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# On exact category of $(m, n)$-ary hypermodules

Najmeh Jafarzadeh and Reza Ameri*

**Abstract.** We introduce and study category of $(m, n)$-ary hypermodules as a generalization of the category of $(m, n)$-modules as well as the category of classical modules. Also, we study various kinds of morphisms. Especially, we characterize monomorphisms and epimorphisms in this category. We will proceed to study the fundamental relation on $(m, n)$-hypermodules, as an important tool in the study of algebraic hyperstructures and prove that this relation is really functorial, that is, we introduce the fundamental functor from the category of $(m, n)$-hypermodules to the category $(m, n)$-modules and prove that it preserves monomorphisms. Finally, we prove that the category of $(m, n)$-hypermodules is an exact category, and, hence, it generalizes the classical case.

## 1 Introduction and preliminaries

The concept of a hypergroup was introduced by Marty in [19]. Afterwards, because of many applications of this theory in both pure and applied sci-

* Corresponding author
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ences, many authors have been doing research in this field. Some review of the hyperstructure theory can be found in [1, 6, 7, 10, 28].

In 1928, Dörnte introduced the concept of $n$-ary groups [12] and since then, $n$-ary system has been studied in different contexts (for instance see [8, 9]).

The research about $n$-ary hyperstructure was initiated by Davvaz and Vougiouklis who introduced these structures in [11]. The notation of $(m, n)$-ary hyperring was defined by Mirvakili et al. in [20]. After that, Anvariyeh et al. in [4] introduced the notion of $(m, n)$-hypermodules over $(m, n)$-ary hyperrings. Ameri and Norouzi introduced [2] the concept of $n$-ary prime and $n$-ary primary hyperideales in Krasner $(m, n)$-hyperring and proved some results in this respect. Some review of $(m, n)$-ary hyperstructures can be found in [3, 5, 17, 23].

Also, Ameri introduced and studied categories of hypergroups and hypermodules [1]; Madanshekaf in [18] proved that the category of hypermodules is an exact category. Recently, in numerous papers, categories of hyperstructures have been studied (for instance, see [1, 2, 13, 14, 18, 22, 24–27]).

In this paper, the authors follow [1, 14, 18] to study the category of $(m, n)$-hypermodules and prove that this category is exact. This paper has been written in 5 sections. In Section 1, we give some basic preliminaries about $(m, n)$-rings and $(m, n)$-hypermodules. In Section 2, we say about monomorphisms and epimorphisms in this category. In Section 3, monomorphisms and epimorphisms in the category of derived $(m, n)$-hypermodules via the fundamental relation are discussed. In Section 4, some properties of the category $R_{(m,n)} - Khmod$ are given and it is proved that this category is an exact category. Finally in Section 5, conclusion of this paper is briefly describe.

In this section we recall some notions and results from [11] and other references for the sake of completeness.

A mapping $f : \underbrace{H \times \cdots \times H}_{n} \to P^*(H)$ is called an $n$-ary *hyperoperation,* where $P^*(H)$ is the set of all nonempty subsets of $H$. An algebraic system $(H, f)$, where $f$ is an $n$-ary hyperoperation defined on $H$, is called an *$n$-ary hypergroupoid.* We should use the following abbreviated notation:

The sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by $x_i^j$. For $j < i$, $x_i^j$ is the empty set. Using this notation,

$$f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n)$$

will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. In the case when $y_{i+1} = \cdots = y_j = y$, the last expression will be written $f(x_1^i, y_{(j-i)}, z_{j+1}^n)$.

If $f$ is an $n$-ary hyperoperation and $t = l(n-1)+1$, for some $l \geq 0$, then $t$-ary hyperoperation $f_l$ is given by

$$f_l(x_1^{l(n-1)+1}) = \underbrace{f(f(\ldots, f(f}_{l}(x_1^n), x_{n+1}^{2n-1}), \ldots, ), x_{(l-1)(n-1)+1}^{l(n-1)+1}).$$

For nonempty subsets $A_1, A_2, \ldots, A_n$ of $H$ we define

$$f(A_1^n) = f(A_1, A_2, \ldots, A_n) = \bigcup \{f(x_1^n) | x_i \in A_i, i = 1, 2, \ldots, n\}.$$

An $n$-ary hyperoperation $f$ is called *associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

hold for every $1 \leq i < j \leq n$ and all $x_1, \ldots, x_{n-1} \in H$. An $n$-ary hypergroupoid with the associative $n$-ary hyperoperation is called an *$n$-ary semihypergroup.*

An $n$-ary hypergroupoid $(H, f)$ in which the equation $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ has a solution, $x_i \in H$ for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$, is called an *$n$-ary quasihypergroup.* If $(H, f)$ is an $n$-ary semihypergroup and $n$-ary quasihypergroup, then $(H, f)$ is called an *$n$-ary hypergroup.* An $n$-ary hypergroupoid $(H, f)$ is *commutative* if for all $\sigma \in \mathbb{S}_n$ and for every $a_1^n \in H$ we have $f(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. If $a_1^n \in H$; then we denote $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ by $a_{\sigma(1)}^{\sigma(n)}$.

**Definition 1.1.** Let $(H, f)$ be an $n$-ary hypergroup and $B$ be a non-empty subset of $H$. $B$ is called an *$n$-ary subhypergroup* of $(H, f)$, if $f(x_1^n) \subseteq B$ for all $x_1^n \in B$, and the equation $b \in f(b_1^{i-1}, x_i, b_{i+1}^n)$ has a solution, $x_i \in B$ for every $b_1^{i-1}, b_{i+1}^n, b \in B$ and $1 \leq i \leq n$.

**Definition 1.2.** [15] Let $(H, f)$ be a commutative $n$-ary hypergroup. $(H, f)$ is called a *canonical $n$-ary hypergroup* if

(1) there exists unique $e \in H$, such that for every $x \in H$,

$$f(x, \underbrace{e, \ldots, e}_{(n-1)}) = x;$$

(2) for all $x \in H$ there exists unique $x^{-1} \in H$, such that

$$e \in f(x, x^{-1} \underbrace{e, \ldots, e}_{(n-2)});$$

(3) if $x \in f(x_1^n)$, then for all $i$, we have

$$x_i \in f(x, x^{-1}, \ldots, x_{i-1}^{-1}, x_{i+1}^{-1}, \ldots, x_n^{-1}),$$

we say that $e$ is the scaler identity of $(H, f)$ and $x^{-1}$ is the inverse of $x$. Note that the inverse of $e$ is $e$.

