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The category of generalized crossed modules

Mahdieh Yavari* and Alireza Salemkar

Abstract. In the definition of a crossed module (T, G, ρ), the actions of the group T and G on themselves are given by conjugation. In this paper, we consider these actions to be arbitrary and thus generalize the concept of ordinary crossed module. Therefore, we get the category **GCM**, of all generalized crossed modules and generalized crossed module morphisms between them, and investigate some of its categorical properties. In particular, we study the relations between epimorphisms and the surjective morphisms, and thus generalize the corresponding results of the category of (ordinary) crossed modules. By generalizing the conjugation action, we can find out what is the superiority of the conjugation to other actions. Also, we can find out that a generalized crossed module with which other actions (other than the conjugation) has the properties similar to a crossed module.

1 Introduction

In the late 1940s, crossed modules were introduced by J.H.C. Whitehead as a means of representing homotopy 2-types. Then S. Mac Lane and Whitehead

* Corresponding author
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subsequently used crossed modules to represent H^3 in group cohomology [5]. They have therefore been used in homotopy theory, homological algebra, non-abelian cohomology, combinatorial group theory, and applications of the related algebra, algebraic K-theory, ring theory, etc.

The study of crossed modules as algebraic objects in their own right is a subject of interest. Recently, a body of research has appeared with this aim, starting with the Ph.D. thesis of Norrie [7]. She shows that the category **CM**, of all crossed modules and crossed module morphisms between them, has many formal properties analogous to those of the category of groups and establishes how much of the group theoretical concepts and results have suitable counterparts for crossed modules.

Loday in [4] showed that the category of crossed modules is equivalent to that of cat^1-groups. Also, many mathematicians investigated the categorical properties of the category **CM** and its important subcategory, the category of crossed P-modules, and characterized important objects of these categories such as products, equalizers, pullbacks, coproducts etc (see [2, 3, 6, 8, 9]). Let us recall the following.

A *crossed module* (T, G, ρ) consists of a group homomorphism $\rho : T \to G$, together with a group action of G on T satisfying

(i) $\rho(gt) = g\rho(t)g^{-1}$,

(ii) $\rho(t)t' = tt't^{-1}$,

for every $t, t' \in T$ and $g \in G$.

A *crossed module morphism* $f = (f_1, f_2) : (T_1, G_1, \rho_1) \to (T_2, G_2, \rho_2)$ is a pair of group homomorphisms $f_1 : T_1 \to T_2$ and $f_2 : G_1 \to G_2$ such that $\rho_2 f_1 = f_2\rho_1$ and $f_1(gt) = f_2(g)f_1(t)$, for every $g \in G_1$ and $t \in T_1$.

Notice that in the definition of crossed module, the action of T on T and that of G on G are the conjugation actions. In this paper, we consider the action of T on T and that of G on G to be arbitrary and generalize the concept of crossed module. So, we get the category **GCM**, of all generalized crossed modules and generalized crossed module morphisms between them, and investigate some important categorical constructions in this category. Also, we study epimorphisms in **GCM** and investigate the relation between epimorphisms and surjective morphisms.

The study of the category of generalized crossed modules helps us to better understand the category of ordinary crossed modules. By generalizing the conjugation action, we can find out what is the superiority of the conjugation to other actions. Also, we can find out a generalized crossed module with which other actions (other than the conjugation) has the properties same as a crossed module. Studying the categorical properties of generalized crossed modules will make it simpler to study within this category and make the research in this category smoother.

2 Preliminary

In this section, we introduce the concept of a generalized crossed module and construct the category **GCM**. Also, we give some definitions which are needed in the sequel. Thence, we discuss some of the general category theoretic ingredients of this category, such as product, equalizer, coequalizer, etc. First, we introduce the concept of a generalized crossed module.

Definition 2.1. A *generalized crossed module* (A, G, ρ) consists of

(i) a group homomorphism $\rho : A \to G$,

(ii) an action of A on A, denoted by $a.a'$, for every $a, a' \in A$,

(iii) an action of G on G, denoted by $g.g'$, for every $g, g' \in G$,

(iv) an action of G on A, denoted by ga, for every $g \in G$ and $a \in A$,

satisfying the conditions:

(1) $\rho(ga) = g.\rho(a)$, for every $g \in G$ and $a \in A$,

(2) $\rho(a)a' = a.a'$, for every $a, a' \in A$.

A *generalized crossed module morphism* $f = (f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ is a pair of group homomorphisms $f_1 : A_1 \to A_2$ and $f_2 : G_1 \to G_2$ such that $\rho_2 f_1 = f_2 \rho_1$ and $f_1(ga) = f_2(g)f_1(a)$, for every $g \in G_1$ and $a \in A_1$:

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & A_2 \\ \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} \\ G_1 & \xrightarrow{f_2} & G_2 \end{array}$$

We denote the category of all generalized crossed modules and generalized crossed module morphisms between them by **GCM**. Note that every crossed module is a generalized crossed module. For another example, consider an arbitrary group G with an arbitrary action of G on itself. Obviously, (G, G, id_G) is a generalized crossed module. If we consider all actions to be trivial then for any group homomorphism $\rho : A \to G$ we have (A, G, ρ) is a generalized crossed module. Also, if A and G are arbitrary groups, the action of A on A is trivial and the actions of G on G and G on A are arbitrary then $(A, G, 1)$, where $1 : A \to G$ is the trivial homomorphism, is a generalized crossed module.

