Dually quasi-De Morgan Stone semi-Heyting algebras II. Regularity

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Abstract. This paper is the second of a two part series. In this Part, we prove, using the description of simples obtained in Part I, that the variety **RDQDStSH₁** of regular dually quasi-De Morgan Stone semi-Heyting algebras of level 1 is the join of the variety generated by the twenty 3-element **RDQDStSH₁**-chains and the variety of dually quasi-De Morgan Boolean semi-Heyting algebras–the latter is known to be generated by the expansions of the three 4-element Boolean semi-Heyting algebras. As consequences of our main theorem, we present (equational) axiomatizations for several subvarieties of **RDQDStSH₁**. The paper concludes with some open problems for further investigation.

1   Introduction

This paper is the second of a two part series. In this Part, we prove, using the description of simples obtained in Part I, that the variety **RDQDStSH₁** of regular dually quasi-De Morgan Stone semi-Heyting algebras of level 1 is the join of the variety generated by the twenty 3-element **RDQDStSH₁**-chains and the variety of dually quasi-De Morgan Boolean semi-Heyting

algebras–the latter is known to be generated by the expansions of the three 4-element Boolean semi-Heyting algebras. Furthermore, as consequences of this theorem, we present (equational) axiomatizations for several subvarieties of **RDQDStSH$_1$**. The paper concludes with some open problems for further investigation.

2   Preliminaries

In this section we recall notations and results from Part I in order to make this paper self-contained.

An algebra $\mathbf{L} = \langle L, \vee, \wedge, \rightarrow, ', 0, 1 \rangle$ is a *dually quasi-De Morgan semi-Heyting algebra* (**DQDSH**-algebra, for short) if $\langle L, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a semi-Heyting algebra, and **L** satisfies:

(a) $0' \approx 1$ and $1' \approx 0$

(b) $(x \wedge y)' \approx x' \vee y'$

(c) $(x \vee y)'' \approx x'' \vee y''$

(d) $x'' \leq x$.

Let **L** be a **DQDSH**-algebra. Then **L** is of level 1 (**DQDSH$_1$**-algebra) if **L** satisfies:

(L1) $x \wedge x'^* \approx (x \wedge x'^*)'^*$ (Level 1).

L is a *dually pseudocomplemented semi-Heyting algebra* (**DPCSH**-algebra) if **L** satisfies:

(e) $x \vee x' \approx 1$.

L is a *De Morgan semi-Heyting algebra* (**DMSH**-algebra) if **L** satisfies:

(DM) $x'' \approx x$.

L *is regular* if **L** satisfies:

(M) $x \wedge x^+ \leq y \vee y^*$, where $x^+ := x'^{*'}$.

L is a *dually quasi-De Morgan Stone semi-Heyting algebra* (**DQDStSH**-algebra) if **L** satisfies:

(St) $x^* \vee x^{**} \approx 1$, where $x^* := x \rightarrow 0$.

L is a *dually quasi-De Morgan Boolean semi-Heyting algebra*
(**DQDBSH**-algebra) if **L** satisfies:

(Bo) $x \vee x^* \approx 1$ where $x^* := x \rightarrow 0$.

L is a *strongly blended dually quasi-De Morgan semi-Heyting
algebra* (**SBDQDSH**) if **L** satisfies:

(SB) $(x \vee y^*)' \approx x' \wedge y^{*'}$ (**Strongly Blended $\vee$-De Morgan law**).

The variety of **DQDSH$_1$**-algebras is denoted by **DQDSH$_1$**, and similar notation applies to other varieties. **DQDStSH$_1$** denotes the subvariety of level 1 of **DQDSH$_1$** defined by (St), and **DQDBSH** denotes the one defined by (Bo), while **RDQDSH$_1$** denotes the variety of regular **DQDSH$_1$**-algebras and **RSBDQDSH$_1$** denotes that of regular, strongly blended **DQDSH$_1$**-algebras, and so on.

If the underlying semi-Heyting algebra is a Heyting algebra, then we replace the part "**SH**" by "**H**" in the names of the varieties that we consider in this sequel.

Let $\mathbf{2^e}$ and $\mathbf{\bar{2}^e}$ be the expansions of the semi-Heyting algebras **2** and $\mathbf{\bar{2}}$ (shown in Figure 1) by adding the unary operation $'$ such that $0' = 1$, $1' = 0$.

Let $\mathbf{L}_i^{dp}$, $i = 1, \ldots, 10$, denote the expansion of the semi-Heyting algebra $\mathbf{L}_i$ (shown in Figure 1) by adding the unary operation $'$ such that $0' = 1$, $1' = 0$, and $a' = 1$.

