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Classification of monoids by Condition (PWP_{ssc}) of right acts

P. Khamechi, H. Mohammadzadeh Saany, and L. Nouri*

Abstract. Condition (PWP) which was introduced by Laan is related to the concept of flatness of acts over monoids. Golchin and Mohammadzadeh introduced Condition (PWP_E) as a generalization of Condition (PWP).

In this paper, we introduce Condition (PWP_{ssc}) which is much easier to check than Conditions (PWP) and (PWP_E) and does not imply them. Also principally weakly flat is a generalization of this condition. At first, general properties of Condition (PWP_{ssc}) will be given. Finally a classification of monoids will be given for which all (cyclic, monocyclic) acts satisfy Condition (PWP_{ssc}) and also a classification of monoids S will be given for which all right S-acts satisfying some other flatness properties have Condition (PWP_{ssc}).

1 Introduction

For a monoid S, with 1 as its identity, a set A (we consider nonempty) is called a right S-act, usually denoted by A_S (or simply A), if S acts on A

* Corresponding author
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unitarian from the right, that is, there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and $s, t \in S$. Left acts are defined dually. The study of flatness properties of S-acts in general began in the early 1970s and a comprehensive survey of this research (up until the year 2000) is found in [14]. In [16], the principally weak form of Condition (P) is defined, called Condition (PWP), and in [8] the weak form of Condition (PWP) is defined, called Condition (PWP_E). In this paper we introduce Condition (PWP_{ssc}) and compare it with principally weakly flat, Condition (PWP) and Condition (PWP_E). At first general properties of this condition will be given and finally a classification of monoids will be given for which all (cyclic, monocyclic) acts satisfy Condition (PWP_{ssc}) and also a classification of monoids S will be given for which all right S-acts satisfying some other flatness properties have Condition (PWP_{ssc}).

From now on by S-act we mean right S-act. Throughout this paper, S will always denote a monoid and A an S-act. For basic definitions and terminologies relating to semigroups and acts over monoids we refer the reader to [12] and [14].

2 General properties

In this section we introduce Condition (PWP_{ssc}) and give some results of it. We show that this condition can be transferred from the product of S-acts to their components and we give equivalences for which S_S^n satisfies Condition (PWP_{ssc}). We show that Condition (PWP_{ssc}) implies principal weak flatness but the converse is not true. For left PSF monoid, we show that the converse is also true. Also we see that Condition (PWP_{ssc}) does not imply Condition (PWP) (Condition (PWP_E)) and vise versa.

An element s of S is called *right e-cancellable*, for an idempotent $e \in S$, if $s = es$ and $ker\rho_s \leq ker\rho_e$ (ρ_x is the right translation on S, for every $x \in S$, that is, $\rho_x : S \to S$, $t \mapsto tx$, for every $t \in S$). S is called *left PP* if every principal left ideal of S is projective as a left S-act. This is equivalent to saying that every element $s \in S$ is right e-cancellable for some idempotent $e \in S$ (see [5]). S is called *left PSF* if every principal left ideal of S is strongly flat as a left S-act. This is equivalent to saying that S is right semi-cancellative, that is, whenever $su = s'u$, for $s, s', u \in S$, there exists $r \in S$ such that $u = ru$ and $sr = s'r$ (see [17]).

We recall, from [14], that A satisfies *Condition* (E) if for all $a \in A$, $s, s' \in S$,

$$as = as' \Rightarrow (\exists a' \in A)\ (\exists u \in S)\ (a = a'u \text{ and } us = us'),$$

and A satisfies *Condition* (P) if for all $a, a' \in A$, $s, s' \in S$,

$$as = a's' \Rightarrow (\exists a'' \in A)\ (\exists u, u' \in S)\ (a = a''u,\ a' = a''u', \text{ and } us = u's'),$$

and A is *strongly flat* if and only if it satisfies both Conditions (P) and (E).

Recall, from [16], that A satisfies *Condition* (PWP) if for all $a, a' \in A$, $s \in S$,

$$as = a's \Rightarrow (\exists a'' \in A)\ (\exists u, v \in S)\ (a = a''u,\ a' = a''v, \text{ and } us = vs).$$

Definition 2.1. We say that A satisfies *Condition strongly semi cancellative-*(PWP) or *Condition* (PWP_{ssc}) if for all $a, a' \in A$, $s \in S$,

$$as = a's \Rightarrow (\exists u \in S)\ (au = a'u, \text{ and } us = s).$$

Theorem 2.2. *The following statements hold*:

(1) *The one element act* Θ_S *satisfies Condition* (PWP_{ssc}).

(2) *If* A *satisfies Condition* (PWP_{ssc}), *then every subact of* A *satisfies it.*

(3) *If* A *satisfies Condition* (PWP_{ssc}), *then every retract of* A *satisfies it.*

(4) *If* $\prod_{i\in I} A_i$, *where* A_i, $i \in I$, *are* S*-acts, satisfies Condition* (PWP_{ssc}), *then* A_i *satisfies it, for every* $i \in I$.

(5) $\coprod_{i\in I} A_i$, *where each* A_i *is an* S*-act, satisfies Condition* (PWP_{ssc}) *if and only if* A_i *satisfies Condition* (PWP_{ssc}), *for every* $i \in I$.

(6) *If* $\{B_i \mid i \in I\}$ *is a chain of subacts of* A *and every* B_i, $i \in I$, *satisfies Condition* (PWP_{ssc}), *then* $\bigcup_{i\in I} B_i$ *satisfies it.*

(7) S_S *satisfies Condition* (PWP_{ssc}) *if and only if* S *is right semi-cancellative.*

Proof. Proofs are obvious. □

Recall that for any nonempty set I, S_S^I is the product of a family of S in **Act**-S.

The converse of part (4) of Theorem 2.2 is not true. Consider the monoid S of Example 1.6 of [23]. S is left PSF and so S_S satisfies Condition (PWP_{ssc}), by part (4) of Theorem 2.2, while S_S^I is not principally weakly flat, by [23, Proposition 1.5], and so it does not satisfy Condition (PWP_{ssc}), by part (1) of Theorem 2.8.

An element $s \in S$ *acts injectively on* A if $as = a's$, $a, a' \in A$, implies $a = a'$. If every $s \in S$ acts injectively on A, then we say that S acts injectively on A (see [14]).

The proof of Propositions 2.3 is clear.

Proposition 2.3. *If S acts injectively on A_i, for every $i \in I$, then $\prod_{i\in I} A_i$ satisfies Condition (PWP_{ssc}).*

Obviously, S acts injectively on S_S if and only if S is right cancellative.