**Definition 1.3.** [21] A *Krasner* $(m, n)$*-hyperring* is an algebraic hyperstructure $(R, h, k)$ which satisfies the following axioms:

(1) $(R, h)$ is a canonical $m$-ary hypergroup;

(2) $(R, k)$ is an $n$-ary semigroup;

(3) the $n$-ary operation $k$ is distributive to the $m$-array hyperoperation $h$, that is, for all $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, and $1 \leq i \leq n$,

$$k(a_1^{i-1}, h(x_1^m), a_{i+1}^n) = h(k(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, k(a_1^{i-1}, x_m, a_{i+1}^n));$$

(4) 0 is a zero element (absorbing element), of the $n$-ary operation $k$, that is, for $x_2^n \in R$ we have

$$k(0, x_2^n) = k(x_2, 0, x_3^n) = \cdots = k(x_2^n, 0).$$

A nonempty subset $S$ of $R$ is called a *subhyperring* of $R$ if $(R, h, k)$ is a Krasner $(m, n)$-hyperring. Let $I$ be a non-empty subset of $R$. We say that $I$ is a *hyperideal* of $(R, h, k)$ if $(I, h)$ is a canonical $m$-array hypergroup of $(R, h)$ and $k(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$, for every $x_1^n \in R$, and $1 \leq i \leq n$.

**Definition 1.4.** [4] Let $M$ be a nonempty set. Then $(M, f, g)$ is an $(m, n)$-*hypermodule* over an $(m, n)$-hyperring $(R, h, k)$, if $(M, f)$ is an $m$-ary hypergroup and the map

$$g : \underbrace{R \times \cdots \times R}_{n-1} \times M \to P^*(M)$$

satisfies the following conditions:

$(i)$ $g(r_1^{n-1}, f(x_1^m)) = f(g(r_1^{n-1}, x_1), \ldots, g(r_1^{n-1}, x_m))$;

$(ii)$ $g(r_1^{i-1}, h(s_1^m), r_{i+1}^{n-1}, x) = f(g(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, g(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x)$;

$(iii)$ $g(r_1^{i-1}, k(r_i^{i+n-1}), r_{i+m}^{n+m-2}, x) = g(r_1^{n-1}, g(r_m^{n+m-2}, x))$;

$(iv)$ $0 \in g(r_1^{i-1}, 0, r_{i+1}^{n-1}, x)$.

If $g$ is an $n$-ary hyperoperation, $S_1, \ldots, S_{n-1}$ are subsets of $R$ and $M_1 \subseteq M$, we set

$$g(S_1^{n-1}, M_1) = \bigcup\{g(r_1^{n-1}, x) | r_i \in S_i, i = 1, \ldots, n-1, x \in M_1\}.$$

If $n = m = 2$, then an $(m, n)$-ary hypermodule $M$ is a hypermodule.

Let $(M, f, g)$ be an $(m, n)$-hypermodule over an $(m, n)$-hyperring $(R, h, k)$. A non-empty subset $N$ of $M$ is called an $(m, n)$-ary subhypermodule of $M$ if $(N, f)$ is an $m$-ary subhypergroup of $(M, f)$ and $g(R^{(n-1)}, N) \in P^*(N)$.

**Definition 1.5.** [4] A *canonical $(m, n)$-hypermodule* $(M, f, g)$ is an $(m, n)$-hypermodule with a canonical $m$-ary hypergroup $(M, f)$ over a Krasner $(m, n)$-hyperring $(R, h, k)$.

A Krasner $(m, n)$-hyperring $(R, h, k)$ is *commutative* if $(R, k)$ is a commutative $n$-ary semigroup. Also, we say that $(R, h, k)$ is with a *scaler identity* if there exists an element $1_R$ such that $x = k(x, 1_R^{(n-1)})$ for all $x \in R$. Let $(R, h, k)$ be a commutative Krasner $(m, n)$-hyperring with a scaler identity $1_R$. For all $r_1^{n-1} \in R$ and $x \in M$ we have

$$g(r_1^{n-1}, 0_M) = \{0_M\}, \quad g(0_R^{n-1}, x) = \{0_M\} \text{ and } \ g(1_R^{n-1}, x) = \{x\}.$$

Moreover, let $g(r_1^{i-1}, -r_i, r_{i+1}^{n-1}, x) = -g(r_1, \ldots, r_{n-1}, x) = g(r_1^{n-1}, -x)$.

**Definition 1.6.** [4] Let $(M_1, f_1, g_1)$ and $(M_2, f_2, g_2)$ be two $(m, n)$-hyperm -odules over an $(m, n)$-hyperring $(R, h, k)$. We say that $\phi : M_1 \to M_2$ is a *homomorphism of $(m, n)$-hypermodules* if for all $x_1^m, x$ of $M_1$ and $r_1^{n-1} \in R$ :

$\phi(f_1(x_1, \ldots, x_m)) = f_2(\phi(x_1), \ldots, \phi(x_m))$;

$\phi(g_1(r_1^{n-1}, x)) = g_2(r_1^{n-1}, \phi(x))$.

In the above definition, if we consider a map $\phi : M_1 \to P^*(M_2)$, then we obtain a *multivalued homomorphism,* shortly we write $m$-homomorphism.

**Definition 1.7.** [16] Let $(H, f)$ be an $n$-ary hypergroup. The relation $\beta^*$ is the smallest equivalence relation such that the quotient $(H/\beta^*, f/\beta^*)$ is an $n$-ary group. where $H/\beta^*$ is the set of equivalence classes. The $\beta^*$ is called *fundamental equivalence relation.*

**Definition 1.8.** [21] Let $(R, h, k)$ be $(m, n)$-hyperring. The relation $\Gamma^*$ is the smallest equivalence relation such that the quotient $(R/\Gamma^*, h/\Gamma^*, k/\Gamma^*)$ is an $(m, n)$-ring, where $R/\Gamma^*$ is the set of equivalence classes. The $\Gamma^*$ is called *fundamental equivalence relation.*

