property in (a) ties down the object Kf in a weak form, and would not give any naturality, by itself. The root of the question is the fact that $\kappa A\colon A_0 \to A$ is determined by the universal property of the adjunction $U \dashv D$ in (1.11); this means that κA, besides being a weak kernel of 1_A, *also satisfies the restricted cancellation property of* 1.2(a).

4.2 Categories of morphisms We begin to study the category of morphisms $\mathsf{E} = \mathsf{C}^{\mathbf{2}}$ of a category C, along the lines above. An object of E is a morphism $x\colon X' \to X''$ of C. A morphism $f = (f', f'')\colon x \to y$ is a commutative square of C. E will always be equipped with the full subcategory E_0 of *null objects*: the isomorphisms of C. We now prove that $\mathsf{E} = (\mathsf{E}, \mathsf{E}_0)$ is pre-pointed.

E_0 is indeed a full replete reflective and coreflective subcategory of E: the embedding $U\colon \mathsf{E}_0 \to \mathsf{E}$ has adjoints $C \dashv U \dashv D$ with

$$\begin{array}{ll} D(x\colon X' \to X'') = 1_{X'}, & C(x\colon X' \to X'') = 1_{X''}, \\ \kappa x\colon 1_{X'} \to x, & \gamma x\colon x \to 1_{X''}, \end{array} \tag{4.5}$$

$$\begin{tikzcd} X' \arrow[r, "1"] \arrow[d, "1"'] & X' \arrow[r, "x"] \arrow[d, "x"] & X'' \arrow[d, "1"] \\ X' \arrow[r, "x"'] & X'' \arrow[r, "1"'] & X'' \end{tikzcd}$$

A morphism $f\colon x \to y$ is null if and only if it factorizes through $\kappa y\colon 1_{Y'} \to y$, which amounts to the existence of a morphism h making the left diagram below commutative

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "x"'] & Y' \arrow[d, "y"] \\ X'' \arrow[r, "f''"'] \arrow[ur, dashed, "h"'] & Y'' \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "x"'] & Y' \arrow[r, "1"] \arrow[d, "1"] & Y' \arrow[d, "y"] \\ X'' \arrow[r, "h"'] & Y' \arrow[r, "y"'] & Y'' \end{tikzcd} \tag{4.6}$$

Equivalently, f factorizes through $\gamma x\colon x \to 1_{X''}$.

Note that $\mathsf{C}^{\mathbf{2}}$ is not multi-pointed, in general: the counit κ need not be pointwise mono, nor the unit γ pointwise epi.

Any category of pairs over C is a full subcategory of $\mathsf{C}^{\mathbf{2}}$, with a consistent choice of null objects and null morphisms.

4.3 Pre-kernels and pre-cokernels If the category C has pullbacks and pushouts, the same holds for its category of morphisms $\mathsf{E} = \mathsf{C}^{\mathbf{2}}$, which is thus a pre-pointed category with all pre-kernels and pre-cokernels (with respect to E_0).

More precisely, without any hypothesis on C, it is easy to see that a morphism $f\colon x \to y$ in E has a pre-kernel if and only if, in the left diagram below, the pair (y, f'') has a pullback (P, u, x''): then we define Kf as the induced map $Kf\colon X' \to P$, and $kf = (1, x'')\colon Kf \to x$. Note that fkf is proved to be null by u

$$
\begin{tikzcd}
X' \arrow[r, "1"] \arrow[d, dashed, "Kf"'] & X' \arrow[r, "f'"] \arrow[d, "x"'] & Y' \arrow[d, "y"] \\
P \arrow[r, "x''"'] \arrow[urr, dashed, "u"] & X'' \arrow[r, "f''"'] & Y''
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"'] & Y' \arrow[r, "y'"] \arrow[d, "y"'] & Q \arrow[d, dashed, "Cf"] \\
X'' \arrow[r, "f''"'] \arrow[urr, dashed, "v"] & Y'' \arrow[r, "1"'] & Y''
\end{tikzcd}
\qquad (4.7)
$$

Dually, the morphism $f\colon x \to y$ has a pre-cokernel cf if and only if, in the right diagram above, the pair (f', x) has a pushout (Q, y', v): then we define Cf as the induced map $Cf\colon Q \to Y''$ and $cf = (y', 1)\colon y \to Cf$. Again $cf.f$ is proved to be null by v.

