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# On $GPW$-flat acts

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**Abstract.** In this article, we present $GPW$-flatness property of acts over monoids, which is a generalization of principal weak flatness. We say that a right $S$-act $A_S$ is $GPW$-flat if for every $s \in S$, there exists a natural number $n = n_{(s,A_S)} \in \mathbb{N}$ such that the functor $A_S \otimes {}_S-$ preserves the embedding of the principal left ideal ${}_S(Ss^n)$ into ${}_SS$. We show that a right $S$-act $A_S$ is $GPW$-flat if and only if for every $s \in S$ there exists a natural number $n = n_{(s,A_S)} \in \mathbb{N}$ such that the corresponding $\varphi$ is surjective for the pullback diagram $P(Ss^n, Ss^n, \iota, \iota, S)$, where $\iota : {}_S(Ss^n) \to {}_SS$ is a monomorphism of left $S$-acts. Also we give some general properties and a characterization of monoids for which this condition of their acts implies some other properties and vice versa.

## 1 Introduction

In 1970, Kilp [7] initiated a study of flatness of acts. In 1983, further investigation of (principal) weak version of flatness was done by Kilp [8]. In 2001, Laan [10] gave equivalents of different flatness properties according to surjectivity of $\varphi$ corresponding to some pullback diagram.

In this article, in Section 2, we introduce a generalization of principal weak flatness, called $GPW$-flatness and will give some general properties.

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In Section 3, we give conditions for a (Rees factor) cyclic act to be $GPW$-flat. In Section 4, we give a characterization of monoids over which all right $S$-acts are $GPW$-flat and also a characterization of monoids $S$ for which this condition of their right $S$-acts implies some other properties and vice versa.

In this paper $S$ will stand for a monoid and $\mathbb{N}$ the set of natural numbers. A nonempty set $A$ is called a *right $S$-act*, denoted $A_S$, if there exists a mapping $A \times S \to A, (a, s) \mapsto as$, such that $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and all $s, t \in S$. An act $A_S$ is called *weakly flat* if the functor $A_S \otimes {}_S-$ preserves all embeddings of left ideals into $S$. An act $A_S$ is called *principally weakly flat* if the functor $A_S \otimes {}_S-$ preserves all embeddings of principal left ideals into $S$. A right $S$-act $A_S$ is called *torsion free* if $ac = a'c$ for any $a, a' \in A_S$ and right cancellable element $c \in S$ implies $a = a'$. A right $S$-act $A_S$ satisfies *Condition* $(P)$ if for every $a, a' \in A_S$, $s, t \in S$, $as = a't$ implies that $a = a''u$, $a' = a''v$ and $us = vt$ for some $a'' \in A_S$, $u, v \in S$. A right $S$-act $A_S$ satisfies *Condition* $(E)$ if for every $a \in A_S$, $s, t \in S$, $as = at$ implies that $a = a'u$ and $us = ut$ for some $a' \in A_S, u \in S$. A right $S$-act $A_S$ satisfies *Condition* $(PWP)$ if for every $a, a' \in A_S$, $s \in S$, $as = a's$ implies that $a = a''u, a' = a''v$ and $us = vs$ for some $a'' \in A_S$, $u, v \in S$. A right $S$-act $A_S$ satisfies *Condition* $(P')$ if for every $a, a' \in A_S$, $s, t, z \in S$, $as = a't$ and $sz = tz$ imply that $a = a''u$, $a' = a''v$ and $us = vt$ for some $a'' \in A_S$, $u, v \in S$.

Let $K$ be a proper right ideal of $S$. If $x, y$, and $z$ denote elements not belonging to $S$, define $A(K) = (\{x, y\} \times (S \setminus K)) \bigcup (\{z\} \times K)$, and define a right $S$-action on $A(K)$ by

$$(x, v)s = \begin{cases} (x, vs) & vs \notin K \\ (z, vs) & vs \in K \end{cases}$$

$$(y, v)s = \begin{cases} (y, vs) & vs \notin K \\ (z, vs) & vs \in K \end{cases}$$

$$(z, v)s = (z, vs).$$

Then clearly $A(K)$ is a right $S$-act.

## 2 General properties

In this section, we introduce $GPW$-flatness property of acts and will give some general properties.

**Definition 2.1.** A right $S$-act $A_S$ is called *GPW-flat* if for every $s \in S$, there exists $n = n_{(s,A_S)} \in \mathbb{N}$, such that the functor $A_S \otimes {}_S-$ preserves the embedding of the principal left ideal ${}_S(Ss^n)$ into ${}_SS$.

Clearly every principally weakly flat right $S$-act is $GPW$-flat, but, by the following example, we see that the converse is not true.

**Example 2.2.** Suppose $S = \{1, x, 0\}$ with $x^2 = 0$, and let $K_S = \{x, 0\}$. Clearly the right Rees factor $S$-act $S/K$ is $GPW$-flat, but it is not principally weakly flat.

