Uniformities and covering properties for partial frames (II)

John Frith and Anneliese Schauerte

Abstract. This paper is a continuation of [2], in which we make use of the notion of a partial frame, which is a meet-semilattice in which certain designated subsets are required to have joins, and finite meets distribute over these. After presenting there our axiomatization of partial frames, which we call $\mathcal{S}$-frames, we added structure, in the form of $\mathcal{S}$-covers and nearness.

Here, in the unstructured setting, we consider regularity, normality and compactness, expressing all these properties in terms of $\mathcal{S}$-covers. We see that an $\mathcal{S}$-frame is normal and regular if and only if the collection of all finite $\mathcal{S}$-covers forms a basis for an $\mathcal{S}$-uniformity on it. Various results about strong inclusions culminate in the proposition that every compact, regular $\mathcal{S}$-frame has a unique compatible $\mathcal{S}$-uniformity.

1   Introduction

In this paper, which is a continuation of [2], we return to the unstructured setting of partial frames to consider regularity and normality. Both properties are naturally expressed in terms of $\mathcal{S}$-covers, but can also be characterized in other familiar ways. We see that an $\mathcal{S}$-frame is normal and regular if and only if the collection of all finite $\mathcal{S}$-covers forms a basis for an $\mathcal{S}$-uniformity on it.

Next we consider the most fundamental covering property of topology: compactness. Various results about strong inclusions culminate in the proposition that every compact, regular $\mathcal{S}$-frame has a unique compatible $\mathcal{S}$-uniformity.

For the convenience of the reader, we conclude by considering how our axioms are related to the formalisms appearing elsewhere in the literature.

The concrete examples of nearness $\mathcal{S}$-frames which occur in this paper are all instances of what could be termed structured κ-frames. It is an open question in the literature (see [8]) whether κ-frames can be characterized by any of the existing axiomatizations of partial frames; the same applies to our axiomatization.

The reader is referred to [2] for a detailed introduction to this material as well as for the relevant preliminaries and definitions.

7   Two covering axioms

We now introduce two further axioms concerning $\mathcal{S}$-covers, for our selection functions. They are particularly useful in making regularity and normality tractable, but will also play a rôle in other situations.

(SCov): If C is an $\mathcal{S}$-cover of an $\mathcal{S}$-frame L and $\emptyset \neq A \subseteq C$ then $A \in \mathcal{S}L$.

(SFin): If C is a finite cover of an $\mathcal{S}$-frame L, then $C \in \mathcal{S}L$.

Informally (SCov) says that non-empty subsets of designated covers are designated and (SFin) says all finite covers are designated. (The restriction in (SCov) to non-empty subsets is not significant. Our selection functions do not automatically select the empty set; it is not needed since our meet-semilattices have a bottom element.)

Note 7.1. (1) We note that if both (SCov) and (SFin) hold, then any finite subset is designated: suppose F is finite; then $F \cup \{1\}$ is a cover, so F is designated since it is a subset of a finite cover.
(2) All the selection functions mentioned in Example 3.5 of [2] satisfy (SCov); only the first selection function mentioned there does not satisfy (SFin).
(3) We observe that Paseka [6] uses an axiom that makes all finite subsets

designated, which is technically stronger than our (SFin). Further, he requires all subsets of designated sets to be designated, which is stronger than our (SCov).

> From now on, we assume that our selection functions satisfy (SCov) and (SFin).

8   Regularity

We take the position in this and the next section that regularity and normality are best viewed as covering properties. This makes the fact that regular partial frames are precisely those with a compatible nearness structure, very clear. In the presence of (SFin) and (SCov) we see that the covering version of regularity is equivalent to the more usual "separating element" notion. (See Definition 8.4 and Lemma 8.5.)

Definition 8.1. Let L be an $\mathcal{S}$-frame.

1. For $a, b \in L$, we say x is *rather below* a, written $x \prec a$, if there exists an $\mathcal{S}$-cover C of L such that $C_x \subseteq \downarrow a$.