Let $\mathbf{L}_i^{dm}$, $i = 1, \ldots, 10$, denote the expansion of $\mathbf{L}_i$ by adding the unary operation $'$ such that $0' = 1$, $1' = 0$, and $a' = a$. We Let $\mathbf{C_{10}^{dp}} := \{\mathbf{L}_i^{dp} : i = 1, \ldots, 10\}$ and $\mathbf{C_{10}^{dm}} := \{\mathbf{L}_i^{dm} : i = 1, \ldots, 10\}$. We also let $\mathbf{C_{20}} := \mathbf{C_{10}^{dm}} \cup \mathbf{C_{10}^{dp}}$.

Each of the three 4-element algebras $\mathbf{D_1}$, $\mathbf{D_2}$ and $\mathbf{D_3}$ has its lattice reduct as the Boolean lattice with the universe $\{0, a, b, 1\}$, b being the complement of a, has the operation $\rightarrow$ as defined in Figure 1, and has the unary operation $'$ defined as follows: $a' = a$, $b' = b$, $0' = 1$, $1' = 0$.

It was shown in [10] that $\mathbf{V(D_1, D_2, D_3)} = \mathbf{DQDBSH}$.

2 :

$\rightarrow$	0	1
0	1	1
1	0	1

2̄ :

$\rightarrow$	0	1
0	1	0
1	0	1

$\mathbf{L_1}$:

$\rightarrow$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

$\mathbf{L_2}$:

$\rightarrow$	0	a	1
0	1	a	1
a	0	1	1
1	0	a	1

$\mathbf{L_3}$:

$\rightarrow$	0	a	1
0	1	1	1
a	0	1	a
1	0	a	1

$\mathbf{L_4}$:

$\rightarrow$	0	a	1
0	1	a	1
a	0	1	a
1	0	a	1

$\mathbf{L_5}$:

$\rightarrow$	0	a	1
0	1	a	a
a	0	1	1
1	0	a	1

$\mathbf{L_6}$:

$\rightarrow$	0	a	1
0	1	1	a
a	0	1	1
1	0	a	1

$\mathbf{L_7}$:

$\rightarrow$	0	a	1
0	1	a	a
a	0	1	a
1	0	a	1

$\mathbf{L_8}$:

$\rightarrow$	0	a	1
0	1	1	a
a	0	1	a
1	0	a	1

$\mathbf{L_9}$:

$\rightarrow$	0	a	1
0	1	0	0
a	0	1	1
1	0	a	1

$\mathbf{L_{10}}$:

$\rightarrow$	0	a	1
0	1	0	0
a	0	1	a
1	0	a	1

$\rightarrow$	0	1	a	b
0	1	0	b	a
1	0	1	a	b
a	b	a	1	0
b	a	b	0	1

$\mathbf{D_1}$:

$\rightarrow$	0	1	a	b
0	1	1	1	1
1	0	1	a	b
a	b	1	1	b
b	a	1	a	1

$\mathbf{D_2}$:

$\rightarrow$	0	1	a	b
0	1	a	1	a
1	0	1	a	b
a	b	a	1	0
b	a	1	a	1

$\mathbf{D_3}$:

Figure 1

The following two results proved in Part I.

Theorem 2.1. *Let* $\mathbf{L} \in \mathbf{RDQDStSH_1}$ *with* $|L| \geq 2$. *Then the following are equivalent:*

(1) $\mathbf{L}$ *is simple*

(2) $\mathbf{L}$ *is subdirectly irreducible*

(3) *For every* $x \in L$, *if* $x \neq 1$, *then* $x \wedge x'^* = 0$

(4) $\mathbf{L} \in \{2^e, \bar{2}^e\} \cup \mathbf{C_{20}} \cup \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$, *up to isomorphism.*

Theorem 2.2. $\mathbf{DQDStSH_1} = \mathbf{SBDQDStSH_1}$.

3   Main result and consequences

Let $\mathbf{V}(\mathbf{K})$ denote the variety generated by the class $\mathbf{K}$ of algebras. Recall $\mathbf{V}(\mathbf{C_{20}})$ is the variety of $\mathbf{DQDSH}$-algebras genrerated by the twenty 3-element algebras mentioned earlier. Let $\mathbf{D}$ denote the variety generated by the three 4-element $\mathbf{DQDSH}$-algebras whose semi-Heyting reducts are $\mathbf{D_1}$,

$\mathbf{D_2}$ and $\mathbf{D_3}$ given in Figure 1. The variety $\mathbf{V}(\mathbf{C_{20}})$ was axiomatized in [10] and also it was shown there that $\mathbf{D} = \mathbf{DQDBSH}$.