Corollary 2.4. *If S is right cancellative, then S_S^I satisfies Condition (PWP_{ssc}), for any nonempty set I.*

Since S acts injectively on $\Theta_i = \{\theta_i\}$, $\prod_{i\in I} \Theta_i$ satisfies Condition (PWP_{ssc}).

Recall, from [2], that for S, the cartesian product $S \times S$, equipped with the right S-action $(s,t)u = (su, tu)$, for $s, t, u \in S$, is called the diagonal act of S and it is denoted by $D(S)$.

In the following theorem we obtain equivalent conditions for S_S^n to satisfy Condition (PWP_{ssc}).

Theorem 2.5. *For a natural number $n \geq 2$, the following statements are equivalent*:

(1) *S_S^n satisfies Condition (PWP_{ssc});*

(2) *$D(S)$ satisfies Condition (PWP_{ssc});*

(3) *S is right semi-cancellative.*

Proof. (1) ⇒ (2) It is clear.

(2) ⇒ (3) This is clear, by parts (4) and (7) of Theorem 2.2.

(3) ⇒ (1) Let $(x_1, x_2, ..., x_n)s = (y_1, y_2, ..., y_n)s$, for $x_1, y_1, ..., x_n, y_n, s \in S$. Then $x_i s = y_i s$, for $1 \leq i \leq n$. Since S is right semi-cancellative, $x_1 s = y_1 s$ implies the existence of $v_1 \in S$ such that $v_1 s = s$ and $x_1 v_1 = y_1 v_1$. Then $x_2 v_1 s = y_2 v_1 s$ implies the existence of $v_2 \in S$ such that $v_2 s = s$ and $x_2 v_1 v_2 = y_2 v_1 v_2$. If $v = v_1 v_2$, then

$$vs = v_1 v_2 s = s,\ x_1 v = x_1 v_1 v_2 = y_1 v_1 v_2 = y_1 v,\ x_2 v = y_2 v.$$

Continuing this procedure, there exists $u \in S$ such that $us = s$ and $x_i u = y_i u$, for $1 \leq i \leq n$. Thus we have $(x_1, x_2, ..., x_n)u = (y_1, y_2, ..., y_n)u$ and so S_S^n satisfies Condition (PWP_{ssc}), as required. $\square$

Now we give an equivalence for cyclic S-act to satisfy Condition (PWP_{ssc}).

Theorem 2.6. *Let ρ be a right congruence on S. Then the S-act S/ρ satisfies Condition (PWP_{ssc}) if and only if*

$$(\forall x, y, s \in S)((xs)\rho(ys) \Longrightarrow (\exists r \in S)((xr)\rho(yr) \wedge rs = s)).$$

Proof. Necessity. Suppose $(xs)\rho(ys)$, for $x, y, s \in S$. Thus $[x]_\rho s = [y]_\rho s$. By assumption, there exists $r \in S$ such that $[x]_\rho r = [y]_\rho r$ and $rs = s$. Hence $(xr)\rho(yr)$, as required.

Sufficiency. Let $[x]_\rho s = [y]_\rho s$, for $x, y, s \in S$. Thus $(xs)\rho(ys)$ and so, by assumption, there exists $r \in S$ such that $(xr)\rho(yr)$ and $rs = s$. Hence $[x]_\rho r = [y]_\rho r$. So S/ρ satisfies Condition (PWP_{ssc}), as required. $\square$

Theorem 2.7. *Let $w \in S$ and $\rho = \rho(w, 1)$. The cyclic S-act S/ρ satisfies Condition (PWP_{ssc}) if and only if for every $x, y, s \in S$ and $m, n \in \mathbb{N}_0$*

$$w^m xs = w^n ys \Longrightarrow (\exists p, q \in \mathbb{N}_0)(\exists r \in S)(w^p xr = w^q yr \wedge rs = s).$$

Proof. Necessity. Let $w^m xs = w^n ys$, for $x, y, s \in S$ and $m, n \in \mathbb{N}_0$. Then we have $(xs)\rho(ys)$, by [14, III, Corollary 8.7]. By Theorem 2.6, there exists $r \in S$ such that $rs = s$ and $(xr)\rho(yr)$. Thus, by [14, III, Corollary 8.7], there exist $p, q \in \mathbb{N}_0$ such that $w^p xr = w^q yr$.

Sufficiency. Let $(xs)\rho(ys)$, for $x, y, s \in S$. Then, by [14, III, Corollary 8.7], there exist $m, n \in \mathbb{N}_0$ such that $w^m xs = w^n ys$. By assumption, there exist $r \in S$ and $p, q \in \mathbb{N}_0$ such that $rs = s$ and $w^p xr = w^q yr$. Hence $(xr)\rho(yr)$, by [14, III, Corollary 8.7], and so S/ρ satisfies Condition (PWP_{ssc}), by Theorem 2.6. $\square$

Notice that in the above theorem, if $w = 1$, then $S/\rho = S/\rho(w, 1) = S/\rho(1, 1) = S/\Delta_S \cong S_S$, which by placing w in the above theorem, the part (7) of Theorem 2.2 is obtained.

Theorem 2.8. *The following statements hold:*

(1) *If A satisfies Condition (PWP_{ssc}), then A is principally weakly flat.*

(2) *For left PSF monoid S, A is principally weakly flat if and only if A satisfies Condition (PWP_{ssc}).*

Proof. (1) Let $as = a's$, for $a \in A$ and $s, s' \in S$. By assumption, there exists $r \in S$ such that $ar = a'r$ and $rs = s$. Hence

$$a \otimes s = a \otimes rs = ar \otimes s = a'r \otimes s = a' \otimes rs = a' \otimes s$$

in $A \otimes Ss$. Thus A is principally weakly flat, as required.

(2) Necessity. Let $as = a's$, for $a \in A$ and $s, s' \in S$. By assumption, there exist $n \in \mathbb{N}$ and elements $a_1, ..., a_n \in A$, $s_1, t_1, ..., s_n, t_n \in S$ such that

$$\begin{array}{rclcrcl} a & = & a_1 s_1 & & & & \\ a_1 t_1 & = & a_2 s_2 & & s_1 s & = & t_1 s \\ a_2 t_2 & = & a_3 s_3 & & s_2 s & = & t_2 s \\ & & \cdots & & \cdots & & \\ a_n t_n & = & a' & & s_n s & = & t_n s. \end{array}$$

Since S is right semi-cancellative, $s_1 s = t_1 s$ implies the existence of $v_1 \in S$ such that $v_1 s = s$ and $s_1 v_1 = t_1 v_1$. Then $s_2 v_1 s = t_2 v_1 s$ implies the existence of $v_2 \in S$ such that $v_2 s = s$ and $s_2 v_1 v_2 = t_2 v_1 v_2$. If $v = v_1 v_2$, then

$$vs = v_1 v_2 s = s, \;\; s_1 v = s_1 v_1 v_2 = t_1 v_1 v_2 = t_1 v, \;\; s_2 v = t_2 v.$$

Continuing this procedure, there exists $u \in S$ such that $us = s$ and $s_i u = t_i u$, for $1 \leq i \leq n$. Thus we have

$$au = (a_1 s_1)u = a_1(s_1 u) = a_1(t_1 u) = (a_1 t_1)u = \ldots = (a_n t_n)u = a'u.$$

So A satisfies Condition (PWP_{ssc}), as required.