**Proposition 2.3.** *For any right $S$-act $A_S$, the following statements are equivalent*:

(1) *$A_S$ is GPW-flat.*

(2) *For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_SS$ implies $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_S(Ss^n)$.*

(3) *For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_S(Ss^n)$.*

(4) *For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies that*

$$\begin{aligned}
a &= a_1 s_1 & & \\
a_1 t_1 &= a_2 s_2 & s_1 s^n &= t_1 s^n \\
a_2 t_2 &= a_3 s_3 & s_2 s^n &= t_2 s^n \\
&\vdots & &\vdots \\
a_k t_k &= a' & s_k s^n &= t_k s^n,
\end{aligned}$$

*for some $k \in \mathbb{N}$ and elements $a_1, \ldots, a_k \in A_S$, $s_1, t_1, \ldots, s_k, t_k \in S$.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are obvious by Definition 2.1.

(2) $\Leftrightarrow$ (4) This is obvious by [9, II, Lemma 5.5]. $\square$

**Corollary 2.4.** *Suppose that $S$ is an idempotent or right cancellative monoid. Then for a right $S$-act $A_S$ the following statements are equivalent*:

(i) $A_S$ *is principally weakly flat.*

(ii) $A_S$ *is GPW-flat.*

*Proof.* This is obvious by Proposition 2.3. □

**Proposition 2.5.** *Every GPW-flat right $S$-act is torsion free.*

*Proof.* This is obvious by Proposition 2.3. □

**Example 2.6.** Let $S = \mathbb{N} \cup G$, where $\mathbb{N}$ is the set of natural numbers and $G$ is a nontrivial group with unit element $e$ and define the multiplication on $S$ as $gn = ng = n$ for every $g \in G$ and $n \in \mathbb{N}$. Clearly all right $S$-acts are torsion free by [9, IV, Theorem 6.1], but not all right $S$-acts are $GPW$-flat, see Theorem 4.5.

The following strict implications exist for different flatness properties of acts:

*Weakly flat* $\Rightarrow$ *Principally weakly flat* $\Rightarrow$ *GPW-flat* $\Rightarrow$ *Torsion free.*

In 2001, Laan [10] gave equivalents of different flatness properties according to surjectivity of $\varphi$ corresponding to some pullback diagram $P(M, N, f, g, Q)$ where $f : {}_SM \to {}_SQ$ and $g : {}_SN \to {}_SQ$ are homomorphisms of left $S$-acts.

Similar to [10, Proposition 2], we have the following proposition.

**Proposition 2.7.** *A right $S$-act $A_S$ is GPW-flat if and only if for every $s \in S$ there exists $n = n_{(s,A_S)} \in \mathbb{N}$ such that the corresponding $\varphi$ is surjective for the pullback diagram $P(Ss^n, Ss^n, \iota, \iota, S)$, where $\iota : {}_S(Ss^n) \to {}_SS$ is a monomorphism of left $S$-acts.*

**Proposition 2.8.** *The following statements hold:*

(1) *Any retract of a GPW-flat right $S$-act is GPW-flat.*

(2) *If $A = \coprod_{i \in I} A_i$ is GPW-flat, then $A_i$ is GPW-flat for every $i \in I$.*

(3) $S_S$ *is GPW-flat.*

(4) $\Theta_S$ *is GPW-flat.*

*Proof.* (3) and (4) are obvious.

(1) Suppose that $B_S$ is a $GPW$-flat right $S$-act and $A_S$ is a retract of $B_S$. Then there exist homomorphisms $f : B_S \to A_S$ and $f' : A_S \to B_S$, such that $ff' = id_{A_S}$. Let $s \in S$. Since $B_S$ is $GPW$-flat, there exists $n \in \mathbb{N}$ such that the equality $b \otimes s^n = b' \otimes s^n$ in $B_S \otimes {}_S S$ implies that $b \otimes s^n = b' \otimes s^n$ in $B_S \otimes {}_S(Ss^n)$, for any $b, b' \in B_S$ by (2) of Proposition 2.3. Let $as^n = a's^n$ for $a, a' \in A_S$. Then $f'(as^n) = f'(a's^n)$ and so $f'(a)s^n = f'(a')s^n$. Since $f'(a), f'(a') \in B_S$, $B_S$ is $GPW$-flat and $f'(a) \otimes s^n = f'(a') \otimes s^n$ in $B_S \otimes {}_S S$, we have

$$\begin{aligned} f'(a) &= b_1 s_1 & & \\ b_1 t_1 &= b_2 s_2 & s_1 s^n &= t_1 s^n \\ b_2 t_2 &= b_3 s_3 & s_2 s^n &= t_2 s^n \\ &\vdots & &\vdots \\ b_k t_k &= f'(a') & s_k s^n &= t_k s^n, \end{aligned}$$

where $b_1, \ldots, b_k \in B_S$, $s_1, t_1, \ldots, s_k, t_k \in S$, by (4) of Proposition 2.3. Thus $f(f'(a)) = f(b_1 s_1)$ and so $a = f(b_1)s_1$. Similarly, $f(b_{i-1})t_{i-1} = f(b_i)s_i$, $2 \leq i \leq k$, and $a' = f(b_k)t_k$. Hence

$$\begin{aligned} a \otimes s^n &= f(b_1)s_1 \otimes s^n = f(b_1) \otimes s_1 s^n = f(b_1) \otimes t_1 s^n = f(b_1)t_1 \otimes s^n \\ &= f(b_2)s_2 \otimes s^n = f(b_2) \otimes s_2 s^n = \ldots = f(b_k)t_k \otimes s^n = a' \otimes s^n \end{aligned}$$

in $A_S \otimes {}_S(Ss^n)$.