We are now ready to give the main result of this paper.

Theorem 3.1. *We have*

$$
\begin{aligned}
\mathbf{RDQDStSH_1} &= \mathbf{V}((\mathbf{C_{20}}) \cup \{\mathbf{D1}, \mathbf{D2}, \mathbf{D3}\}) \\
&= \mathbf{V}(\mathbf{C_{20}}) \vee \mathbf{D} \\
&= \mathbf{RSBDQDStSH_1} \\
&= \mathbf{RDmsStSH_1}.
\end{aligned}
$$

Proof. Since $\mathbf{2^e}$ and $\mathbf{\bar{2}^e}$ are subalgebras of some of the other simple algebras listed in Theorem 2.1, the first equation is immediate from Theorem 2.1, using well known results from universal algebra (see [2]). The second equation follows from the first, using the definition of the join of two varieties, and the third equation is immediate from Theorem 2.2. To prove the last equation, it suffices to verify that all 25 simple algebras in $\mathbf{RDQDStSH_1}$ satisfy the identity: $(x \vee y)' \approx x' \wedge y'$. □

If we restrict the underlying semi-Heyting algebras to Heyting algebras, Theorem 3.1 reduces to the following

Corollary 3.2. *Let $\mathbf{RDQDStH_1}$ denote the subvariety of $\mathbf{RDQDStSH_1}$ defined by: $(x \wedge y) \to x \approx 1$. Then*

$$
\begin{aligned}
\mathbf{RDQDStH_1} &= \mathbf{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{D_2}\}) \\
&= \mathbf{V}(\mathbf{L_1^{dm}}) \vee \mathbf{V}(\mathbf{L_1^{dp}}) \vee \mathbf{V}(\mathbf{D_2}) \\
&= \mathbf{RSBDQDStH_1} \\
&= \mathbf{RDmsStH_1}.
\end{aligned}
$$

Proof. Verify that $\mathbf{2^e}, \mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{D_2}$ satisfy the identity: $(x \wedge y) \to x \approx 1$, while the remaining simple algebras among the twenty five listed in Theorem 2.1 do not. Also, observe that $\mathbf{2^e}$ is a subalgebra of $\mathbf{L_1^{dm}}$ (or $\mathbf{L_1^{dp}}$). Then the corollary follows from Theorem 3.1. □

The following corollaries, which give (equational) bases to several subvarieties of $\mathbf{RDQDStSH_1}$, can also be similarly deduced from Theorem 2.1 and Theorem 3.1.

In these corollaries the reader should interpret "defined by" as "defined, modulo **RDQDStSH₁**, by".

Corollary 3.3. RDQDStH₁ *is also defined by the "linearity" identity:*

$$(x \to y) \vee (y \to x) \approx 1.$$

It is also defined by:

$$x \vee (x \to y) = (x \to y)^* \to x.$$

Corollary 3.4. *We have*

(a) $\mathbf{RDMStSH_1} = \mathbf{V}(\mathbf{C_{10}^{dm}}) \vee \mathbf{D}$

(b) $\mathbf{RDPCStSH_1} = \mathbf{V}(\mathbf{C_{10}^{dp}})$.

(c) $\mathbf{RDMStH_1} = \mathbf{V}(\{\mathbf{L_1^{dm}}, \mathbf{D_2}\}) = \mathbf{V}(\mathbf{L_1^{dm}}) \vee \mathbf{V}(\mathbf{D_2})$

(d) $\mathbf{RDPCStH_1} = \mathbf{V}(\mathbf{L_1^{dp}})$.

Corollary 3.5. *Let* **RDQDcmStSH₁** *be the subvariety of* **RDQDStSH₁** *defined by the commutative law:* $x \to y \approx y \to x$. *Then*

(a) $\mathbf{RDQDcmStSH_1} = \mathbf{V}(\{\mathbf{L_{10}^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{D_1}\})$
$\qquad\qquad = \mathbf{V}(\mathbf{L_{10}^{dm}}) \vee \mathbf{V}(\mathbf{L_{10}^{dp}}) \vee \mathbf{V}(\mathbf{D_1})$

(b) $\mathbf{RDMcmStSH_1} = \mathbf{V}(\{\mathbf{L_{10}^{dm}}, \mathbf{D_1}\})$

(c) $\mathbf{RDPCcmStSH_1} = \mathbf{V}(\mathbf{L_{10}^{dp}})$

(d) $\mathbf{RDMcmStSH_1} \cap \mathbf{RDPCcmStSH_1} = \mathbf{V}(\mathbf{\bar{2}^e})$.