2. L is *regular* if, for all $a \in L$, there exists $T \in \mathcal{S}L$ such that $a = \bigvee T$ and $t \prec a$ for all $t \in T$.

Lemma 8.2. *The following are equivalent for an $\mathcal{S}$-frame L:*

1. *L is regular.*

2. *There is an $\mathcal{S}$-nearness compatible with L.*

3. *The collection of all $\mathcal{S}$-covers of L forms an $\mathcal{S}$-nearness compatible with L, called the fine $\mathcal{S}$-nearness on L.*

Proof. The implications (a) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are all clear, using the fact that $x \prec a$ if and only if $x \lhd a$ with the $\mathcal{S}$-nearness consisting of all $\mathcal{S}$-covers. $\square$

Proposition 8.3. *Let L be an $\mathcal{S}$-frame. An $\mathcal{S}$-frame is regular if and only if the collection of all finite covers forms a base for an $\mathcal{S}$-nearness on L, called the finitely fine $\mathcal{S}$-nearness on L.*

Proof. ($\Rightarrow$): Suppose that L is a regular $\mathcal{S}$-frame and $a \prec b$ in L. There exists an $\mathcal{S}$-cover D of L such that $D_a = \{d \in D : d \wedge a \neq 0\} \subseteq \downarrow b$. Using (SCov), let $s = \bigvee\{d \in D : d \wedge a \neq 0\}$ and $t = \bigvee\{d \in D : d \wedge a = 0\}$. Then $s \vee t = 1$. Further, $E = \{s, t\}$, being finite, is an $\mathcal{S}$-cover by (SFin). Also $E_a = \{e \in E : e \wedge a \neq 0\} = \{s\}$, since $a \wedge t = \bigvee\{a \wedge d : d \in D, d \wedge a = 0\} = 0$, using the fact that $\{d \in D : d \wedge a = 0\}$ is designated. So $E_a = \{s\} \subseteq \downarrow b$. ($\Leftarrow$): This uses Lemma 8.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 8.4. We define $a \prec_s b$ if there exists c (called a *separating element*) satisfying $a \wedge c = 0$ and $c \vee b = 1$.

Lemma 8.5. *Let L be an $\mathcal{S}$-frame. Then for any a, b in an $\mathcal{S}$-frame, $a \prec b$ if and only if $a \prec_s b$.*

Proof. Suppose $a \prec b$. Then there exists an $\mathcal{S}$-cover D such that $D_a = \{d \in D : d \wedge a \neq 0\} \subseteq \downarrow b$. Let $c = \bigvee\{d \in D : d \wedge a = 0\}$, which exists by (SCov). Then $a \wedge c = 0$, again by (SCov), and $b \vee c = 1$. So $a \prec_s b$.

Conversely, suppose that there exists c such that $a \wedge c = 0$ and $b \vee c = 1$. By (SFin), $D = \{b, c\}$ is an $\mathcal{S}$-cover and $D_a = \{b\} \subseteq \downarrow b$, so $a \prec b$. $\qquad\square$

Remark 8.6. From Lemma 8.5 it follows that $a \prec b \Rightarrow a \leq b$.

In the next example, we examine regularity in familiar categories of $\mathcal{S}$-frames.

Example 8.7.

- Meet-semilattices: here the only regular $\mathcal{S}$-frame is the two-element chain. This is because $\{1\}$ is the only designated cover, so only $0 \prec 0, 0 \prec 1$ and $1 \prec 1$. Consequently, this example will be of no further interest in our study of structured partial frames.

- Bounded distributive lattices: here, an $\mathcal{S}$-frame L is regular if and only if it is Boolean (See [4].) This is because if a can be expressed as a finite join $a = t_1 \vee t_2 \vee \ldots \vee t_n$ with $t_i \prec a$ for each $i = 1, \ldots, n$, then $a \prec a$ making it complemented.

- σ-frames, κ-frames and frames: here the notion of regularity introduced in this paper corresponds exactly with the usual notions for these structures provided in the literature. (See [3] and [5], [4], [7].)