**Definition 1.9.** [4] Let $(M, f, g)$ be an $(m, n)$-ary hypermodule over an $(m, n)$-ary hyperring $(R, h, k)$. Then $\hat{\epsilon}$ denotes the transitive closure of the relation $\epsilon = \bigcup_{\alpha \geq 0} \epsilon_\alpha$, where $\epsilon_0$ is the diagonal, that is, $\epsilon_0 = \{(x, x) | x \in M\}$ and for every integer $\alpha \geq 1, \epsilon_\alpha$ is the relation defined as follows:

$$x\epsilon_\alpha y \quad \text{if and only if} \quad \{x, y\} \subseteq f_{(\alpha)},$$

for some $\alpha \in \mathbb{N}$. If $x\epsilon_0 y$ (that is, $x = y$) then we write $\{x, y\} \subseteq u_{(0)}$. We define $\epsilon^*$ as the smallest equivalence relation such that the quotient $(M/\epsilon^*, f/\epsilon^*, g/\epsilon^*)$ is an $(m, n)$-ary module over an $(m, n)$-ary hyperring $R$, where $M/\epsilon^*$ is the set of equivalence classes. The $\epsilon^*$ is called *fundamental equivalence relation.*

**Theorem 1.10.** [4] *The fundamental relation $\epsilon^*$ is the transitive closure of the relation $\epsilon$, that is, $(\epsilon^* = \hat{\epsilon})$.*

**Theorem 1.11.** [4] *Let $(M, f, g)$ be an $(m, n)$-ary hypermodule over an $(m, n)$-ary hyperring $(R, h, k)$. Then, $(M/\epsilon^*, f/\epsilon^*)$ is an $(m, n)$-ary module over on $(m, n)$-ary ring $(R/\Gamma^*, h/\Gamma^*, k/\Gamma^*)$.*

Recall that, in the fundamental $R/\Gamma^*$-module $(M/\epsilon^*, f/\epsilon^*, g/\epsilon^*)$, the hyperoperations $f/\epsilon^*, g/\epsilon^*$ are defined as follows:

$$f/\epsilon^*(\epsilon^*(a_1), \ldots, \epsilon^*(a_m)) := \{\epsilon^*(a) | a \in f(a_1, \ldots, a_m)\} = \epsilon^*(f(a_1^m)),$$
$$g/\epsilon^*(\Gamma^*(r_1), \ldots, \Gamma^*(r_{n-1}), \epsilon^*(x)) := g(\Gamma^*(r_1), \ldots, \Gamma^*(r_{n-1}), \epsilon^*(x)).$$

## 2 Monomorphisms and epimorphisms in the categories of $R_{(m,n)} - KHmod$

In this section, we recall the definition of the categories of $(m, n)$-ary hypermodules. Then we study the relationship between monomorphisms, epimorphisms, isomorphisms, monics, epics, and iso arrows in these categories.

**Definition 2.1.** [14] The *category $R_{(m,n)} - Hmod$ of $(m,n)$-ary hypermodules* is defined by

(i) the objects of $R_{(m,n)} - Hmod$ are $(m,n)$- hypermodules,

(ii) for the objects $M$ and $K$, the set of all morphisms from $M$ to $K$ is defined as

$$Hom_R(M,K) = \{f | f : M \to P^*(K) \text{ is an m-homomorphism}\},$$

(iii) the composition $gf$ of morphisms $f : M \to P^*(K)$ and $g : K \to P^*(L)$ is defined by

$$gf : H \to P^*(K), \quad gf(x) = \bigcup_{t \in f(x)} g(t),$$

(iv) for any object $H$, the morphism $1_H : H \to P^*(H)$, defined by $1_H(x) = \{x\}$, is the identity morphism.

**Remark 2.2.** [14] Consider the category whose objects are all $(m,n)$-hypermodules and whose morphisms are all $R$-homomorphisms denoted by $R_{(m,n)} - hmod$. The class of all $R$-homomorphisms from $A$ into $B$ is denoted by $hom_R(A,B)$. In addition, $R_{s(m,n)} - hmod$, is the category of all $(m,n)$-hypermodules whose morphisms are all strong $R$-homomorphisms. The class of all strong $R$-homomorphisms from $A$ into $B$ is denoted by $hom_{R_S}(A,B)$. It is easy to observe that $R_{s(m,n)} - hmod$ is a subcategory of $R_{(m,n)} - hmod$.

**Remark 2.3.** [14] Later in this paper, we consider the category of all $(m,n)$-hypermodules over a $(m,n)$-hyperring $R$, in the sense of canonical $(m,n)$-hypermodules over Krasner $(m,n)$-hyperring $R$ with a scaler identity. We denote this category by $R_{(m,n)} - KHmod$. Hence the objects of $R_{(m,n)} - KHmod$ are the canonical $(m,n)$-hypermodules over Krasner $(m,n)$-hyperring and all morphisms are multivalued homomorphisms.

**Theorem 2.4.** *$\mathcal{F} : R_{s(m,n)} - hmod \to R_{(m,n)}/\Gamma^* - mod$, defined by $\mathcal{F}(M) = M/\epsilon^*$ and $\mathcal{F}(\phi : M_1 \to M_2) = \phi^* : M_1/\epsilon^* \to M_2/\epsilon^*$, is a functor, where $R_{(m,n)}/\Gamma^* - mod$ is the category of all $(m,n)$-modules over $R/\Gamma^*$.*

The next results characterize monomorphisms and epimorphisms in the category $R_{(m,n)} - KHmod$. First

**Definition 2.5.** Let $\phi : A \to B$ is a morphism of $(m, n)$-hypermodules. We say that $\phi$ is *weakly injective* if

$$\forall a, b \in A, \phi(a) \cap \phi(b) \neq \phi \Rightarrow a = b.$$

We say that $\phi$ is *strongly injective* if

$$\forall a, b \in A, \phi(a) = \phi(b) \Rightarrow a = b.$$

**Remark 2.6.** Clearly, every weakly injective morphisms is also strongly injective.

**Proposition 2.7.** *In the category $R_{(m,n)} - KHmod$, if $\phi : B \to C$ is a monomorphism, then it is strongly injective.*