Sufficiency. This is true, by (1). □

The following example shows that the converse of part (1) of Theorem 2.8 is not true in general.

Example 2.9. Let J be a proper right ideal of S. Let x, y, z be different symbols not belonging to S. Define $A(J) := (\{x, y\} \times (S \setminus J)) \cup (\{z\} \times J)$ and a right S-action on $A(J)$ by

$$(x, u)s = \begin{cases} (x, us) & \text{if } us \notin J, \\ (z, us) & \text{if } us \in J, \end{cases}$$

$$(y, u)s = \begin{cases} (y, us) & \text{if } us \notin J, \\ (z, us) & \text{if } us \in J, \end{cases}$$

$$(z,u)s = (z,us).$$

Clearly $A(J)$ is an S-act. Let $S = \{0,1,e,s\}$ with the table

	0	1	e	s
0	0	0	0	0
1	0	1	e	s
e	0	e	e	0
s	0	s	s	0

If $J = \{0,e\}$, then J is a right ideal of S. Because $0 \in J0$ and $e \in Je$, $A(J) = \{(x,1),(x,s),(y,1),(y,s),(z,0),(z,e)\}$, by [1, Proposition 2.2], is a flat S-act and so is principally weakly flat. But it does not satisfy Condition (PWP_{ssc}). Otherwise $(x,s)s = (y,s)s$ implies that there exists $u \in S$ such that $(x,s)u = (y,s)u$ and $us = s$. Because 1 is the only element such that $1s = s$, $(x,s) = (y,s)$, which is a contradiction.

Recall, from [8], that A satisfies *Condition* (PWP_E) if for all $a, a' \in A$, $s \in S$

$$as = a's \Rightarrow (\exists a'' \in A)\ (\exists u,v \in S)(e,f \in E(S))\ (ae = a''ue,\ a'f = a''vf,\\ es = s = fs, \text{ and } us = vs).$$

Clearly Condition (PWP) implies Condition (PWP_E).

Suppose S is not semi-cancellative. Then S_S does not satisfy Condition (PWP_{ssc}) while S_S satisfies Condition (PWP). Therefore Condition (PWP) does not imply Condition (PWP_{ssc}).

Now, in the following example, we show that Condition (PWP_{ssc}) does not imply Condition (PWP_E) and so does not imply Condition (PWP). In this example we use the monoid S of Example 1.6 of [23].

Example 2.10. Let $(I,\leq)$ be a totally ordered set with no successor for each element (as $\mathbb{R}$). Consider the commutative monoid $S = \{x_i^m | i \in I, m \in \mathbb{N}\} \cup \{1\}$ in which $x_i^m x_j^n$ equals to x_j^n if $i < j$ and equals to x_i^{m+n} if $i = j$. Then S is left PSF and, since S does not have any idempotent except 1, it is not left PP. Now let $A = \{a_1,a_2,a_3\}$ be a set such that $a_i 1 = a_i$ and $a_i s = a_3$, for $i = 1,2,3$ and any $s \in S$. Obviously A is an S-act which does not satisfy Condition (PWP_E). Otherwise $a_1 x_i^m = a_2 x_i^m$ implies that there exist $u, v, f^2 = f, e^2 = e \in S$, and $b \in A$ such that $a_1 e = bue$, $a_2 f = bvf$,

$ex_i^m = x_i^m = fx_i^m$, and $ux_i^m = vx_i^m$. Because 1 is the only idempotent such that $1x_i^m = x_i^m$, $a_1 = bu$ implies $b = a_1$ and $u = 1$. Then $a_2 \neq a_1 v$ for every $v \in S$, which is a contradiction. Therefore A does not satisfy Condition (PWP_E) and so does not satisfy Condition (PWP). On the other hand, $a_j x_i^m = a_3 = a_k x_i^m$, for $j, k = 1, 2, 3$ and any $x_i^m \in S$. Let $l < i$. Then $a_j x_l^m = a_3 = a_k x_l^m$ and $x_l^m x_i^m = x_i^m$, that is, A satisfies Condition (PWP_{ssc}).

3 Classification by Condition (PWP_{ssc}) of acts

In this section we give a classification of monoids for which all (cyclic, monocyclic) acts satisfy Condition (PWP_{ssc}) and give a classification of monoids for which all acts satisfying some other flatness properties have Condition (PWP_{ssc}).

Theorem 3.1. *The following statements are equivalent*:

(1) *All S-acts satisfy Condition (PWP_{ssc});*

(2) *all S-acts generated by exactly two elements satisfy Condition (PWP_{ssc});*

(3) *all cyclic S-acts satisfy Condition (PWP_{ssc});*

(4) *all monocyclic S-acts of the form $S/\rho(s, s^2)$ $(s \in S)$ satisfy Condition (PWP_{ssc});*

(5) *all Rees factor S-acts satisfy Condition (PWP_{ssc});*

(6) *all divisible S-acts satisfy Condition (PWP_{ssc});*

(7) *all cofree S-acts satisfy Condition (PWP_{ssc});*

(8) *S is regular.*

Proof. Implications (1) ⇒ (2), (1) ⇒ (3) ⇒ (4), (3) ⇒ (5) are obvious.

Implications (1) ⇒ (6) ⇒ (7) are obvious because cofree ⇒ divisible.

(2) ⇒ (8) Let $s \in S$. If $sS = S$, then there exists $z \in S$ such that $sz = 1$. Therefore $szs = s$ and s is regular. Now let $sS \neq S$. Put

$$A = S \coprod^{sS} S = \big\{(l, x) |\ l \in S \setminus sS\big\} \mathbin{\dot\cup} sS \mathbin{\dot\cup} \{(t, y) |\ t \in S \setminus sS\}.$$

Indeed $A =< (1, x), (1, y) >= (1, x)S \cup (1, y)S$. Since A is generated by two different elements, by assumption, A satisfies Condition (PWP_{ssc}). Now $s = (1, x)s = (1, y)s$ implies that there exists $r \in S$ such that $(1, x)r = (1, y)r$ and $rs = s$. Thus $r \in sS$ and so there exists $x \in S$ such that $r = sx$. Then $s = rs = sxs$, that is, s is regular and so S is regular.