Remark 8.8. Suppose a selection function $\mathcal{S}_1$ is finer than a selection function $\mathcal{S}_2$, meaning that for any meet-semilattice A, $G \in \mathcal{S}_2 A \Rightarrow G \in \mathcal{S}_1 A$. A straightforward argument shows that if L is an $\mathcal{S}_1$-frame which is a regular $\mathcal{S}_2$-frame, then L is a regular $\mathcal{S}_1$-frame. This pattern is clearly visible in Example 8.7.

Definition 8.9. An $\mathcal{S}$-frame map $h : L \to M$ is called *dense* if, for all $a \in L$, $h(a) = 0$ implies that $a = 0$.
An $\mathcal{S}$-frame map $h : L \to M$ is called *codense* if, for all $a \in L$, $h(a) = 1$ implies that $a = 1$.

Proposition 8.10. *Let L be an $\mathcal{S}$-frame. In the full subcategory of all regular $\mathcal{S}$-frames*

- *every dense map is a monomorphism and*

- *every codense map is $1 - 1$.*

Proof. Suppose that $h : L \to M$ is a dense $\mathcal{S}$-frame map. Suppose $k, l : N \to L$ are $\mathcal{S}$-frame maps from the regular $\mathcal{S}$-frame N to L such that $hk = hl$.

Consider $x \prec a$ in N. There exists an $\mathcal{S}$-cover C of N such that $C_x \subseteq {\downarrow} a$. Then $l[C]$ is an $\mathcal{S}$-cover of L. One shows routinely that $l[C]_{k(x)} \subseteq {\downarrow} l(a)$, and hence that $k(x) \prec l(a)$.

Since N is regular, any $a \in N$ can be expressed as $a = \bigvee X$, for some designated set X in N such that $x \in X \Rightarrow x \prec a$. Then $k(a) = \bigvee k[X]$. Now $k(x) \prec l(a) \Rightarrow k(x) \le l(a)$ by Remark 8.6; so $k(a) \le l(a)$. By symmetry, $l(a) \le k(a)$, and so $k(a) = l(a)$.

For the second claim, suppose that $h : L \to M$ is a codense morphism, with L and M regular $\mathcal{S}$-frames and that $h(a) = h(b)$. We have that $a = \bigvee X$, where X is designated and $x \in X \Rightarrow x \prec a$. For such x we claim that $x \le b$ which yields $a \le b$. By symmetry, $a = b$. For the claim: suppose that $C_x \subseteq {\downarrow} a$ for some $\mathcal{S}$-cover C. Then $h[C]_{h(x)} \subseteq {\downarrow} h(a) = {\downarrow} h(b)$. Let $s = \bigvee \{c \in C : c \wedge x = 0\}$. Then $h(s) \vee h(b) = h(s) \vee h(a) \ge \bigvee h[C] = 1$, so $s \vee b = 1$. Then, by (SFin), $D = \{s, b\}$ is a designated cover and $x = x \wedge \bigvee D = (x \wedge s) \vee (x \wedge b) = x \wedge b$, proving that $x \le b$. $\square$

We remark that the argument for every dense map being monic in the proposition above rests on the regularity of N, rather than that of L or M.

9   Normality

In this section, we define normality in terms of binary $\mathcal{S}$-covers. It is then characterized in terms of shrinkings. We also show that an $\mathcal{S}$-frame is normal and regular if and only if the finitely fine $\mathcal{S}$-nearness is an $\mathcal{S}$-uniformity.

Definition 9.1. An $\mathcal{S}$-frame L is called *normal* if, whenever $\{a, b\}$ is an $\mathcal{S}$-cover of L, there exist $\mathcal{S}$-covers $\{a, c\}$ and $\{b, d\}$ of L with $c \wedge d = 0$.

Definition 9.2. We say that $A = \{a_1, \ldots, a_n\}$ is a *shrinking* of the $\mathcal{S}$-cover $B = \{b_1, \ldots, b_n\}$ if A is an $\mathcal{S}$-cover and $a_j \prec b_j$ for $j = 1, \ldots, n$.

Since we are assuming (SFin), there is no distinction between finite covers and finite $\mathcal{S}$-covers; we just use the shorter term "finite covers" from now on.