*Proof.* Suppose that $\phi : B \to C$ is a monomorphism. For $b_1, b_2 \in B$, let $\phi(b_1) = \phi(b_2)$. Define the mappings $\hat{b_1}, \hat{b_2} : \underbrace{R \times \cdots \times R}_{n-1} \to P^*(B)$ by $\hat{b_1}(r_1^{n-1}) = \{g_1(r_1^{n-1}, b_1)\}$ and $\hat{b_2}(r_1^{n-1}) = \{g_1(r_1^{n-1}, b_2)\}$ (here $R$ is viewed as a $(m, n)$-hypermodule). By 2.1(ii), $\hat{b_1}, \hat{b_2}$ are well-defined morphisms of $(m, n)$-hypermodules. Moreover,
$\phi \circ (\hat{b_1}(r_1^{n-1})) = \phi(g_1(r_1^{n-1}, b_1)) = g_2(r_1^{n-1}, \phi(b_1)) = g_2(r_1^{n-1}, \phi(b_2)) = \phi(g_1(r_1^{n-1}, b_2)) = \phi \circ (\hat{b_2}(r_1^{n-1}))$, and thus $\hat{b_1} = \hat{b_2}$. In particular $\hat{b_1}(1_1^{n-1}) = \hat{b_2}(1_1^{n-1})$, that is, $b_1 = b_2$. □

**Proposition 2.8.** *In the category $R_{(m,n)} - KHmod$, if $\phi : B \to C$ is weakly injective then it is a monomorphism.*

*Proof.* It easily follows from Proposition 4 of [13]. □

**Definition 2.9.** Let $\phi : A \to B$ be a morphism of $(m, n)$-hypermodules. We say that $\phi$ is *weakly surjective* if

$$\forall b \in B, \exists a \in A, b \in \phi(a).$$

We say that $\phi$ is *strongly surjective* if

$$\forall \acute{B} \in P^*(B), \exists a \in A, \acute{B} = \phi(a).$$

**Remark 2.10.** Clearly, strongly surjective morphisms are also weakly surjective.

| $+$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | $\{0, 1\}$ |

Table 1: $(A, +)$

| $+'$ | 0 | 1 | -1 |
|---|---|---|---|
| 0 | 0 | 1 | -1 |
| 1 | 1 | 1 | $\{0, 1, -1\}$ |
| -1 | -1 | $\{0, 1, -1\}$ | -1 |

Table 2: $(B, +')$

**Example 2.11.** We shall provide an example of a weakly surjective morphism which is not strongly surjective. Let $A$ and $B$ be two canonical hypergroup as Tabels 1 and 2:

Define $0 * x = 0$ and $1 * x = x$ for all $x \in A, B$. Then, it is easy to check that $(A, +, *)$ is a Krasner hyperring, and $A$ and $B$ are also $A$-hypermodule with the external multiplication $*$. Let $\varphi : B \to A$ with $\varphi(1) = \varphi(-1) = 0$ and $\varphi(0) = 0$. Clearly, $\varphi$ is weakly surjective, but not strongly surjective (for example $\varphi(1), \varphi(-1), \varphi(0) \neq \{0, 1\}$).

**Proposition 2.12.** *In the category $R_{(m,n)} - KHmod$, if $\phi : B \to C$ is an epimorphism then it is weakly surjective.*

*Proof.* It easily follows from Proposition 5 of [13].

□

**Proposition 2.13.** *In the category $R_{(m,n)} - KHmod$, if $\phi : B \to C$ is strongly surjective, then it is an epimorphism.*

*Proof.* It easily follows from Proposition 5 of [13]. □

**Remark 2.14.** Clearly, an isomorphism of $(m, n)$-hypermodules is strongly injective and weakly surjective.

**Proposition 2.15.** *In the category $R_{(m,n)} - KHmod$, a morphism $\phi : A \to B$ is an isomorphism if and only if it is a single-valued bijective morphism.*

*Proof.* Firstly, assume that $\phi : A \to B$ is an isomorphism. By a similar way to Proposition 6 of [13], $\phi$ is single-valued bijective morphism.

Next, suppose that $\phi : A \to B$ is a single-valued bijective morphism. Define the map $\psi : B \to A$ by

$$\psi(b) = \{a\} \text{ if and only if } \phi(a) = \{b\}.$$

Clearly, $\phi \circ \psi = id_B$ and $\psi \circ \phi = id_A$, and to check that $\psi$ is a morphism fix $b_1^m \in B$ and $a_1^m \in A$ satisfy for all $i \in 1, 2, \ldots, m,\ \phi(a_i) = b_i$. Observe that

$$\begin{aligned} a \in \psi(f_2(b_1^m)) &\Leftrightarrow a = \psi(b) \wedge b \in f_2(b_1^m) \Leftrightarrow b = \varphi(a) \wedge b \in f_2(b_1^m) \\ &\Leftrightarrow \varphi(a) \in f_2(\varphi(a_1, \ldots, a_m)) = \varphi(f_1(a_1^m)) \Leftrightarrow a \in f_1(a_1^m) \\ &\Leftrightarrow a \in f_1(\psi(b_1), \ldots, \psi(b_m)). \end{aligned}$$

Thus $\psi(f_2(b_1^m)) = f_1(\psi(b_1), \ldots, \psi(b_m))$, and

$$\begin{aligned} a \in \psi(g_2(r_1^{n-1}, b_1)) &\Leftrightarrow a = \psi(b) \wedge b \in g_2(r_1^{n-1}, b_1) \\ &\Leftrightarrow b = \varphi(a) \wedge b \in g_2(r_1^{n-1}, b_1) \\ &\Leftrightarrow \varphi(a) \in \varphi(g_1(r_1^{n-1}, a_1)) = g_2(r_1^{n-1}, \varphi(a_1)) \\ &\Leftrightarrow a \in g_1(r_1^{n-1}, a_1) \\ &\Leftrightarrow a \in g_1(r_1^{n-1}, \psi(b_1)). \end{aligned}$$

□

## 3 Monomorphisms and epimorphisms in categories of $R_{(m,n)}/\Gamma^* - mod$

In this section, we characterize monomorphisms and epimorphisms in the category of $R_{(m,n)}/\Gamma^* - mod$.