(4) $\Rightarrow$ (8) By part (1) of Theorem 2.8, all monocyclic S-acts of the form $S/\rho(s, s^2)$ $(s \in S)$ are principally weakly flat. Thus, by [14, IV, Theorem 6.6], S is regular.

(5) $\Rightarrow$ (8) By part (1) of Theorem 2.8, all Rees factor S-acts are principally weakly flat. Thus, by [14, IV, Theorem 6.6], S is regular.

(7) $\Rightarrow$ (8) Every S-act can be embedded in a cofree S-act. By assumption, every S-act is a subact of an S-act satisfying Condition (PWP_{ssc}). Thus every S-act satisfies Condition (PWP_{ssc}), by part (2) of Theorem 2.2, and so every S-act is principally weakly flat, by part (1) of Theorem 2.8. Therefore, by [14, IV, Theorem 6.6], S is regular.

(8) $\Rightarrow$ (1) Every S-act is principally weakly flat, by [14, IV, Theorem 6.6]. Every regular monoid is left PP and so is left PSF. Thus every S-act satisfies Condition (PWP_{ssc}), by part (2) of Theorem 2.8. $\square$

Recall, from [10], [9], and [15], that A satisfies *Condition* $(E'P)$ if for all $a \in A$, $s, s', z \in S$,

$$as = as',\ sz = s'z \Rightarrow (\exists a' \in A)\ (\exists u, u' \in S)\ (a = a'u = a'u' \text{ and } us = u's').$$

A satisfies *Condition* (EP) if for all $a \in A$, $s, s' \in S$

$$as = as' \Rightarrow (\exists a' \in A)\ (\exists u, u' \in S)\ (a = a'u = a'u' \text{ and } us = u's').$$

A satisfies *Condition* (E') if for all $a \in A$, $s, s', z \in S$

$$as = as',\ sz = s'z \Rightarrow (\exists a' \in A)\ (\exists u \in S)\ (a = a'u \text{ and } us = us').$$

Theorem 3.2. *The following statements are equivalent*:

(1) *All S-acts satisfy Condition (PWP_{ssc});*

(2) *all S-acts satisfying Condition (E) satisfy Condition (PWP_{ssc});*

(3) *all S-acts satisfying Condition $(E'P)$ satisfy Condition (PWP_{ssc});*

(4) *S is regular.*

Proof. Implications (1) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious, because $(E) \Rightarrow (E'P)$.

(2) $\Rightarrow$ (4) Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $xs = 1$. Thus $sxs = s$, that is, s is regular and so S is regular. Let $sS \neq S$. Put

$$A = S \coprod^{sS} S = \big\{(l, x) |\ l \in S \setminus sS\big\} \,\dot\cup\, sS \,\dot\cup\, \{(t, y) |\ t \in S \setminus sS\}.$$

Indeed

$$B = \{(l, x) |\ l \in S \setminus sS\} \dot\cup sS \cong S_S \cong \{(t, y) |\ t \in S \setminus sS\} \dot\cup sS = C.$$

B and C are subacts of A which are generated by $(1, x)$ and $(1, y)$, respectively. A is generated by $(1, x)$ and $(1, y)$ because $A = B \cup C$. By the above isomorphisms, B and C satisfy Condition (E) and so A satisfies Condition (E). Thus, by assumption, A satisfies Condition (PWP_{ssc}). Then $(1, x)s = (1, y)s$ implies that there exists $r \in S$ such that $rs = s$ and $(1, x)r = (1, y)r$. Therefore $r \in sS$ and so there exists $x \in S$ such that $r = sx$. Hence $s = rs = sxs$, that is, s is regular. Thus S is regular, as required.

(4) ⇒ (1) This is true, by Theorem 3.1. □

By the proof of Theorem 3.2, we conclude that the above theorem is true for finitely generated S-acts.

Theorem 3.3. *Let $(*)$ be a property on S-acts such that every S-act satisfying Property $(*)$ is principally weakly flat and S_S satisfies in Property $(*)$. Then the following statements are equivalent*:

(1) *All S-acts satisfying Property $(*)$ satisfy Condition (PWP_{ssc});*

(2) *all finitely generated S-acts satisfying Property $(*)$ satisfy Condition (PWP_{ssc});*

(3) *all cyclic S-acts satisfying Property $(*)$ satisfy Condition (PWP_{ssc});*

(4) *S is left PSF.*

Proof. Implications (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (4) S is a cyclic S-act satisfying Property $(*)$ and so, by assumption, satisfies Condition (PWP_{ssc}). Therefore S is left PSF, by part (7) of Theorem 2.2.

(4) ⇒ (1) By assumption, every S-act satisfying Property $(*)$ is principally weakly flat. Since S is left PSF, principally weakly flat is equivalent to Condition (PWP_{ssc}), by Theorem 2.8. Therefore all S-acts satisfying Property $(*)$ satisfy Condition (PWP_{ssc}), as required. □

Theorem 3.4. *The following statements are equivalent*:

(1) *All torsion free S-acts satisfy Condition (PWP_{ssc});*

(2) *all torsion free cyclic S-acts satisfy Condition (PWP_{ssc});*

(3) *all torsion free Rees factor* S*-acts satisfy Condition* (PWP_{ssc})*;*

(4) S *is left almost regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) All torsion free Rees factor S-acts are principally weakly flat, by part (1) of Theorem 2.8. Thus S is left almost regular, by [14, IV, Theorem 6.5].

(4) $\Rightarrow$ (1) All torsion free S-acts are principally weakly flat, by [14, IV, Theorem 6.5]. Also every left almost regular monoid is left PP, by the duality of [14, IV, Proposition 1.3], and so is left PSF. Thus all torsion free S-acts satisfy Condition (PWP_{ssc}), by part (2) of Theorem 2.8. $\square$

Recall, from [21], that A is called $\mathfrak{R}$*-torsion free* if for any $a, b \in A$ and $c \in S$, c right cancellable, $ac = bc$ and $a \;\mathfrak{R}\; b$ ($\mathfrak{R}$ is Green's equivalence) imply that $a = b$.