Therefore a pre-kernel of any morphism $f\colon x \to y$ of E is necessarily of the form $(1, x'')\colon x' \to x$ where $x = x''x'$ is a factorization of x in C. More precisely, it is of this form up to an arbitrary isomorphism $(i, j)\colon \overline{x} \to x'$ of E

$$
\begin{tikzcd}
\overline{X} \arrow[r, "i"] \arrow[d, "\overline{x}"'] & X' \arrow[r, "1"] \arrow[d, "x"'] & X' \arrow[d, "x"] \\
\overline{E} \arrow[r, "j"'] & E \arrow[r, "x''"'] & X''
\end{tikzcd}
\qquad (4.8)
$$

This remark clearly shows that the 'special form' $(1, x'')\colon x' \to x$ does *not* give, by itself, a choice of pre-kernels: it only fixes $i = 1$ but leaves free the other isomorphism j. (To obtain a 'distinguished' form of pre-kernels one should also choose a distinguished form for any factorization $x = x''x'$ in C, which cannot be done in a natural way in the usual categories of structured sets.)

4.4 Short pre-exact sequences For every object $x\colon X' \to X''$ of the pre-pointed category $\mathsf{E} = \mathsf{C}^{\mathbf{2}}$, a factorization $x = x''.x'$ in C gives a *short pre-exact sequence* of $\mathsf{C}^{\mathbf{2}}$, that is, a sequence $(f, g)\colon \bullet \to x \to \bullet$ where f is the pre-kernel of g and the latter is the pre-cokernel of f

$$\begin{tikzcd}
X' \arrow[r, "1"] \arrow[d, dashed, "x'"'] & X' \arrow[r, "x'"] \arrow[d, "x"'] & E \arrow[d, "x''"] \\
E \arrow[r, "x''"'] \arrow[urr, dashed, "1"'] & X'' \arrow[r, "1"'] & X''
\end{tikzcd} \tag{4.9}$$

Conversely, we have seen that *any* short pre-exact sequence of $\mathsf{C}^{\mathbf{2}}$ is of this type, up to isomorphism in E.

Without any hypothesis on the category C, a sequence $\bullet \to x \to \bullet$ of the form (4.9) will be called a *special short pre-exact sequence* of $\mathsf{C}^{\mathbf{2}}$ (with respect to E_0, the full subcategory of isomorphisms of C).

5 Functorial factorizations as torsion operators in categories of morphisms

We show now that a functorial factorization in a category C can be viewed as a torsion operator in the pre-pointed category $\mathsf{E} = \mathsf{C}^{\mathbf{2}}$. Such a torsion operator is algebraic, or is a torsion theory, if and only if the functorial factorization is algebraic, or is a factorization system.

For the sake of uniformity, as expressed in the tables of the Introduction, we are slightly adapting the existing terminology on weak factorization systems, from [2, 4, 10, 14, 21]. 'Algebraic factorizations', in the present sense, were introduced in [14] under the name of 'natural weak factorization systems', and studied in [4, 5, 10] as 'algebraic weak factorization systems'. We also note that a 'functorial realisation of a weak factorization system', in the sense of [21], is intermediate between functorial and algebraic factorizations, in the present sense.

The domain and codomain functors of a category of morphisms are written as Dom and Cod.

5.1 Torsion operators and torsion theories (a) Extending Definition 2.5, a *torsion operator* (ε, η) in the pre-pointed category $\mathsf{E} = (\mathsf{E}, \mathsf{E}_0)$ will consist of:

(to.1) two endofunctors $L, R \colon \mathsf{E} \to \mathsf{E}$ and two natural transformations $\varepsilon \colon L \to 1$ and $\eta \colon 1 \to R$ such that, for every object A, the pair $(\varepsilon A, \eta A)$ is a short pre-exact sequence of E (with central object A)

$$LA \xrightarrow{\varepsilon A} A \xrightarrow{\eta A} RA \tag{5.1}$$

Equivalently, we can assign

(to.1′) two functors $\varepsilon, \eta \colon \mathsf{E} \to \mathsf{E}^{\mathbf{2}}$ with $\mathrm{Cod}.\varepsilon = \mathrm{id}\mathsf{E} = \mathrm{Dom}.\eta$, that satisfy the 'pre-exactness' condition above.