**Definition 3.1.** [4] Let $B$ be a sub-hypermodule of an $(m, n)$- hypermodule over an $(m, n)$-hyperring $(R, h, k)$. Then the set

$$A/B = \{f(x_1^{i-1}, B, x_{i+1}^m) | x_1^{i-1}, x_{i+1}^m \in A\}$$

endowed with the $m$-array hyperoperation $F$ is defined as follows: for all $x_{11}^{1m}, \ldots, x_{m1}^{mm} \in A$

$$\begin{aligned} &F(f(x_{11}^{1(i-1)}, B, x_{1(i+1)}^{1m}), \ldots, f(x_{m1}^{m(i-1)}, B, x_{m(i+1)}^{mm})) \\ &= \{f(t_1^{i-1}, B, t_{i+1}^{m}) | t_1 \in f(x_{11}^{m1}), \ldots, t_m \in f(x_{1m}^{mm})\} \end{aligned}$$

and with the $n$-ary hyperoperation $G : \underbrace{R \times \cdots \times R}_{n-1} \times A/B \to P^*(A/B)$ defined by: for all $x_1^{i-1}, x_{i+1}^{m} \in A$ and $r_1^{n-1} \in R$,

$$\begin{aligned} &G(r_1^{n-1}, f(x_1^{i-1}, B, x_{i+1}^{m})) \\ &= \{f(z_1^{i-1}, B, z_{i+1}^{m}) | z_1 \in g(r_1^{n-1}, x), \ldots, z_m \in g(r_1^{n-1}, x_m)\} \end{aligned}$$

is an $(m, n)$- hypermodule over an $(m, n)$-hyperring $(R, h, k)$, and $(A/B, F, G)$ is called the quotient $(m, n)$-ary hypermodule of $A$ by $B$.

**Definition 3.2.** [4] Let $(A, f, g)$ is a canonical $(m, n)$-hypermodule over $R$ and $B$ be a subhypermodule of $A$. The mapping $\pi : A \to A/B$ defined by $x \to f(x, B, 0^{(m-2)})$ is called the projection map of $A$ by $B$.

**Theorem 3.3.** [4] *The projection map $\pi$ is a homomorphism of $(m, n)$-hypermodules.*

**Proposition 3.4.** *Let $(A/\epsilon^*, f_1/\epsilon^*, g_1)/\epsilon^*, (B/\epsilon^*, f_2/\epsilon^*, g_2/\epsilon^*)$ be $(m, n)$-modules over the $R_{(m,n)}/\Gamma^*$-ring $R$. For any homomorphism $\varphi^* : A/\epsilon^* \to B/\epsilon^*$ the following are equivalent:*

(1) *$\varphi^*$ is injective;*

(2) *for any module $C/\epsilon^*$ and for homomorphisms $\psi^*, \gamma^* : C/\epsilon^* \to A/\epsilon^*$ if $\varphi^* \circ \psi^* = \varphi^* \circ \gamma^*$, then $\psi^* = \gamma^*$.*

*Proof.* (1) $\Rightarrow$ (2) If $\varphi^*$ is injective and and $\psi^*, \gamma^* : C/\epsilon^* \to A/\epsilon^*$ are homomorphisms such that $\psi^* \neq \gamma^*$ then, for some $\epsilon^*(c) \in C/\epsilon^*$, $\psi^*(\epsilon^*(c)) \neq \gamma^*(\epsilon^*(c))$. Since $\varphi^*$ is injective, it follows that $\varphi^*(\psi^*(\epsilon^*(c))) \neq \varphi^*(\gamma^*(\epsilon^*(c)))$, whence $\varphi^*\psi^* \neq \varphi^*\gamma^*$.

(2) $\Rightarrow$ (1) Suppose $\varphi^* : A/\epsilon^* \to B/\epsilon^*$. Let $\epsilon^*(a), \epsilon^*(\acute{a}) \in A/\epsilon^*$, and let $\{\epsilon^*(x)\}$ be any given one-element class. Consider the homomorphism

$$\bar{a}, \bar{\acute{a}} \quad : \{\epsilon^*(x)\} \to A/\epsilon^*$$

where

$$\bar{a}(\epsilon^*(x)) = \epsilon^*(a), \quad \bar{\acute{a}}(\epsilon^*(x)) = \epsilon^*(\acute{a}).$$

Since $\bar{a} \neq \bar{\acute{a}}$, it follows, since $\varphi^*$ is monomorphism, that $\varphi^*\bar{a} \neq \varphi^*\bar{\acute{a}}$. Thus, $\varphi^*(\epsilon^*(a)) = (\varphi^*\bar{a})(\epsilon^*(x)) \neq (\varphi^*\bar{\acute{a}})(\epsilon^*(x)) = \varphi^*(\epsilon^*(\acute{a}))$. Whence $\varphi^*$ is injective. $\square$

**Proposition 3.5.** *Let $(A/\epsilon^*, f_1/\epsilon^*, g_1/\epsilon^*), (B/\epsilon^*, f_2/\epsilon^*, g_2/\epsilon^*)$ be $(m,n)$-modules over the $(m,n)$-ring $R$. For any homomorphisms $\varphi^* : A/\epsilon^* \to B/\epsilon^*$ the following are equivalent:*

(1) *$\varphi^*$ is surjective;*

(2) *for any module $C/\epsilon^*$ and for any homomorphisms $\psi^*, \gamma^* : B/\epsilon^* \to C/\epsilon^*$ if $\psi^* \circ \varphi^* = \gamma^* \circ \varphi^*$, then $\psi^* = \gamma^*$.*

*Proof.* (1) $\Rightarrow$ (2) Assume that $\varphi^* : A/\epsilon^* \to B/\epsilon^*$ is surjective and $\psi^*, \gamma^* : B/\epsilon^* \to C/\epsilon^*$ are morphisms such that $\psi^* \circ \varphi^* = \gamma^* \circ \varphi^*$. For a fixed $\epsilon^*(b) \in B/\epsilon^*$ let $\epsilon^*(a) \in A/\epsilon^*$ be such that $\varphi^*(\epsilon^*(a)) = \epsilon^*(b)$. Then $\psi^*(\epsilon^*(b)) = \psi^*(\varphi^*(\epsilon^*(a))) = \psi^* \circ \varphi^*(\epsilon^*(a)) = \gamma^* \circ \varphi^*(\epsilon^*(a)) = \gamma^*(\varphi^*(\epsilon^*(a))) = \gamma^*(\epsilon^*(b))$. So $\varphi^*$ is an epimorphism.