Theorem 3.5. *The following statements are equivalent*:

(1) *All* $\mathfrak{R}$*-torsion free* S*-acts satisfy Condition* (PWP_{ssc})*;*

(2) *all* $\mathfrak{R}$*-torsion free* S*-acts generated by exactly two elements satisfy Condition* (PWP_{ssc})*;*

(3) S *is regular.*

Proof. Implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$ and so s is regular. Let $sS \neq S$. Put

$$A = S \coprod^{sS} S = \big\{(l, x)|\; l \in S \setminus sS\big\} \,\dot\cup\, sS \,\dot\cup\, \{(t, y)|\; t \in S \setminus sS\}.$$

By the proof of part (2) $\Rightarrow$ (4) of Theorem 3.2, A is an S-act which is generated by two different elements $(1, x)$ and $(1, y)$ and also satisfy Condition (E). Every S-act satisfying Condition (E) is $\mathfrak{R}$-torsion free, by [21, Proposition 1.2]. Thus A is $\mathfrak{R}$-torsion free and so, by assumption, satisfies Condition (PWP_{ssc}). Hence, by the proof of part (2) $\Rightarrow$ (4) of Theorem 3.2, s is regular. Therefore S is regular, as required.

(3) $\Rightarrow$ (1) Every $\mathfrak{R}$-torsion free S-act is principally weakly flat, by [21, Theorem 4.5]. Since every regular monoid is left PP and so is left PSF, principally weakly flat is equivalent to Condition (PWP_{ssc}), by part (2) of Theorem 2.8. Therefore every $\mathfrak{R}$-torsion free S-act satisfies Condition (PWP_{ssc}). $\square$

We recall, from [14], that A is (*strongly*) *faithful* if for $s, t \in S$ the equality $as = at$, for (some) all $a \in A$, implies that $s = t$.

Theorem 3.6. *The following statements are equivalent*:

(1) *All faithful S-acts satisfy Condition (PWP_{ssc});*

(2) *all faithful S-acts generated by exactly two elements satisfy Condition (PWP_{ssc});*

(3) *S is regular.*

Proof. Implication (1) ⇒ (2) is obvious.

(2) ⇒ (3) Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$ and so s is regular. Let $sS \neq S$. Put $A = S \coprod^{sS} S$. By the proof of part (2) ⇒ (4) of Theorem 3.2, A is an S-act which is generated by two different elements $(1, x)$ and $(1, y)$. Indeed A is faithful and so, by assumption, satisfies Condition (PWP_{ssc}). Hence, by the proof of part (2) ⇒ (4) of Theorem 3.2, s is regular. Therefore S is regular, as required.

(3) ⇒ (1) All S-acts satisfy Condition (PWP_{ssc}), by Theorem 3.1. Thus all faithful S-acts satisfy Condition (PWP_{ssc}). □

Notation: C_l (C_r) is the set of all left (right) cancellable elements of S.

Lemma 3.7. *The following statements are equivalent*:

(1) *There exists at least one strongly faithful S-act;*

(2) *sS as an S-act is strongly faithful, for every $s \in S$;*

(3) *S as an S-act is strongly faithful;*

(4) *$sS \subseteq C_l$, for every $s \in S$;*

(5) *S is left cancellative.*

Proof. Implications (2) ⇒ (1), (5) ⇒ (4) and (3) ⇒ (1) are obvious.

(1) ⇒ (5) Let A be a strongly faithful S-act and $sl = st$, for $s, l, t \in S$. Then $asl = ast$, for $a \in A$. Since A is strongly faithful and $as \in A$, $l = t$. Therefore S is left cancellative.

(5) ⇒ (3) Let S be left cancellative and $sl = st$, for $l, t, s \in S$. By assumption, $l = t$ and so S is strongly faithful, as an S-act.

(4) ⇒ (5) Let $sS \subseteq C_l$, for $s \in S$ and $rt = rl$, for $r, t, l \in S$. Then $(sr)t = (sr)l$. By assumption, $t = l$ and so S is left cancellative.

(5) ⇒ (2) Let $skt = skl$, for $sk \in sS$ and $t, l \in S$. By assumption, $t = l$ and so sS is strongly faithful as an S-act. □

By the above lemma, for S there exists no strongly faithful S-act if and only if S is not left cancellative.

Theorem 3.8. *The following statements are equivalent*:

(1) *All strongly faithful* S*-acts satisfy Condition* (PWP_{ssc})*;*

(2) *all strongly faithful* S*-acts generated by exactly two elements satisfy Condition* (PWP_{ssc})*;*

(3) S *is not left cancellative or it is a group.*

Proof. Implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) If S is not left cancellative, then (3) is satisfied. Let S be left cancellative and $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$ and so s is regular. Now let $sS \neq S$. Put

$$A = S \coprod^{sS} S = \big\{(l, x)|\ l \in S \setminus sS\big\} \dot\cup sS \dot\cup \{(t, y)|\ t \in S \setminus sS\}.$$

We have

$$B = \{(l, x)|\ l \in S \setminus sS\big\} \dot\cup sS \cong S_S \cong \{(t, y)|\ t \in S \setminus sS\} \dot\cup sS = C$$

and

$$A =< (1, x), (1, y) >= B \cup C.$$

Since S is left cancellative, S_S is strongly faithful, by Lemma 3.7. By the above isomorphisms, B and C are strongly faithful as subacts of A. Thus A is strongly faithful. Since A is generated by two different elements $(1, x)$ and $(1, y)$, by assumption, A satisfies Condition (PWP_{ssc}). By the proof of part (2) $\Rightarrow$ (4) of Theorem 3.2, s is regular and so S is regular. Thus for every $s \in S$ there exists $x \in S$ such that $sxs = s$. Since S is left cancellative, $xs = 1$. Thus every element in S has left inverse and so S is a group.

(3) $\Rightarrow$ (1) If S is not left cancellative, then there exists no strongly faithful S-act, by Lemma 3.7. Thus (1) is satisfied. If S is left cancellative, then there exists at least a strongly faithful S-act, by Lemma 3.7. Since S is a group, it is regular and so (1) is satisfied, by Theorem 3.1. $\square$

Lemma 3.9. *Let* ρ *be a right congruence on* S*. Then the following statements are equivalent*:

(1) S/ρ *is a strongly faithful cyclic* S*-act;*

(2) $\rho = \Delta_S$ *and* S *is left cancellative.*

Proof. (1) $\Rightarrow$ (2) Since S/ρ is strongly faithful as an S-act, there exists at least one strongly faithful S-act. Hence S is left cancellative, by Lemma 3.7. Now let $(s,t) \in \rho$, for $s,t \in S$. Then $[1]_\rho \cdot s = [s]_\rho = [t]_\rho = [1]_\rho \cdot t$. Thus $s = t$, since S/ρ is strongly faithful, and so $\rho = \Delta_S$.

