Injectivity in a category: an overview on smallness conditions

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Abstract. Some of the so called smallness conditions in algebra as well as in category theory, are important and interesting for their own and also tightly related to injectivity, are essential boundedness, cogenerating set, and residual smallness.

In this overview paper, we first try to refresh these smallness condition by giving the detailed proofs of the results mainly by Bernhard Banaschewski and Walter Tholen, who studied these notions in a much more categorical setting. Then, we study these notions as well as the well behavior of injectivity, in the class $mod(\Sigma, \mathcal{E})$ of models of a set Σ of equations in a suitable category, say a Grothendieck topos $\mathcal{E}$, given by M.Mehdi Ebrahimi. We close the paper by some examples to support the results.

1   Introduction

The notions of *essential boundedness, cogenerating set*, and *residual smallness* are important notions in Algebra and Category Theory and, as given in [10], they are as well tightly related to the *Well Behaviour* of injectivity.

In this overview, we discuss the relation between these notions and some others related to them and to injectivity with respect to an arbitrary subclass $\mathcal{M}$ of morphisms, not necessarily monomorphisms, in an arbitrary category $\mathcal{A}$, mainly referring to Bernhard Banaschewski [3] and Walter Tholen [30].

We then describe the relationship between the class $mod\Sigma$ of models of a set Σ of equations in the category **Set** and the corresponding class $mod(\Sigma, \mathcal{E})$ of models of Σ in a suitable category $\mathcal{E}$ regarding the notions mentioned above, given by M.Mehdi Ebrahimi [11]. The basic nature of these results is that, for any given Σ, whatever a property holds in $mod\Sigma$, the corresponding property holds in $mod(\Sigma, \mathcal{E})$, provided $\mathcal{E}$ satisfies some special properties, in particular when $\mathcal{E}$ is a Grothendieck topos. The last section contains some examples to support the results.

We hope this paper and [10] will help and encourage, in particular, young mathematicians working in many different fields of mathematics to further study injectivity and related notions with respect to many different classes $\mathcal{M}$ of morphisms, not necessarily monomorphisms, which they encounter in their study.

In the rest of this section we recall some definitions needed in the sequel. For more information, see [1, 8, 11–13, 21, 23, 24, 28, 30, 31].

Definition 1.1. One says that $\mathcal{M}$ has *good properties* with respect to composition if it is:

(1) *Isomorphism closed*; that is, contains all isomorphisms and is closed under composition with isomorphisms.

(2) *Left regular*; that is, for $f \in \mathcal{M}$ with $fg = f$ we have g is an isomorphism.

(3) *Composition closed*; that is, for $f : A \to B$ and $g : B \to C$ in $\mathcal{M}$, gf is also in $\mathcal{M}$.

(4) *Left cancellable*; that is, $gf \in \mathcal{M}$ implies $f \in \mathcal{M}$.

(5) *Right cancellable*; that is, $gf \in \mathcal{M}$ implies $g \in \mathcal{M}$.

(6) *Cancellable*; that is, left and right cancellable.

Now let us recall what we mean by a subobject and an extension with respect to an arbitrary class $\mathcal{M}$ of morphisms, not necessarily monomorphisms.

Definition 1.2. We say that A is an $\mathcal{M}$-*subobject* of B, or B is an $\mathcal{M}$-*extension* of A, whenever there exists an $\mathcal{M}$-morphism (although not necessarily a monomorphism) $m : A \to B$. Sometimes, we use $m : A \hookrightarrow B$ to emphasize this convention and also we say that (A, m) (or (m, B)) is an $\mathcal{M}$-subobject of B (or an $\mathcal{M}$-extension of A).

Remark 1.3. One can define the following pre-order relation on the class of all $\mathcal{M}$-subobjects of an object A: For $\mathcal{M}$-subobjects $m : A_1 \hookrightarrow A$ and $n : A_2 \hookrightarrow A$ of A,

$$m \leq n \Leftrightarrow \exists f : A_1 \rightarrow A_2 \text{ such that } nf = m$$

Now one can see that if $\mathcal{M}$ is left regular then the relation

$$m \sim n \Leftrightarrow m \leq n \ \& \ n \leq m$$

is an equivalence relation and $m \sim n$ if and only if $A_1 \simeq A_2$ (as two objects in the category). Also the above pre-order gives a partial order on the class of all $\mathcal{M}$-subobjects of A (modulo isomorphism). So, from now on, $\mathcal{M}$-subobjects refer to this partially ordered class.

In a similar way, we can make the class of $\mathcal{M}$-extensions of an object A, modulo isomorphism, into a partially ordered class.

Recall that in classical Universal Algebra a subdirectly irreducible object is defined as follows:

Definition 1.4. We say that A is *subdirectly irreducible* if for any monomorphism $f : A \rightarrow \prod_{i \in I} A_i$ with all $\pi_i f$ epimorphisms, there exists an index $i_0 \in I$ for which $\pi_{i_0} f$ is an isomorphism.

The following definition generalizes the above and will be seen in Theorem 3.7 to be equivalent to the above for the classical case:

Definition 1.5. An object S in a category is called $\mathcal{M}$-*subdirectly irreducible* if there are an object X and two different morphisms $f, g : X \rightarrow S$ such that any morphism h with domain S and $hf \neq hg$ belongs to $\mathcal{M}$.

When the class of $\mathcal{M}$-subdirectly irreducible objects in a category $\mathcal{A}$ form just only a set we say that $\mathcal{A}$ is $\mathcal{M}$-*residually small*.

Similar to what we have with respect to monomorphisms, we can define $\mathcal{M}$-well poweredness and $\mathcal{M}$-cowell poweredness as follows:

Definition 1.6. Let $\mathcal{M}$ be a class of morphisms of a category $\mathcal{A}$. We say that $\mathcal{A}$ is $\mathcal{M}$-*well powered* ($\mathcal{M}$-*cowell powered*) if the class of all $\mathcal{M}$-subobjects ($\mathcal{M}$-extensions) of A (modulo isomorphism) forms a set, for any $A \in \mathcal{A}$.

Definition 1.7. We say that the category $\mathcal{A}$ fulfills the $\mathcal{M}$-*chain condition* if for every (so called *small well-ordered chain*) directed system $(f_{\alpha\beta} : X_\alpha \to X_\beta)_{0 \leq \alpha \leq \beta \leq \lambda}$ whose index set is a well-ordered chain with the least element 0, and for all α, $f_{0\alpha} \in \mathcal{M}$ (and of course, $f_{\alpha\alpha} = id$ and $f_{\beta\gamma}f_{\alpha\beta} = f_{\alpha\gamma}$), there is a (so called *upper bound*) family $(h_\alpha : X_\alpha \to X)_{0 \leq \alpha \leq \lambda}$ with $h_0 \in \mathcal{M}$ and $h_\beta f_{\alpha\beta} = h_\alpha$.

Note that every chain of the above form is actually a chain in the partially ordered class of $\mathcal{M}$-extensions of X_0.

2   Essential Boundedness and Residual Smallness

In this section we study essential boundedness and residual smallness and their relation with injective hulls.

Definition 2.1. An $\mathcal{M}$-morphism $f : A \hookrightarrow B$ is called an $\mathcal{M}$-*essential extension* of A if any morphism $g : B \to C$ is in $\mathcal{M}$ whenever $gf \in \mathcal{M}$. The class of all $\mathcal{M}$-essential extensions (of A) is denoted by $\mathcal{M}^*$ ($\mathcal{M}^*_A$).

The following is about the size of the class of $\mathcal{M}$-essential extensions of an object A.

Definition 2.2. $\mathcal{A}$ is said to be $\mathcal{M}$-*essentially bounded* if for every $A \in \mathcal{A}$ there is a set $\{m_i : A \hookrightarrow B_i : i \in I\} \subseteq \mathcal{M}$ such that for any $\mathcal{M}$-essential extension $n : A \hookrightarrow B$ there exists $i_0 \in I$ and $h : B \to B_{i_0}$ with $m_{i_0} = hn$.

In fact $\mathcal{A}$ is $\mathcal{M}$-essentially bounded whenever for every $A \in \mathcal{A}$ there exists a subset of the class of all $\mathcal{M}$-extensions of A, elements of which form "collectively" an upper bound for the partially ordered class $\mathcal{M}^*_A$.

The following theorem shows the relation between $\mathcal{M}$-essentially boundedness and another important notion related to $\mathcal{M}$-injectivity, namely, $\mathcal{M}^*$-cowell poweredness.

Theorem 2.3. [30] $\mathcal{M}^*$-*cowell poweredness implies $\mathcal{M}$-essential boundedness. Conversely, if $\mathcal{M}$ consists of monomorphisms only, $\mathcal{A}$ is $\mathcal{M}$-well powered and $\mathcal{M}$-essentially bounded, then $\mathcal{A}$ is $\mathcal{M}^*$-cowell powered.*

Proof. If $\mathcal{A}$ is $\mathcal{M}^*$-cowell powered then it is enough to consider $\mathcal{M}^*_A$ itself as the upper bound set of $\mathcal{M}$-essential extensions of A, for every A.