(2) $\Rightarrow$ (1) Will be proved by showing that the negation of (1) leads us to the negation of (2). Indeed, if $\mathrm{Im}\varphi^* \neq B/\epsilon^*$, then $B/\epsilon^*/\mathrm{Im}\varphi^*$ is a module with zero $\mathrm{Im}\varphi^*$ and $|B/\epsilon^*/\mathrm{Im}\varphi^*| \geq 2$. Consider the homomorphism $\psi^* = \pi_{\mathrm{Im}\varphi^*} : B/\epsilon^* \to B/\epsilon^*/\mathrm{Im}\varphi^*$ and $\gamma^* : B/\epsilon^* \to B^*/\mathrm{Im}\varphi^*$ given by $\gamma^*(\epsilon^*(b)) = \mathrm{Im}\varphi^*$ for any $\epsilon^*(b) \in B/\epsilon^*$. They are obviously different, but we have for any $\epsilon^*(a) \in A/\epsilon^*$,

$$\psi^* \circ \varphi^*(\epsilon^*(a)) = \psi^*(\varphi^*(\epsilon^*((a)) = f_2/\epsilon^*(\varphi^*(\epsilon^*(a)), \mathrm{Im}\varphi^*, o^{(m-2)}) = \mathrm{Im}\varphi^*,$$

$$\gamma^* \circ \varphi^*(\epsilon^*(a)) = \gamma^*(\varphi^*(\epsilon^*(a))) = \mathrm{Im}\varphi^*.$$

Thus $\psi^* \circ \varphi^* = \gamma^* \circ \varphi^*$. $\square$

The following theorem is an immediate consequence of Propositions 3.1 and 3.2.

**Theorem 3.6.** (1) *In the category $R_{(m,n)}/\Gamma^* -$mod all monomorphisms are the homomorphisms which are injective.*

(2) *In the category $R_{(m,n)}/\Gamma^*$ $-$ mod the epimorphisms are all homomorphisms which are projective.*

**Corollary 3.7.** *The category $R_{(m,n)}/\Gamma^* - mod$ is a balanced category.*

**Theorem 3.8.** *The fundamental functor $\mathcal{F}$ preserves monomorphisms.*

*Proof.* Let $(A, f_1, g_1), (B, f_2, g_2)$ be $(m,n)$-hypermodules over the $(m,n)$-hyperring $R$ and $\varphi : A \to B$ be a monomorphism, we prove $\varphi^* : A/\epsilon^* \to B/\epsilon^*$ is monic. For $a, b \in A$, let $\varphi^*(\epsilon^*(a)) = \varphi^*(\epsilon^*(b))$ since, $\varphi(a)\epsilon^*\varphi(b)$, then there exists $f_{2i} \in \mathcal{F}_2$ such that $\{\varphi(a), \varphi(b)\} \subseteq f_{2i}$. Since $\phi$ is a monomorphism, we have

$$\{a, b\} \subseteq \{\varphi^{-1}(\varphi(a))\, \varphi^{-1}(\varphi(b))\} = \varphi^{-1}\{\varphi(a), \varphi(b)\} \subseteq \varphi^{-1}(f_{2i}) \in \mathcal{F}_1.$$

Thus $\varphi^*$ is monic. □

## 4 Categorical properties of $R_{(m,n)} - Khmod$

In this section, we give some properties of the category $R_{(m,n)} - Khmod$. First of all,we characterize subobjects and quotient objects in this category.

**Theorem 4.1.** *Let $(A, f_1, g_1), (B, f_2, g_2), (C, f_3, g_3)$ be $(m,n)$-hypermodules over the $(m,n)$-hyperring $R$ and $\varphi : A \to B, \psi : A \to C$ homomorphisms. If $\psi$ is onto, then*

*(1) $ker\psi \subseteq ker\varphi$ implies the existence of a homomorphism $\gamma : C \to B$ and $\varphi = \gamma \circ \psi$;*

*(2) $\varphi$ is onto implies that $\gamma$ is onto;*

*(3) $ker\varphi = ker\psi$ implies that $\gamma$ is one to one;*

*(4) $\varphi$ is onto and $ker\varphi = ker\psi$ implies that $\gamma$ is an isomorphism.*

*Proof.* (1) Since $\psi$ is onto for any $c \in C$ there exists $a \in A$ such that $\psi(a) = c$. Define $\gamma(c) := \varphi(a)$. Then $\varphi$ is well defined, because if there exist $a_1, a_2 \in A$ such that $\psi(a_1) = \psi(a_2)$, then $0 \in f_3(\psi(a_1), -\psi(a_2), o^{m-2}) = \psi(f_1(a_1, -a_2, o^{m-2}))$ and so there exists $a \in f_1(a_1, -a_2, o^{m-2})$ such that $0 = \psi(a)$. That is $a \in ker\psi$ so, $a \in ker\varphi$, that is, $0 \in \varphi(f_1(a_1, -a_2, o^{m-2})) = f_2(\varphi(a_1), -\varphi(a_2), o^{m-2})$. Hence $\varphi(a_1) = \varphi(a_2)$.
(2) It follows easily from definition of $\gamma$. Statements (3) and (4) are direct applications of (1) and (2). □

**Corollary 4.2.** *Let $A, B$ be $(m,n)$-hypermodules over the $(m,n)$-hyperring $R$ and $\varphi : A \to B$ be a homomorphism, and $C$ a subhypermodule of $A$. Then,* *(1) $C \subseteq ker\varphi$, then there is a unique homomorphism $\bar{\varphi} : A/C \to B$ such that $\bar{\varphi}(f(a, C, 0^{(m-2)})) = \varphi(a)$; for all $a \in A$, that is, the diagram*

$$\begin{tikzcd} A \arrow[r, two heads, "\pi"] \arrow[dr] & A/C \arrow[d, dotted, "\bar{\varphi}"] \\ & B \end{tikzcd}$$

*commutes,*

(2) *$\varphi$ is onto then $\bar{\varphi}$ is onto,*

(3) *$C = ker\varphi$, then $\bar{\varphi}$ is one to one,*

(4) *$\varphi$ is onto and $C = ker\varphi$, then $\bar{\varphi}$ is an isomorphism.*

**Theorem 4.3.** *Let $A$ be an $(m, n)$-hypermodule over the $(m, n)$-hyperring $R$. Then*

(1) *there exists a bijection between the subhypermodules of $A$ and the subobjects of $A$ in $R_{(m,n)} - Khmod$;*

(2) *there exists a bijection between the quotient hypermodules of $A$ over subhypermodules and the quotient objects of $A$ in $R_{(m,n)} - Khmod$.*

*Proof.* Let $S_0$ be the class of subhypermodules of $A$ and $S$ the class of subobjects of $A$ in $R_{(m,n)} - Khmod$ and let us consider the function $\varphi : S_0 \to S, \varphi(B) = [B, i_B].(i_B : B \to A, i_b(b) = b$, is the inclusion homomorphism).