Definition 2.2. A generalized crossed module (A_1, G_1, ρ_1) is a *generalized crossed submodule* of a generalized crossed module (A_2, G_2, ρ_2) if
(i) A_1 is a subgroup of A_2 and G_1 is a subgroup of G_2,
(ii) $\rho_1 = \rho_2|_{A_1}$,
(iii) The actions of A_1 on A_1, G_1 on G_1, and G_1 on A_1 are induced by those of A_2 on A_2, G_2 on G_2, and G_2 on A_2, respectively.

Definition 2.3. A generalized crossed submodule (A_1, G_1, ρ_1) of a generalized crossed module (A_2, G_2, ρ_2) is a *normal generalized crossed submodule*, denoted by $(A_1, G_1, \rho_1) \trianglelefteq (A_2, G_2, \rho_2)$, if
(i) $A_1 \trianglelefteq A_2$ and $G_1 \trianglelefteq G_2$,
(ii) $ga \in A_1$, for $g \in G_2, a \in A_1$,
(iii) $a^{-1}(ga) \in A_1$, for $g \in G_1, a \in A_2$.

Let (A_1, G_1, ρ_1) be a normal generalized crossed submodule of a generalized crossed module (A_2, G_2, ρ_2). Then the generalized crossed module morphism $i = (i_1, i_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$, where i_1 and i_2 are the inclusion maps, is a normal monomorphism. So, a normal generalized crossed submodule is a normal subobject in the category **GCM**.

In the following results, we discuss some standard limits and colimits ([1]) in the category **GCM**.

Theorem 2.4. *The category* **GCM** *is complete. In fact, it is closed under products, pullbacks, equalizers, and has terminal and also initial objects.*

Proof. Easily proved. Consider the family $\{(A_i, G_i, \rho_i)\}_{i \in I}$ of generalized crossed modules. Then the generalized crossed module $(\prod_{i \in I} A_i, \prod_{i \in I} G_i, \alpha)$ where the action of $\prod_{i \in I} G_i$ on $\prod_{i \in I} A_i$ ($\prod_{i \in I} G_i$ on $\prod_{i \in I} G_i$, $\prod_{i \in I} A_i$ on $\prod_{i \in I} A_i$) is componentwise and $\alpha : \prod_{i \in I} A_i \to \prod_{i \in I} G_i$ is defined by $\alpha((a_i)_{i \in I}) = (\rho_i(a_i))_{i \in I}$ is the product of them.

Let $(f_1, f_2) : (A_1, G_1, \rho_1) \to (A_3, G_3, \rho_3)$ and $(f_1', f_2') : (A_2, G_2, \rho_2) \to (A_3, G_3, \rho_3)$ be two morphisms. Then $(P_A, P_G, \alpha' = \alpha|_{P_A})$, where α introduced in the case of products and $P_A = \{(a_1, a_2) : f_1(a_1) = f_1'(a_2)\}$ and

$P_G = \{(g_1, g_2) : f_2(g_1) = f_2'(g_2)\}$ are, respectively, the pullbacks of f_1, f_1' and f_2, f_2' in the category **Grp** of groups, is the pullback of (f_1, f_2) and (f_1', f_2') in **GCM**.

Let $f = (f_1, f_2), g = (g_1, g_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ be two morphisms. Then $(E_A, E_G, \rho' = \rho_1|_{E_A})$, where $E_A = \{a \in A_1 : f_1(a) = g_1(a)\}$ and $E_G = \{g \in G_1 : f_2(g) = g_2(g)\}$ are, respectively, the equalizers of f_1, g_1 and f_2, g_2 in **Grp**, is the equalizer of f and g in **GCM**.

Finally, $(\{e\}, \{e\}, id)$ is the initial as well as the terminal object.

$\square$

Definition 2.5. Let $f = (f_1, f_2)$ be a morphism in the category **GCM**. The morphism f is called *injective* if f_1 and f_2 are monomorphisms in the category **Grp**.

