Dually quasi-De Morgan Stone semi-Heyting algebras I. Regularity

Hanamantagouda P. Sankappanavar

Abstract. This paper is the first of a two part series. In this paper, we first prove that the variety of dually quasi-De Morgan Stone semi-Heyting algebras of level 1 satisfies the strongly blended ∨-De Morgan law introduced in [20]. Then, using this result and the results of [20], we prove our main result which gives an explicit description of simple algebras(=subdirectly irreducibles) in the variety of regular dually quasi-De Morgan Stone semi-Heyting algebras of level 1. It is shown that there are 25 nontrivial simple algebras in this variety.

In Part II, we prove, using the description of simples obtained in this Part, that the variety **RDQDStSH₁** of regular dually quasi-De Morgan Stone semi-Heyting algebras of level 1 is the join of the variety generated by the twenty 3-element **RDQDStSH₁**-chains and the variety of dually quasi-De Morgan Boolean semi-Heyting algebras–the latter is known to be generated by the expansions of the three 4-element Boolean semi-Heyting algebras. As consequences of this theorem, we present (equational) axiomatizations for several subvarieties of **RDQDStSH₁**. The Part II concludes with some open problems for further investigation.

1   Introduction

The concept of regularity has played an important role in the theory of pseudocomplemented De Morgan algebras (see [15]). (A similar notion was earlier considered for double p-algebras in [22].) The main purpose of this paper is to carry it over to the variety **DQDSH** of dually quasi-De Morgan semi-Heyting algebras studied in [20] and to initiate the investigation of regularity in the level 1 subvariety **DQDStSH$_1$** of **DQDSH** whose members have Stone semi-Heyting reducts.

   This paper, the first of a two-part series, is organized as follows: We first prove in Section 3 that the variety **DQDStSH$_1$** of dually quasi-De Morgan Stone semi-Heyting algebras of level 1 satisfies the strongly blended ∨-De Morgan law introduced in [20]. Then, using this result and the results of [20], we prove, in Section 4, our main result which gives an explicit description of simple algebras(=subdirectly irreducibles) in the variety **RDQDStSH$_1$** of regular dually quasi-De Morgan Stone semi-Heyting algebras of level 1. It is shown that there are 25 (nontrivial) simple algebras in this variety.

   In Part II, we prove, using the description of simples, obtained in this Part, that **RDQDStSH$_1$** is the join of the variety generated by the twenty 3-element **RDQDStSH$_1$**-chains and the variety of dually quasi-De Morgan Boolean semi-Heyting algebras–the latter is known to be generated by the expansions of the three 4-element Boolean semi-Heyting algebras. Furthermore, as consequences of this theorem, we present, (equational) axiomatizations for several subvarieties of **RDQDStSH$_1$**. The Part II concludes with some open problems for further investigation.

2   Dually quasi-De Morgan semi-Heyting algebras

The following definition is taken from [18].

   An algebra $\mathbf{L} = \langle L, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a *semi-Heyting algebra* if $\langle L, \vee, \wedge, 0, 1 \rangle$ is a bounded lattice and $\mathbf{L}$ satisfies:

(SH1) $x \wedge (x \rightarrow y) \approx x \wedge y$

(SH2) $x \wedge (y \rightarrow z) \approx x \wedge ((x \wedge y) \rightarrow (x \wedge z))$

(SH3) $x \rightarrow x \approx 1$.

Let **L** be a semi-Heyting algebra. **L** is a *Heyting algebra* if **L** satisfies:

(SH4) $(x \wedge y) \to y \approx 1$.

L is a *Stone semi-Heyting algebra* if **L** satisfies:

(St) $x^* \vee x^{**} \approx 1$, where $x^* := x \to 0$.

L is a *Boolean semi-Heyting algebra* if **L** satisfies:

(Bo) $x \vee x^* \approx 1$ (where $x^* := x \to 0$).

Semi-Heyting algebras are distributive and pseudocomplemented, with a^* as the pseudocomplement of an element a. We will use these and other properties (see [18]) of semi-Heyting algebras, frequently without explicit mention, throughout this paper.

The following definition is taken from [20].

Definition 2.1. An algebra $\mathbf{L} = \langle L, \vee, \wedge, \to, ', 0, 1 \rangle$ is a *semi-Heyting algebra with a dual quasi-De Morgan operation* or *dually quasi-De Morgan semi-Heyting algebra* (**DQDSH**-algebra, for short) if $\langle L, \vee, \wedge, \to, 0, 1 \rangle$ is a semi-Heyting algebra, and **L** satisfies:

(a) $0' \approx 1$ and $1' \approx 0$

(b) $(x \wedge y)' \approx x' \vee y'$

(c) $(x \vee y)'' \approx x'' \vee y''$

(d) $x'' \leq x$.

A **DQDSH**-algebra **L** is a *dually pseudocomplemented semi-Heyting algebra* (**DPCSH**-algebra) if **L** satisfies:

(e) $x \vee x' \approx 1$.

A **DQDSH**-algebra **L** is a *De Morgan semi-Heyting algebra* or *symmetric semi-Heyting algebra* (**DMSH**-algebra) if **L** satisfies:

(DM) $x'' \approx x$.