First we prove that $\varphi$ is one to one. To do so, considering $B_1, B_2 \in S_0, \varphi(B_1) = \varphi(B_2)$ implies that $[B_1, i_{B_1}] = [B_2, i_{B_2}]$, that is, $[B_1, i_{B_1}] \sim [B_2, i_{B_2}]$. Thus there exists an isomorphism $\phi : B_1 \to B_2$ such that $i_{B_1} = i_{B_2} \circ \phi$ for any $b \in B_1, b = \phi(b) \in B_2$. Hence, $B_1 \subseteq B_2$.
But we also have $i_{B_1} \circ \phi^{-1} = i_{B_2}$, which follows that $B_2 \subseteq B_1$ and that is why $B_1 = B_2$.

The function $\varphi$ is also onto. In fact, for any $[B, \gamma] \in S$ knowing that $\gamma$ is a homomorphism it follows that $\gamma(B) \in S_0$. We will show that $\varphi(\gamma(B)) = [B, \gamma]$, that is $(\gamma(B), i_{\gamma(B)}) \sim (B, \gamma)$.
The mapping $\phi : B \to \gamma(B), \phi(b) = \gamma(b)$ is a bijection and it is a homomorphism. Thus $\phi$ is an isomorphism and we have $\gamma = i_{\gamma(B)} \circ \phi$ which means

that $(\gamma(B), i_{\gamma(B)}) \sim (B,\gamma)$ :

$$\begin{tikzcd} B \arrow[r, "\gamma"] \arrow[d, dotted, "\phi"'] & A \\ \gamma(B) \arrow[ur, "i_{\gamma(B)}"'] & \end{tikzcd}$$

The first part of the theorem is proved now.
(2) Let $Q_0$ be the class of quotient hypermodules of $A$ over the subhypermodules and $Q$ the class of quotient objects of $A$ in $h_{(m,n)} - Rmod$ and let us consider the function $\eta : Q_0 \to Q, \eta(A/B) = [\pi_B, A/B]$, where $\pi_B : A \to A/B$ is the canonical projection.

First we prove that $\eta$ is one to one. To do so, considering $A/B_1, A/B_2 \in Q_0, \eta(A/B_1) = \eta(A/B_2)$ then $[\pi_{B_1}, A/B_1] = [\pi_{B_2}, A/B_2]$, that is, $(\pi_{B_1}, A/B_1) \sim (\pi_{B_2}, A/B_2)$. Thus there exists an isomorphism $\psi : A/B_1 \to A/B_2$ such that $\pi_{B_1} = \psi \circ \pi_{B_2}$, that is the diagram

$$\begin{tikzcd} A \arrow[r, "\pi_{B_2}"] \arrow[dr, "\pi_{B_1}"'] & A/B_2 \arrow[d, dotted, "\psi"] \\ & A/B_1 \end{tikzcd}$$

commutes. We have
$B_1 = \{a \in A | \pi_{B_1}(a) = B_1\} = \{a \in A | \psi(\pi_{B_1}(a)) = B_1\} = \{a \in A | \pi_{B_2}(a) = B_2\} = B_2$, which leads to $A/B_1 = A/B_2$. The function $\eta$ is also onto. In fact, for any $[\gamma, C] \in Q]$ knowing that $\gamma$ is a homomorphism which is onto (that is, an epimorphism in $h_{(m,n)} - Rmod$; by Theorem 2.3), if we set $B = ker\gamma$, in view of Theorem 4.1, there exists an isomorphism $\alpha : A/B \to C$ such that $\alpha \circ \pi_B = \gamma$ (indeed $\alpha$ is an isomorphism because $\alpha$ is onto and $ker\gamma = B = ker\pi_B$). This means that $[\gamma, C] = [\pi_B, A/B] = \eta(A/B)$, which completes the bijective of $\eta$. □

In the next theorem we show that the category $R_{(m,n)} - Khmod$ has images and coimages, too.

**Theorem 4.4.** (1) *The category $R_{(m,n)} - Khmod$ is a category with* images.

(2) The category $R_{(m,n)} - Khmod$ is a category with *coimages.*

*Proof.* (1) For any $\varphi : A \to B$ among $(m,n)$-hypermodules, $[\varphi(A), i_{\varphi(A)}] \in S(B)$, where $S(B)$ is the class of subobjects of $B$ and $i_{\varphi(A)} : \varphi(A) \to B$ is the inclusion mapping.

We will prove that $\mathrm{Im}\varphi = [\varphi(A), i_{\varphi(A)}]$. If we consider $\theta : A \to \varphi(A), \theta(a) = \varphi(a)$ for every $a \in A$, then $\theta$ is a homomorphism and we have $\varphi = i_{\varphi(A)} \circ \theta$. Next, let $[K, u] \in S(B)$ and there exist homomorphism $\theta' : A \to K$ in such a way that $\varphi = u \circ \theta'$. Then $u$ is clearly a monomorphism, and $\varphi(a) = u(\theta'(a))$, for any $a \in A$, implies that $\varphi(A) \subseteq u(K)$. But the mapping $u_1 : K \to u(K), u_1(a) = u(a)$, is an isomorphism, so $u_1^{-1} : u(K) \to K$ is also an isomorphism. Therefore, $\gamma = u_1|_{\varphi(A)}$ is a homomorphism and $i_{\varphi(A)} = u \circ \gamma$. Thus, $[\varphi(A), i_{\varphi(A)}] \leq [K, u]$ and this means that $\mathrm{Im}\varphi = [\varphi(A), i_{\varphi(A)}]$ :

$$\begin{tikzcd} \varphi(A) \arrow[r, "i_{\varphi(A)}"] \arrow[d, dotted, "\gamma"'] & B \arrow[d, "id_B"] \\ K \arrow[r, "u"] & B \end{tikzcd}$$

(2) For any $\varphi : A \to B$ let $C = ker\varphi$. Then $[\pi_C, A/C] \in Q(A)$ where $Q(A)$ denotes the class of quotient objects of $A$ in the sense of previous theorem. We claim that coim $(\varphi) = [\pi_C, A/C]$. To prove the claim, let $\theta : A/C \to B$ be defined by $\theta(f(a, C, 0^{(m-2)})) = \varphi(a)$, for any $a \in A$. Then $\theta$ is a homomorphism and $\theta \circ \pi_C = \varphi$. Next, let $[\nu, D] \in Q(A)$. Also there exists a homomorphism $\theta : D \to B$ such that $\varphi = \theta \circ \nu$. Then, according to the fact that $\nu$ is a homomorphism which is onto and $ker\nu \subseteq ker\varphi = C$, we can see that there exists a homomorphism $\gamma : D \to A/C$ such that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "\nu"] \arrow[dr, "\pi_C"'] & D \arrow[r, "\theta"] \arrow[d, dotted, "\gamma"] & B \\ & A/C \arrow[ur, "\theta"'] & \end{tikzcd}$$

Thus $[\pi_C, A/C] \leq [\nu, D]$ which leads us to the equality $\mathrm{coim}(\varphi) = [\pi_C, A/C]$. □

Now we come to the concepts of kernel and cokernel.