Corollary 3.6. *The variety* $\mathbf{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_3^{dm}}, \mathbf{L_3^{dp}}, \mathbf{D_2}\})$ *is defined by the identity:*

$$(x \to y) \to (0 \to y) \approx (x \to y) \to 1.$$

The variety generated by $\mathbf{D_1}$ was axiomatized in [10]. Here are two more bases for it.

Corollary 3.7. $\mathbf{V}(\mathbf{D_1})$ *is defined by*

$$x \rightarrow (y \rightarrow z) \approx z \rightarrow (x \rightarrow y).$$

It is also defined by

$$(x \rightarrow y) \rightarrow (u \rightarrow w) \approx (x \rightarrow u) \rightarrow (y \rightarrow w) \quad \text{(Medial Law)}.$$

Corollary 3.8. *The variety* $\mathbf{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{D_2}\})$ *is defined by:*

$$y \leq x \rightarrow y.$$

It is also defined by:

$$[(x \rightarrow y) \rightarrow y] \rightarrow (x \rightarrow y) \approx x \rightarrow y.$$

It is also defined by

$$x \rightarrow (y \rightarrow z) \approx (x \rightarrow y) \rightarrow (x \rightarrow z) \quad \text{(Left distributive law)}.$$

Corollary 3.9. *The variety*
$\mathbf{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{D_2}\})$ *is defined by:*

$$[x \rightarrow (y \rightarrow x)] \rightarrow x \approx x.$$

Corollary 3.10. $\mathbf{V}(\{\mathbf{L_1^{dp}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dp}}\})$ *is defined by:*

(1) $[x \rightarrow (y \rightarrow x)] \rightarrow x \approx x$

(2) $x \vee x' \approx 1.$

Corollary 3.11. $\mathbf{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{D_2}\})$ *is defined by:*

(1) $[x \rightarrow (y \rightarrow x)] \rightarrow x \approx x$

(2) $x'' \approx x.$

Corollary 3.12. *The variety*
$\mathbf{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_3^{dm}}, \mathbf{L_3^{dp}}, \mathbf{L_4^{dm}}, \mathbf{L_4^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}},$
$\mathbf{D_2}, \mathbf{D_3}\})$ *is defined by the identity:*

$$(0 \rightarrow 1)^+ \rightarrow (0 \rightarrow 1)' \approx 0 \rightarrow 1.$$

Corollary 3.13. *The variety* $\mathbf{V}(\{\mathbf{L_1^{dp}}, \mathbf{L_2^{dp}}, \mathbf{L_3^{dp}}, \mathbf{L_4^{dp}}\})$ *is defined by the identities:*

(1) $(0 \to 1)^+ \to (0 \to 1)' \approx 0 \to 1$

(2) $x \vee x' \approx 1$.

Corollary 3.14. *The variety*
$\mathbf{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_2}, \mathbf{D_3}\})$ *is defined by the identities:*

(1) $(0 \to 1)^+ \to (0 \to 1)' \approx 0 \to 1$

(2) $x'' \approx x$.

Corollary 3.15. *The variety* $\mathbf{V}(\{\mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_3}\})$ *is defined by the identity:*

$$(0 \to 1)^+ \to (0 \to 1) \approx (0 \to 1)'.$$

$\mathbf{V}(\mathbf{D_3})$ was axiomatized in [10]. Here is another base for it.

Corollary 3.16. $\mathbf{V}(\mathbf{D_3})$ *is defined by the identities:*

(1) $(0 \to 1)^+ \to (0 \to 1) \approx (0 \to 1)'$

(2) $x \vee x^* \approx 1$.

Corollary 3.17. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}$, $\mathbf{D_2}, \mathbf{D_3}$ *is defined by the identities:*

(1) $(0 \to 1)^+ \to (0 \to 1)' \approx (0 \to 1)$

(2) $(0 \to 1)^+ \to (0 \to 1)^{*'*} \approx (0 \to 1)$

(3) $x'' \approx x$.

Corollary 3.18. *The variety generated by the algebras*
$\mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{L_7^{dm}}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{L_9^{dp}}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{L_{10}^{dm}}$,
$\mathbf{D_1}, \mathbf{D_3}$ *is defined by the identity:*

$$(0 \to 1)^+ \to (0 \to 1)' \approx (0 \to 1)'.$$

Corollary 3.19. *The variety generated by the algebras*
$\mathbf{L_5^{dp}}, \mathbf{L_6^{dp}}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dp}}, \mathbf{L_9^{dp}}, \mathbf{L_{10}^{dp}}$ *is defined by the identities:*

(1) $(0 \rightarrow 1)^+ \rightarrow (0 \rightarrow 1)' \approx (0 \rightarrow 1)'$

(2) $x \vee x' \approx 1$.