(2) $\Rightarrow$ (1) $S/\rho = S/\Delta_S \cong S_S$. Since S is left cancellative, $S_S \cong S/\rho$ is strongly faithful, by Lemma 3.7. □

Theorem 3.10. *The following statements are equivalent*:

(1) *Every strongly faithful cyclic S-act satisfies Condition (PWP_{ssc});*

(2) *S is not left cancellative or it is left PSF.*

Proof. (1) $\Rightarrow$ (2) If S is not left cancellative, then (2) is satisfied. Let S be left cancellative. Then S, as a cyclic S-act, is strongly faithful, by Lemma 3.7. Thus, by assumption, S_S satisfies Condition (PWP_{ssc}) and so S is left PSF, by part (7) of Theorem 2.2.

(2) $\Rightarrow$ (1) If S is not left cancellative, then there exists no strongly faithful S-act, by Lemma 3.7. Thus (1) is satisfied. If S is left cancellative, then there exists at least one strongly faithful cyclic S-act, by Lemma 3.7. If S/ρ is a strongly faithful cyclic S-act, then $\rho = \Delta_S$, by Lemma 3.9, and so $S/\rho \cong S_S$. By assumption, S is left PSF and so $S/\rho \cong S_S$ satisfies Condition (PWP_{ssc}), by part (7) of Theorem 2.2. □

Theorem 3.11. *The following statements are equivalent*:

(1) *There exists at least one strongly faithful cyclic S-act satisfying Condition (PWP_{ssc});*

(2) *S is left cancellative and every strongly faithful cyclic S-act satisfies Condition (PWP_{ssc});*

(3) *S is left cancellative and left PSF.*

Proof. (1) $\Rightarrow$ (2) Since there exists at least one strongly faithful cyclic S-act, S is left cancellative, by Lemma 3.7. If S/ρ is a strongly faithful cyclic S-act satisfying Condition (PWP_{ssc}), then $\rho = \Delta_S$, by Lemma 3.9. Thus $S/\rho \cong S_S$ satisfies Condition (PWP_{ssc}) and so S is left PSF, by part (7) of Theorem 2.2. Therefore every strongly faithful cyclic S-act satisfies Condition (PWP_{ssc}), by Theorem 3.10.

(2) $\Rightarrow$ (3) This is true, by Theorem 3.10.

(3) $\Rightarrow$ (1) Since S is left cancellative, there exists at least one strongly faithful cyclic S-act, by Lemma 3.7. If S/ρ is a strongly faithful cyclic S-act,

then $\rho = \Delta_S$, by Lemma 3.9. Thus $S/\rho = S/\Delta_S \cong S_S$. Since S is left PSF, $S/\rho \cong S_S$ satisfies Condition (PWP_{ssc}), by part (7) of Theorem 2.2. $\square$

We recall, from [22], that an act A is called *strongly torsion free* if for $a, b \in A$ and any $s \in S$, the equality $as = bs$ implies $a = b$. It is obvious that every strongly torsion free S-act satisfies Condition (PWP_{ssc}).

Let K_S be a right ideal of S. Then the Rees factor S-act S/K_S is simple if and only if $K_S = S$, by [14, I, Proposition 5.28]. Thus S/K_S is simple if and only if $S/K_S = S/S_S \cong \Theta_S$. Since Θ_S satisfies Condition (PWP_{ssc}), by part (1) of Theorem 2.2, every simple Rees factor S-act satisfies Condition (PWP_{ssc}).

Recall, from [14], that a right ideal K_S of S satisfies *Condition* (LU) if for every $k \in K_S$ there exists $l \in K_S$ such that $lk = k$.

Lemma 3.12. *Let* $S \neq C_r$. *Then the following statements hold*:

(1) $I = S \setminus C_r$ *is a proper right ideal of* S.

(2) S/I $(I = S \setminus C_r)$ *is a torsion free* S*-act.*

(3) *If* S *is left PSF, then the right ideal* $I = S \setminus C_r$ *satisfies Condition* (LU).

Proof. (1) Since $S \neq C_r$ and $1 \in C_r$, $\emptyset \neq I \subset S$. Let $i \in I$ and $s \in S$. Then there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1 i = l_2 i$. Thus $l_1 is = l_2 is$. If $is \in C_r$, then $l_1 = l_2$, which is a contradiction. Therefore $is \in I$ and so I is a proper right ideal of S, as required.

(2) Let $sc \in I$, for $s \in S$ and $c \in C_r$. We claim that $s \in I$. Since $sc \in I$, there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1 sc = l_2 sc$. Thus $l_1 s = l_2 s$. If $s \notin I$, then $s \in C_r$ and so $l_1 = l_2$, which is a contradiction. Thus $s \in I$ and so the S-act S/I is torsion free, by [14, III, Proposition 8.10].

(3) Let $i \in I$. Then i is not right cancellable. Thus there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1 i = l_2 i$. Since S is left PSF, there exists $r \in S$ such that $l_1 r = l_2 r$ and $ri = i$, by part (7) of Theorem 2.2. If $r \notin I$, then $l_1 = l_2$, which is a contradiction. Thus $r \in I$ and $ri = i$. Therefore I satisfies Condition (LU). $\square$

We recall, from [20], that A is called *GP-flat* if for every $s \in S$ and $a, a' \in A$, $a \otimes s = a' \otimes s$ in $A \otimes S$ implies the existence of a natural number n such that $a \otimes s^n = a' \otimes s^n$ in $A \otimes Ss^n$.

Lemma 3.13. *Let* S *be right cancellative. Then for every* S*-act we have* *strongly torsion free $\Leftrightarrow$ torsion free $\Leftrightarrow$ GP-flat $\Leftrightarrow$ principally weakly flat $\Leftrightarrow$ Condition (PWP) $\Leftrightarrow$ Condition (P') $\Leftrightarrow$ Condition (PWP_E) $\Leftrightarrow$ Condition (PWP_{ssc}) $\Leftrightarrow$ translation kernel flat $\Leftrightarrow$ principally weakly kernel flat.*