Conversely, we show that for every $A \in \mathcal{A}$, $\mathcal{M}_A^*$ is a set. Since $\mathcal{A}$ is $\mathcal{M}$-essentially bounded, for every $A \in \mathcal{A}$ there exists a set $\{m_i : A \hookrightarrow X_i : i \in I\}$ of $\mathcal{M}$-morphisms such that for every $\mathcal{M}$-essential extension $n : A \hookrightarrow B$ there exist $i_0 \in I$ and a morphism $h_n : B \to X_{i_0}$ with $m_{i_0} = h_n n$. Thus, $h_n \in \mathcal{M}$ and hence B is an $\mathcal{M}$-subobject of X_{i_0}. But, since X_i,s form a set and $\mathcal{A}$ is $\mathcal{M}$-well powered, the union of the classes $\mathcal{M}-\mathrm{Sub}X_i$, of all $\mathcal{M}$-subobjects of X_i, s, $i \in I$ is a set. Now we define $\alpha : \mathcal{M}_A^* \to \bigcup_{i \in I}(\mathcal{M}-\mathrm{Sub}X_i)$ which maps every $\mathcal{M}$-essential extension n of A to h_n. Since $\mathcal{M} \subseteq \mathcal{M}ono$, we can easily see that α is one-one and so $\mathcal{M}_A^*$ is a set, for every $A \in \mathcal{A}$. $\qquad\square$

The next result shows that $\mathcal{M}$-essential boundedness is a necessary condition to having enough $\mathcal{M}$-injectives, which later, in 2.7, helps us to show that residual smallness is also necessary to having enough injectives.

Recall that it is said that a category $\mathcal{A}$ has enough $\mathcal{M}$-injectives if any $A \in \mathcal{A}$ is an $\mathcal{M}$-subobject of an $\mathcal{M}$-injective object.

Theorem 2.4. *If $\mathcal{A}$ has enough $\mathcal{M}$-injectives then $\mathcal{A}$ is $\mathcal{M}$-essentially bounded.*

Proof. Let $m : A \hookrightarrow I$ be an $\mathcal{M}$-injective extension of A. One can easily see that any $\mathcal{M}$-essential extension of A is in I, and hence the result. $\quad\square$

The next notion helps us to provide the relation between residual smallness and $\mathcal{M}^*$-cowell poweredness.

Definition 2.5. Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in a category $\mathcal{A}$. The pair $(\mathcal{E}, \mathcal{M})$ is called a *factorization structure* for morphisms in $\mathcal{A}$ if:

(1) Both $\mathcal{E}$ and $\mathcal{M}$ are isomorphism closed.

(2) $\mathcal{A}$ has $(\mathcal{E}, \mathcal{M})$-factorization of morphism (that is, each morphism f in $\mathcal{A}$ has a factorization of the form $f = me$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$).

Also, $\mathcal{A}$ is said to have $(\mathcal{E}, \mathcal{M})$-*factorization diagonalization* if, in addition to the above, the following condition also holds:
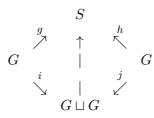
(3) $\mathcal{A}$ has a unique $(\mathcal{E}, \mathcal{M})$-diagonalization property; that is for each commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{e} & B \\
f\downarrow & & \downarrow g \\
C & \xrightarrow{m} & D
\end{array}
$$

with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists a unique diagonal morphism $d : B \to C$ making the resulting triangles commutative.

Theorem 2.6. [30] *Let $\mathcal{M} = \mathcal{M}ono$ and $\mathcal{E}$ be another class of morphisms of $\mathcal{A}$ such that $\mathcal{A}$ has $(\mathcal{E}, \mathcal{M})$-factorization. Also, let $\mathcal{A}$ be $\mathcal{E}$-cowell powered and have a generating set $\mathcal{G}$ such that for all $G \in \mathcal{G}$, $G \sqcup G \in \mathcal{A}$. Then $\mathcal{M}^*$-cowell poweredness implies $\mathcal{M}$-residual smallness.*

Proof. Suppose S is $\mathcal{M}$-subdirectly irreducible, and so there exist an object X and morphisms $x, y : X \to S$ such that $x \neq y$ and any morphism $f : S \to D$ with $fx \neq fy$ is in $\mathcal{M}$. But, since $\mathcal{A}$ has a generating set $\mathcal{G}$, there are an object $G \in \mathcal{G}$ and a morphism $f : G \to X$ such that $xf \neq yf$. Now, we take $g := xf, h := yf$. On the other hand, since $G \sqcup G$ exists, the universal property of coproducts gives a unique morphism $t : G \sqcup G \to S$ making the following diagram commutative:

$$
\begin{array}{ccc}
 & S & \\
{}^{g}\nearrow & \uparrow & \nwarrow{}^{h} \\
G & \vert & G \\
{}_{i}\searrow & \vert & \swarrow{}_{j} \\
 & G \sqcup G &
\end{array}
$$

Now, by hypothesis, we can factor t as $t = me$. We claim that $m \in \mathcal{M}^*$. To show this, assume that $l : S \to P$ is such that $lm \in \mathcal{M}$. Note that $lx \neq ly$. For if $lx = ly$, then $lxf = lyf$, so $lmei = lmej$ and hence, by left cancellability of monomorphisms, we have $ei = ej$ and therefore $mei = mej$, that is, $h = xf = yf = g$, which is a contradiction. Now, $\mathcal{M}$-subdirect irreducibility of S implies that $l \in \mathcal{M}$. That is, $m \in \mathcal{M}^*$, and so S is an $\mathcal{M}^*$-extension of an object X depending on $G \in \mathcal{G}$. Since $\mathcal{G}$ is a set and $\mathcal{A}$ is $\mathcal{M}^*$-cowell powered, these S,s form a set. $\square$

Now, using Theorems 2.4, 2.3 and 2.6, respectively, we can see that residual smallness is also a necessary condition to having enough $\mathcal{M}$-injectives, when $\mathcal{M} = \mathcal{M}ono$ and the hypotheses of these results are present:

Corollary 2.7. *Under the hypotheses of Theorems 2.4, 2.3 and 2.6, residual smallness is a necessary condition for having enough $\mathcal{M}$-injectives when $\mathcal{M} = \mathcal{M}ono$.*

The next notions to be considered in this section are defined as follows:

Definition 2.8. $\mathcal{A}$ has $\mathcal{M}$-*transferability property* if for every pair f, u of morphisms with $f \in \mathcal{M}$ one has a commutative square

$$
\begin{array}{ccc}
A & \overset{f}{\hookrightarrow} & B \\
u \downarrow & & \\
C & &
\end{array}
\qquad \Longrightarrow \qquad
\begin{array}{ccc}
A & \overset{f}{\hookrightarrow} & B \\
u \downarrow & & \downarrow v \\
C & \overset{g}{\hookrightarrow} & D
\end{array}
$$

with $g \in \mathcal{M}$.

Note 2.9. Note that if pushouts exist in $\mathcal{A}$ and $\mathcal{M}$ is left cancellable, then it is easily seen that pushouts transfer (or preserve) $\mathcal{M}$-morphisms if and only if $\mathcal{A}$ has the $\mathcal{M}$-transferability property.

The next result shows the relation between having enough injectives and the M-transferability condition.

Lemma 2.10. *If $\mathcal{A}$ has enough $\mathcal{M}$-injectivs then, $\mathcal{A}$ fulfills $\mathcal{M}$-transferability property.*

Proof. Cosider the following diagram:

$$
\begin{array}{ccc}
A & \overset{f}{\hookrightarrow} & B \\
u \downarrow & & \\
C & &
\end{array}
$$

Having enough $\mathcal{M}$-injectives gives an $\mathcal{M}$-injective extension $g : C \hookrightarrow D$, and hence a morphism $v : B \to D$ with $vf = gu$, which proves the lemma. $\qquad \square$

Definition 2.11. We say that $\mathcal{A}$ has $\mathcal{M}$-*bounds* if for any small family $(h_\alpha : A \hookrightarrow B_\alpha)_{\alpha \in I}$ of $\mathcal{M}$-morphisms there exists an $\mathcal{M}$-morphism $h : A \hookrightarrow B$ which factors over all h_α,s.

Note that for $\mathcal{A}$ having $\mathcal{M}$-bounds means that every subset of the class of $\mathcal{M}$-extensions of an object A has an upper bound in the partially ordered class of $\mathcal{M}$-extensions of A.

The following lemma shows that, up to the existence of colimits in $\mathcal{A}$, the condition of having $\mathcal{M}$-bounds is equivalent to being closed under $\mathcal{M}$-multiple pushouts, provided $\mathcal{M}$ is left cancellable.

Lemma 2.12. *Up to the existence of colimits, having $\mathcal{M}$-bounds for $\mathcal{A}$ is equivalent to being closed under $\mathcal{M}$-multiple pushouts, provided $\mathcal{M}$ is closed under composition and is left cancellable.*

Proof. ($\Leftarrow$) This implication is trivial.

($\Rightarrow$) Suppose $\{m_i : X_0 \hookrightarrow X_i\}_{i \in I}$ is a small family of $\mathcal{M}$-morphisms. We show that this family has an $\mathcal{M}$-multiple pushout. Since $\mathcal{A}$ has colimits, there exists a multiple pushout of $\{m_i\}_{i \in I}$, say $(\{n_i\}_{i \in I}, P)$, where $n_i : X_i \to P$, for every $i \in I$. On the other hand, $\{m_i\}_{i \in I}$ has an $\mathcal{M}$-bound, say $m : X_0 \hookrightarrow Y$, such that for every $i \in I$ there exists $h_i : X_i \to Y$ with $h_i m_i = m$. Now, the universal property of pushouts gives a unique morphism $t : P \to Y$ such that $t n_i = h_i$, for every $i \in I$. But we have $h_i m_i = m$, and so $t n_i m_i = m$ belongs to $\mathcal{M}$. Therefore, by left cancellability of $\mathcal{M}$ we have $n_i m_i \in \mathcal{M}$. $\square$

Theorem 2.13. [30] *Let $\mathcal{A}$ satisfy the $\mathcal{M}$-transferability and the $\mathcal{M}$-chain condition, and let $\mathcal{M}$ be closed under composition. Then, $\mathcal{A}$ has $\mathcal{M}$-bounds.*

Proof. Consider a nonempty small family $(m_\alpha : X \to X_\alpha)_{\alpha \in I}$. We should find an $\mathcal{M}$-morphism $n : X \hookrightarrow Y$ over which each m_α factors. Using the Well-Ordering Theorem, we take I as an initial segment of the ordinal numbers, and proceed to construct n by ordinal induction. Put $n_0 = m_0$ as the initial step. Now suppose that for every $j \leq i$, the family $(m_\alpha)_{\alpha \leq j \leq i}$ has an $\mathcal{M}$-bound $n_j : X \hookrightarrow Y_j$. Now consider the following two cases for i.