Lemma 9.3. *In any normal $\mathcal{S}$-frame the relation $\prec$ interpolates.*

Proof. Suppose $a \prec b$; by Lemma 8.5, there exists s such that $a \wedge s = 0$ and $b \vee s = 1$. Applying normality to $\{b, s\}$ gives covers $\{b, c\}, \{s, d\}$ with $c \wedge d = 0$. Then $a \prec d \prec b$. □

The proof of the following result follows traditional lines and so is omitted. (See [7] IX Lemma 1.2.1. We note that they use pseudocomplements, but this can easily be circumvented by using the "rather below" relation as defined in this paper.)

Lemma 9.4. *An $\mathcal{S}$-frame L is normal if and only if every finite cover of L has a shrinking.*

Remark 9.5. We feel that the partial frame context illuminates the following: Regular bounded distributive lattices and regular σ-frames are known to be normal (see [4], [1] and [3]), whereas this is not the case for frames and κ-frames in general. This reflects the pattern that regularity becomes a weaker condition as more and more sets become designated (see Example 8.7 and Remark 8.8), whereas normality, depending only on binary covers, is unaffected.

We conclude this section with a result linking uniform structures to the notions of regularity and normality.

Proposition 9.6. *An $\mathcal{S}$-frame L is normal and regular if and only if the collection of all finite covers forms a basis for an $\mathcal{S}$-uniformity on L; that is, the finitely fine $\mathcal{S}$-nearness is an $\mathcal{S}$-uniformity.*

Proof. In Proposition 8.3 it was shown that L is regular if and only if the finite covers of L form a base for an $\mathcal{S}$-nearness on L. We thus consider the relationship between normality and the existence of star-refinements.

($\Rightarrow$) Suppose that L is normal and regular and that $A = \{a_1, \ldots, a_n\}$ is a finite cover of L. By Lemma 9.4, A has a shrinking $S = \{s_1, \ldots, s_n\}$. Let $t_1, \ldots, t_n$ be the respective separating elements, that is, $s_j \wedge t_j = 0, t_j \vee a_j = 1$ for $j = 1, \ldots, n$. Now let $C = \{a_1, t_1\} \wedge \ldots \wedge \{a_n, t_n\} \wedge \{s_1, \ldots, s_n\}$. Then C is a finite cover of L. We need only show that $C <^* A$.

A typical non-zero element of C has the form $b_E \wedge s_i$ where $E \subseteq \{1, \ldots, n\}$, $b_E = \bigwedge_{j \in E} a_j \wedge \bigwedge_{j \notin E} t_j$ and $i \in E$. Taking two such elements,

$$(b_E \wedge s_i) \wedge (b_F \wedge s_k) \neq 0 \Rightarrow i \in F \Rightarrow b_F \wedge s_k \leq a_i,$$

so $C(b_E \wedge s_i) \leq a_i$, showing that $C <^* A$.

($\Leftarrow$) Suppose that $\{a, b\}$ is a cover of L. By assumption, there is a finite cover E of L such that $E <^* \{a, b\}$.
Let $c = \bigvee\{e \in E : e \not\leq a\}$ and $d = \bigvee\{e \in E : e \not\leq b\}$. (These joins exist by (SCov).) Then $1 = \bigvee E = \bigvee\{e \in E : e \not\leq a\} \vee \bigvee\{e \in E : e \leq a\} \leq c \vee a$. Similarly, $1 = d \vee b$. We now show that $c \wedge d = 0$: take $e, f \in E$ with $e \not\leq a, f \not\leq b$. Then $Ee \not\leq a$, so $Ee \leq b$. Similarly $Ef \leq a$. So if $e \wedge f \neq 0$, then $f \leq Ee \leq b$ and $e \leq Ef \leq a$ – a contradiction. $\square$

Remark 9.7. It is feasible to impose a useful condition on selection functions that is formally weaker than (SFin). One may consider the condition

(SBin): If C is a binary cover of an $\mathcal{S}$-frame L, then $C \in \mathcal{S}L$.