Proposition 2.6. *In the category* **GCM**, *monomorphisms are exactly injective morphisms.*

Proof. Let $f = (f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ be a monomorphism, $K_1 = ker f_1$ and $K_2 = ker f_2$. Evidently, $\rho' = \rho_1|_{K_1} : K_1 \to K_2$ is a group homomorphism and (K_1, K_2, ρ') is a generalized crossed submodule of (A_1, G_1, ρ_1). Now, consider the morphisms $i = (i_1, i_2), 1 = (1_1, 1_2) : (K_1, K_2, \rho') \to (A_1, G_1, \rho_1)$, where i_1 and i_2 are the inclusion maps and 1_1 and 1_2 are the trivial group maps. It is clear that $fi = f1$. Since f is a monomorphism, we get $i = 1$ which implies that $K_1 = \{e_{A_1}\}$ and $K_2 = \{e_{G_1}\}$. Hence, f is injective. The converse is true because if $f = (f_1, f_2)$ is injective then the morphisms f_1 and f_2 are monomorphisms in the category **Grp**. Then it is straightforward to see that f is a monomorphism in the category **GCM**. $\square$

Remark 2.7. Let (A, G, ρ) be a generalized crossed module. Also, let N be a normal subgroup of A and N' be a normal subgroup of G satisfying

(i) $(N, N', \rho' = \rho|_N) \trianglelefteq (A, G, \rho)$,

(ii) $(N, N, id_N) \trianglelefteq (A, A, id_A)$,

(iii) $(N', N', id_{N'}) \trianglelefteq (G, G, id_G)$.

We show that $(A/N, G/N', \bar{\rho})$, where $\bar{\rho} : A/N \to G/N'$ is defined by $\bar{\rho}(xN) = \rho(x)N'$, with the following actions is a generalized crossed module. Notice that

(I) The action of A/N on A/N is defined by $xN.x'N = (x.x')N$, for every $x, x' \in A$. To prove that this action is well defined, let $x_1N = x_2N$ and $x_3N = x_4N$, where $x_i \in A$ for $i = 1, \cdots, 4$. We show that $(x_1.x_3)N = (x_2.x_4)N$. Since $x_1N = x_2N$ and $x_3N = x_4N$, there exist $n_1, n_2 \in N$ such that $x_1 = x_2n_1$ and $x_3 = x_4n_2$. Therefore, $(x_1.x_3)N = x_1.(x_4n_2)N = (x_1.x_4)(x_1.n_2)N = (x_1.x_4)N = ((x_2n_1).x_4)N = x_2.(n_1.x_4)N$, where the second equality is because the action of A on A is a group homomorphism. Also, the third equality is because $(x_1.n_2) \in N$. Now, since $(N, N, id_N) \trianglelefteq (A, A, id_A)$ and $(x_2.x_4)^{-1}(x_2.(n_1.x_4)) = x_2.(x_4^{-1}(n_1.x_4)) \in N$, we get the result.

(II) The action of G/N' on A/N is defined by $gN'xN = (gx)N$, for every $g \in G$ and $x \in A$. Since $(N, N', \rho' = \rho|_N) \trianglelefteq (A, G, \rho)$, by similar proof of (I), we can show that this action is well defined.

(III) The action of G/N' on G/N' is defined by $gN'.g'N' = (g.g')N'$, for every $g, g' \in G$. Since $(N', N', id_{N'}) \trianglelefteq (G, G, id_G)$, by similar proof of (I), we can show that this action is well defined.

(IV) $\bar{\rho}$ is a group homomorphism.

• $\bar{\rho}$ is well defined: Let $x_1N = x_2N$, for $x_1, x_2 \in A$. Then $x_2^{-1}x_1 \in N$. Hence, $\rho(x_2)^{-1}\rho(x_1) = \rho'(x_2^{-1}x_1) \in N'$ and $\rho(x_1)N' = \rho(x_2)N'$.

• $\bar{\rho}(xNx'N) = \rho(xx')N' = \rho(x)N'\rho(x')N' = \bar{\rho}(xN)\bar{\rho}(x'N)$.

(V) We have

• $\bar{\rho}(gN'xN) = \bar{\rho}(gxN) = \rho(gx)N' = (g.\rho(x))N' = gN'.\rho(x)N' = gN'.\bar{\rho}(xN)$.

• $\bar{\rho}(x_1N)x_2N = \rho(x_1)N'x_2N = \rho(x_1)x_2N = (x_1.x_2)N = x_1N.x_2N$.

By (I-V), we have $(A/N, G/N', \bar{\rho})$ is a generalized crossed module. We call $(A/N, G/N', \bar{\rho})$ a *qoutient generalized crossed module.*

Proposition 2.8. *Coequalizers exist in the category* **GCM**.

Proof. Let $f = (f_1, f_2), g = (g_1, g_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ be two morphisms. Also, let N be the normal subgroup of A_2 generated by all the elements of the form $f_1(a)^{-1}g_1(a)$ and N' be the normal subgroup of G_2 generated by all the elements of the form $f_2(g)^{-1}g_2(g)$ satisfying

(i) $(N, N', \rho' = \rho_2|_N) \trianglelefteq (A_2, G_2, \rho_2)$,

(ii) $(N, N, id_N) \trianglelefteq (A_2, A_2, id_{A_2})$,

(iii) $(N', N', id_{N'}) \trianglelefteq (G_2, G_2, id_{G_2})$.