(2) Suppose that $A = \coprod_{i \in I} A_i$ is $GPW$-flat right $S$-act and let $s \in S$. Let $j \in I$. By assumption, there exists $n \in \mathbb{N}$ such that $as^n = a's^n$ for $a, a' \in A_S$ implies $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_S(Ss^n)$. Let $as^n = a's^n$ for $a, a' \in A_j$. Thus $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_S(Ss^n)$, by assumption. Hence $a \otimes s^n = a' \otimes s^n$ in $A_j \otimes {}_S(Ss^n)$, by [2, Corollary 2.3]. □

A monoid $S$ is called *left almost regular* if for every $s \in S$

$$\begin{aligned} s_1 c_1 &= s r_1 \\ s_2 c_2 &= s_1 r_2 \\ &\vdots \\ s_m c_m &= s_{m-1} r_m \\ s &= s_m r s, \end{aligned}$$

for some $r, r_1, \ldots, r_m$, $s_1$, $s_2$, $\ldots$, $s_m \in S$ and right cancellable elements $c_1, c_2, \ldots, c_m \in S$. Clearly every regular and right cancellative monoid is left almost regular.

As, by [9, IV, Theorem 6.5], over a left almost regular monoid every torsion free right $S$-act is principally weakly flat, the following proposition is easily checked.

**Proposition 2.9.** *Let $S$ be a left almost regular monoid, and $A_S$ be a right $S$-act. Then the following statements are equivalent*:

(1) *$A_S$ is principally weakly flat.*

(2) *$A_S$ is GPW-flat.*

(3) *$A_S$ is torsion free.*

**Theorem 2.10.** *For a proper right ideal $K$ of monoid $S$ the following statements are equivalent*:

(1) $\big(\forall s \in S, \exists n \in \mathbb{N}\big)\big(\forall l \in S \setminus K\big)\Big(ls^n \in K \Rightarrow \big(\exists k \in K, ls^n = ks^n\big)\Big).$

(2) *$A(K)$ is GPW-flat.*

*Proof.* (1) $\Rightarrow$ (2) Let $s \in S$. Then, by assumption, there exists $n \in \mathbb{N}$ such that (1) is established. Let $as^n = a's^n$ for $a, a' \in A(K)$. Since $(x,1)S \cong S_S \cong (y,1)S$ (every free right $S$-act is $GPW$-flat), without loss of generality, we can take $a = (x, r_1), a' = (y, r_2)$, where $r_1, r_2 \in S \setminus K$. Since $(x, r_1)s^n = (y, r_2)s^n$, we have $r_1 s^n = r_2 s^n \in K$, and so there exists $k \in K$ such that $r_1 s^n = ks^n = r_2 s^n$. Hence

$$(x, r_1) \otimes s^n = (x, 1) \otimes r_1 s^n = (x, 1) \otimes ks^n = (y, 1) \otimes ks^n = (y, r_2) \otimes s^n$$

in $A(K) \otimes {}_S(Ss^n)$.

(2) $\Rightarrow$ (1) Let $A(K)$ be $GPW$-flat and suppose $s \in S$. Then there exists $n \in \mathbb{N}$ such that $A(K) \otimes {}_S-$ preserves the embedding $\iota : {}_S(Ss^n) \to {}_SS$. Now let $l \in S \setminus K$ such that $ls^n \in K$. Then clearly $(x, l)s^n = (y, l)s^n$. By Proposition 2.3, we have

$$\begin{aligned} (x, l) &= (w_1, u_1)s_1 & & \\ (w_1, u_1)t_1 &= (w_2, u_2)s_2 & s_1 s^n &= t_1 s^n \\ &\vdots & &\vdots \\ (w_{m-1}, u_{m-1})t_{m-1} &= (w_m, u_m)s_m & s_{m-1}s^n &= t_{m-1}s^n \\ (w_m, u_m)t_m &= (y, l) & s_m s^n &= t_m s^n, \end{aligned}$$

for some $m \in \mathbb{N}$, $u_1, \ldots, u_m \in S$, $s_1, t_1, \ldots, s_m, t_m \in S$, and $w_1, \ldots, w_m \in \{x, y, z\}$. By definition of $A(K)$, there exists $i \in \{1, \ldots, m-1\}$ such that $w_i \neq w_{i+1}$, and so there exists $k \in K$ such that $u_i t_i = u_{i+1} s_{i+1} = k$. Hence we have

$$ls^n = u_1 s_1 s^n = u_1 t_1 s^n = u_2 s_2 s^n = \ldots = u_i t_i s^n = ks^n,$$

as required. □

Recall from [9, III, Definition 10.14] that an element $s$ of a monoid $S$ is called *right $e$-cancellable* for an idempotent $e \in S$ if $s = es$ and $\ker \rho_s \leq \ker \rho_e$. A monoid $S$ is called *left PP* if every element $s \in S$ is right $e$-cancellable for some idempotent $e \in S$. It is obvious that every regular and every right cancellative monoid is left $PP$. An element $s \in S$ is *right semi-cancellable* if the equality $xs = ys$ for any $x, y \in S$ implies that there exists $r \in S$ such that $rs = s$ and $xr = yr$. A monoid $S$ is called *left PSF* if every element $s \in S$ is right semi-cancellable. Clearly every left $PP$ monoid is left $PSF$.