Corollary 3.20. *The variety generated by the algebras*
$\mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}}, \mathbf{D_1}, \mathbf{D_3}$ *is defined by the identities:*

(1) $(0 \rightarrow 1)^+ \rightarrow (0 \rightarrow 1)' \approx (0 \rightarrow 1)'$

(2) $x'' \approx x$.

Corollary 3.21. *The variety generated by the algebras*
$\mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_3}$ *is defined by the identities:*

(1) $(0 \rightarrow 1)^+ \rightarrow (0 \rightarrow 1)' \approx (0 \rightarrow 1)'$

(2) $(0 \rightarrow 1)^+ \rightarrow (0 \rightarrow 1)' \approx (0 \rightarrow 1)$.

It is also defined by

$$(0 \rightarrow 1)' \approx 0 \rightarrow 1.$$

Corollary 3.22. *The variety generated by the algebras*
$\mathbf{D_1}, \mathbf{D_3}$ *is defined by the identities:*

(1) $(0 \rightarrow 1)^+ \rightarrow (0 \rightarrow 1)' \approx (0 \rightarrow 1)'$

(2) $x \vee x^* \approx 1$.

Corollary 3.23. *The variety generated by the algebras*
$\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_3^{dm}}, \mathbf{L_3^{dp}}, \mathbf{L_4^{dm}}, \mathbf{L_4^{dp}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dp}}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dp}},$
$\mathbf{L_9^{dm}}, \mathbf{L_9^{dp}}, \mathbf{L_{10}^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{D_1}, \mathbf{D_2}$ *is defined by the identity:*

$$(0 \rightarrow 1)' \rightarrow (0 \rightarrow 1) \approx 0 \rightarrow 1.$$

Corollary 3.24. *The variety generated by the algebras*
$\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}}, \mathbf{D_1}, \mathbf{D_2}$ *is defined by the identities:*

(1) $(0 \rightarrow 1)' \rightarrow (0 \rightarrow 1) \approx 0 \rightarrow 1$

(2) $x'' \approx x$.

Corollary 3.25. *The variety generated by the algebras*
$\mathbf{D_1}, \mathbf{D_2}$ *is defined by the identities:*

(1) $(0 \to 1)' \to (0 \to 1) \approx 0 \to 1$

(2) $x \vee x^* \approx 1$.

Corollary 3.26. *The variety generated by the algebras* $\mathbf{L}_1^{\mathbf{dm}}, \mathbf{L}_1^{\mathbf{dp}}, \mathbf{L}_3^{\mathbf{dm}}, \mathbf{L}_3^{\mathbf{dp}}, \mathbf{L}_6^{\mathbf{dm}}, \mathbf{L}_6^{\mathbf{dp}}, \mathbf{L}_8^{\mathbf{dm}}, \mathbf{L}_8^{\mathbf{dp}}, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3$ *is defined by the identity:*

$$x \vee (y \to (x \vee y)) \approx (0 \to x) \vee (x \vee y).$$

Corollary 3.27. *The variety generated by the algebras* $\mathbf{L}_2^{\mathbf{dm}}, \mathbf{L}_2^{\mathbf{dp}}, \mathbf{L}_5^{\mathbf{dm}}, \mathbf{L}_5^{\mathbf{dp}}, \mathbf{D}_2$ *is defined by the identity:*

$$x \vee (y \to x) \approx [(x \to y) \to y] \to x.$$

Corollary 3.28. *The variety generated by the algebras* $\mathbf{L}_3^{\mathbf{dm}}, \mathbf{L}_3^{\mathbf{dp}}, \mathbf{L}_4^{\mathbf{dm}}, \mathbf{L}_4^{\mathbf{dp}}, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3$ *is defined by the identity:*

$$x \vee (x \to y) \approx x \to (x \vee (y \to 1)).$$

Corollary 3.29. *The variety generated by the algebras* $\mathbf{L}_5^{\mathbf{dm}}, \mathbf{L}_6^{\mathbf{dp}}, \mathbf{L}_7^{\mathbf{dm}}, \mathbf{L}_8^{\mathbf{dp}}, \mathbf{D}_3$ *is defined by the identity:*