One recovers $L = \mathrm{Dom}.\varepsilon$ and $R = \mathrm{Cod}.\eta$. Then, for every morphism $f \colon A \to B$ in E, εf and ηf are the 'naturality squares' of the transformations ε and η

$$\begin{tikzcd} LA \arrow[r, "\varepsilon A"] \arrow[d, "Lf"'] & A \arrow[r, "\eta A"] \arrow[d, "f"'] & RA \arrow[d, "Rf"] \\ LB \arrow[r, "\varepsilon B"'] & B \arrow[r, "\eta B"'] & RB \end{tikzcd} \tag{5.2}$$

(b) A *torsion theory* in the pre-pointed category $\mathsf{E} = (\mathsf{E}, \mathsf{E}_0)$ is a torsion operator (ε, η) where (L, ε) and (R, η) are, respectively, an idempotent comonad and an idempotent monad. In other words, we are requiring that:

(to.2) $\quad L\varepsilon = \varepsilon L$ is invertible, $\qquad\qquad R\eta = \eta R$ is invertible.

If E is multi-pointed this agrees with the characterisation of torsion theories given at the end of 2.5.

(c) As an intermediate notion, an *algebraic torsion operator* $(\varepsilon, \delta, \eta, \mu)$ in the pre-pointed category $\mathsf{E} = (\mathsf{E}, \mathsf{E}_0)$ will consist of

(i) a torsion operator (ε, η), as defined above,

(ii) a comonad $L = (L, \varepsilon, \delta)$ on E that extends $\varepsilon \colon L \to 1$,

(iii) a monad $R = (R, \eta, \mu)$ on E that extends $\eta \colon 1 \to R$.

Again this is a torsion theory when the comonad L and the monad R are idempotent (that is, when δ and μ are invertible).

Proposition 5.1. *If E is a multi-pointed category, an algebraic torsion operator $(\varepsilon, \delta, \eta, \mu)$ in E gives a torsion theory (ε, η), with trivial additional data:*

$$\delta = (\varepsilon L)^{-1}, \qquad\qquad \mu = (\eta R)^{-1}.$$

Proof. Each component $\eta RA\colon RA \twoheadrightarrow R^2A$ is epi and a section of μA, whence ηR is invertible, and the monad (R, η, μ) is idempotent, with $\mu = (\eta R)^{-1}$. Similarly, εL is invertible, and the comonad (L, ε, δ) is idempotent, with $\delta = (\varepsilon L)^{-1}$. □

5.2 Functorial factorizations Let C be any category and $\mathsf{E} = \mathsf{C}^{\mathbf{2}}$ its pre-pointed category of morphisms (see 4.2).

A *functorial factorization* (L, R) in the category C consists of two endofunctors $L, R\colon\ \mathsf{C}^{\mathbf{2}} \to \mathsf{C}^{\mathbf{2}}$ such that:

(i) $\mathrm{Dom}.L = \mathrm{Dom}$, $\quad \mathrm{Cod}.L = \mathrm{Dom}.R\ \ (= E)$, $\quad \mathrm{Cod}.R = \mathrm{Cod}$,

(ii) for every morphism $x\colon X' \to X''$ of C, $x = Rx.Lx$.

Equivalently, we have a functor $E\colon \mathsf{C}^{\mathbf{2}} \to \mathsf{C}$ and two natural transformations $L\colon\ \mathrm{Dom} \to E$ and $R\colon E \to \mathrm{Cod}$ satisfying (ii).