**Proposition 2.11.** *Suppose that $S$ is a left PP monoid. An act $A_S$ is GPW-flat if and only if for every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies $es^n = s^n$ and $ae = a'e$ for some $e^2 = e \in S$.*

*Proof.* This is obvious by [9, III, Theorem 10.16]. □

For a left $PSF$ monoid, similar to argument used in [11, Proposition 2.5], we can show the following proposition.

**Proposition 2.12.** *Suppose that $S$ is a left PSF monoid. An act $A_S$ is GPW-flat if and only if for every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies $rs^n = s^n$ and $ar = a'r$ for some $r \in S$.*

**Corollary 2.13.** *For a left PSF monoid $S$, the following statements are equivalent:*

(1) *$\prod_{i=1}^{k} A_i$ is GPW-flat.*

(2) *For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $\alpha_i, \alpha_i' \in A_i, 1 \leq i \leq k$, if $(\alpha_1, \alpha_2, \ldots, \alpha_k)s^n = (\alpha_1', \alpha_2', \ldots, \alpha_k')s^n$, then $us^n = s^n$ and $(\alpha_1, \alpha_2, \ldots, \alpha_k)u = (\alpha_1', \alpha_2', \ldots, \alpha_k')u$ for some $u \in S$.*

*Proof.* This is obvious by Proposition 2.12. □

**Proposition 2.14.** *For any family* $\{A_i\}_{i\in I}$ *of right* $S$*-acts, if* $\prod_{i\in I} A_i$ *is* $GPW$*-flat, then* $A_i$ *is* $GPW$*-flat, for every* $i \in I$.

*Proof.* Let $\prod_{i\in I} A_i$ be $GPW$-flat and let $s \in S$ and $i \in I$. By assumption there exists $n \in \mathbb{N}$ such that the functor $\prod_{i\in I} A_i \otimes {}_S-$ preserves the embedding $\iota : {}_S(Ss^n) \to {}_SS$. Let $a_i s^n = a'_i s^n$ for any $a_i, a'_i \in A_i$ and suppose $a_j \in (A_j)_S$ for $j \neq i$. If

$$a_k = \begin{cases} a_i & \text{if } k = i \\ a_j & \text{if } k \neq i \end{cases}$$

$$a'_k = \begin{cases} a'_i & \text{if } k = i \\ a_j & \text{if } k \neq i \end{cases}$$

Then $(a_k)_I s^n = (a'_k)_I s^n$ and so $(a_k)_I \otimes s^n = (a'_k)_I \otimes s^n$ in $\prod_{i\in I} A_i \otimes_S (Ss^n)$, by Proposition 2.3. Now we have $a_i \otimes s^n = a'_i \otimes s^n$ in $A_i \otimes_S (Ss^n)$, by [14, Remrak 3.1], and so $A_i$ is $GPW$-flat. □

Golchin in [3] showed that if $S = G\dot{\cup} I$ where $G$ is a group and $I$ is an ideal of $S$ and $A$ is a right $S$-act that is ((principally) weakly) flat, torsion free, satisfies Condition $(P)$ or $(P_E)$ as a right $I^1$ -act, then it has these properties as a right $S$-act. Similarly, we can show the following theorem for $GPW$-flatness.

**Theorem 2.15.** *Let* $S = G\dot{\cup} I$ *and let* $A$ *be a right* $S$ *-act. If* $A$ *is* $GPW$*-flat as a right* $I^1$ *-act, then it is* $GPW$*-flat as a right* $S$ *-act.*

*Proof.* This is obvious by Proposition 2.3. □

## 3 $GPW$-flatness of (Rees factor) cyclic acts

In this section, we give conditions for a (Rees factor) cyclic act to be $GPW$-flat.

**Proposition 3.1.** *Suppose that* $\rho$ *is a right congruence on a monoid* $S$*. Then the following statements are equivalent*:

(i) $S/\rho$ *is* $GPW$*-flat.*

(ii) $\big(\forall s \in S\big)\big(\exists n \in \mathbb{N}\big)\big(\forall u, v \in S\big)\Big((us^n)\rho(vs^n) \Rightarrow u(\rho \vee \ker \rho_{s^n})v\Big)$.

*Proof.* (i) $\Rightarrow$ (ii) Let $s \in S$. Since the right $S$-act $S/\rho$ is $GPW$-flat, there exists $n \in \mathbb{N}$ such that the functor $(S/\rho)_S \otimes {}_S-$ preserves the embedding $\iota : {}_S(Ss^n) \to {}_SS$. Now suppose that $(us^n)\rho(vs^n)$ for $u, v \in S$. Thus $[v]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $(S/\rho)_S \otimes {}_SS$ and so $[v]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $(S/\rho)_S \otimes {}_S(Ss^n)$. Hence $u(\rho \vee \ker \rho_{s^n})v$, by [9, III, Lemma 10.6].