(1) i is the immediate successor of i', that is $i = i' + 1$. In this case we apply $\mathcal{M}$-transferability to get the commutative diagram

$$m_i = m_{i'+1} \quad \begin{array}{ccc} X & \overset{n_{i'}}{\longrightarrow} & Y_{i'} \\ \downarrow & & \downarrow \quad f_{i'} \\ X_{i'+1} & \overset{g_{i'+1}}{\longrightarrow} & Y_{i'+1} \end{array}$$

Now taking $n_i (= n_{i'+1}) = f_{i'+1} n_{i'} = g_{i'+1} m_{i'+1}$, we see that n_i factors through all m_j, $j \leq i$.

(2) i is a limit ordinal, and so it is not an immediate successor. In this case we apply the $\mathcal{M}$-chain condition to n_j,s ($j < i$) already created, to obtain $h : X \to Z$ which factors through all n_i, and then apply the $\mathcal{M}$-transferability property to m_i and h, and define n_i as in the successor step. $\square$

3 Cogenerating Set and Residual Smallness

In this section we see the relation between $\mathcal{M}$-residual smallness and having a cogenerating set in a category, and among other things we give the generalized Birkhoff's Subdirect Representation Theorem.

First recall the following definitions:

Definition 3.1. (1) A set $\mathcal{G}$ of objects of a category is a *generating set* if for every parallel pair of different morphisms $m, n : X \to Y$ there exist $G \in \mathcal{G}$ and a morphism $h : G \to X$ such that $mh \neq nh$.

(2) A set $\mathcal{C}$ of objects of a category is a *cogenerating set* if for every parallel different morphisms $m, n : X \to Y$ there exist an object $C \in \mathcal{C}$ and a morphism $f : Y \to C$ such that $fm \neq fn$.

The following definition generalizes the notion of a cogenerating set, which will be shown in Theorem 3.13 to be equivalent to the above when $\mathcal{M}$ is the class of all monomorphisms.

Definition 3.2. A set $\mathcal{C}$ of objects in a category is called an $\mathcal{M}$-*cogenerating set* if it is closed under products (including the empty one which is the terminal object), and any object A admits some $\mathcal{M}$-morphism $A \hookrightarrow \prod_{i \in I} C_i$ with $C_i \in \mathcal{C}$.

Now we see the relation between $\mathcal{M}$-residual smallness and having a cogenerating set in a category.

Theorem 3.3. [30] *Let $\mathcal{A}$ have a cogenerating set $\mathcal{C}$ and $\mathcal{A}$ be $\mathcal{M}$-well powered. Then $\mathcal{A}$ is $\mathcal{M}$-residually small.*

Proof. Suppose S is an $\mathcal{M}$-subdirectly irreducible object of $\mathcal{A}$. Thus, there exist an object X and morphisms $x, y : X \to S$ such that any morphism f with $fx \neq fy$ is in $\mathcal{M}$. Since $\mathcal{C}$ is a cogenerating set, there are $C \in \mathcal{C}$ and $f : S \to C$ with $fx \neq fy$, and so $f \in \mathcal{M}$. That is, S is an $\mathcal{M}$-subobject of an element of $\mathcal{C}$. But, since $\mathcal{C}$ is a set and $\mathcal{A}$ is $\mathcal{M}$-well powered, the class of $\mathcal{M}$-subobjects of elements of $\mathcal{C}$ forms a set, and every $\mathcal{M}$-subdirectly irreducible object belongs to this set. Hence we are done. □

The following important theorem helps us to prove that the two definitions of a subdirectly irreducible object in the category of algebras of a

type τ are equivalent, and suggests conditions under which the converse of Theorem 3.3 is true. In fact it is the generalization of Birkhoff's Subdirectly Representation Theorem.

Definition 3.4. A generating set $\mathcal{G}$ is called *chain faithful* if for any $G \in \mathcal{G}$, any pair of different morphisms $x, y : G \to X_0$, and any well ordered chain $(f_{\alpha\beta} : X_\alpha \to X_\beta)_{0 \leq \alpha \leq \beta < \lambda}$ with $f_{0\alpha}x \neq f_{0\alpha}y$ for all α, there is an upper bound $(h_\alpha : X_\alpha \to Y)_{\alpha < \lambda}$ with $h_0x \neq h_0y$.

Theorem 3.5. [30] (Birkhoff's Subdirect Representation Theorem) *Let $\mathcal{E}$ be a class of morphisms in $\mathcal{A}$ such that $\mathcal{A}$ has $(\mathcal{E}, \mathcal{M})$-factorization diagonalization, and $\mathcal{A}$ possess a chain faithful generating set $\mathcal{G}$. Then, for every $A \in \mathcal{A}$ there is a set indexed A-monocone $(f_i : A \to S_i)_{i \in I}$ (that is, for any $u \neq v$ there exists $i_0 \in I$ such that $f_{i_0}u \neq f_{i_0}$) with all S_i being $\mathcal{M}$-subdirectly irreducible and $f_i \in \mathcal{E}$.*

Proof. For every $A \in \mathcal{A}$, $G \in \mathcal{G}$, and two different morphisms $x, y : G \to A$, let $\mathcal{B}_G := \{f : A \to X : f \in \mathcal{E}, \ fx \neq fy\}$. Since $\mathcal{A}$ is $\mathcal{E}$-well powered, $\mathcal{B}_G$ is actually a set. The set $\mathcal{B}_G$ is a partially ordered set as in Remark 1.7, and we know that every well-ordered chain $\mathcal{C}_G = (f_{\alpha\beta} : X_\alpha \to X_\beta)_{0 \leq \alpha \leq \beta \leq \gamma}$ in $\mathcal{B}_G$ has an upper bound $(h_\alpha : X_\alpha \to Y)$ with $h_0x \neq h_0y$, $h_0 = h_1m_1$ where $m_\alpha := f_{0\alpha}$. On the other hand, we know that $h_1 = n_1e_1$ with $n_1 \in \mathcal{M}$ and $e_1 \in \mathcal{E}$, and so $h_0 = h_1m_1 = n_1e_1m_1$. But $h_0x \neq h_0y$ and hence $n_1e_1m_1x \neq n_1e_1m_1y$, and so $e_1m_1x \neq e_1m_1y$. We also know that for every $\alpha \leq \gamma$ we have $h_\alpha f_{1\alpha} = h_1$ and, by diagonalizaton property, there exists a family $(d_\alpha : X_\alpha \to Z)$ such that $d_\alpha f_{1\alpha} = e_1, n_1d_\alpha = h_\alpha$, that is $e_1 \geq f_{1\alpha}$. But $e_1m_1 = d_\alpha f_{1\alpha}m_1$ and so $e_1m_1 \geq m_1$, $e_1m_1x \neq e_1m_1y$. So e_1m_1 is an upper bound for the chain $\mathcal{C}_G$ in $\mathcal{B}_G$. Therefore, $\mathcal{B}_G$ has a maximal element, say $e_{xy} : A \to S_{xy}$. We claim that the family $(e_{xy} : A \to S_{xy})_{x,y}$ is an A-monocone of subdirectly irreducible objects. Note that $(e_{xy} : A \to S_{xy})_{x,y}$ is a set, since $\mathcal{G}$ is a set and x, y depends on $\mathcal{G}$. Now, to show that $(e_{xy})_{x,y}$ is an A-monocone, suppose $u, v : C \to A$ and $u \neq v$. Since $\mathcal{G}$ is a generating set, there exist $G \in \mathcal{G}$ and $w : G \to C$ such that $uw \neq vw$. Now take $x := uw, y := vw$. We have $e_{xy}u \neq e_{xy}v$ because otherwise we have $e_{xy}x = e_{xy}y$ which is a contradiction. Thus $(e_{xy} : A \to S_{xy})$ is a monocone. Now we show that the objects S_{xy},s are $\mathcal{M}$-subdirectly irreducible. We know that for every S_{xy} there are $e_{xy}x, e_{xy}y : G \to S_{xy}$ such that $e_{xy}x \neq e_{xy}y$. Now suppose $fe_{xy}x \neq fe_{xy}y$. By $(\mathcal{E}, \mathcal{M})$-factorization diagonalization, we have

$f = me$, and so $mee_{xy}x \neq mee_{xy}y$, hence $ee_{xy}x \neq ee_{xy}y$. But $ee_{xy} \in \mathcal{E}$ implies $ee_{xy} \in \mathcal{B}_G$. Also $e \leq ee_{xy}$. Thus, maximality of e makes e to be an isomorphism and therefore e belongs to $\mathcal{M}$, and so $f = me$ belongs to $\mathcal{M}$. $\qquad\square$

The following is a characterization of $\mathcal{M}$-subdirectly irreducible objects for $\mathcal{M}$ being the class of all monomorphisms.