These data can be reorganised as a pair (ε, η) of natural transformations $\varepsilon\colon L \to 1\colon \mathsf{E} \to \mathsf{E}$ and $\eta\colon 1 \to R\colon \mathsf{E} \to \mathsf{E}$ such that, for every object $x\colon X' \to X''$ in $\mathsf{C}^{\mathbf{2}}$, the pair $(\varepsilon x, \eta x)$ is a short pre-exact sequence of E, and actually a special one (in the sense of 4.4)

$$Lx \xrightarrow{\varepsilon x} x \xrightarrow{\eta x} Rx$$

$$\begin{tikzcd}
X' \arrow[r, "1"] \arrow[d, "Lx"'] & X' \arrow[r, "Lx"] \arrow[d, "x"] & Ex \arrow[d, "Rx"] \\
Ex \arrow[r, "Rx"'] & X'' \arrow[r, "1"'] & X''
\end{tikzcd} \tag{5.3}$$

$$\varepsilon x = (1, Rx)\colon Lx \to x, \qquad\qquad \eta x = (Lx, 1)\colon x \to Rx.$$

But every short exact sequence in $\mathsf{E} = \mathsf{C}^{\mathbf{2}}$ is of this type, up to isomorphism. We have thus proved the following characterisation.

Theorem 5.2 (Functorial factorizations as torsion operators)**.** *A functorial factorization* (L, R) *in the category* C *amounts to a torsion operator* (ε, η) *in* $\mathsf{C}^{\mathbf{2}}$ *(as defined in* 5.1*), where the natural transformations* $\varepsilon\colon L \to 1$ *and* $\eta\colon 1 \to R$ *are defined as in* (5.3)*.*

The action of the functors $L, R\colon \mathsf{C}^{\mathbf{2}} \to \mathsf{C}^{\mathbf{2}}$ on a morphism $f = (f', f'')\colon x \to y$ of $\mathsf{C}^{\mathbf{2}}$ is

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "Lx"'] & Y' \arrow[d, "Ly"] \\ Ex \arrow[r, "{E(f',f'')}"] \arrow[d, "Rx"'] & Ey \arrow[d, "Ry"] \\ X'' \arrow[r, "f''"'] & Y'' \end{tikzcd} \qquad \begin{aligned} &Lf = (f', E(f', f''))\colon Lx \to Ly, \\ &Rf = (E(f', f''), f'')\colon Rx \to Ry. \end{aligned} \tag{5.4}$$

Proof. The first part was proved above; the second plainly follows from 5.2(i). □

Definition 5.3 (Algebraic factorizations)**.** An *algebraic factorization* $(L, R, \varepsilon, \delta, \eta, \mu)$ in the category C is a functorial factorization (L, R) where the functors $L, R\colon \mathsf{C}^{\mathbf{2}} \to \mathsf{C}^{\mathbf{2}}$ and the derived natural transformations $\varepsilon\colon L \to 1$ and $\eta\colon 1 \to R$ defined as in (5.3) are enriched to a comonad structure (L, ε, δ) and a monad structure (R, η, μ), respectively.

Writing down the axioms of the comonad L and the monad R, we have

$$\varepsilon Lx \cdot \delta x = \mathrm{id}Lx = L\varepsilon x \cdot \delta x, \quad \mu x \cdot \eta Rx = \mathrm{id}Rx = \mu x \cdot R\eta x, \tag{5.5}$$

$$\delta Lx \cdot \delta x = L\delta x \cdot \delta x, \qquad \mu x \cdot \mu Rx = \mu x \cdot R\mu x. \tag{5.6}$$

This notion was introduced in [14] as a 'natural weak factorization system'. We shall see in 5.3 that these data automatically have a distributive law $\lambda\colon LR \to RL$, as considered in [4, 5, 10].

Theorem 5.4 (Algebraic factorizations as algebraic torsion operators)**.** *Let (L, R) be a functorial factorization in C and (ε, η) the corresponding torsion operator in $\mathsf{E} = \mathsf{C}^{\mathbf{2}}$.*

(a) *Enriching the first to an algebraic factorization $(L, R, \varepsilon, \delta, \eta, \mu)$ in C, as defined above, is the same as enriching the second to an algebraic torsion operator $(\varepsilon, \delta, \eta, \mu)$, as defined in 5.1(c).*