Now, consider the qoutient generalized crossed module $(A_2/N, G_2/N', \bar{\rho_2})$ (introduced in Remark 2.7) and the natural epimorphism

$h = (h_1, h_2) : (A_2, G_2, \rho_2) \to (A_2/N, G_2/N', \bar{\rho_2})$. It is clear that $hf = hg$. Suppose that there exists a morphism $k = (k_1, k_2) : (A_2, G_2, \rho_2) \to (A_3, G_3, \rho_3)$ such that $kf = kg$. Since $N \subseteq kerk_1$ and $N' \subseteq kerk_2$, there exist unique group homomorphisms $\alpha_1 : A_2/N \to A_3$ and $\alpha_2 : G_2/N' \to G_3$ such that $\alpha_1 h_1 = k_1$ and $\alpha_2 h_2 = k_2$. Finally, we prove (α_1, α_2) is a morphism in the category **GCM** and the proof is complete.

- $\rho_3\alpha_1(xN) = \rho_3\alpha_1(h_1(x)) = \rho_3 k_1(x) = k_2\rho_2(x) = \alpha_2(h_2(\rho_2(x))) = \alpha_2(\rho_2(x)N') = \alpha_2\bar{\rho_2}(xN)$, for $xN \in A_2/N$.
- $\alpha_1(gN'xN) = \alpha_1(gxN) = \alpha_1 h_1(gx) = k_1(gx) = k_2(g)k_1(x) = \alpha_2(h_2(g))\alpha_1(h_1(x)) = \alpha_2(gN')\alpha_1(xN)$, for $gN' \in G_2/N', xN \in A_2/N$. □

3 Epimorphisms in the category GCM

It is evident that every surjective map in a concrete category is an epimorphism. But the converse is not true in general. For example, in [8], it is shown that in the category **CM** an epimorphism need not be surjective. Since **CM** is a full subcategory of **GCM**, we get that in **GCM** epimorphisms and surjectives are not the same. In this section, we investigate epimorphisms in **GCM** and give some conditions under which we can recognize that a morphism is an epimorphism or not. Finally, we show that in some cases every epimorphism is in fact surjective.

Investigating epimorphisms of a category will help us to characterize projective and quotient objects which are the useful objects of a category.

Proposition 3.1. *Let $(f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ be an epimorphism in the category* **GCM**. *Then f_2 is surjective if and only if $Imf_2 = J' \trianglelefteq G_2$ and $\rho_2(A_2) \subseteq J'$.*

Proof. Suppose that $(f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ is an epimorphism, $J' \trianglelefteq G_2$ and $\rho_2(A_2) \subseteq J'$. Let, on the contrary, $J' \neq G_2$. Then consider the two morphisms $(1, \epsilon), (1, 1) : (A_2, G_2, \rho_2) \to (\{e\}, G_2/J', 1)$, where 1 is the trivial homomorphism and ϵ is the natural homomorphism. We show that $(1, \epsilon)$ and $(1, 1)$ are morphisms in the category **GCM**.

- $11(b) = 1(e) = J' = \rho_2(b)J' = \epsilon\rho_2(b)$, for every $b \in A_2$. The last equality is because, by the assumption, $\rho_2(A_2) \subseteq J'$.
- $1(gb) = e = \epsilon(g)1(b)$.

• $11(b) = 1(e) = J' = 1\rho_2(b)$, for every $b \in A_2$.
• $1(gb) = e = 1(g)1(b)$.

It is clear that $(1, \epsilon)(f_1, f_2) = (1, 1)(f_1, f_2)$. Now, since (f_1, f_2) is an epimorphism, we have $(1, \epsilon) = (1, 1)$, which is a contradiction.

For the converse, let (f_1, f_2) be an epimorphism such that f_2 is surjective. Then $J' = G_2 \trianglelefteq G_2$ and $\rho_2(A_2) \subseteq J' = G_2$. □

In the proof of Lemma 3.3, we construct special subgroups of the group of permutations on an arbitrary group G (P_G) which are useful in this paper. First, in the next remark, we recall the properties of some special elements of P_G.

Remark 3.2. Let G be a group and $g \in G$. Then the left translation map $\lambda_g : G \to G$, defined by $\lambda_g(x) = gx$, for every $x \in G$, is clearly an element of P_G.

Now, suppose J is a subgroup of G which is not normal. So, $|G : J| \geq 3$. Therefore, we can choose three different cosets J, Jh_1 and Jh_2. Consider the map $\sigma : G \to G$ defined by

$$\sigma(y) = \begin{cases} x_1h_2 & \text{if} \quad y = x_1h_1 \\ x_1h_1 & \text{if} \quad y = x_1h_2 \\ y & \text{otherwise.} \end{cases}$$

Note that
• we fixed these cosets and elements h_1, h_2, that is, σ is well defined and evidently, $\sigma \in P_G$.
• $\sigma^2 = id_G$ which implies $\sigma = \sigma^{-1}$.
• there does not exist $g \in J$ such that $\lambda_g = \sigma$. This is because, if $\lambda_g = \sigma$ for some $g \in J$ then $\lambda_g(e_G) = \sigma(e_G)$. Hence, $g = e_G$ and $\sigma = \lambda_g = \lambda_{e_G} = id_G$, a contradiction.