(ii) $\Rightarrow$ (i) Let $s \in S$. By assumption, there exists $n \in \mathbb{N}$ such that $(us^n)\rho(vs^n)$, for every $u, v \in S$, implies that $u(\rho \vee \ker \rho_{s^n})v$. Suppose $[v]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $(S/\rho)_S \otimes {}_SS$, thus $(us^n)\rho(vs^n)$. Now, by assumption, $u(\rho \vee \ker \rho_{s^n})v$ and so, by [9, III, Lemma 10.6], $[v]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $(S/\rho)_S \otimes {}_S(Ss^n)$. Hence $S/\rho$ is $GPW$-flat, by Proposition 2.3. □

**Corollary 3.2.** *The principal right ideal $zS$ is $GPW$-flat if and only if for every $s \in S$, there exists $n \in \mathbb{N}$ such that for any $x, y \in S$, $zxs^n = zys^n$ implies that $x(\ker \lambda_z \vee \ker \rho_{s^n})y$.*

*Proof.* Since $zS \cong S/\ker \lambda_z$, by Proposition 3.1, it suffices to take $\rho = \ker \lambda_z$. □

**Theorem 3.3.** *Suppose that $K$ is a right ideal of $S$. Then $S/K$ is $GPW$-flat if and only if for every $s \in S$ there exists a natural number $n \in \mathbb{N}$ such that $ls^n \in K$, for $l \in S \setminus K$ implies that $ls^n = ks^n$, for some $k \in K$.*

*Proof.* If $K = S$, then $S/K \cong \Theta_S$ is $GPW$-flat by (3) of Proposition 2.8. Thus suppose that $K$ is a proper right ideal of $S$.

*Necessity.* Suppose that $S/K$ is $GPW$-flat for the proper right ideal $K$ of $S$ and let $s \in S$. Then there exists $n \in \mathbb{N}$ such that the functor $A_S \otimes {}_S-$ preserves the embedding $\iota : {}_S(Ss^n) \to {}_SS$. Now suppose $ls^n \in K$ for $l \in S \setminus K$. Then $[l] \otimes s^n = [j] \otimes s^n$ in $S/K \otimes {}_SS$, for any $j \in K$ and so, by Proposition 2.3, there exist $m \in \mathbb{N}$, $p_1, \ldots, p_m, s_1, t_1, \ldots, s_m, t_m \in S$ such that

$$\begin{aligned} [l] &= [p_1]s_1 & & \\ [p_1]t_1 &= [p_2]s_2 & s_1s^n &= t_1s^n \\ [p_2]t_2 &= [p_3]s_3 & s_2s^n &= t_2s^n \\ &\vdots & &\vdots \\ [p_m]t_m &= [j] & s_ms^n &= t_ms^n. \end{aligned}$$

Since $j \in K$, we have $p_m t_m \in K$. Let q be the least number such that $q \in \{1, \ldots, m\}$ and $p_q t_q \in K$. Let $k = p_q t_q$, then $p_{q-1} t_{q-1} = p_q s_q$, and so

$$ls^n = p_1 s_1 s^n = p_1 t_1 s^n = p_2 s_2 s^n = \ldots = \\ p_{q-1} t_{q-1} s^n = p_q s_q s^n = p_q t_q s^n = ks^n.$$

*Sufficiency.* Let $K$ be a right ideal of $S$ and let $s \in S$. Thus there exists $n \in \mathbb{N}$ such that $ls^n \in K$, for $l \in S \setminus K$ implies that $ls^n = ks^n$ for some $k \in K$, by assumption. Let for any $u, v \in S$, $[u] \otimes s^n = [v] \otimes s^n$ in $S/K \otimes_S S$. Thus there are four cases as follows:

*Case* 1. $u, v \in K$. Then it is clear that $[u] = [v]$ in $S/K$ and so $[u] \otimes s^n = [v] \otimes s^n$ in $S/K \otimes_S (Ss^n)$.

*Case* 2. $u \in K, v \in S \setminus K$. Then there exists $k \in K$ such that $vs^n = ks^n$, by assumption. Then

$$[u] \otimes s^n = [k] \otimes s^n = [1] \otimes ks^n = [1] \otimes vs^n = [v] \otimes s^n$$

in $S/K \otimes_S (Ss^n)$.

*Case* 3. $u \in S \setminus K, v \in K$. It is similar to the Case 2.

*Case* 4. $u, v \in S \setminus K$. Then from $[u] \otimes s^n = [v] \otimes s^n$ in $S/K \otimes_S S$, we have either $us^n = vs^n$ or $us^n, vs^n \in K$. If $us^n = vs^n$, the result follows. Otherwise, $us^n = ks^n$ and $vs^n = ls^n$ for some $k, l \in K$, by assumption. So

$$[u] \otimes s^n = [1] \otimes us^n = [1] \otimes ks^n = [k] \otimes s^n = \\ [l] \otimes s^n = [1] \otimes ls^n = [1] \otimes vs^n = [v] \otimes s^n$$

in $S/K \otimes_S (Ss^n)$. □

## 4 Characterization of monoids by $GPW$-flatness of acts

Now we classify monoids over which all right $S$-acts are $GPW$-flat and also monoids over which some other properties imply $GPW$-flatness and vice versa.

A monoid $S$ is called *regular* if for every $s \in S$ there exists $x \in S$ such that $s = sxs$.

**Definition 4.1.** An element $s \in S$ is called *eventually regular* if $s^n$ is regular for some $n \in \mathbb{N}$. That is, $s^n = s^n x s^n$ for some $n \in \mathbb{N}$ and $x \in S$. A monoid $S$ is called *eventually regular* if every $s \in S$ is eventually regular.

Obviously every regular monoid is eventually regular.