This is natural because normality is defined in terms of binary covers. For instance, it can be shown that, if (SBin) replaces (SFin), then an $\mathcal{S}$-frame is normal if and only if every binary cover has a shrinking. Moreover, one can replace the finitely fine $\mathcal{S}$-nearness by one generated by all binary covers. Then an $\mathcal{S}$-frame is regular and normal if and only if this particular $\mathcal{S}$-nearness is in fact an $\mathcal{S}$-uniformity. In the next section, however, we need the full strength of (SFin), so have not pursued the use of (SBin).

10   Compactness

The main result of this section is the fact that every compact, regular $\mathcal{S}$-frame has a unique compatible $\mathcal{S}$-nearness which is in fact an $\mathcal{S}$-uniformity. En route, we show how to construct a totally bounded $\mathcal{S}$-uniformity from a strong inclusion.

Definition 10.1. An $\mathcal{S}$-frame is called *compact* if every $\mathcal{S}$-cover has a finite sub $\mathcal{S}$-cover.

Proposition 10.2. *Every compact regular $\mathcal{S}$-frame is normal.*

Proof. Let $\{a, b\}$ be a cover of the compact, regular $\mathcal{S}$-frame L. By regularity, there exist designated sets S, T such that $a = \bigvee S, b = \bigvee T, s \in S \Rightarrow s \prec a$ and $t \in T \Rightarrow t \prec b$. Then $S \cup T$ is designated, by an application of (S3), and so an $\mathcal{S}$-cover, since $a \vee b = 1$. By compactness, there exist $x \prec a, y \prec b$ with $x \vee y = 1$. So there exist $c, d \in L$ such that $x \wedge c = 0, a \vee c = 1, y \wedge d = 0, d \vee b = 1$. Then $\{a, c\}$ and $\{b, d\}$ are covers of L with $c \wedge d = 0$, as required. $\square$

Definition 10.3. A *strong inclusion* on an $\mathcal{S}$-frame L is a binary relation $\blacktriangleleft$ on L such that, for all $a, b, c, d \in L$:
(SI1) $a \leq b \blacktriangleleft c \leq d \Rightarrow a \blacktriangleleft d$.
(SI2) $\blacktriangleleft$ is a sublattice of $L \times L$.
(SI3) $a \blacktriangleleft b \Rightarrow a \prec b$.
(SI4) $\blacktriangleleft$ interpolates on L; that is, if $a \blacktriangleleft b$ then there exists $c \in L$ with $a \blacktriangleleft c \blacktriangleleft b$.
(SI5) If $a \blacktriangleleft b$, there exist $c, d \in L$ with $c \blacktriangleleft d, a \wedge d = 0$ and $b \vee c = 1$.
(SI6) For $a \in L, a = \bigvee X$ for some designated set X in L such that $x \in X \Rightarrow x \blacktriangleleft a$.

We remark that, by Note 7.1, all finite joins do exist. Also, condition (SI3) and (SI6) show that any $\mathcal{S}$-frame with a strong inclusion on it, is regular.

Reassuringly, we now show that any uniform $\mathcal{S}$-frame has an underlying strong inclusion, after which we show that any strong inclusion is induced by a totally bounded $\mathcal{S}$-uniformity.

Lemma 10.4. *Let* $(L, \mathcal{K}L)$ *be a uniform* $\mathcal{S}$*-frame. The uniformly below relation given by* $a \lhd b \Leftrightarrow C_a \subseteq {\downarrow} b$ *for some* $C \in \mathcal{K}L$*, is a strong inclusion on* L.

Proof. The proofs of (SI1) and (SI2) are routine, and (SI3) follows from Definition 8.1.

(SI4): Suppose $a \lhd b$. There exists $D \in \mathcal{K}L$ such that $D_a \subseteq {\downarrow} b$. Let $C \in \mathcal{K}L$ such that $C <^* D$. Then $a \lhd Ca \lhd b$, where $Ca = \bigvee \{c \in C : c \wedge a \neq 0\}$, as usual.