Let (f_1, f_2) be an epimorphism in the category **GCM**. Then, by Proposition 3.1, it is important to know that when $Imf_2 \trianglelefteq G_2$. In the following lemma, we see conditions under which $Imf_2 \trianglelefteq G_2$.

Lemma 3.3. *Suppose that $(f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ is an epimorphism in the category* **GCM** *such that the action of G_2 on A_2 is trivial and $\rho_2(A_2) \subseteq J' = Imf_2$. Then $J' \trianglelefteq G_2$.*

Proof. Let $(f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ be an epimorphism with the property mentioned in the assumption. Since (A_2, G_2, ρ_2) is a generalized crossed module and the action of G_2 on A_2 is trivial, for every $a, a' \in A_2$ we have $a.a' = \rho_2(a)a' = a'$. So, the action of A_2 on A_2 is trivial too. Now assume, on the contrary, that J' is not a normal subgroup of G_2. Consider the subgroup H of P_{G_2} generated by the set $\{\lambda_g : g \in G_2\} \cup \{\sigma\}$, where, for $g \in G_2$, λ_g and σ are as in Remark 3.2. Define the map $\gamma : A_2 \to H$ by $\gamma(b) = \lambda_{\rho_2(b)}$, for $b \in A_2$. By the definition of γ, it is clear that γ is a group homomorphism. We claim that if all the actions are trivial then (A_2, H, γ) is a generalized crossed module. This is because
• $\gamma(xb) = \gamma(b) = x.\gamma(b)$, for every $x \in H$ and $b \in A_2$.
• $\gamma(b)b' = b' = b.b'$, for every $b, b' \in A_2$.

Now, define the morphisms

$$\alpha = (id_{A_2}, \alpha_2), \beta = (id_{A_2}, \beta_2) : (A_2, G_2, \rho_2) \to (A_2, H, \gamma)$$

by $\alpha_2(g) = \lambda_g$ and $\beta_2(g) = \sigma^{-1}\lambda_g\sigma$, for every $g \in G_2$. It is clear that α_2 and β_2 are group homomorphisms. We show that α and β are morphisms in the category **GCM**.
• $\gamma id_{A_2}(b) = \gamma(b) = \lambda_{\rho_2(b)} = \alpha_2(\rho_2(b))$, for every $b \in A_2$.
• $id_{A_2}(gb) = b = \alpha_2(g)id_{A_2}(b)$ and $id_{A_2}(gb) = b = \beta_2(g)id_{A_2}(b)$, for every $g \in G_2$ and $b \in A_2$.
• $\beta_2(\rho_2(b)) = \sigma^{-1}\lambda_{\rho_2(b)}\sigma = \lambda_{\rho_2(b)} = \gamma id_{A_2}(b)$, for every $b \in A_2$. To see this, take $y \in G_2$. Three cases may occur:

Case (i) If $y = xh_1$, $x \in J'$ then $\lambda_{\rho_2(b)}(xh_1) = \rho_2(b)xh_1$. On the other hand, $\sigma^{-1}\lambda_{\rho_2(b)}\sigma(xh_1) = \sigma^{-1}\lambda_{\rho_2(b)}(xh_2) = \sigma(\rho_2(b)xh_2) = \rho_2(b)xh_1$. The last equality is because, by the assumption, $\rho_2(b) \in J'$ and so $\rho_2(b)x \in J'$.

Case (ii) If $y = xh_2$, $x \in J'$ then, by similar proof of Case (i), we get the result.

Case (iii) If $y \neq xh_1, xh_2$, for every $x \in J'$ then $\rho_2(b)y \neq xh_1, xh_2$, for every $x \in J'$. This is because, on the contrary, if there exists $x \in J'$ such that $\rho_2(b)y = xh_1$ then $y = \rho_2(b)^{-1}xh_1$. Since $\rho_2(b) \in J'$, we have $\rho_2(b)^{-1} \in J'$ and so $\rho_2(b)^{-1}x \in J'$. It contradicts $y \neq xh_1$, for every $x \in J'$. By similar argument, $\rho_2(b)y \neq xh_2$, for every $x \in J'$. Hence, $\lambda_{\rho_2(b)}(y) = \rho_2(b)(y) = \sigma^{-1}\lambda_{\rho_2(b)}\sigma(y)$.

Finally, it is clear that $\alpha f = \beta f$. Since f is an epimorphism, we get $\alpha = \beta$, which is a contradiction. This is because, if $\alpha = \beta$ then $\alpha_2 = \beta_2$.