**Definition 4.2.** An element $s \in S$ is called *eventually left almost regular* if

$$\begin{aligned} s_1 c_1 &= s^n r_1 \\ s_2 c_2 &= s_1 r_2 \\ &\vdots \\ s_m c_m &= s_{m-1} r_m \\ s^n &= s_m r s^n, \end{aligned}$$

for some $n \in \mathbb{N}$, elements $s_1, s_2, \ldots, s_m, r, r_1, \ldots, r_m \in S$ and right cancellable elements $c_1, c_2, \ldots, c_m \in S$. In other words $s \in S$ is called *eventually left almost regular* if $s^n$ is left almost regular for some $n \in \mathbb{N}$.

If every element of a monoid $S$ is eventually left almost regular, then $S$ is called *eventually left almost regular.*

It is clear that every left almost regular monoid is eventually left almost regular, and also every eventually regular monoid is eventually left almost regular.

**Example 4.3.** Let $S = \{1, 0, e, f, a\}$ be the monoid with the following table

| | $1$ | $0$ | $e$ | $f$ | $a$ |
|---|---|---|---|---|---|
| $1$ | $1$ | $0$ | $e$ | $f$ | $a$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e$ | $e$ | $0$ | $e$ | $a$ | $a$ |
| $f$ | $f$ | $0$ | $0$ | $f$ | $0$ |
| $a$ | $a$ | $0$ | $0$ | $a$ | $0$ |

Clearly $S$ is eventually regular and so it is eventually left almost regular. But $S$ is not regular, because $a \in S$ is not regular. Also $S$ is not left almost regular, since $a \in S$ is not left almost regular.

**Theorem 4.4.** *The following statements are equivalent:*

(1) *$S$ is an eventually left almost regular monoid.*
(2) *All torsion free right Rees factor acts over $S$ are GPW-flat.*
(3) *All torsion free cyclic right $S$-acts are GPW-flat.*
(4) *All torsion free finitely generated right $S$-acts are GPW-flat.*
(5) *All torsion free right $S$-acts are GPW-flat.*

*Proof.* (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are clear.

(2) $\Rightarrow$ (1) Suppose that all torsion free right Rees factor $S$-acts are $GPW$-flat and let $s \in S$. Let $K(s)$ be the subset of $S$ consisting of all elements $t \in S$ such that

$$\begin{aligned} s_1c_1 &= s^n r_1 \\ s_2c_2 &= s_1r_2 \\ &\vdots \\ s_{m-1}c_{m-1} &= s_{m-2}r_{m-1} \\ tc_m &= s_{m-1}r_m, \end{aligned}$$

for some $n \in \mathbb{N}$, the elements $s_1, s_2, \ldots, s_m, r, r_1, \ldots, r_m \in S$ and the right cancellable elements $c_1, c_2, \ldots, c_m \in S$. We see that $s^n \in K(s)$ for some $n \in \mathbb{N}$, and so $K(s)$ is non-empty, because, if $m = 1$ and $c_1 = r_1 = 1$, then $t = s^n$ has the required property mentioned. Now let $J = \bigcup_{t \in K(s)} tS$. Let $s'c \in J$, for $s' \in S$ and $c$ right cancellable. Then $s'c \in tS$ for some $t \in K(s)$, and so we have

$$\begin{aligned} s_1c_1 &= s^n r_1 \\ s_2c_2 &= s_1r_2 \\ &\vdots \\ s_{m-1}c_{m-1} &= s_{m-2}r_{m-1} \\ tc_m &= s_{m-1}r_m \\ s'c &= tr_{m+1}, \end{aligned}$$

for some $n \in \mathbb{N}$, the elements $s_1, s_2, \ldots, s_m, r, r_1, \ldots, r_m, r_{m+1} \in S$ and the right cancellable elements $c_1, c_2, \ldots, c_m \in S$. Thus $s' \in J$, and so $S/J$ is torsion free by [9, III, Proposition 8.10]. Hence $S/J$ is $GPW$-flat by assumption and so by Theorem 3.3, for $s^n \in J$, there exists $tr \in J$ such that $s^n = trs^n$, where $t \in K(s)$, and $r \in S$. Now $s^n = trs^n$ and $t \in K(s)$ implies that $s$ is eventually left almost regular.

(1) $\Rightarrow$ (5) Let $S$ be an eventually left almost regular monoid and suppose $A_S$ is a torsion free right $S$-act and let $s \in S$. Since $s$ is eventually left almost

regular, we have

$$
\begin{aligned}
s_1c_1 &= s^n r_1 \\
s_2c_2 &= s_1 r_2 \\
&\vdots \\
s_m c_m &= s_{m-1} r_m \\
s^n &= s_m r s^n,
\end{aligned}
$$

for some $n \in \mathbb{N}$, the elements $s_1, s_2, \ldots, s_m, r, r_1, \ldots, r_m \in S$ and the right cancellable elements $c_1, c_2, \ldots, c_m \in S$. Let $as^n = a's^n$ for $a, a' \in A_S$. Using torsion freeness, it can easily be seen that $as_m r = a's_m r$. Hence we have

$$a \otimes s^n = a \otimes s_m r s^n = as_m r \otimes s^n = a's_m r \otimes s^n = a' \otimes s_m r s^n = a' \otimes s^n$$

in $A_S \otimes {}_S(Ss^n)$ and so $A_S$ is $GPW$-flat, as required. □

A right $S$-act $A_S$ is a generator if for any distinct homomorphisms $\alpha, \beta : X_S \to Y_S$, there exists a homomorphism $f : A_S \to X_S$ such that $\alpha f \neq \beta f$. Equivalently, a right $S$-act $A_S$ is a generator if and only if there exists an epimorphism $\pi : A_S \to S_S$ ([9, II, Theorem 3.16]).