(SI5): Suppose $a \lhd b$. By (SI4) above, there exists $t \in L$ with $a \lhd t \lhd b$. Take $C \in \mathcal{K}L$ such that $C_a \subseteq {\downarrow} t$ and $C_t \subseteq {\downarrow} b$. Let $e = \bigvee \{c \in C : c \wedge t = 0\}$ and $f = \bigvee \{c \in C : c \wedge a = 0\}$, using (SCov). Then $a \wedge f = 0$. Also $b \vee e \geq \bigvee \{c \in C : c \wedge t \neq 0\} \vee \bigvee \{c \in C : c \wedge t = 0\} = 1$.

We show that $e \lhd f$, by showing that $C_e \subseteq {\downarrow} f$. Take $k \in C$ with $k \wedge e \neq 0$. Then there exists $c \in C$ with $k \wedge c \neq 0$ and $c \wedge t = 0$. Then $k \nleq t$: if $k \leq t$, then $k \wedge c = 0$; a contradiction. Also $k \wedge a = 0$: if $k \wedge a \neq 0$, then $k \leq t$; a contradiction. But then $k \leq f$, by the definition of f, as required.

(SI6) follows from the compatibility condition of a uniform $\mathcal{S}$-frame. $\square$

Remark 10.5. We call the uniformly below relation of Lemma 10.4 the strong inclusion *induced by* the $\mathcal{S}$-uniformity in question.

Proposition 10.6. *Any strong inclusion on an* $\mathcal{S}$*-frame is induced by a totally bounded* $\mathcal{S}$*-uniformity.*

Proof. Let $\blacktriangleleft$ be a strong inclusion on an $\mathcal{S}$-frame L. Let $\mathcal{B}$ be the collection of all finite covers $C = \{c_1, \ldots, c_n\}$ such that there exists a finite cover $W_C = \{w_1, \ldots, w_n\}$ (called a "witness") such that $w_j \blacktriangleleft c_j$ for $j = 1, \ldots, n$. We claim that $\mathcal{B}$ is a base for an $\mathcal{S}$-uniformity on L, which is then clearly totally bounded.

$\mathcal{B}$ is closed under finite meets because $a \blacktriangleleft b$ and $c \blacktriangleleft d$ implies $a \wedge c \blacktriangleleft b \wedge d$. For compatibility, consider $a \blacktriangleleft d$ and take $b, c \in L$ with $a \blacktriangleleft b \blacktriangleleft c \blacktriangleleft d$. There exist, by (SI5), $p, q, r, s, t, u \in L$ such that $p \blacktriangleleft q \leq r \blacktriangleleft s \leq t \blacktriangleleft u$ and $p \vee d = 1, q \wedge c = 0, r \vee c = 1, s \wedge b = 0, t \vee b = 1, u \wedge a = 0$.

Let $C = \{u, d\}$ and $D = \{s, c\}$. Now $u \vee d \geq p \vee d = 1$ and $s \vee c \geq r \vee c = 1$, so C and D are $\mathcal{S}$-covers of L. Further, $s \blacktriangleleft u$ and $c \blacktriangleleft d$, so D witnesses C being in $\mathcal{B}$. We note that $Ca = d$, so compatibility follows from (SI6).

For star-refinements, start with $C = \{c_1, \ldots, c_n\} \in \mathcal{B}$ and $W_C = \{w_1, \ldots, w_n\}$

a witness. For each j take $d_j \in L$ with $w_j \blacktriangleleft d_j \blacktriangleleft c_j$. Let $D = \{d_1, \ldots, d_n\}$; then $D \in \mathcal{B}$. By (SI5) we have $s_j \blacktriangleleft t_j, t_j \wedge d_j = 0, s_j \vee c_j = 1$. Let $D_j = \{c_j, t_j\}$. By repeated interpolation, as in the compatibility argument above, $D_j \in \mathcal{B}$. Finally, let $F = D \wedge D_1 \wedge \ldots \wedge D_n$. We show that $F <^* C$: A typical member of F has the form $d_j \wedge z_1 \wedge \ldots \wedge z_n$ where $d_j \in D$ and $z_i \in D_i$ for $i = 1, \ldots, n$.