Therefore for every $g \in G_2$, we have $\lambda_g = \sigma^{-1}\lambda_g\sigma$ and so $\sigma\lambda_g = \lambda_g\sigma$. Now, let $g = xh_1, x \in J'$. Then $\sigma\lambda_{xh_1}(e_{G_2}) = \sigma(xh_1) = xh_2$ but $\lambda_{xh_1}\sigma(e_{G_2}) = \lambda_{xh_1}(e_{G_2}) = xh_1$. Hence, for $g = xh_1$, we have $\sigma\lambda_g \neq \lambda_g\sigma$. So, $J' \trianglelefteq G_2$. $\square$

Using Proposition 3.1 and Lemma 3.3 we have the following corollary.

Corollary 3.4. *Suppose that $(f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ is an epimorphism in the category* **GCM** *such that the action of G_2 on A_2 is trivial and $\rho_2(A_2) \subseteq J' = Imf_2$. Then f_2 is surjective.*

In the following theorem, we give simpler condition under which we can recognize when a morphism is an epimorphism.

Theorem 3.5. *Suppose $(f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ is a morphism in the category* **GCM** *where f_2 is surjective and for $g, g' \in G_1$, if $f_2(g) = f_2(g')$ we have $ga = g'a$ for every $a \in A_1$. Then (f_1, f_2) is an epimorphism if and only if*

$$(f_1, id_{G_2}) : (A_1, G_2, f_2\rho_1) \to (A_2, G_2, \rho_2)$$

is an epimorphism in **GCM**.

Proof. Suppose $f = (f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ is a morphism with the properties mentioned in the assumption. First, we show that $(A_1, G_2, f_2\rho_1)$ with the following actions is a generalized crossed module:

act (**i**) The action of A_1 on itself is the same as the action of A_1 on itself in (A_1, G_1, ρ_1).

act (**ii**) The action of G_2 on itself is the same as the action of G_2 on itself in (A_2, G_2, ρ_2).

act (**iii**) The action of G_2 on A_1 is defined by $g_2a_1 = g_1a_1$, where $g_2 = f_2(g_1) \in G_2$ and $a_1 \in A_1$. By the assumption, it is straightforward to see that this action is well defined.

Notice that:

(I) If $g_2 = f_2(g_1) \in G_2$ and $a_1 \in A_1$ then $g_2.f_2\rho_1(a_1) = g_2.\rho_2 f_1(a_1)$ $= \rho_2(g_2f_1(a_1)) = \rho_2(f_2(g_1)f_1(a_1)) = \rho_2(f_1(g_1a_1)) = f_2\rho_1(g_1a_1) = f_2\rho_1(g_2a_1)$.

(II) $f_2\rho_1(a)a' = \rho_1(a)a' = a.a'$, for $a, a' \in A_1$.

Hence, $(A_1, G_2, f_2\rho_1)$ is a generalized crossed module. Now, we prove that $(f_1, id_{G_2}) : (A_1, G_2, f_2\rho_1) \to (A_2, G_2, \rho_2)$ is a morphism in the category **GCM**.

• $\rho_2 f_1 = f_2\rho_1 = id_{G_2} f_2\rho_1$.
• Let $g_2 = f_2(g_1) \in G_2$ and $a_1 \in A_1$. Then $f_1(g_2a_1) = f_1(g_1a_1) = f_2(g_1)f_1(a_1) = g_2f_1(a_1) = id_{G_2}(g_2)f_1(a)$.

Now, let (f_1, id_{G_2}) be an epimorphism. Then consider the morphisms $h = (h_1, h_2)$, $g = (g_1, g_2) : (A_2, G_2, \rho_2) \to (A_3, G_3, \rho_3)$ where $hf = gf$. Hence $h_2f_2 = g_2f_2$ and since f_2 is surjective, we have $h_2 = g_2$. Also, $h_1f_1 = g_1f_1$ and so $(h_1, h_2)(f_1, id_{G_2}) = (g_1, g_2)(f_1, id_{G_2})$. Since (f_1, id_{G_2}) is an epimorphism, we get $(h_1, h_2) = (g_1, g_2)$ and $h_1 = g_1$. Therefore, (f_1, f_2) is an epimorphism.

For the converse, assume that (f_1, f_2) is an epimorphism with the properties mentioned in the hypothesis. To prove that (f_1, id_{G_2}) is an epimorphism, consider the morphisms $(h_1, h_2), (g_1, g_2) : (A_2, G_2, \rho_2) \to (A_3, G_3, \rho_3)$ such that

$$(h_1, h_2)(f_1, id_{G_2}) = (g_1, g_2)(f_1, id_{G_2}).$$

Therefore, $h_1f_1 = g_1f_1$ and $h_2 = h_2id_{G_2} = g_2id_{G_2} = g_2$. Hence $h_2f_2 = g_2f_2$, and so

$$(h_1, h_2)(f_1, f_2) = (g_1, g_2)(f_1, f_2).$$

Now, since (f_1, f_2) is an epimorphism, we get $(h_1, h_2) = (g_1, g_2)$, and the proof is complete. □

By Theorem 3.5, we get

Corollary 3.6. *Suppose $(f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ is a morphism in the category* **GCM** *where f_2 is surjective and f_1 is injective. Then (f_1, f_2) is an epimorphism if and only if $(f_1, id_{G_2}) : (A_1, G_2, f_2\rho_1) \to (A_2, G_2, \rho_2)$ is an epimorphism in* **GCM**.