Theorem 3.6. [30] *Assume that $\mathcal{A}$ has a generating set and $(\mathcal{E}, \mathcal{M})$-factorization diagonalizastion for $\mathcal{M} = \mathcal{M}ono$ and $\mathcal{E} = \mathcal{E}pic$. Then $S \in \mathcal{A}$ is $\mathcal{M}$-subdirectly irreducible if and only if every S-monocone has an $\mathcal{M}$-morphism (monomorphism).*

Proof. ($\Rightarrow$) Suppose S is an $\mathcal{M}$-subdirectly irreducible object and $(m_i : S \rightarrow A_i)_{i \in I}$ is an S-monocone. Because S is subdirectly irreducible, there exist an object X and different morphisms $x, y : X \rightarrow S$ such that any f with $fx \neq fy$ is in $\mathcal{M}$. But, since $(m_i : S \rightarrow A_i)_{i \in I}$ is a monocone, there is $i_0 \in I$ such that $m_{i_0}x \neq m_{i_0}y$, and hence $m_{i_0} \in \mathcal{M}$.

($\Leftarrow$) Let every S-monocone contain an $\mathcal{M}$-morphism. Since $\mathcal{A}$ satisfies the conditions of the above theorem, S has a monocone $(f_i : S \rightarrow S_i)_{i \in I}$ such that S_i,s are $\mathcal{M}$-subdirectly irreducible. Thus, using the hypothesis, we can find an object $G \in \mathcal{G}$ and two different morphisms $x, y : G \rightarrow X$ such that f_{xy} is a monomorphism. Now, assume that there is a morphism $g : S \rightarrow X$ such that $gx \neq gy$. We claim that g is a monomorphism. By $(\mathcal{E}, \mathcal{M})$-factorization we have $g = me$, so $mex \neq mey$, and since m is a monomorphism, $ex \neq ey$. On the other hand, by the definition of a monocone, $f_{xy}x \neq f_{xy}y$ and $e \leq f_{xy}$. Thus, there exists $h : Y \rightarrow S_{xy}$ such that $he = f_{xy} \in \mathcal{M}$, and hence $e \in \mathcal{M}$. So, $g = me \in \mathcal{M}$, that is S is $\mathcal{M}$-subdirectly irreducible. $\qquad\square$

Now, the following theorem shows that the two Definitions 1.4 and 1.5 of subdirectly irreducible objects are equivalent in a category of algebras.

Theorem 3.7. *Let $\mathcal{A}$ be the category of algebras of type τ. Then S is a subdirectly irreducible object of $\mathcal{A}$ if and only if every S-monocone contains an isomorphism.*

Proof. ($\Rightarrow$) Suppose that every S-monocone contains an isomorphism and S has the subdirect product representation

$$
\begin{array}{ccc}
S & \overset{f}{\hookrightarrow} & \prod_{i\in I} A_i \\
f_i & \searrow & \downarrow \quad \pi_i \\
& & A_i
\end{array}
$$

with f monic and f_i epic for all $i \in I$. Since f is a monomorphism and $(\pi_i)_{i\in I}$ are collectively mono, $(f_i : S \to A_i)_{i\in I}$ is an S-monocone, and so there is $i_0 \in I$ such that f_{i_0} is an isomorphism.

($\Leftarrow$) Let S be subdirectly irreducible and $(f_i : S \to A_i)_{i\in I}$ be an S-monocone. Then, by the universal property of products, we have the commutative diagram

$$
\begin{array}{ccc}
S & \overset{\exists\,!f}{\longrightarrow} & \prod_{i\in I} A_i \\
f_i & \searrow & \downarrow \quad \pi_i \\
& & A_i
\end{array}
$$

The morphism f is a monomorphism, because if there are morphisms $x, y : X \to S$ such that $fx = fy$, then for every $i \in I$ we have $\pi_i f x = \pi_i f y$. Therefore, for every $i \in I$ we have $f_i x = f_i y$. But, since $(f_i)_{i\in I}$ is an S-monocone, $x = y$. That is, f is a monomorphism. Now, by subdirect irreducibility of S, there exists $i_0 \in I$ such that f_{i_0} is an isomorphism. This proves the assertion. $\qquad\square$

Considering the hypotheses of Theorem 3.5, we get the converse of Theorem 3.3 as follows.

Lemma 3.8. [30] *Let $\mathcal{A}$ satisfy the conditions of Theorem 3.5. Then $\mathcal{M}$-residual smallness implies $\mathcal{A}$ has a cogenerating set.*

Proof. We claim that the set $\mathcal{C}$ of $\mathcal{M}$-subdirectly irreducible objects forms a cogenerating set. To prove this, consider different parallel morphisms $x, y : X \to Y$. Now, by Theorem 3.5, there exists a set index Y-monocone $(f : Y \to S_i)_{i\in I}$ with $S_i \in \mathcal{C}$, and so there exists $i_0 \in I$ such that $f_{i_0} x \neq f_{i_0} y$. $\qquad\square$

Banaschewski in [3] says that the notion of $\mathcal{M}$- injectivity in a category $\mathcal{A}$ *Well Behaves* if and only if the following three propositions hold.

Proposition 3.9. *For any $A \in \mathcal{A}$ the following conditions are equivalent.*

(I1) *A is $\mathcal{M}$- injective*

(I2) *A is an $\mathcal{M}$- absolute retract*

(I3) *A has no proper $\mathcal{M}$- essential extensions.*

Proposition 3.10. *Every $A \in \mathcal{A}$ has an injective hull.*

Proposition 3.11. *For any $\mathcal{M}$- morphism $f : A \to B$ the following conditions are equivalent.*

(H1) *B is an $\mathcal{M}$- injective hull of A*

(H2) *B is a maximal $\mathcal{M}$- essential extension of A*

(H3) *B is a minimal $\mathcal{M}$- injective $\mathcal{M}$- extension of A.*

The following is a collection of sufficient conditions for the Well Behaviour of injectivity, given in the above mentioned papers:

- $\mathcal{M}$ is closed under composition, isomorphism closed, left regular, $\mathcal{A}$ fulfills Banaschewski's $\mathcal{M}$-condition (for any $f \in \mathcal{M}$ there exists a morphism $g \in \mathcal{A}$ such that $fg \in \mathcal{M}^*$), satisfies the $\mathcal{M}$-transferability condition, has $\mathcal{M}$-direct limits of well-ordered directed systems, and is $\mathcal{M}^*$-cowell powered.

The next result gives the relationship between the behavior of $\mathcal{M}$-injectivity and subdirect irreducible objects in an equational class of algebras, when $\mathcal{M} = \mathcal{M}ono$.

Proposition 3.12. [3] *For an equational class $\mathcal{A}$, the following conditions are equivalent:* (1) *Injectivity is well-behaved.*

(2) *$\mathcal{A}$ has enough injectives.*

(3) *Every subdirectly irreducible algebra in $\mathcal{A}$ has an injective extension.*

(4) *$\mathcal{A}$ satisfies $\mathcal{M}$-transferability and $\mathcal{M}^*$-cowell poweredness.*

Proof. (1)$\Rightarrow$(2) $\Rightarrow$(3) are trivial.

(3)$\Rightarrow$(4) To show transferability, consider the diagram:

$$A \quad \overset{m}{\hookrightarrow} \quad B$$
$$g \downarrow$$
$$C$$

Now, Birkhoff's Representation Theorem gives a monomorphism $n : C \to \prod_{i \in I} S_i$ with subdirectly irreducible S_i, s and (3) gives an injective extension $p : \prod_{i \in I} S_i \to I$; note that the product of injective objects is an injective object. Now we have the following complete diagram with pn a monomorphism:

$$
\begin{array}{ccc}
A & \overset{m}{\hookrightarrow} & B \\
g \downarrow & & \downarrow \overline{png} \\
C & \overset{pn}{\hookrightarrow} & I
\end{array}
$$

Now we prove $\mathcal{M}^*$-cowell poweredness. By (3) and Birhkoff's Representation Theorem, every $A \in \mathcal{A}$ has an injective extension I. Since every essential extension of A is in I, there exists a set of essential extensions for A.

(4)$\Rightarrow$(1) With (4), $\mathcal{A}$ satisfies all the sufficient conditions for the Well-Behaviour Theorems. $\qquad \square$

Now we prove the claim we made earlier that having a cogenerating set is equivalent to having an $\mathcal{M}$-cogenerating set.

Lemma 3.13. *Let $\mathcal{M} = \mathcal{M}ono$ and $\mathcal{A}$ have products. Then, $\mathcal{A}$ has an $\mathcal{M}$-cogenerating set if and only if $\mathcal{A}$ has a cogenerating set.*

Proof. ($\Leftarrow$) Suppose $\mathcal{A}$ has a cogenerating set $\mathcal{C}$. Then, for every $A \in \mathcal{A}$ we consider the following two cases:

(1) Let there be different morphisms $x, y : X \to A$. Then, there exist $C_{xy} \in \mathcal{C}$ and a morphism $m_{xy} : A \to C_{xy}$ such that $m_{xy}x \neq m_{xy}y$. Now, consider $\mathcal{B} = \{m_{xy} : A \to C_{xy} : x, y : X \to A; X \in \mathcal{A}, x \neq y\}$. So, by the universal property of products, we have the following commutative diagram:

$$
\begin{array}{ccc}
A & \overset{m_{xy}}{\longrightarrow} & C_{xy} \\
\exists!m \searrow & & \uparrow \quad \pi_{x,y} \\
& \prod_{x,y} C_{xy} &
\end{array}
$$

We claim that m is a monomorphism. To prove this, let $\alpha, \beta : Y \to A$ be such that $\alpha \neq \beta$ and $m\alpha = m\beta$. Then, we have $m\alpha = \pi_{xy}m_{xy}\alpha = \pi_{xy}m_{xy}\beta = m\beta$ for every x, y. Since $\{\pi_{xy}\}_{xy}$ is a monocone, we have $m_{xy}\alpha = m_{xy}\beta$ for every $m_{xy} \in B$. In particular we have $m_{\alpha\beta}\alpha = m_{\alpha\beta}\beta$, which is a contradiction.