As we know, $S \times A_S$ is a generator for each right $S$-act $A_S$.

**Theorem 4.5.** *The following statements are equivalent*:

(1) *$S$ is an eventually regular monoid.*
(2) *A right $S$-act $A_S$ is GPW-flat if $Hom(A_S, S_S) \neq \emptyset$.*
(3) *$S \times A_S$ is GPW-flat for every generator right $S$-act $A_S$.*
(4) *$S \times A_S$ is GPW-flat for every right $S$-act $A_S$.*
(5) *All generator right $S$-acts are GPW-flat.*
(6) *All right Rees factor $S$-acts are GPW-flat.*
(7) *All cyclic right $S$-acts are GPW-flat.*
(8) *All right $S$-acts are GPW-flat.*

*Proof.* (8) ⇒ (7) ⇒ (6), (8) ⇒ (5), (8) ⇒ (4) ⇒ (3), (2) ⇒ (4), (5) ⇒ (4) and (8) ⇒ (2) are obvious.

(4) ⇒ (8) This is valid by Proposition 2.14.

(6) ⇒ (1) If all right Rees factor acts over $S$ are $GPW$-flat, then all right Rees factor acts over $S$ are torsion free. So every right cancellable element

of $S$ is right invertible, by [9, IV, Theorem 6.1], but by Theorem 4.4, $S$ is eventually left almost regular. Now let $s \in S$. Then

$$\begin{aligned} s_1 c_1 &= s^n r_1 \\ s_2 c_2 &= s_1 r_2 \\ &\vdots \\ s_m c_m &= s_{m-1} r_m \\ s^n &= s_m r s^n, \end{aligned}$$

Multiplying both sides of the equalities in the above scheme by $c_i^{-1}$ for $i \in \{1, \ldots, m\}$, respectively, we get

$$\begin{aligned} s_1 &= s^n r_1 c_1^{-1} \\ s_2 &= s_1 r_2 c_i^{-1} \\ &\vdots \\ s_m &= s_{m-1} r_m c_m^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} s^n = s_m r s^n = s_{m-1} r_m c_m^{-1} r s^n &= s_{m-2} r_{m-1} c_{m-1}^{-1} r_m c_m^{-1} r s^n = \ldots \\ &= s^n r_1 c_1^{-1} \ldots r_{m-1} c_{m-1}^{-1} r_m c_m^{-1} r s^n, \end{aligned}$$

and so $s$ is eventually regular, as required.

(1) $\Rightarrow$ (8) Suppose that $A_S$ is a right $S$-act and let $s \in S$. By Proposition 2.3, we have to show that there exists $m \in \mathbb{N}$ such that for any $a, a' \in A_S$, if $a \otimes s^m = a' \otimes s^m$ in $A_S \otimes {}_S S$, then $a \otimes s^m = a' \otimes s^m$ in $A_S \otimes {}_S (Ss^m)$. Since $s$ is eventually regular, there exist $n \in \mathbb{N}$ and $t \in S$, such that $s^n = s^n t s^n$. If $m = n$. Let $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_S S$ for any $a, a' \in A_S$, then

$$a \otimes s^n = a \otimes s^n t s^n = a s^n \otimes t s^n = a' s^n \otimes t s^n = a' \otimes s^n t s^n = a' \otimes s^n$$

in $A_S \otimes {}_S (Ss^n)$ and so $A_S$ is $GPW$-flat.

(3) $\Rightarrow$ (4) Suppose that $A_S$ is a right act over $S$. As we show in the proof of (5) $\Rightarrow$ (4), $S \times A_S$ is a generator and so, by assumption, $S \times (S \times A_S)$ is $GPW$-flat, which means that $S \times A_S$ is $GPW$-flat, by Proposition 2.14. $\square$

It is obvious that Condition $(P)$ implies $GPW$-flatness, but the following example shows that this is not the case for Condition $(E)$.

**Example 4.6.** Let $S = (\mathbb{N}, .)$ be the monoid of natural numbers with multiplication and let $A_{\mathbb{N}} = \mathbb{N} \coprod^{2\mathbb{N}} \mathbb{N}$. Then $A_{\mathbb{N}}$ satisfies Condition $(E)$, but it is not $GPW$-flat.

Now, the question is that: What is the structure of monoids over which Condition $(E)$ of their acts implies $GPW$-flatness?

**Theorem 4.7.** *For a right cancellative monoid $S$ the following statements are equivalent*:

(1) $\prod_{i \in I} A_i$ *is principally weakly flat, for any family* $\{A_i\}_{i \in I}$ *of right $S$-acts.*

(2) $\prod_{i \in I} A_i$ *is GPW-flat, for any family* $\{A_i\}_{i \in I}$ *of right $S$-acts.*

(3) $\prod_{i \in I} A_i$ *is torsion free, for any family* $\{A_i\}_{i \in I}$ *of right $S$-acts.*

(4) *$S$ is a group.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are obvious.