Proof. Let (f_1, f_2) be a morphism with the properties mentioned in the hypothesis. Also, let $f_2(g) = f_2(g')$, for $g, g' \in G_1$. Then for $a \in A_1$ we have $f_1(ga) = f_2(g)f_1(a) = f_2(g')f_1(a) = f_1(g'a)$. Since f_1 is injective, we have $ga = ga'$. So, by Theorem 3.5, we get the result. □

By Theorem 3.5, it is important to know that when (f_1, id_{G_2}), introduced in Theorem 3.5, is an epimorphism.

Theorem 3.7. *The morphism $(f_1, id_{G_2}) : (A_1, G_2, f_2\rho_1) \to (A_2, G_2, \rho_2)$, introduced in Theorem 3.5, is an epimorphism in* **GCM** *if and only if* *for any two morphisms* $(\alpha_1, \beta), (\alpha_2, \beta) : (A_2, G_2, \rho_2) \to (A_3, G_3, \rho_3)$, *with* $\alpha_1(f_1(A_1)) = \alpha_2(f_1(A_1))$, *we have* $\alpha_1 = \alpha_2$.

Proof. Let $(f_1, id_{G_2}) : (A_1, G_2, f_2\rho_1) \to (A_2, G_2, \rho_2)$ be an epimorphism and $(\alpha_1, \beta), (\alpha_2, \beta) : (A_2, G_2, \rho_2) \to (A_3, G_3, \rho_3)$ be such that $\alpha_1(f_1(A_1)) = \alpha_2(f_1(A_1))$. Then

$$(\alpha_1, \beta)(f_1, id_{G_2}) = (\alpha_2, \beta)(f_1, id_{G_2}).$$

Since (f_1, id_{G_2}) is an epimorphism, we have $\alpha_1 = \alpha_2$. For the converse, consider two morphisms $(\alpha_1, \beta_1), (\alpha_2, \beta_2) : (A_2, G_2, \rho_2) \to (A_3, G_3, \rho_3)$, where

$$(\alpha_1, \beta_1)(f_1, id_{G_2}) = (\alpha_2, \beta_2)(f_1, id_{G_2}).$$

Hence $\beta_1 = \beta_2$ and $\alpha_1(f_1(A_1)) = \alpha_2(f_1(A_1))$. By the assumption, $\alpha_1 = \alpha_2$ and we get the result. □

Obviously, Theorems 3.5 and 3.7 are true for every full subcategory of the category **GCM**. In the following, suppose that $\mathfrak{C}$ is the full subcategory of **GCM** where every object $(X, G, \lambda) \in \mathfrak{C}$ has the property that for every $x, x' \in X$, $(x.x')(x')^{-1} = x(x'.x^{-1})$. Now, as a consequence of Theorems 3.5 and 3.7, we express conditions under which a morphism in $\mathfrak{C}$ is an epimorphism.

Corollary 3.8. *If the morphism* $(f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ *in* $\mathfrak{C}$ *has the properties that*

f_2 is surjective,

the kernel of f_2 is contained in the kernel of action of G_1 on A_1,

$Imf_1 \trianglelefteq A_2$, and

$A_2 = f_1(A)M$, where M is the normal subgroup of A_2 generated by $\{(b.b')(b')^{-1} : b, b' \in A_2\}$,

then (f_1, f_2) is an epimorphism.

Proof. By Theorem 3.5, we must show that the morphism

$$(f_1, id_{G_2}) : (A_1, G_2, f_2\rho_1) \to (A_2, G_2, \rho_2)$$

is an epimorphism in $\mathfrak{C}$. Using Theorem 3.7, it is enough to show that for morphisms $(\alpha_1, \beta), (\alpha_2, \beta) : (A_2, G_2, \rho_2) \to (A_3, G_3, \rho_3)$ where $\alpha_1(f_1(A_1)) =$

$\alpha_2(f_1(A_1))$ we have $\alpha_1 = \alpha_2$. To see this, by the assumption, we only prove that $\alpha_1(m) = \alpha_2(m)$, for every $m \in \{(b.b')(b')^{-1} : b, b' \in B\}$. We have $\rho_3\alpha_1 = \beta\rho_2 = \rho_3\alpha_2$. Hence