$$(0 \to 1)^* \to (0 \to 1) \approx (0 \to 1)'.$$

Corollary 3.30. *The variety generated by the algebras* $\mathbf{L}_1^{\mathbf{dm}}, \mathbf{L}_1^{\mathbf{dp}}, \mathbf{L}_2^{\mathbf{dm}}, \mathbf{L}_2^{\mathbf{dp}}, \mathbf{L}_3^{\mathbf{dm}}, \mathbf{L}_3^{\mathbf{dp}}, \mathbf{L}_4^{\mathbf{dm}}, \mathbf{L}_4^{\mathbf{dp}}, \mathbf{D}_2$ *is defined by the identity:*

$$0 \to 1 \approx 1 \text{ (FTT identity)}.$$

Corollary 3.31. *The variety generated by the algebras* $\mathbf{L}_1^{\mathbf{dm}}, \mathbf{L}_1^{\mathbf{dp}}, \mathbf{L}_3^{\mathbf{dm}}, \mathbf{L}_3^{\mathbf{dp}}, \mathbf{L}_6^{\mathbf{dm}}, \mathbf{L}_6^{\mathbf{dp}}, \mathbf{L}_8^{\mathbf{dm}}, \mathbf{L}_8^{\mathbf{dp}}, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3$ *is defined by the identity:*

$$x \vee (y \to x) \approx (x \vee y) \to x.$$

Corollary 3.32. *The variety generated by the algebras* $\mathbf{L}_1^{\mathbf{dp}}, \mathbf{L}_3^{\mathbf{dp}}, \mathbf{L}_6^{\mathbf{dp}}, \mathbf{L}_8^{\mathbf{dp}}$ *is defined by the identities:*

(1) $x \vee (y \to x) \approx (x \vee y) \to x$

(2) $x \vee x' \approx 1$.

Corollary 3.33. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identities:*

(1) $x \vee (y \to x) \approx (x \vee y) \to x$

(2) $x'' \approx x.$

Corollary 3.34. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{L_9^{dm}}, \mathbf{L_9^{dp}},$ $\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identity:*

$$x^* \vee (x \to y) \approx (x \vee y) \to y.$$

Corollary 3.35. $\mathbf{V}(\{\mathbf{L_1^{dp}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dp}}, \mathbf{L_9^{dp}}\})$ *is defined by the identity:*

(1) $x^* \vee (x \to y) \approx (x \vee y) \to y$

(2) $x \vee x' \approx 1.$

Corollary 3.36. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_9^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identity:*

(1) $x^* \vee (x \to y) \approx (x \vee y) \to y$

(2) $x'' \approx x.$

Corollary 3.37. *The variety generated by the algebras* $\mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{D_2}$ *is defined by the identity:*

$$x \vee (0 \to x) \vee (y \to 1) \approx x \vee [(x \to 1) \to (x \to y)].$$

Corollary 3.38. *The variety generated by the algebras* $\mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{D_2}$ *defined by the identity:*

$$x \vee y \vee (x \to y) \approx x \vee [(x \to y) \to 1].$$

Corollary 3.39. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_7^{dm}}, \mathbf{L_7^{dp}}, \mathbf{D_2}$ *is defined by the identity:*

$$x \vee [(0 \to y) \to y) \approx x \vee [(x \to 1) \to y].$$

Corollary 3.40. *The variety generated by the algebras* $\mathbf{L_7^{dm}}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *defined by the identity:*

$$x \vee [x \to (y \wedge (0 \to y))] \approx x \to [(x \to y) \to y].$$

Corollary 3.41. *The variety generated by the algebras* $\mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *defined by the identity:*

$$x \vee y \vee [y \to (y \to x)] \approx x \to [x \vee (0 \to y)].$$

It is also defined by the identity:

$$x \vee [y \to (0 \to (y \to x))] \approx x \vee y \vee (y \to x).$$

Corollary 3.42. *The variety generated by the algebras* $\mathbf{L_7^{dm}}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{L_9^{dm}}, \mathbf{L_9^{dp}}, \mathbf{L_{10}^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identity:*

$$x \vee (x \to y) \approx x \vee [(x \to y) \to 1].$$

Corollary 3.43. *The variety generated by the algebras* $\mathbf{2^e}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dp}}, \mathbf{L_9^{dp}}, \mathbf{L_{10}^{dp}}$ *is defined by the identities:*

(1) $x \vee (x \to y) \approx x \vee [(x \to y) \to 1]$

(2) $x \vee x' \approx 1.$

Corollary 3.44. *The variety generated by the algebras* $\mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{L_9^{dm}},, \mathbf{L_{10}^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identities:*

(1) $x \vee (x \to y) \approx x \vee [(x \to y) \to 1]$

(2) $x'' \approx x.$

Corollary 3.45. *The variety generated by the algebras* $\mathbf{L_9^{dm}}, \mathbf{L_9^{dp}}, \mathbf{L_{10}^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{D_1}$ *is defined by the identity:*

$$0 \to 1 \approx 0. \ (\text{FTF identity})$$

Corollary 3.46. *The variety generated by the algebras* $\mathbf{L_{10}^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{D_1}$ *is defined by the identity:*

$$x \to y \approx y \to x. \ (\text{commutative identity})$$

A base for $\mathbf{V}(\mathbf{C_{10}^{dp}})$ was given in [10]. We give some new ones below.