For any j, $F(d_j \wedge z_1 \wedge \ldots \wedge z_n) \leq D_j(d_j \wedge z_1 \wedge \ldots \wedge z_n) \leq D_j(d_j \wedge z_j) \leq c_j$. Denote by $\lhd$ the strong inclusion on L induced by the $\mathcal{S}$-uniformity just constructed. We show that $\lhd = \blacktriangleleft$.

We have already seen that, if $a \blacktriangleleft b$, then $a \lhd b$ by the compatibility argument above. Suppose now that $a \lhd b$. Then there is $C = \{c_1, \ldots, c_n\} \in \mathcal{B}$ with the associated witness $W_C = \{w_1, \ldots, w_n\}$ such that $Ca \leq b$. Then, using the finiteness of W_C we obtain:

$a \leq W_C a = \bigvee\{w_j : w_j \in W_C, w_j \wedge a \neq 0\} \blacktriangleleft \bigvee\{c_j : w_j \in W_C, w_j \wedge a \neq 0\} \leq Ca \leq b$, so $a \blacktriangleleft b$ as required. $\qquad\square$

Proposition 10.7. *Every compact, regular $\mathcal{S}$-frame has a unique compatible $\mathcal{S}$-nearness, which is in fact an $\mathcal{S}$-uniformity.*

Proof. In a compact, regular $\mathcal{S}$-frame, the rather below relation is a strong inclusion. This follows from a standard argument where the interpolation condition (SI4) follows from Lemma 9.3 and Proposition 10.2. Then Proposition 10.6 applies.

To show uniqueness, fix an $\mathcal{S}$-nearness $\mathcal{K}L$ on a compact, regular $\mathcal{S}$-frame L. We show that every finite cover is in $\mathcal{K}L$. Then every $\mathcal{S}$-cover is in $\mathcal{K}L$, by compactness.

So let C be a finite cover of L. Each $c \in C$ can be written as $c = \bigvee T_c$ where T_c is a designated subset of L and $t \lhd c$ for each $t \in T_c$. Then, by Axiom (S3), $S = \bigcup\{T_c : c \in C\}$ is an $\mathcal{S}$-cover of L. By compactness, S has a finite sub $\mathcal{S}$-cover $\{t_1, \ldots, t_n\}$. Let $j \in \{1, \ldots, n\}$; then $t_j \lhd c_j$ for some $c_j \in C$. So there exists $D_j \in \mathcal{K}L$ such that $D_j t_j \leq c_j$. Let $E = D_1 \wedge \ldots \wedge D_n$. Then $E \in \mathcal{K}L$ and $E \leq C$, as required. $\qquad\square$

Remark 10.8. If $\mathcal{S}$ selects only finite subsets, so that $\mathcal{S}$-frames are bounded distributive lattices, it follows from the above that each $\mathcal{S}$-frame has a unique $\mathcal{S}$-nearness on it (which consists of all finite covers). So the theory of $\mathcal{S}$-nearness and $\mathcal{S}$-uniformity is simply not interesting in this case.

11   Alternative formalisms

Partial frames have been presented in the literature using various different formalisms. In this section, we present a brief review of these. We note, however, that a general treatment of compatible covering structures for partial frames has not appeared previously. Of course, nearnesses and uniformities in the specific cases of frames and σ-frames, are well known.

The "selection function" description we have used in this paper most closely resembles that of Paseka [6].

Paseka's axioms (A1) to (A4) correspond to our (S1) to (S4). In addition, he has one further axiom, (A5), which we state now, translated into our notation. For a meet-semilattice A, write $\mathcal{D}_{\mathcal{S}}A = \{\downarrow G : G \in \mathcal{S}A\}$, where $\downarrow G = \bigcup\{\downarrow g : g \in G\}$. This is a meet-semilattice, using intersection as the (finite) meet operation.