(3) $\Rightarrow$ (4) This is obvious, by [14, Remark 3.1] and [9, IV. Theorem 6.1]. □

An element $a \in A_S$ is called divisible by $s \in S$ if there exists $b \in A_S$, such that $bs = a$. An act $A_S$ is said to be divisible if $Ac = A$, for any left cancellable element $c \in S$. It is clear that $A_S$ is divisible if and only if every element of $A_S$ is divisible by any left cancellable element of $S$.

**Theorem 4.8.** *The following statements are equivalent*:

(1) *All right $S$-acts are divisible.*

(2) *All GPW-flat right $S$-acts are divisible.*

(3) *All GPW-flat finitely generated right $S$-acts are divisible.*

(4) *All GPW-flat cyclic right $S$-acts are divisible.*

(5) *All GPW-flat monocyclic right $S$-acts are divisible.*

(6) *All left cancellable elements of $S$ are left invertible.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (6) For every $s \in S$ we have $S/\rho(s,s) = S_S/\Delta_S \cong S_S$, by (3) of Proposition 2.8, $S_S$ is $GPW$-flat, and so it is divisible by assumption. Thus $Sc = S$, for any left cancellable element $c \in S$. Thus, there exists $s \in S$ such that $sc = 1$, and so $c$ is left invertible, as required.

(6) $\Rightarrow$ (1) It is clear from [9, III, Proposition 2.2]. □

Recall, from [15], that a right $S$-act $A_S$ is called *strongly torsion free* if the equality $as = a's$, for $a, a' \in A_S$ and $s \in S$ implies $a = a'$. It is clear that every strongly torsion free right $S$-act is $GPW$-flat, but not the converse.

**Theorem 4.9.** *The following statements are equivalent*:
(1) *All $GPW$-flat right $S$-acts are strongly torsion free.*
(2) *All $GPW$-flat finitely generated right $S$-acts are strongly torsion free.*
(3) *All $GPW$-flat cyclic right $S$-acts are strongly torsion free.*
(4) *$S$ is right cancellative monoid.*

*Proof.* This follows from [15, Theorem 3.1]. □

Recall from [6] that a right $S$-act $A_S$ is *$E$-torsion free* if for any $a, a' \in A_S$ and $e \in E(S)$, $ae = a'e$ implies $a = a'$.

**Theorem 4.10.** *The following statements are equivalent:*
(1) *All $GPW$-flat right $S$-acts are $E$-torsion free.*
(2) *All $GPW$-flat finitely generated right $S$-acts are $E$-torsion free.*
(3) *All $GPW$-flat cyclic right $S$-acts are $E$-torsion free.*
(4) *$E(S) = \{1\}$.*

*Proof.* This is obvious by [6, Theorem 3.1]. □

Recall from [1, Definition 1] that a right $S$-act $A_S$ is called *principally weakly kernel flat* (PWKF) if the corresponding $\varphi$ is bijective for the pullback diagram $P(Ss, Ss, f, f, S)$ ($s \in S$), and $A_S$ is *translation kernel flat (TKF)* if the corresponding $\varphi$ is bijective for the pullback diagram $P(S, S, f, f, S)$.

**Theorem 4.11.** *The following statements on a monoid $S$ are equivalent*:
(1) *All $GPW$-flat right $S$-acts are $PWKF$ and $S$ is left $PSF$.*
(2) *All $GPW$-flat right $S$-acts are $TKF$ and $S$ is left $PSF$.*
(3) *All $GPW$-flat right $S$-acts satisfy Condition $(PWP)$ and $S$ is left $PSF$.*
(4) *All $GPW$-flat right $S$-acts satisfy Condition $(P')$ and $S$ is left $PSF$.*
(5) *$S$ is right cancellative.*

*Proof.* (1) ⇒ (2) ⇒ (3) and (4) ⇒ (3) are clear.

(5) ⇒ (1) By [13, Theorem 2.12] and Corollary 2.4, it is clear.

(3) ⇒ (5) This is obvious, by [12, Theorem 2.8].

(5) ⇒ (4) Since $S$ is right cancellative, $S$ is left $PSF$. By [4, Theorem 2.8] and Corollary 2.4, it is clear. □

**Theorem 4.12.** *The following statements on a monoid $S$ are equivalent*:

(1) *All GPW-flat right $S$-acts are $PWKF$ and there exists a regular left $S$-act.*

(2) *All GPW-flat right $S$-acts are $TKF$ and there exists a regular left $S$-act.*

(3) *All GPW-flat right $S$-acts satisfy Condition $(PWP)$ and there exists a regular left $S$-act.*

(4) *All GPW-flat right $S$-acts satisfy Condition $(P')$ and there exists a regular left $S$-act.*

(5) $|E(S)| = 1$ *and there exists a regular left $S$-act.*

(6) *$S$ is right cancellative.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) are clear.

(6) $\Rightarrow$ (4) This is clear by [4, Theorem 2.9] and Corollary 2.4.

(5) $\Leftrightarrow$ (6) This is clear by [12, Theorem 2.9].

(3) $\Rightarrow$ (6) This is clear by [12, Theorem 2.9].

(6) $\Rightarrow$ (1) This is clear by [13, Theorem 2.18] and Corollary 2.4. □

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