Proof. Since S is right cancellative, strongly torsion free is equivalent to torsion free, by definition. We always have strongly torsion free $\Rightarrow$ Condition (PWP) (Condition (PWP_E), Condition (PWP_{ssc}), Condition (P'))$\Rightarrow$ principally weakly flat $\Rightarrow$ GP-flat $\Rightarrow$ torsion free. Thus strongly torsion free $\Leftrightarrow$ torsion free $\Leftrightarrow$ GP-flat $\Leftrightarrow$ principally weakly flat $\Leftrightarrow$ Condition (PWP) $\Leftrightarrow$ Condition (PWP_{ssc}) $\Leftrightarrow$ Condition (PWP_E) $\Leftrightarrow$ Condition (P'). Also we always have principally weakly kernel flat $\Rightarrow$ translation kernel flat $\Rightarrow$ Condition (PWP). Since S is right cancellative, it is left PP and so translation kernel flat $\Leftrightarrow$ principally weakly kernel flat, by [3, Proposition 27]. Let $a, a' \in A$ and $s, s' \in S$. $as = a's'$ in A if and only if $a \otimes s = a' \otimes s'$ in $A \otimes S$ (see [14, II, Proposition 5.13]). It is obvious that $a \otimes s = a' \otimes s'$ in $A \otimes S$ if and only if $a \otimes (s, s) = a' \otimes (s', s')$ in $A \otimes \Delta$. Since S is right cancellative, $\ker \rho_z = {}_S\Delta$ for every $z \in S$. Now Condition (PWP) is equivalent to translation kernel flat, by [3, Proposition 5]. $\square$

Theorem 3.14. *Let $(*)$ be a property on S-acts such that flat $\Rightarrow$ Property $(*)$ $\Rightarrow$ torsion free. Then the following statements are equivalent:*

(1) *S is left PSF and Property $(*)$ in S-acts implies principally weakly kernel flat;*

(2) *S is left PSF and Property $(*)$ in S-acts implies translation kernel flat;*

(3) *S is left PSF and Property $(*)$ in S-acts implies Condition (PWP);*

(4) *S is left PSF and Property $(*)$ in S-acts implies Condition (P');*

(5) *S is right cancellative.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious because principally weakly kernel flat $\Rightarrow$ translation kernel flat $\Rightarrow$ Condition (PWP), also implication (4) $\Rightarrow$ (3) is obvious because Condition (P') $\Rightarrow$ Condition (PWP).

(3) $\Rightarrow$ (5) Let S be not right cancellative and $I = S \setminus C_r$. Then I is a proper right ideal of S which satisfies Condition (LU), by Lemma 3.12. Put

$$A = S \coprod^{I} S = \{(l, x) |\ l \in S \setminus I\} \,\dot\cup\, I \,\dot\cup\, \{(t, y) |\ t \in S \setminus I\}.$$

So A is flat, by [14, III, Proposition 12.19]. By assumption, A satisfies Condition (PWP). Let $i \in I$, then the equality $(1,x)i = (1,y)i$ implies there exist $a \in A$ and $u, v \in S$ such that $(1,x) = au$, $(1,y) = av$, and $ui = vi$. Hence there exist $l, t \in S \setminus I$ such that $(l,x) = a = (t,y)$, which is a contradiction. Thus S is right cancellative, as required.

(5) $\Rightarrow$ (1) Since S is right cancellative, it is left PP and so it is left PSF. Also torsion free is equivalent to principally weakly kernel flat, by Lemma 3.13. Thus, by assumption, every S-act satisfying $(*)$ is principally weakly kernel flat.

(5) $\Rightarrow$ (4) Since S is right cancellative, it is left PP and so it is left PSF. Also torsion free is equivalent to Condition (P'), by Lemma 3.13. Thus, by assumption, every S-act satisfying the Property $(*)$ satisfies Condition (P'). □

Using a similar argument as in the proof of the above theorem, we conclude that Theorem 3.14 is true for finitely generated S-acts. Furthermore the Property $(*)$ in Theorem 3.14 can be any property as flat, weakly flat, principally weakly flat, and GP-flat.

Notice that in general [6, Theorem 2.8], [18, Theorems 2.6 and 2.8], [19, Lemma 2.12], and [20, Theorem 3.11] follow for every S-act, by putting the Property $(*)$ in Theorem 3.14 with any property as flat, weakly flat, principally weakly flat, and GP-flat.

Corollary 3.15. *Let* $(*)$ *be a property on* S*-acts such that flat* $\Rightarrow$ *Property* $(*)$ $\Rightarrow$ *torsion free. Then the following statements are equivalent*:

(1) *All* S*-acts satisfying Property* $(*)$ *are principally weakly kernel flat and satisfy Condition* (PWP_{ssc})*;*

(2) *all* S*-acts satisfying Property* $(*)$ *are translation kernel flat and satisfy Condition* (PWP_{ssc})*;*

(3) *all* S*-acts satisfying Property* $(*)$ *satisfy Conditions* (PWP) *and* (PWP_{ssc})*;*

(4) *all* S*-acts satisfying Property* $(*)$ *satisfy Conditions* (P') *and* (PWP_{ssc})*;*

(5) S *is right cancellative.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious, because principally weakly kernel flat $\Rightarrow$ translation kernel flat $\Rightarrow$ Condition (PWP), also implication (4) $\Rightarrow$ (3) is obvious because Condition (P') $\Rightarrow$ Condition (PWP).

(3) $\Rightarrow$ (5) Since S_S is flat, by assumption, it satisfies Property $(*)$ and so it satisfies Condition (PWP_{ssc}). Therefore S is left PSF, by part (7) of Theorem 2.2. Also, by assumption, every S-act satisfying Property $(*)$ satisfies Condition (PWP) and thus S is right cancellative, by Theorem 3.14.

(5) $\Rightarrow$ (1) Since S is right cancellative, every S-act satisfying Property $(*)$ is principally weakly kernel flat, by Theorem 3.14. Also, since S is right cancellative, S is left almost regular and so every torsion free S-act satisfies Condition (PWP_{ssc}), by Theorem 3.4. Therefore every S-act satisfying Property $(*)$ satisfies Condition (PWP_{ssc}).

(5) $\Rightarrow$ (4) Since S is right cancellative, torsion free is equivalent to Condition (P'), by Lemma 3.13. Therefore, by assumption, every S-act satisfying Property $(*)$ satisfies Condition (P'). Also, since S is right cancellative, S is left almost regular and so every torsion free S-act satisfies Condition (PWP_{ssc}), by Theorem 3.4. Therefore every S-act satisfying Property $(*)$ satisfies Condition (PWP_{ssc}), as required. □

Note that, since Theorem 3.14 holds for finitely generated S-acts, Corollary 3.15 is also true for finitely generated S-acts. Furthermore the Property $(*)$ in the above corollary can be any property as flat, weakly flat, principally weakly flat, and GP-flat.

Lemma 3.16. *Let S be right cancellative. Then for every S-act we have*

weakly pullback flat $\Leftrightarrow$ weakly kernel flat $\Leftrightarrow$ (WP) $\Leftrightarrow$ Condition (P) $\Leftrightarrow$ Condition (P_E) $\Leftrightarrow$ flat $\Leftrightarrow$ weakly flat.