Corollary 3.47. *The variety* $\mathbf{V}(\mathbf{C_{10}^{dp}})$ *is defined by :*

$$x' \wedge x'' \approx 0 \quad \text{(dual Stone identity).}$$

It is also defined by :

$$x' \approx x'^{*'}.$$

A base for $\mathbf{V}(\mathbf{C_{20}})$ was given in [10]. We give a new one below.

Corollary 3.48. *The variety* $\mathbf{V}(\mathbf{C_{20}})$ *is defined by :*

$$x^* \leq x'.$$

A base for $\mathbf{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$ was given in [10]. We give a new one below.

Corollary 3.49. *The variety* $\mathbf{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$ *is defined by :*

$$x^* \approx x'.$$

Corollary 3.50. *The variety generated by the algebras in*
$\{\mathbf{L_i^{dp}} : i = 1, \ldots, 8\} \cup \{\mathbf{L_i^{dm}} : i = 1, \ldots, 8\} \cup \{\mathbf{D_2}\}$ *is defined by the identity:*

$$(x \rightarrow y)^* \approx (x \wedge y^*)^{**}.$$

It is also defined by

$$(0 \rightarrow 1)^* \approx 0.$$

Corollary 3.51. *The variety generated by the algebras in*
$\{\mathbf{L_i^{dp}}, i = 1, \ldots, 8\}$, *is defined by the identities:*

(1) $(x \rightarrow y)^* \approx (x \wedge y^*)^{**}$

(2) $x' \wedge x'' \approx 0 \quad$ *(dual Stone identity).*

Corollary 3.52. *The variety generated by the algebras*
$\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{D_2}$ *is defined by the identity:*

$$x \wedge z \leq y \vee (y \rightarrow z) \text{ (strong Kleene identity).}$$

Corollary 3.53. *The variety generated by*
$\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{D_2}$ *is defined by the identity:*

$$x \vee y \leq (x \rightarrow y) \rightarrow y.$$

Corollary 3.54. *The variety generated by*
$\mathbf{L_1^{dp}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dp}}$ *is defined by the identity:*

(1) $x \vee y \leq (x \rightarrow y) \rightarrow y$

(2) $x \vee x' \approx 1.$

Corollary 3.55. *The variety generated by*
$\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{D_2}$ *is defined by the identity:*

(1) $x \vee y \leq (x \rightarrow y) \rightarrow y$

(2) $x'' \approx x.$

The variety $\mathbf{D} = \mathbf{V}\{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$ was axiomatized in [10]. Here are two more bases for it.

Corollary 3.56. *The variety* $\mathbf{D}$ *is defined by the identity:*

$$x \vee (y \rightarrow z) \approx (x \vee y) \rightarrow (x \vee z).$$

It is also defined by the identity:

$$x^{2('*)} \approx x.$$

We would like to mention here that in the case of either of the two bases in the preceding corollary the identities (St) and (L1) are consequences of the rest of the identities and hence are redundant.

Corollary 3.57. *The variety generated by* $\mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{D_2}$ *is defined by the identity:*

$$(x \rightarrow y) \rightarrow x \approx x.$$

$\mathbf{V}(\mathbf{D_2})$ was axiomatized in [10]. Here are some more bases for it. This variety has an interesting property in that $\vee$ is definable in terms of $\rightarrow$.

Corollary 3.58. *The variety generated by* $\mathbf{D_2}$ *is defined by the identity:*

$$x \vee y \approx (x \rightarrow y) \rightarrow y.$$

It is also defined by the identities:

(1) $x \vee (y \to z) \approx (x \vee y) \to (x \vee z)$

(2) $(x \to y) \to x \approx x.$

It is also defined by the identity:

$$x \vee (x \to y) \approx x \vee ((x \vee y) \to 1).$$

Corollary 3.59. *The variety generated by*
$\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_9^{dm}}, \mathbf{L_9^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identity:*

$$x \to (y \to z) \approx y \to (x \to z).$$

Corollary 3.60. *The variety generated by*
$\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{D_2}$ *is defined by the identity:*

$$(x \to y) \to z \leq ((y \to x) \to z) \to z. \ (Prelinearity)$$

We conclude this section by mentioning that one can easily write down the bases for intersections of the varieties mentioned in this section. Similarly, one can also easily determine the subvarieties of the varieties considered in this section, obtained by adding the identity $x'' \approx x$, or the identity: $x \vee x' \approx 1$, to their bases occurring in the preceding corollaries.