(2) Suppose that there dose not exist two different parallel morphisms with codomain A. In this case A is a preterminal object. On the other hand, we know that $\prod_{i \in \emptyset} C_i$ exists and is the terminal object T. So there exists a unique morphism $m : A \to T$. But, since A is a preterminal object, m is a monomorphism.

($\Leftarrow$) Suppose $\mathcal{A}$ has an $\mathcal{M}$-cogenerating set $\mathcal{C}$ and $x, y : X \to A$ are two different morphisms. Then, there exists an $\mathcal{M}$-morphism $m : A \to \prod_{i \in I} C_i$ where $\{C_i\}_{i \in I} \subseteq \mathcal{C}$. Now consider $\pi_i : \prod_{i \in I} C_i \to C_i$. Since m is a monomorphism, $mx \neq my$ and so there exists an index $i_0 \in I$ such that $\pi_{i_0} mx \neq \pi_{i_0} my$. Now, consider $\pi_{i_0} m : A \to C_{i_0}$. $\qquad \square$

Theorem 3.14. *Let* $\mathcal{M} = \mathcal{M}ono$ *and* $\mathcal{A}$ *be well powered with products and a generating set* $\mathcal{G}$. *Then, having an* $\mathcal{M}$-*cogenerating set implies* $\mathcal{M}^* - cowell$ *poweredness.*

Proof. Suppose $\mathcal{A}$ has an $\mathcal{M}$-cogenerating set $\mathcal{C}$. Consider an essential extension $f : A \hookrightarrow B$ of A for any $A \in \mathcal{A}$. We know that there exists a monomorphism $m : B \to \prod_{\alpha \in I} C_\alpha$ with $C_\alpha \in \mathcal{C}$. Then, for any $G \in \mathcal{G}$ and a pair of different morphisms $x, y : G \to A$ we have $mfx \neq mfy$, and hence $\pi_\alpha mfx \neq \pi_\alpha mfy$ for some projection morphism $\pi_\alpha : \prod_{\alpha \in I} C_\alpha \to C_\alpha$. Pick α_{xy} as one such for each pair $x \neq y$. Then $A \to \prod_{\beta \in J} C_\beta$ is a monomorphism, where $J = \{\alpha_{xy} : x \neq y : G \to A\}$ and $Card J \leq Card \bigcup_{G \in \mathcal{G}} (G, A)^2$. Essentialness of f implies that $B \to \prod_{\beta \in J} C_\beta$ is a monomorphism, and since there exists only a set of the products $\prod_{\beta \in J} C_\beta$, we are done. $\qquad \square$

Theorem 3.15. [30] *Let* $\mathcal{M} = \mathcal{M}ono$ *and* $\mathcal{A}$ *be well powered and have direct products and a faithful generating set* $\mathcal{G}$ *which* $G \sqcup G \in \mathcal{A}$. *In addition, suppose* $\mathcal{A}$ *has* $(\mathcal{E}, \mathcal{M})$-*factorization diagonalization for a class* $\mathcal{E}$ *of morphisms for which* $\mathcal{A}$ *is* $\mathcal{E}$-*well powered. Then, the following statements are equivalent:*

   (i) $\mathcal{A}$ *is* $\mathcal{M}$-*essentially bounded.*
   (ii) $\mathcal{A}$ *is* $\mathcal{M}^*$-*cowell powered.*
   (iii) $\mathcal{A}$ *is residually small.*
   (iv) $\mathcal{A}$ *has a cogenerating set.*

Proof. By Theorem 2.3, (i) and (ii) are equivalent. Also $\mathcal{M}^*$-cowell poweredness implies residual smallness by Theorem 2.6. Moreover Lemma 3.8

and Theorem 3.3 ensure that residual smallness implies having an $\mathcal{M}$-cogenerating set and Theorem 3.3 ensures the converses. $\mathcal{M}^*$-cowell poweredness is implied by Theorem 3.14. Thus, $\mathcal{M}^*$-cowell poweredness, residual smallness, and having a cogenerating set are equivalent. □

4 Injectivity of Algebras in a Grothendieck Topos

The main purpose of this section is to describe the relationship between the class $mod\Sigma$ of models of a set Σ of equations in the category **Set** and the corresponding class $mod(\Sigma, \mathcal{E})$ of models of Σ in a suitable category $\mathcal{E}$ with respect to the notions mentioned in the previous sections.

The basic nature of our results is that, for any given Σ, whatever a property holds in $mod\Sigma$, the corresponding property holds in $mod(\Sigma, \mathcal{E})$, provided $\mathcal{E}$ satisfies some special properties, in particular when $\mathcal{E}$ is a Grothendieck topos. For more information, see the references in particular [7, 11–13, 29]

A *Grothendieck topos* $\mathcal{E}$ is a reflective subcatgory of a functor category $\mathbf{Set}^{\mathcal{C}^{op}} = \hat{\mathcal{C}}$ ($i : \mathcal{E} \overset{\rightarrow}{\leftarrow} \hat{\mathcal{C}} : R$, $R \dashv i$), for some small category $\mathcal{C}$, whose reflection functor R preserves finite limits.

For example, for a topological space X, the category $PreshX$, of presheaves on X, and the category ShX, of sheaves on X, are Grothendieck topoi. Also, the category **Set** of all sets with functions between them is a Grothendieck topos.

Definition 4.1. Let $\mathcal{A}$ be a finitely complete category. An *algebra* in $\mathcal{A}$ is an entity $(A, (\lambda_A)_{\lambda \in \Lambda})$, where A is an object of $\mathcal{E}$, Λ is a set and for each $\lambda \in \Lambda$, the λth operation $\lambda_A : A^{n_\lambda} \to A$ is a morphism in $\mathcal{A}$ where each n_λ is a finite cardinal number and A^{n_λ} is the n_λth power of A. The family $\tau = (n_\lambda)_{\lambda \in \Lambda}$ is called the *type* of this algebra. The algebra $(A, (\lambda_A)_{\lambda \in \Lambda})$ is simply denoted by A.

A *homomorphism* $h : A \to B$ from an algebra $(A, (\lambda_A)_{\lambda \in \Lambda})$ to an algebra $(B, (\lambda_B)_{\lambda \in \Lambda})$, both in $\mathcal{A}$, of type τ is a morphism in $\mathcal{A}$ such that for each $\lambda \in \Lambda$, $\lambda_B h^{n_\lambda} = h\lambda_A$.

The collection of all algebras (of type τ) in $\mathcal{A}$ and homomorphisms between them form a category denoted by $\mathcal{A}lg(\tau)\mathcal{A}$ (or by $\mathcal{A}lg(\tau)$ if $\mathcal{A} = \mathbf{Set}$).

Definition 4.2. Given any $A \in Alg(\tau)\mathcal{A}$ and any finite set $X = \{x_1, ..., x_n\}$, an *equation* (or *identity*) $p = q$ in X is any pair $(p, q) \in FX \times FX$, where FX is the free (set based) algebra of type τ on X. It is said that A *satisfies an equation* $p = q$ if $p_A = q_A$, where $p_A = \phi(p)$, $q_A = \phi(q)$ and $\phi : FX \to Hom_{\mathcal{A}}(A^n, A)$ is the homomorphism which freely extends the map $h : X \to Hom_{\mathcal{A}}(A^n, A)$ given by $x_i \mapsto \pi_i$ (the ith projection). Notice that $Hom_{\mathcal{A}}(A^n, A)$ is a set based algebra of type τ with the λth operation given by $(f_1, ..., f_{n_\lambda}) \mapsto \lambda_A \prod_{i=1}^{n_\lambda} f_i$.

The full subcategory of $Alg(\tau)\mathcal{A}$ whose objects are all algebras in $\mathcal{A}$ satisfying a set Σ of equations is denoted by $mod(\Sigma, \mathcal{A})$ (or by $mod\Sigma$, if $\mathcal{A} = \mathbf{Set}$, which is called an *equational category of algebras*).

Remark 4.3. Let $K : \mathcal{A} \to \mathcal{B}$ be a functor, preserving finite limits. Then
(1) K induces another functor

$$\overline{K} : Alg(\tau)\mathcal{A} \to Alg(\tau)\mathcal{B}$$

defined on objects by $\overline{K}A = (KA, (K\lambda_A)_{\lambda \in \Lambda})$ and on homomorphisms $f : A \to B$, by $\overline{K}(f) = K(f)$.
(2) K can be lifted to a functor

$$\overline{K} : mod(\Sigma, \mathcal{A}) \to mod(\Sigma, \mathcal{B})$$

for any given set Σ of equations. This is because if $\sigma = (p, q)$ is an equation, then $A \models (p = q)$ implies that $p_A = q_A$, and hence $Kp_A = Kq_A$ which implies that $p_{\overline{K}A} = q_{\overline{K}A}$. To show that this implies $\overline{K}A \models (p = q)$, let h be given as in Definition 4.2 and h' be the corresponding map related to KA: $h' : X \to Hom((KA)^n, KA)$. Then, since K preserves finite limits, and so finite products, there is an isomorphism $\beta : (KA)^n \to KA^n$, we get a map $\alpha : Hom(A^n, A) \to Hom((KA)^n, KA)$ given by $\alpha(f) = Kf\beta$. Thus, $\alpha(\pi_i) = \pi'_i$ and so $\alpha(h) = h'$. This means $\alpha\phi_A|_X = \phi_{KA}|_X$, where ϕ_A and ϕ_{KA} are the homomorphisms existing by the freeness of FX (see Definition 4.2). Therefore, $\alpha\phi_A = \phi_{KA}$, and hence

$$\phi_{KA}(p) = \alpha\phi_A(p) = \alpha(p_A) = \alpha(q_A) = \alpha\phi_A(q) = \phi_{KA}(q)$$

as required.