**Theorem 4.5.** (1) *The category* $R_{(m,n)} - Khmod$ *has* kernels and *cokernels.*

(2) *The category* $R_{(m,n)} - Khmod$ *is a* normal and *conormal category.*

*Proof.* $R_{(m,n)} - Khmod$ is a category with kernels. Let $\varphi : A \to B$ be a morphism in $R_{(m,n)} - Khmod$ and let $C = ker\varphi$. We know that $i_C : C \to A$ is a monomorphism in $R_{(m,n)} - Khmod$. We will prove that $ker\varphi = [C, i_C.]$

(1) Let $0_{CB}$ be the zero morphism from $C$ into $B, 0_{CB} : C \to B, 0_{CB}(c) = 0$ for any $c \in C$ we have $\varphi \circ i_C = \varphi(c) = 0 = 0_{CB}(c)$, thus $\varphi \circ i_C = 0_{CB}$. If $D \in Obj\ R_{(m,n)} - Khmod$ and $\mu \in hom(D, A)$ such that $\varphi \circ \mu = 0_{DB}$ then $\varphi(\mu(d)) = 0$, for all $d \in D$, and hence $\mu(D) \subseteq C$. It follows that the mapping $\gamma : D \to C, \gamma(d) = \mu(d)$, is a monomorphism and it verifies the equality $\mu = i_C \circ \gamma$ :

$$\begin{tikzcd} D \arrow[d, dotted] \arrow[dr, "\mu"] & & \\ C \arrow[r, hook, "i_C"'] & A \arrow[r, "f"'] & B \end{tikzcd}$$

Now, let $\varphi : A \to B$ be a morphism in $R_{(m,n)} - Khmod$ and let $\pi : B \to B/\varphi(A)$ be the projection. We will show that Coker$\varphi = [\pi, B/\varphi(A)]$. We have $\pi\varphi(a) = \pi(\varphi(a)) = f(\varphi(a), \varphi(A), 0^{(m-2)}) = \varphi(A) = o_{A,B/\varphi(A)}$. For any $C \in Obj\ R_{(m,n)} - Khmod$ and $\mu \in hom(B, C)$ such that $\mu \circ \varphi = 0_{AC}$. We have $(\mu \circ \varphi)(a) = \mu(\varphi(a)) = 0$, for all $a \in A$. Therefore, $\varphi(A) \subseteq ker\mu$. So, there exists a unique homomorphism $\gamma\ \ : B/\varphi(A) \to C$, such that $\gamma \circ \pi = \mu$ :

$$\begin{tikzcd} A \arrow[r, "\varphi"] & B \arrow[r, two heads, "\pi"] \arrow[dr, "\mu"'] & B/\varphi(A) \arrow[d, dotted, "\gamma"] \\ & & C \end{tikzcd}$$

(2) $Kh_{(m,n)} - Rmod$ is a normal category. Let $A$ be an $Obj\ R_{(m,n)} - mod$ and $[B, \varphi]$ a subobject of $A$. Then we have $[B, \varphi] = [\varphi(B), i_{\varphi(B)}]$ and if $\pi_{\varphi(B)} : A \to A_{\varphi(B)}$ is the canonical projection, then, according to $(i)$ of part 1, $ker\pi_{\varphi(B)} = [\varphi(B), i_{\varphi(B)}] = [B, \varphi]$.

Also, $Kh_{(m,n)} - Rmod$ is a conormal category. Let $A$ be an $Obj\ R_{(m,n)} - Khmod$ and $[\psi, B]$ a quotient object of $A$. then $[\psi, B] = [\pi_C, A/C]$, where $C = ker\psi, \pi_C$ is the canonical projection. According to $(ii)$ of part 1 we have $coker i_C = [\pi_C, A/C] = [\psi, B]$. □

Recall from [18] that normal and conormal categories with kernels and cokernels are exact if every morphism $\alpha : A \to B$ can be written as a composition $A \to I \to B$ where $q$ is an epimorphism and $\nu$ is a monomorphism. Now we can easily prove the following theorem from the above theorem.

**Theorem 4.6.** *The category $R_{(m,n)} - Khmod$ is an exact category.*

## 5 Conclusions and future works

In this paper, some aspects of $(m,n)$-hypermodules, denoted by $R_{(m,n)} - Khmod$, were studied. We constructed the category of $(m,n)-$hypermodules and proved that it is an exact category in the sense that it is normal and conormal with kernels and cokernels in which every morphism $\alpha$ has a factorization $\alpha = \nu q$, where $q$ is an epimorphism and $\nu$ is a monomorphism. Two of the most used results of the paper were those which state that the monomorphisms of $R_{(m,n)} - Khmod$ (in the categorical sense) are the one-one-homomorphisms and the epimorphisms of $R_{(m,n)} - Khmod$ are the onto homomorphisms. Also, we proved that the fundamental relation on $(m,n)$-hypermodules induces a functor from category $R_{(m,n)} - Khmod$ into category $(m,n)$-hypermodules and this functor preserves monomorphisms. Therefore, this paper leads to a better study of algebraic hyperstructures theory in view of category theory and investigates the relationship between this category and its related classical category.

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***Najmeh Jafarzadeh***, *Department of Mathematics, Payamenoor University, P.O. Box 19395-3697, Tehran, Iran.*
*Email: jafarzadeh@phd.pnu.ac.ir*

***Reza Ameri***, *School of Mathematics, Statistics and Computer Science, College of Sciences, University of Tehran, P.O. Box 14155-6455, Tehran, Iran.*
*Email: rameri@ut.ac.ir*