4 Conclusion and some open problems

It should be pointed out that, based on the results from [10], all varieties appearing in Section 3.1 are discrimnator varieities. It was also shown in [10] that all the twenty 3-element algebras in **RDQDStSH₁** are semiprimal (see [2] or [10] for definition). A similar argument proves that $\mathbf{D_1}$ and $\mathbf{D_2}$ are semiprimal as well; and $\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}}, \mathbf{D3}$ are, in fact, primal. Thus the variety **RDQDStSH₁** is generated by semiprimal algebras. We would also like to note here that the algebras in $\{\mathbf{L_i^{dm}} : i = 5, \ldots, 8\} \cup \{\mathbf{L_i^{dp}} : i = 5, \ldots, 8\}$ are also primal. From these observations and from the results of [10] we conclude that the algebras in $\{\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}}, \mathbf{D3}\} \cup \{\mathbf{L_i^{dm}} : i = 5, \ldots, 8\} \cup \{\mathbf{L_i^{dp}} : i = 5, \ldots, 8\}$ are the only atoms in the lattice of subvarieties of the variety **RDQDStSH₁**.

It is our view that each of the varieties mentioned in Section 3 is worthy of further study, both algebraically and logically.

We conclude this paper with some more open problems for further investigation.

Problem 1. Find equational bases for the remaining subvarieties of **RDQDStSH$_1$**.

Problem 2. Give an explicit description of simple algebras in the variety of regular dually Stone semi-Heyting algebras of level 1.

Problem 3. Axiomatize logically each of the subvarieties of **RDQDStSH$_1$**. (In other words, for each subvariety **V** of **RDQDStSH$_1$** find a propositional logic **P** such that **V** is an equivalent algebraic semantics for **P**.)

In particular, the following problem is of interest.

Problem 4. The 2-element, 3-element, 4-element algebras in Figure 1 can be viewed respectively as 2, 3 and 4-valued logical matrices. Axiomatize these algebras logically (with 1 as the only designated truth value), using $\rightarrow$ and $'$ as implication and negation respectively. (For the algebra **2** in Figure 1, the answer is, of course, well known: the classical propositional logic.)

Problem 5. Investigate the lattice of subvarieties of **DPCStSH$_1$**.

Problem 6. Investigate the lattice of subvarieties of **DMStSH$_1$**.

Problem 7. Investigate the lattice of subvarieties of the variety of commutative **DMSH$_1$**-algebras.

Problem 8. Find a duality for **RDQDStSH$_1$** and for each of its subvarieties.

References

[1] R. Balbes and PH. Dwinger, "Distributive Lattices", Univ. of Missouri Press, Columbia, 1974.

[2] S. Burris and H.P. Sankappanavar, "A Course in Universal Algebra", Springer-Verlag, New York, 1981. The free, corrected version (2012) is available online as a PDF file at math.uwaterloo.ca/∼snburris.

[3] B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand. 21 (1967), 110-121.

[4] W. McCune, *Prover9 and Mace 4*, http://www.cs.unm.edu/mccune/prover9/.

[5] H. Rasiowa, "An Algebraic Approach to Non-Classical Logics", North–Holland Publ.Comp., Amsterdam, 1974.

[6] H.P. Sankappanavar, *Heyting algebras with dual pseudocomplementation*, Pacific J. Math. 117 (1985), 405-415.

[7] H.P. Sankappanavar, *Heyting algebras with a dual lattice endomorphism*, Zeitschr. f. math. Logik und Grundlagen d. Math. 33 (1987), 565–573.

[8] H.P. Sankappanavar, *Semi-De Morgan algebras*, J. Symbolic. Logic 52 (1987), 712-724.

[9] H.P. Sankappanavar, *Semi-Heyting algebras: An abstraction from Heyting algebras*, Actas del IX Congreso Dr. A. Monteiro (2007), 33-66.

[10] H.P. Sankappanavar, *Expansions of semi-Heyting algebras. I: Discriminator varieties*, Studia Logica 98 (1-2) (2011), 27-81.

[11] H.P. Sankappanavar, *Dually quasi-De Morgan Stone semi-Heyting algebras I. Regularity*, Categ. General Alg. Struct. Appl. 2(1) (2014), 47-64.

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