As a corollary of the above remark, we have:

Lemma 4.4. *If Σ is a set of equations and $A \in mod(\Sigma, \mathcal{A})$ then $\overline{h}_U(A) \in mod\Sigma$, for all $U \in \mathcal{A}$, where $h_U = Hom_{\mathcal{A}}(U, -) : \mathcal{A} \to \mathbf{Set}$ is the hom-functor.*

For the converse of the above lemma, we need the following:

Theorem 4.5. *Let the finitely complete category $\mathcal{A}$ have a generating set $\mathcal{G}$. Then, for any $A \in Alg(\tau)\mathcal{A}$ and a set Σ of equations, $A \in mod(\Sigma, \mathcal{A})$ if and only if $\overline{h}_G(A) \in mod\Sigma$ for each $G \in \mathcal{G}$, where $h_G = Hom_{\mathcal{A}}(G, -)$.*

Proof. ($\Rightarrow$) follows from the above lemma. For the converse, let $A \in Alg(\tau)\mathcal{A}$ and $\overline{h}_G(A) \in mod\Sigma$ for each $G \in \mathcal{G}$. Let $\sigma = (p, q)$ be an equation in Σ. By hypothesis, for all $G \in \mathcal{G}$, $\overline{h}_G(A) \models \sigma$, that is, $p_{\overline{h}_G(A)} = q_{\overline{h}_G(A)}$. So, similar to Remark 4.3(2), we have $\phi_{h_G A}(p) = \phi_{h_G A}(q)$ and so $\alpha\phi_A(p) = \alpha\phi_A(q)$ which means $\alpha(p_A) = \alpha(q_A)$, and hence by the definition of α, $h_G(p_A) = h_G(q_A)$ for all $G \in \mathcal{G}$. Then, since $\mathcal{G}$ is a set of generators, the preceding equalities yield that $p_A = q_A$. Thus $A \models \sigma$, and hence $A \in mod(\Sigma, \mathcal{A})$. $\square$

Corollary 4.6. *For a Grothendieck topos $\mathcal{E} \hookrightarrow \hat{\mathcal{C}}$ and a set Σ of equations, $A \in mod(\Sigma, \mathcal{E})$ if and only if $AU \in mod\Sigma$ for all $U \in \mathcal{C}$.*

Proof. Let $\mathcal{E}$ be a Grothendieck topos, with the reflection functor $R : \mathbf{Set}^{\mathcal{C}^{op}} = \hat{\mathcal{C}} \to \mathcal{E}$. Then, the set $\{R(h_U) : U \in \mathcal{C}\}$ is a generating set for $\mathcal{E}$. So, by Theorem 4.5, $A \in mod(\Sigma, \mathcal{E})$ if and only if $\overline{h}_{R(h_U)}(A) \in mod\Sigma$ for each $U \in \mathcal{C}$. But, since R is a left adjoint to the inclusion functor $i : \mathcal{E} \to \hat{\mathcal{C}}$, using Yoneda Lemma we have $\overline{h}_{R(h_U)}(A) = Hom(R(h_U), A) \cong Hom(h_U, i(A) = Hom(h_U, A) \cong A(U)$, and so get the result. $\square$

Now a natural question to ask would be, what is the relationship between the behaviour of a certain classical algebraic notion in $mod\Sigma$ and in $mod(\Sigma, \mathcal{E})$. In the following, we consider the notions mentioned in the previous sections, and show that the properties of $mod\Sigma$, regarding these notions, survive the passage to $mod(\Sigma, \mathcal{E})$, for a set Σ of equations and an arbitrary Grothendieck topos $\mathcal{E}$ (fixed from now on). First, we state an adjoint pair between $mod(\Sigma, \mathcal{E})$ and $mod(\Sigma)$.

Adjoint situations

(1) **Free objects**.
First recall that the free functor $F_1 : \hat{\mathcal{C}} \to mod(\Sigma, \hat{\mathcal{C}})$ is given by $P \mapsto F_1 P$ where $(F_1 P)U$ is the free algebra on PU in $mod\Sigma$, for each $U \in \mathcal{C}$, and for $f : U \to V$, $(F_1 P)f$ is obtained using the universal property of free objects.

Now the free functor $F : \mathcal{E} \to mod(\Sigma, \mathcal{E})$ is defined to be the composition

$$\mathcal{E} \overset{i}{\to} \hat{\mathcal{C}} \overset{F_1}{\to} mod(\Sigma, \hat{\mathcal{C}}) \overset{\overline{R}}{\to} mod(\Sigma, \mathcal{E})$$

where i is the inclusion functor, $\overline{R}$ is the lifted functor of the reflection functor.

So, F is a left adjoint to the underlying functor $|-| : mod(\Sigma, \mathcal{E}) \to \mathcal{E}$.

(2) **Adjoint relation of $\hat{\mathcal{C}}$ and $\mathbf{Set}^{|\mathcal{C}|}$**.
Define the functor $G : \hat{\mathcal{C}} \to \mathbf{Set}^{|\mathcal{C}|}$, where $|\mathcal{C}|$ is a discrete category whose set of objects is the set of objects of the category C, by $GP = (PU)_{U \in \mathcal{C}}$, for $P \in \hat{\mathcal{C}}$, and for $f : P \to Q$, $Gf = (f_U)_{U \in \mathcal{C}}$. This functor has a left adjoint $H : \mathbf{Set}^{|\mathcal{C}|} \to \hat{\mathcal{C}}$ given by $HX = P_X$ for $X = (X_U)_{U \in \mathcal{C}}$ where

$$P_X U = \prod_{W \in \mathcal{C}} X_W^{Hom_{\mathcal{C}}(W,U)}$$

in which $X_W^{Hom_{\mathcal{C}}(W,U)}$ is the set of all functions from the set $Hom_{\mathcal{C}}(W,U)$ to the set X_W; and for $f : U \to V$, $P_X f : P_X U \to P_X V$ is the product map $\prod_{W \in \mathcal{C}} id^{Hom(id_W, f)}$ where $Hom(id_W, f) : Hom_{\mathcal{C}}(W, U) \to Hom_{\mathcal{C}}(W, V)$ maps $g : W \to U$ to $fg : W \to V$, and

$$id^{Hom(id_W, f)} : X_W^{Hom_{\mathcal{C}}(W,V)} \to X_W^{Hom_{\mathcal{C}}(W,U)}$$

maps $Hom_{\mathcal{C}}(W, V) \to X_W$ to $Hom_{\mathcal{C}}(W, U) \overset{Hom(id_W, f)}{\longrightarrow} Hom_{\mathcal{C}}(W, V) \to X_W$. The definition of the functor H on arrows is as follows: for a given family $f = (f_U)_{U \in \mathcal{C}} : X \to Y$ of maps $f_U : X_U \to Y_U$ we define $Hf : P_X \to P_Y$ by $Hf = ((Hf)_U)_{U \in \mathcal{C}}$ with $(Hf)_U = \prod_{W \in \mathcal{C}} f_W^{id}$, where $f_W^{id} : X_W^{Hom_{\mathcal{C}}(W,U)} \to Y_W^{Hom_{\mathcal{C}}(W,U)}$ maps $Hom_{\mathcal{C}}(W, U) \to X_W$ to $Hom_{\mathcal{C}}(W, U) \to X_W \overset{f_W}{\to} Y_W$.

(3) **Adjoint relation between** $mod(\Sigma, \hat{\mathcal{C}})$ **and** $(mod\Sigma)^{|\mathcal{C}|}$.
The above adjunction is lifted to the adjunction:

$$\overline{G} : mod(\Sigma, \hat{\mathcal{C}}) \rightleftarrows mod(\Sigma, \mathbf{Set}^{|\mathcal{C}|}) : \overline{H}$$

But, $mod(\Sigma, \mathbf{Set}^{|\mathcal{C}|}) \cong (mod\Sigma)^{|\mathcal{C}|}$. So, we get an adjoint pair between $mod(\Sigma, \hat{\mathcal{C}})$ and $(mod\Sigma)^{|\mathcal{C}|}$ which we denote them by the same notations $\overline{G}$, $\overline{F}$.