(b) *Then the four natural transformations $\varepsilon, \delta, \eta, \mu$ have components as in the diagram below*

$$L^2x \xleftarrow{\delta x} Lx \xrightarrow{\varepsilon x} x \xrightarrow{\eta x} Rx \xleftarrow{\mu x} R^2x \tag{5.7}$$

$$\begin{tikzcd}
X' \arrow[r, equal] \arrow[d, "L^2x"'] & X' \arrow[r, equal] \arrow[d, "Lx"'] & X' \arrow[r, "Lx"] \arrow[d, "x"] & Ex \arrow[d, "Rx"] & ERx \arrow[l, "\mu'x"'] \arrow[d, "R^2x"] \\
ELx & Ex \arrow[l, "\delta''x"] \arrow[r, "Rx"'] & X'' \arrow[r, equal] & X'' \arrow[r, equal] & X''
\end{tikzcd} \tag{5.8}$$

$$\delta x = (1, \delta''x)\colon Lx \to L^2x, \quad \mu x = (\mu'x, 1)\colon R^2x \to Rx.$$

(c) *An algebraic factorization in* C *amounts to*

- *three functors* $L, R\colon \mathsf{C}^{\mathbf{2}} \to \mathsf{C}^{\mathbf{2}}$ *and* $E\colon \mathsf{C}^{\mathbf{2}} \to \mathsf{C}$,

- *two natural transformations* $\delta''\colon E \to EL$ *and* $\mu'\colon ER \to E$,

such that, for every morphism $x\colon X' \to X''$ *of* C,

(i) $Lx\colon X' \to Ex, \quad Rx\colon Ex \to X''$,

(ii) $x = Rx \cdot Lx, \quad \delta''x \cdot Lx = L^2x, \quad Rx \cdot \mu'x = R^2x$,

(iii) $RLx \cdot \delta''x = \mathrm{id}Ex = E(1_{X'}, Rx) \cdot \delta''x$,
$\mu'x \cdot LRx = \mathrm{id}Ex = \mu'x \cdot E(Lx, 1_{X''})$,

(iv) $\delta''Lx \cdot \delta''x = E(1_{X'}, \delta''x) \cdot \delta''x, \quad \mu'x \cdot \mu'Rx = \mu'x \cdot E(\mu'x, 1_{X''})$,

$$\begin{tikzcd}
EL^2x & ELx \arrow[l, "\delta''Lx"'] \arrow[r, "RLx"] & Ex \\
ELx \arrow[u, "E\delta x"] & Ex \arrow[l, "\delta''x"] \arrow[u, "\delta''x"] \arrow[ur, "1"'] \arrow[r, "\delta''x"'] & ELx \arrow[u, "E\varepsilon x"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Ex \arrow[r, "LRx"] \arrow[d, "E\eta x"'] \arrow[dr, "1"'] & ERx \arrow[d, "\mu'x"] & ER^2x \arrow[l, "E\mu x"'] \arrow[d, "R^2x"] \\
ERx \arrow[r, "\mu'x"'] & Ex & ERx \arrow[l, "\mu'x"]
\end{tikzcd}$$

Proof. Point (a) is obvious. As to (b), we already know from 5.2 that the components of ε and δ are as shown in diagram (5.8). Since the first component of ε is an identity, the counit axiom of L (see (5.5)) implies that the same holds for δ, as in the diagram. Dually for μ.

For (c), conditions (i), (ii) allow us to reconstruct the natural transformations $\varepsilon, \delta, \eta, \mu$, as in diagram (5.8). The left part of condition (iii) is the counit axiom of L on the non-trivial components $\varepsilon'' = R$ and δ'', taking into account that, by (5.3):

$$(L(\varepsilon x))'' = E(\varepsilon x) = E(1_{X'}, Rx).$$

Dually, the right-hand part of condition (iii) is the unit axiom of R on the non-trivial components $\eta' = L$ and μ'.