Paseka's (A5): For any meet-semilattice A, $\mathcal{G} \in \mathcal{S}\mathcal{D}_{\mathcal{S}}A$ implies that $\bigcup \mathcal{G} \in \mathcal{D}_{\mathcal{S}}A$.

We did not need Paseka's axiom (A5) in this paper. However, if one assumes it, then $\mathcal{D}_{\mathcal{S}}$ becomes a functor from meet-semilattices to $\mathcal{S}$-frames, which is left adjoint to the inclusion functor. (See [9].)

Zhao [9] and Zenk [8] present partial frames using a formalism that uses downsets. The following is called a "set system" by Zhao and a "subset selector" by Zenk.

Definition 11.1. A *set system* is a function which assigns to each meet-semilattice A a collection ZA of downsets of A such that the following conditions hold, for all meet-semilattices A and B:

(D1) For all $x \in A, \downarrow x \in ZA$.

(D2) For $S, T \in ZA, S \cap T \in ZA$.

(D3) For $\mathcal{A} \in ZZA, \bigcup \mathcal{A} \in ZA$.

(D4) For any meet-semilattice map $f : A \to B$, $\{\downarrow f[S] : S \in ZA\} \subseteq ZB$.

Both Zhao and Zenk then define Z-frames and Z-frame maps as follows:

Definition 11.2. Let Z be a set system.

1. A *Z-frame*, L, is a meet-semilattice such that
   (i) for all $S \in ZL$, S has a join in L and
   (ii) for all $x \in L$, for all $S \in ZL$, $x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$.

2. Let L and M be Z-frames. A *Z-frame map* $f : L \to M$ is a meet-semilattice map such that, for all $S \in ZL$, $f(\bigvee S) = \bigvee f[S]$.

For the convenience of the reader used to using downsets in this context, we now list some correspondences between our axioms and those of Zhao and Zenk.

Remark 11.3. Let $\mathcal{S}$ be a rule which assigns to each meet-semilattice A a collection $\mathcal{S}A$ of subsets of A. Then $\mathcal{D}_\mathcal{S}$ is a function which assigns to each meet-semilattice A a collection of downsets of A.
If $\mathcal{S}$ satisfies (S1) then $\mathcal{D}_\mathcal{S}$ satisfies (D1).
If $\mathcal{S}$ satisfies (S2) then $\mathcal{D}_\mathcal{S}$ satisfies (D2).
If $\mathcal{S}$ satisfies (S3) then $\mathcal{D}_\mathcal{S}$ satisfies the condition: "For any meet-semilattice A, if $G \in \mathcal{D}_\mathcal{S}A$ and, for each $y \in G$, $y = \bigvee G_y$ for some $G_y \in \mathcal{D}_\mathcal{S}A$, then $\bigcup_{y \in G} G_y \in \mathcal{D}_\mathcal{S}A$."
If $\mathcal{S}$ satisfies (S4), then $\mathcal{D}_\mathcal{S}$ satisfies the condition: "For any meet-semilattice map $f : A \to B$, $\{\downarrow f[G] : G \in \mathcal{D}_\mathcal{S}A\} = \mathcal{D}_\mathcal{S}(f[A]) \subseteq \mathcal{D}_\mathcal{S}(B)$."
If $\mathcal{S}$ satisfies Paseka's Axiom (A5), then $\mathcal{D}_\mathcal{S}$ satisfies (D3).

Remark 11.4. The $\mathcal{S}$-frames and the $\mathcal{D}_\mathcal{S}$-frames coincide.
The $\mathcal{S}$-frame maps and the $\mathcal{D}_\mathcal{S}$-frame maps coincide.

We conclude with a remark on covers in partial frames: As mentioned earlier, Paseka, in [6], considers a notion of cover which is equivalent to our $\mathcal{S}$-cover. However, our axioms (SFin) and (SCov) are somewhat less restrictive than Paseka's here, since he assumes that every subset of a designated set is designated.

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Anneliese Schauerte, *Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag Rondebosch, 7701, South Africa.*
  Email: anneliese.schauerte@uct.ac.za

John Frith, *Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag Rondebosch, 7701, South Africa.*
  Email: john.frith@uct.ac.za