(4) **Adjoint relation between** $mod(\Sigma, \mathcal{E})$ **and** $mod\Sigma$.
Define a functor $\Gamma = Hom_{\mathcal{E}}(T, -) : mod(\Sigma, \mathcal{E}) \to mod\Sigma$ by $A \mapsto Hom_{\mathcal{E}}(T, A)$ where T is the terminal object of $\mathcal{E}$. Notice that $Hom_{\mathcal{E}}(T, A)$ is naturally a $mod\Sigma$-algebra, in fact for each operation $\lambda_A : A^n \to A$ on A, the corresponding operation on $Hom_{\mathcal{E}}(T, A)$ is given by $(Hom_{\mathcal{E}}(T, A))^n \cong Hom_{\mathcal{E}}(T, A^n) \stackrel{Hom(id, \lambda_A)}{\longrightarrow} Hom_{\mathcal{E}}(T, A)$. This functor has a left adjoint $\Delta : mod\Sigma \to mod(\Sigma, \mathcal{E})$ which is the composite:

$$mod\Sigma \stackrel{\Delta_0}{\to} (mod\Sigma)^{|\mathcal{C}|} \stackrel{\overline{H}}{\to} mod(\Sigma, \hat{\mathcal{C}}) \stackrel{\overline{R}}{\to} mod(\Sigma, \mathcal{E})$$

where Δ_0 is the constant functor which maps $A \in mod\Sigma$ to the constant family $(A)_{U \in \mathcal{C}}$.

Now, applying the above adjunction, we get the following results:

Proposition 4.7. *The category $Alg(\tau)\hat{\mathcal{C}}$ is isomorphic to the category of all $Alg(\tau)$-valued presheaves on $\mathcal{C}$.*

Proof. Consider the functor $G : \hat{\mathcal{C}} \to \mathbf{Set}^{|\mathcal{C}|}$ given above. This functor preserves finite limits, and so by Remark 4.3(1) can be lifted to a functor $\overline{G} : Alg(\tau)\hat{\mathcal{C}} \to Alg(\tau)\mathbf{Set}^{|\mathcal{C}|}$. Then, the composition of this functor together with the isomorphism $Alg(\tau)\mathbf{Set}^{|\mathcal{C}|} \cong Alg(\tau)^{|\mathcal{C}|}$ gives the desired isomorphism. $\qquad \square$

Lemma 4.8. *$mod\Sigma$ has a cogenerating set if and only if $mod(\Sigma, \hat{\mathcal{C}})$ has one such.*

Proof. If $mod\Sigma$ has a cogenerating set, then clearly so does $(mod\Sigma)^{|\mathcal{C}|} \cong mod(\Sigma, \mathbf{Set}^{|\mathcal{C}|})$. Now, since the functor $\overline{H} : mod(\Sigma, \mathbf{Set}^{|\mathcal{C}|}) \to mod(\Sigma, \hat{\mathcal{C}})$ transfers a cogenerating set to one such, we get that $mod(\Sigma, \hat{\mathcal{C}})$ has a cogenerating set.

For the converse, apply the functor Γ for the case where $\mathcal{E} = \hat{\mathcal{C}}$. Then noting that Γ transfers a cogenerating set of $mod(\Sigma, \hat{\mathcal{C}})$ to one such set in $mod\Sigma$ we get the result. □

Proposition 4.9. *$mod(\Sigma, \mathcal{E})$ has a cogenerating set if and only if $mod\Sigma$ has one such a set.*

Proof. The "if part" is proved similar to the corresponding part of the above lemma. To prove the "only if" part, first applying the above lemma we get that $mod(\Sigma, \hat{\mathcal{C}})$ has a cogenerating set. Then, $mod(\Sigma, \hat{\mathcal{C}})$ has a set of essential extensions, by Theorem 3.14. Secondly, we see that essential monomorphisms in $mod(\Sigma, \mathcal{E})$ are also essential monomorphisms in $mod(\Sigma, \hat{\mathcal{C}})$, since the reflection functor $\overline{R} : mod(\Sigma, \hat{\mathcal{C}}) \to mod(\Sigma, \mathcal{E})$ preserves monomorphisms. Therefore, $mod(\Sigma, \mathcal{E})$ has a set of essential extensions and hence has a cogenerating set, by Theorem 2.6 and Lemma 3.8; notice that $mod(\Sigma, \mathcal{E})$ has a generating set consisting of $mod(\Sigma, \mathcal{E})$-free algebras on the generating set $\{R(h_U) : U \in \mathcal{C}\}$ of $\mathcal{E}$, where $R : \mathbf{Set}^{\mathcal{C}^{op}} = \hat{\mathcal{C}} \to \mathcal{E}$ is the reflection fucntor. □

Corollary 4.10. *$mod(\Sigma, \mathcal{E})$ is residually small if and only if $mod\Sigma$ is residually small.*

Proof. This follows from the above proposition, Lemma 3.8, and Theorem 3.3. □

Corollary 4.11. *$mod(\Sigma, \mathcal{E})$ is essentially bounded if and only if $mod\Sigma$ is essentially bounded.*

Proof. Apply the above corollary, and the fact that, by Theorems 2.6 and 3.14, essential boundedness for $mod(\Sigma, \mathcal{E})$ is equivalent to residual smallness. □

Corollary 4.12. *$mod(\Sigma, \mathcal{E})$ is $\mathcal{M}^*$-cowell powered if and only if $mod\Sigma$ is $\mathcal{M}^* - cowellpowered$.*

Proof. The above corollary and Theorem 2.3 give the result. □

Now, applying theorems given in the previous sections, we consider the well-behaviour theorems in $mod(\Sigma, \mathcal{E})$.

Lemma 4.13. *In $mod(\Sigma, \mathcal{E})$, we have*

(i) *any composite of essential monomorphisms is an essential monomorphism, and*

(ii) *any direct limit of essential monomorphisms is an essential monomorphism.*

Proof. (i) is clear. To prove (ii), first we note that monomorphisms in $\hat{\mathcal{C}}$ are exactly of the form $f = (f_U)_{U \in \mathcal{C}}$ where each f_U is a monomorphism in **Set**. Also, notice that the reflection functor $R : \hat{\mathcal{C}} \to \mathcal{E}$ preserves monomorphisms. These facts, gives the result. □

Theorem 4.14. *Banaschewski's condition holds in $mod(\Sigma, \mathcal{E})$.*

Proof. Let $h : A \to B$ be a monomorphism in $mod(\Sigma, \mathcal{E})$. Take all the congruences Θ on B such that $B/\Theta \in mod(\Sigma, \mathcal{E})$ and $A \to B \to B/\Theta$ is a monomorphism. Then, by the note given in the proof of the above lemma, any join of a chain of such congruences on B is again such a congruence. So, by Zorn's Lemma, there exists a maximal such congruence, namely Θ_0. Then $A \to B/\Theta_0$ is an essential monomorphism. □

Again applying an adjunction we get:

Lemma 4.15. *Pushouts transfer monomorphisms in $mod(\Sigma, \mathcal{E})$ if and only if they do in $mod\Sigma$.*

Proof. Applying the adjunction $\Delta \dashv \Gamma$, and using the fact that Δ is faithful, we get the implication ($\Rightarrow$). To get the converse, a discussion similar to the proof of Lemma 4.13 is applied. □

To prove Theorem 4.18, we should recall a theorem about the existence of injective hulls.

Theorem 4.16. [30] *For $\mathcal{M} = \mathcal{M}ono$, let $\mathcal{A}$ be $\mathcal{M}$-well powered and fulfill the $\mathcal{M}$-chain condition. Then, $\mathcal{A}$ has $\mathcal{M}$-injective hulls if and only if*

(i) $\mathcal{A}$ *is $\mathcal{M}^*$-cowell powered.*

(ii) $\mathcal{A}$ *satisfies the $\mathcal{M}$-transferability condition.*

(iii) $\mathcal{A}$ *satisfies Banaschewski's condition.*

Theorem 4.17. *The category $mod(\Sigma, \mathcal{E})$ has enough injectives if and only if it is essentially bounded and pushouts transfer monomorphisms.*

Proof. If $mod(\Sigma, \mathcal{E})$ is essentially bounded and pushouts transfer monomorphisms then, by Theorem 4.16 (applying Theorems 2.3 and 4.14), we get the result. For the converse, by Lemma 2.10 and Note 2.9, we have pushouts transfer monomorphisms in $mod(\Sigma, \mathcal{E})$. To show that this category is essentially bounded, applying Theorem 2.3 we show that it has a set of essential extensions. But, having enough injectives we have that all the essential extensions of $A \in mod(\Sigma, \mathcal{E})$ can be embedded in any injective extension of A, and so form a set. $\qquad\square$

Proposition 4.18. [11] *The category $mod(\Sigma, \mathcal{E})$) has enough injectives if and only if $mod\Sigma$ has enough injectives.*

Proof. It follows from the above theorem, Lemma 4.15, and Corollary 4.11. $\qquad\square$

Finally as a corollary of the above results we have

Theorem 4.19. *For $mod(\Sigma, \mathcal{E})$, the following are equivalent:*
   (i) *Injectivity is well-behaved.*
   (ii) *$mod(\Sigma, \mathcal{E})$ has enough injectives.*
   (iii) *$mod(\Sigma, \mathcal{E})$ is essentially bounded and pushouts transfer monomorphisms.*

Proof. (ii) and (iii) are equivalent by the above proposition. Also, (i) clearly implies (ii). To see (iii)$\Rightarrow$(i), it is enough to check the collection of the sufficient conditions for the Well Behaviour of injectivity, mentioned after Proposition 3.11. But Lemma 4.13 (ii), Theorem 3.15, and $\mathcal{M} = \mathcal{M}ono$, give the result. $\qquad\square$

In particular, the above results give that:

Corollary 4.20. *Injectivity is well-behaved in $mod(\Sigma, \mathcal{E})$ if and only if it is well-behaved in $mod\Sigma$.*

5   Some Examples

In this section we recall the behaviour of $\mathcal{M}$-injectivity for some classes of $\mathcal{M}$ of monomorphisms in some categories to support the results mentioned

above. No proof is given in this section. For more information see, for example, [17–20, 22, 25, 27].