Finally, condition (iv) is the (co)associativity axiom of L and R (see (5.6)) on the non-trivial components δ'' and μ', after noting that

$$(L(\delta x))'' = E(\delta x) = E(1_{X'}, \delta'' x), \quad (R(\mu x))' = E(\mu x) = E(\mu' x, 1_{X''}).$$

□

5.3 The distributive law Continuing the analysis of the algebraic case, these components $\delta'' x$ and $\mu' x$ give raise to a natural transformation $\lambda \colon LR \to RL \colon \mathsf{C}^{\mathbf{2}} \to \mathsf{C}^{\mathbf{2}}$, with $\lambda' x = \delta'' x$ and $\lambda'' x = \mu' x$

$$\begin{tikzcd} Ex \arrow[r, "\delta'' x"] \arrow[d, "LRx"'] \arrow[dr, dashed, "1"] & ELx \arrow[d, "RLx"] \\ ERx \arrow[r, "\mu' x"'] & Ex \end{tikzcd} \tag{5.9}$$

The latter is automatically a distributive law of the comonad L over the monad R [1], that is, it makes the following diagrams commutative

$$\begin{tikzcd} L \arrow[r, equal] \arrow[d, "L\eta"'] & L \arrow[d, "\eta L"] \\ LR \arrow[r, "\lambda"] \arrow[d, "\varepsilon R"'] & RL \arrow[d, "R\varepsilon"] \\ R \arrow[r, equal] & R \end{tikzcd} \qquad \begin{tikzcd} LR^2 \arrow[r, "\lambda R"] \arrow[d, "L\mu"'] & RLR \arrow[r, "R\lambda"] & R^2 L \arrow[d, "\mu L"] \\ LR \arrow[rr, "\lambda"] \arrow[d, "\delta R"'] & & RL \arrow[d, "R\delta"] \\ L^2 R \arrow[r, "L\lambda"'] & LRL \arrow[r, "\lambda L"'] & RL^2 \end{tikzcd} \tag{5.10}$$

In fact, the upper part of the first diagram comes out of:

$$\begin{aligned} (\lambda x)' \cdot (L\eta x)' &= \delta'' x \cdot (\eta x)' = \delta'' x \cdot Lx = \delta'' x \cdot E(1, x) \\ &= EL(1, x) \cdot \delta''(\mathrm{id}X') = L^2 x = (\eta L x)', \\ (\lambda x)'' \cdot (L\eta x)'' &= \mu' x \cdot E(Lx, 1_{X''}) = \mathrm{id}Ex = (\eta L x)'', \end{aligned}$$

because of the naturality of $\delta'' \colon E \to EL$ over the map $(1, x) \colon \mathrm{id}X' \to x$, and the unit axiom of μ, as expressed in 5.4(iii).

The lower part of the same diagram is dealt in a dual way. The commutativity of the second diagram comes from the (co)associativity axioms of δ and μ.

Theorem 5.5 (Factorisations systems as torsion theories). *A torsion operator (ε, η) in the pre-pointed category $\mathsf{E} = \mathsf{C}^{\mathbf{2}}$ is a torsion theory if and only if the associated functorial factorization (L, R) gives an (orthogonal) factorization system $(\mathcal{E}, \mathcal{M})$ in C, with*

$$\begin{aligned} \mathcal{E} &= \{f \in \mathrm{Mor}\mathsf{C} \mid R(f) \text{ is an isomorphism}\}, \\ \mathcal{M} &= \{f \in \mathrm{Mor}\mathsf{C} \mid L(f) \text{ is an isomorphism}\}. \end{aligned} \tag{5.11}$$

Proof. This is essentially Theorem 3.2 of [14].

Let (ε, η) be a torsion operator in E, with $\varepsilon \colon L \to 1$ and $\eta \colon 1 \to R$. We can assume that ε, η are as in diagram (5.3), determined by two functors $L, R \colon \mathsf{C}^{\mathbf{2}} \to \mathsf{C}^{\mathbf{2}}$ which form a functorial factorization in C. If (L, ε) is an idempotent comonad and (R, η) an idempotent monad, then, applying the theorem mentioned above, L and R produce a factorization system as in (5.11). The converse is obvious. □

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While finishing this paper, we learnt from S. Mantovani that she spoke about "Torsion theories for crossed modules" in a talk at Louvain-la-Neuve, in 2015 [18].

The approach is similar to the present one, based on the [15] framework of a category equipped with a reflective-coreflective subcategory of 'trivial objects'. Her talk included results on factorization systems in a category C with pullbacks and pushouts, characterised as torsion theories in the category of morphisms in C.

Our results are also related to a paper by B.J. Gardner [9], based on radical theory instead of torsion theories, and to a paper by W. Tholen [22] focused on the opposition connected-totally disconnected.

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