$$\begin{aligned}\alpha_1((b.b')b'^{-1}) &= \alpha_1(\rho_2(b)b')\alpha_1(b')^{-1}\\ &= (\beta\rho_2(b)\alpha_1(b'))\alpha_1(b')^{-1}\\ &= (\rho_3\alpha_1(b)\alpha_1(b'))\alpha_1(b')^{-1}\\ &= (\rho_3\alpha_2(b)\alpha_1(b'))\alpha_1(b')^{-1}\\ &= (\alpha_2(b).\alpha_1(b'))\alpha_1(b')^{-1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\alpha_2((b.b')b'^{-1}) &= \alpha_2(b(b'.b^{-1}))\\ &= \alpha_2(b)\alpha_2(\rho_2(b')b^{-1})\\ &= \alpha_2(b)(\beta\rho_2(b')\alpha_2(b^{-1}))\\ &= \alpha_2(b)(\rho_3\alpha_2(b')\alpha_2(b^{-1}))\\ &= \alpha_2(b)(\rho_3\alpha_1(b')\alpha_2(b^{-1}))\\ &= \alpha_2(b)(\alpha_1(b').\alpha_2(b)^{-1}).\end{aligned}$$

By the property of the objects of $\mathfrak{C}$, we have

$$\alpha_1((b.b')b'^{-1}) = \alpha_2((b.b')b'^{-1}),$$

and so the result. □

Using Corollaries 3.6 and 3.8 we have

Corollary 3.9. *If the morphism* $(f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ *in* $\mathfrak{C}$ *has the properties that*

f_2 *is surjective,*

f_1 *is injective,*

$Imf_1 \trianglelefteq A_2$, *and* $A_2 = f_1(A)M$, *where* M *is the normal subgroup of* A_2 *generated by* $\{(b.b')(b')^{-1} : b, b' \in A_2\}$,

then (f_1, f_2) *is an epimorphism.*

Definition 3.10. A morphism in the category $\mathfrak{D}$ is called a *regular epimorphism provided it is a coequalizer of some pair of morphisms.*

By Proposition 2.8, coequalizers exist in the category **GCM** and so we can investigate regular epimorphisms in this category. We show that, a special kind of regular epimorphisms are exactly surjectives. For this purpose, we use the following lemma.

Lemma 3.11. *Let* $(f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ *be a regular epimorphism in the category* **GCM**. *Then*

$$(f_m, id_{G_2}) : (f_1(A_1), G_2, \rho' = \rho_2|_{f_1(A_1)}) \to (A_2, G_2, \rho_2)$$

is a regular epimorphism.

Proof. Let $(f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ be a regular epimorphism. Since f_1 is a group homomorphism and $f_1(A_1)$ is a subgroup of A_2, we have $f_1 = f_m f_e$, where $f_e : A_1 \to f_1(A_1)$, and $f_m : f_1(A_1) \to A_2$ is the inclusion map. It is clear that $(f_1(A_1), G_2, \rho' = \rho_2|_{f_1(A_1)})$ is a generalized crossed submodule of (A_2, G_2, ρ_2) and $(f_e, f_2) : (A_1, G_1, \rho_1) \to (f_1(A_1), G_2, \rho')$, $(f_m, id_{G_2}) : (f_1(A_1), G_2, \rho') \to (A_2, G_2, \rho_2)$ are morphisms in **GCM**, where

$$(f_1, f_2) = (f_m, id_{G_2})(f_e, f_2).$$

Now, suppose (f_1, f_2) is a regular epimorphism. So, there exist morphisms $(h_1, h_2), (g_1, g_2) : (A_3, G_3, \rho_3) \to (A_1, G_1, \rho_1)$ such that (f_1, f_2) is the coequalizer of them. It is straightforward to see that (f_m, id_{G_2}) is the coequalizer of $(f_e, f_2)(h_1, h_2)$ and $(f_e, f_2)(g_1, g_2)$. □

Theorem 3.12. *In the category* **GCM**,

(i) *every surjective morphism is a regular epimorphism.*

(ii) *if* (f_1, f_2) *is a regular epimorphism, where* f_2 *is surjective, then* (f_1, f_2) *is surjective.*

Proof. (i) Let $(f_1, f_2) : (A_1, G_1, \rho_1) \to (A_2, G_2, \rho_2)$ be a surjective map. It is straightforward to show that (f_1, f_2) is the coequalizer of $(i, i), (1, 1) : (ker f_1, ker f_2, \rho' = \rho_1|_{ker f_1}) \to (A_1, G_1, \rho_1)$.

(ii) Suppose that (f_1, f_2) is a regular epimorphism and f_2 is surjective. By Lemma 3.11, we have $(f_1, f_2) = (f_m, id_{G_2})(f_e, f_2)$, where (f_m, id_{G_2}) is a regular epimorphism and also a monomorphism. Therefore, (f_m, id_{G_2}) is an isomorphism. By the assumption, (f_e, f_2) is surjective and so (f_1, f_2) is surjective. □

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Mahdieh Yavari, *Department of Mathematics, Shahid Beheshti University, G.C., Tehran 19839, Iran.*

Email: m_yavari@sbu.ac.ir

Alireza Salemkar, *Department of Mathematics, Shahid Beheshti University, G.C., Tehran 19839, Iran.*

Email: salemkar@sbu.ac.ir