Example (I)

In this set of examples, $\mathcal{M}$ is taken to be the class of all monomorphisms.

(1) The Baer Criterion for modules over a commutative ring R with identity says that $\mathcal{M}$-injectivity is equivalent to $\mathcal{M}_1$-injectivity, where $\mathcal{M}_1$ is the set of all monomorphisms from ideals of R to R. Also, recall that the category of R-modules has enough injectives (hulls) and hence injectivity well-behaves. (See also, for example, [2]).

(2) Recall that the category **Set** has enough injectives. In fact every non-empty set is injective. So, if we take Σ to be the empty set, then $mod\Sigma$, as the full subcategory of all algebras of type $\tau = \emptyset$, is **Set** and hence has enough injectives. Using Proposition 4.18, this implies that for any Grothendieck topos $\mathcal{E}$, $mod(\Sigma, \mathcal{E}) = \mathcal{E}$ has enough injectives. In particular, the category **MSet**, of M-sets for a monoid M, has enough injectives. Thus injectivity is well-behaved in such categories. The direct proof of this fact is given in [6]. For more information of the case where M is the monoid $(\mathbb{N}^\infty, \min, \infty)$, whose acts are called *projection algebras* see also [6, 15].

(3) For the category of acts over a monoid, $\mathcal{M}$-injectivity is the same as $\mathcal{M}_1$-injectivity, for $\mathcal{M}_1$ to be the class of all monomorphisms to cyclic acts and $\mathcal{M}$ to be the class of all monomorphisms (see also [22]).

(4) Unlike the category of modules over commutative rings with identity (see (1) above), Baer Criterion does not generally hold for M-sets (see [22]). But, for some classes of monoids M it holds, and so $\mathcal{M}$-injectivity is equivalent to $\mathcal{M}_1$-injectivity, where $\mathcal{M}_1$ is the set of all monomorphisms from ideals of M to M. Some classes of monoids such that the Baer Criterion holds for acts over them are given in [19], also see [17].

(5) The category **Boo** of Boolean algebras has enough injectives (the power set of every set is injective in **Boo**). So, the category of Boolean algebras in any Grothendieck topos has enough injectives.

(6) The category **Ab** of abelian groups has enough injectives (recall that here injectivity means divisibility). So, the category of abelian groups in any Grothendieck topos, in particular in **ShX**, has enough injectives.

Example (II)

In this example, $\mathcal{M}$ is taken to be the class of *sequentially dense* monomorphisms in the category **Act-S** of S-acts for a semigroup S. For the details of the results, see also [20, 26, 27].

The *sequential closure* $C = (C_B)_{B \in \mathbf{Act-S}}$ on **Act-S** is defined as

$$C_B(A) = \{b \in B : bS \subseteq A\}$$

for any subact A of an S-act B.

It is easily shown that C is a closure operator on **Act-S** in the sense of [9], which means: $C_B(A)$ is a subact of B, and (i) $A \leq C_B(A)$, (ii) $A_1 \leq A_2 \leq B$ implies $C_B(A_1) \leq C_B(A_2)$, (iii) for every S-map $f : B \to D, f(C_B(A)) \subseteq C_D(f(A))$ for each subact A of B.

Notice that the above closure operator also satisfies $C_B(A \cap A') = C_B(A) \cap C_B(A')$ for $A, A' \leq B$, and for $A \leq B \leq D$, $C_B(A) = C_D(A) \cap B$. Also, if $S^2 = S$ then C is idempotent.

We say that A is *C-dense* in B if $C_B(A) = B$. An S-map $f : A \to B$ is said to be C-dense if $f(A)$ is a C-dense subact of B.

Now, taking $\mathcal{M}_d$ to be the class of all C-dense monomorphisms **Act-S**, we have:

Lemma 5.1. (1)   *The class $\mathcal{M}_d$ is: isomorphism closed, closed under composition with isomorphisms, left regular, left cancellable, almost right cancellable (in the sense that $gf \in \mathcal{M}_d$ implies $g \in \mathcal{M}_d$ provided g is a monomorphism), closed under direct limits. And, the category **Act-S** satisfies the $\mathcal{M}_d$-transferability property, fulfills Banaschewski's $\mathcal{M}_d$-condition, has $\mathcal{M}_d$-bounds, fulfills the $\mathcal{M}_d$-chain condition, and is $\mathcal{M}_d$-essentially bounded.*

(2)   *The class $\mathcal{M}_d$ is closed under composition if and only if $S^2 = S$.*

Theorem 5.2. *For an S-act A, the following are equivalent:*

(1) *A is $\mathcal{M}_d$-injective.*

(2) *For every C-dense monomorphism $h : B \to cS^1$ to a cyclic act and every S-map $f : B \to A$ there exists an S-map $g : cS^1 \to A$ such that $gh = f$.*

(3) *Every S-map $f : cS \to A$ from a cyclic act can be extended to $\overline{f} : cS^1 \to A$.*

(4) *For every C-dense monomorphism $h : B \to B \sqcup cS^1$ to a singly generated extension of B and every S-map $f : B \to A$ there exists an S-map $g : B \sqcup cS^1 \to A$ such that $gh = f$.*

Now we mention what is true about the Well-Behaviour Theorems. The First Theorem is always true:

Theorem 5.3. (The First Theorem of Well-Behaviour) *For a semigroup S and every S-act A, the following are equivalent:*
   (i) *A is $\mathcal{M}_d$-injective.*
   (ii) *A is $\mathcal{M}_d$-absolute retract.*
   (iii) *A has no proper $\mathcal{M}_d$-essential extension.*

The Second and The Third Theorems are not true in general, but we have:

Theorem 5.4. (The Second Theorem of Well-Behaviour) *If $S^2 = S$ then for each S-act A, the $\mathcal{M}_d$-injective hull of A exists (and it is unique up to isomorphism).*

Theorem 5.5. (The Third Theorem of Well-Behaviour) *If $S^2 = S$ then for every extension B of an S-act A, the following are equivalent:*
   (i) *B is the $\mathcal{M}_d$-injective hull of A.*
   (ii) *B is a maximal $\mathcal{M}_d$-essential extension of A.*
   (iii) *B is a minimal $\mathcal{M}_d$-injective C-dense extension of A.*

Notice that for the cases where S is the monoid $(\mathbb{N}, \infty)$, or S is a left zero semigroup, we have $S^2 = S$ and so $\mathcal{M}_d$-injectivity is well-behaved for projection algebras (see [15] and [17]) and for acts over a left zero semigroup (see [18]).

The following results show that $S^2 = S$ is not a necessary condition for the well-behaviour of $\mathcal{M}_d$-injectivity:

Theorem 5.6. *If $(Id_r(S), \cap, \cup)$ is a Boolean algebra or S is a null semigroup, then* **S-Act** *has $\mathcal{M}_d$-injective hulls.*

But, also we have:

Theorem 5.7. *If S is a right cancellative semigroup and for each S-act A, the $\mathcal{M}_d$-injective hull of A exists then $S^2 = S$.*

Example (III)

In this example, $\mathcal{M}$ is taken to be the class of sequentially (or s)-pure monomorphisms in the category of S-acts for a semigroup S. What we have so far is given in the following. (see also [4, 5]).

Definition 5.8. Let A be a subact of an S-act B. Then A is said to be *sequentially pure*, or *s-pure*, in B if every *sequential system of equations* $xs = a_s$, $s \in S$ over A is solvable in A whenever it is solvable in B.

A homomorphism $f : A \to B$ is called *s-pure* if $f(A)$ is *s-pure* in B.

Proposition 5.9. *The class $\mathcal{M}_p$ of s-pure monomorphisms is: isomorphism closed, closed under composition, left regular, left cancellable, and* **Act-S** *has $\mathcal{M}_p$-transferability property, Banaschewski condition is true for weak s-pure essentialness (see the following definition), but does not satisfy the $\mathcal{M}_p$-chain condition.*

Definition 5.10. An s-pure monomorphism $f : A \to B$ of S-acts is said to be:

(1) *essential s-pure* if for every homomorphism $g : B \to C$ such that gf is a monomorphism we have g itself is also a monomorphism.

(2) *weak s-pure essential* if for every homomorphism $g : B \to C$ such that gf is an s-pure monomorphism we have g is a monomorphism.

(3) *s-pure essential* if for every homomorphism $g : B \to C$ such that gf is an s-pure monomorphism we have g is also an s-pure monomorphism.

Theorem 5.11. (The First Theorem of Well-Behaviour) *For a finitely generated semigroup S and an S-act A, the following are equivalent:*

(1) *A is s-pure injective.*

(2) *A is retract of each of its s-pure extension.*

(3) *A has no proper weak s-pure essential extension.*

Lemma 5.12. *If S is a finitely generated semigroup, and B is a maximal weak s-pure essential extension of A, then B is s-pure injective.*

Theorem 5.13. *If S is a finitely generated semigroup and B is a maximal weak s-pure essential extension of A, then it is an s-pure injective hull of A.*

Theorem 5.14. *If each S-act A has a set of weak s-pure essential extensions, then every S-act has a maximal weak s-pure essential extension.*

Corollary 5.15. (The Second Theorem of Well-Behaviour) *If S is a finitely generated semigroup, then each S-act has an s-pure injective hull.*

The converse of the former theorem is also true provided that *s*-pure injective hull exists:

For The Third Theorem, we have:

Theorem 5.16. (1) *If S is a finitely generated semigroup, and each S-act A has a set of weak s-pure essential extensions, then every s-pure injective hull is a maximal weak s-pure essential extension.*

(2) *The s-pure injective hull is unique up to isomorphism.*

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