Proof. Since S is right cancellative, by [4, Proposition 1], Condition (P) $\Leftrightarrow$ flat $\Leftrightarrow$ weakly flat. Also, since Condition (P) $\Rightarrow$ (WP) $\Rightarrow$ weakly flat, Condition (WP) $\Leftrightarrow$ weakly flat. We always have weakly pullback flat $\Rightarrow$ weakly kernel flat $\Rightarrow$ weakly flat. Thus weakly pullback flat $\Leftrightarrow$ weakly kernel flat $\Leftrightarrow$ weakly flat, by [19, Theorem 2.14]. Since S is right cancellative, it is left PP and so Condition (P_E) $\Leftrightarrow$ weakly flat, by [11, Theorem 2.5]. □

Theorem 3.17. *Let $(*)$ be a property on S-acts such that flat $\Rightarrow$ Property $(*)$ $\Rightarrow$ weakly flat. Then the following statements are equivalent*:

(1) *S is left PSF and Property $(*)$ in S-acts implies weakly pullback flat;*

(2) *S is left PSF and Property $(*)$ in S-acts implies Condition (P);*

(3) *S is right cancellative.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious because weakly pullback flat $\Rightarrow$ Condition (P).

(2) $\Rightarrow$ (3) All flat S-acts satisfy Condition (WP), since flat implies Property $(*)$, by assumption, Property $(*)$ implies Condition (P) and Condition (P) implies Condition (WP). Thus S is right cancellative, by [19, Theorem 2.14].

(3) $\Rightarrow$ (1) Since S is right cancellative, S is left PSF and every weakly flat S-act is weakly pullback flat, by [19, Theorem 2.14]. Since Property $(*)$ implies weakly flat, Property $(*)$ for S-acts implies weakly pullback flat, as required. □

Using a similar argument as in the proof of the above theorem, we conclude that Theorem 3.17 is true for finitely generated S-acts. Furthermore Property $(*)$ in Theorem 3.17 can be properties as flat and weakly flat.

Notice that we have Theorem 2 of [4] and Theorem 5.4 of [13] as a corollary of Theorem 3.17. Furthermore these theorems are true for finitely generated S-acts.

Corollary 3.18. *Let* $(*)$ *be a property on* S*-acts such that flat* $\Rightarrow$ *Property* $(*)$ $\Rightarrow$ *weakly flat. Then the following statements are equivalent*:

(1) *All* S*-acts satisfying Property* $(*)$ *are weakly pullback flat and satisfy Condition* (PWP_{ssc})*;*

(2) *all* S*-acts satisfying Property* $(*)$ *are weakly kernel flat and satisfy Condition* (PWP_{ssc})*;*

(3) *all* S*-acts satisfying Property* $(*)$ *satisfy Conditions* (WP) *and* (PWP_{ssc})*;*

(4) *all* S*-acts satisfying Property* $(*)$ *satisfy Conditions* (P) *and* (PWP_{ssc})*;*

(5) S *is right cancellative.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious, because weakly pullback flat $\Rightarrow$ weakly kernel flat $\Rightarrow$ Condition (WP), also implications (1) $\Rightarrow$ (4) $\Rightarrow$ (3) are obvious, because weakly pullback flat $\Rightarrow$ Condition (P) $\Rightarrow$ Condition (WP).

(3) $\Rightarrow$ (5) Since S_S is flat, by assumption, satisfies Property $(*)$ and so satisfies Condition (PWP_{ssc}). Therefore S is left PSF, by part (7) of Theorem 2.2. Also, by assumption, every S-act satisfying Property $(*)$ satisfies Conditions (WP) and thus S is right cancellative, by Theorem 3.17.

(5) $\Rightarrow$ (1) Since S is right cancellative, every S-act satisfying Property $(*)$ is weakly pullback flat, by Theorem 3.17. Also S is left PSF, by Theorem

3.17. Since Property $(*)$ implies principally weakly flat, every S-act satisfying Property $(*)$ satisfies Condition (PWP_{ssc}), by Theorem 2.8(2). $\square$

Using a similar argument as in the proof of the above corollary, we conclude that Corollary 3.18 is true for finitely generated S-acts. Furthermore Property $(*)$ in Corollary 3.18 can be properties as flat and weakly flat.

Theorem 3.19. *The following statements are equivalent*:

(1) *Every Rees factor S-act satisfying Condition (P) satisfies Condition (PWP_{ssc});*

(2) *every free Rees factor S-act satisfies Condition (PWP_{ssc});*

(3) *S does not contain a left zero or it is left PSF.*

Proof. Implications (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) If S does not contain a left zero, then (3) is true. Let z be a left zero of S. Then $K_S = zS = \{z\}$ is a right ideal of S. Since $|K_S| = 1$, $S/K_S \cong S_S$ is free, by [14, I, Proposition 5.22], and so, by assumption, $S/K_S \cong S_S$ satisfies Condition (PWP_{ssc}). Thus S is left PSF, by part (7) of Theorem 2.2.

(3) $\Rightarrow$ (1) Let K_S be a right ideal of S such that S/K_S satisfies Condition (P). If $K_S = S$, then $S/K_S = S/S_S \cong \Theta_S$ and so $S/K_S \cong \Theta_S$ satisfies Condition (PWP_{ssc}), by part (1) of Theorem 2.2. If $K_S \neq S$, then $|K_S| = 1$, by [14, III, Proposition 13.9]. If $z \in K_S$, then $K_S = zS = \{z\}$, that is, z is a left zero of S. Since S contains a left zero, S is left PSF, by assumption, and so $S/K_S \cong S_S$ satisfies Condition (PWP_{ssc}), by part (7) of Theorem 2.2. $\square$

Remark

The referee had told the new property lies between strong torsion freeness and principal flatness, and it looks very similar to the right semi-cancellativity of a monoid. So why not, for example, to say "A is semi-cancellative" instead of "A satisfies Condition (PWP_{ssc})"? We did not change the naming, since in our definition the first part ($as = a's$) is similar to Condition (PWP) and the last part ($au = a'u$ and $us = s$) is similar to semi-cancellative. Also semi-cancellativity of monoids is equivalent to PSF monoids, while the property PSF is not defined for S-acts. So we thought that the assertion "A is semi-cancellative" is ambiguous.

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Pouyan Khamechi, *Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran.*

Email: Pouyan_Khamechi@pgs.usb.ac.ir

Hossein Mohammadzadeh Saany, *Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran.*

Email: hmsdm@math.usb.ac.ir

Leila Nouri, *Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran.*

Email: Leila_Nouri@math.usb.ac.ir