The varieties of **DQDSH**-algebras, **DPCSH**-algebras and **DMSH**-algebras are denoted, respectively, by **DQDSH**, **DPCSH** and **DMSH**, while **DQDStSH** denotes the subvariety of **DQDSH** defined by the Stone identity (St), and **DQDBSH** denotes the one defined by (Bo). Let $\mathbf{L} \in \mathbf{DQDSH}$. $\mathbf{L}$ is a *semi-Heyting algebra with a blended dual quasi-De Morgan operation* or a *blended dually quasi-De Morgan semi-Heyting algebra* (**BDQDSH**-algebra, for short) if $\mathbf{L}$ satisfies:

(B) $(x \vee x^*)' \approx x' \wedge x^{*\prime}$ (**Blended $\vee$-De Morgan law**).

$\mathbf{L}$ is a *semi-Heyting algebra with a strongly blended dual quasi-De Morgan operation* or *strongly blended dually quasi-De Morgan semi-Heyting algebra* (**SBDQDSH**, for short) if $\mathbf{L}$ satisfies:

(SB) $(x \vee y^*)' \approx x' \wedge y^{*\prime}$ (**Strongly Blended $\vee$-De Morgan law**).

L is a semi-Heyting algebra with a dual ms-operation (**DmsSH**-algebra) if $\mathbf{L}$ satisfies:

$(x \vee y)' \approx x' \wedge y'$ ($\vee$-**De Morgan law**).

The varieties of **BDQDSH**-algebras, **SBDQDSH**-algebras and **DmsSH**-algebras are denoted, respectively, by **BDQDSH**, **SBDQDSH** and **DmsSH**. It is clear that $\mathbf{DmsSH} \subseteq \mathbf{SBDQDSH} \subseteq \mathbf{BDQDSH}$. If the underlying semi-Heyting algebra is a Heyting algebra, then we replace the part "**SH**" by "**H**" in the names of the varieties that we consider.

Two important classes of examples of **DQDStSH**-algebras are Stone Heyting algebras with a dual pseudocomplement (**DPCStH**) and De Morgan Stone Heyting algebras (symmetric Stone Heyting algebras) (**DMStH**, for short).

In the sequel, $a'^{*\prime}$ will be denoted by a^+, for $a \in \mathbf{L} \in \mathbf{DQDSH}$. The following lemma will be used, often without explicit reference to it. Most of the items in this lemma were proved in [20] and the others are left to the reader.

Lemma 2.2. *Let* $\mathbf{L} \in \mathbf{DQDSH}$ *and let* $x, y, z \in L$. *Then*

(i) $1'^* = 1$

  (ii) $x \leq y$ *implies* $x' \geq y'$

 (iii) $(x \wedge y)'^* = x'^* \wedge y'^*$

 (iv) $x'^* \leq x^{*'}$

  (v) $x''' = x'$

 (vi) $(x \vee y)' = (x'' \vee y'')'$

(vii) $(x \vee y)' = (x'' \vee y)'$

(viii) $x \leq (x \vee y) \rightarrow x$

 (ix) $x \leq x \rightarrow 1$

  (x) $x \wedge [(x \rightarrow y) \rightarrow z] = x \wedge (y \rightarrow z)$

 (xi) $x \vee x^+ = 1$.

Recall that $\mathbf{L} \in \mathbf{DQDSH}$ is a $\mathbf{DQDSH}$-*chain* if the lattice reduct of $\mathbf{L}$ is a chain. In what follows we are interested in the $\mathbf{DQDSH}$-chains of size 2 and 3 which we describe below. It was observed in [18] that $\mathbf{2}$ and $\bar{\mathbf{2}}$, shown in Figure 1, are, up to isomorphism, the only two 2-element semi-Heyting algebras.

$$\mathbf{2} \ : \quad \begin{array}{c} 1 \ \bullet \\ \big| \\ 0 \ \bullet \end{array} \qquad \begin{array}{c|cc} \rightarrow & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \end{array} \qquad\qquad \bar{\mathbf{2}} \ : \quad \begin{array}{c} 1 \ \bullet \\ \big| \\ 0 \ \bullet \end{array} \qquad \begin{array}{c|cc} \rightarrow & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$$

<div align="center">Figure 1</div>
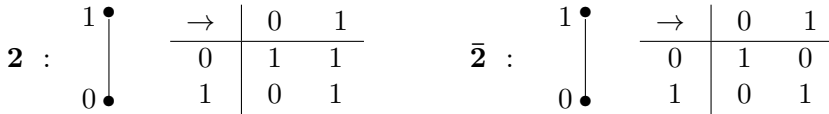
Let $\mathbf{2}^e$ and $\bar{\mathbf{2}}^{\mathbf{e}}$ denote their expansions obtained by adding the unary operation $'$ defined by: $0' = 1$ and $1' = 0$. It is clear that $\mathbf{2}^e$ and $\bar{\mathbf{2}}^{\mathbf{e}}$ are the only 2-element $\mathbf{DQDStSH}$-algebras.

It was also observed in [18] that there are, up to isomorphism, ten 3-element semi-Heyting chains, shown in Figure 2.

$\mathbf{L}_1$ (lattice 1–a–0):

$\rightarrow$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

$\mathbf{L}_2$ (lattice 1–a–0):

$\rightarrow$	0	a	1
0	1	a	1
a	0	1	1
1	0	a	1

$\mathbf{L}_3$ (lattice 1–a–0):

$\rightarrow$	0	a	1
0	1	1	1
a	0	1	a
1	0	a	1

$\mathbf{L}_4$ (lattice 1–a–0):

$\rightarrow$	0	a	1
0	1	a	1
a	0	1	a
1	0	a	1

$\mathbf{L}_5$ (lattice 1–a–0):

$\rightarrow$	0	a	1
0	1	a	a
a	0	1	1
1	0	a	1

$\mathbf{L}_6$ (lattice 1–a–0):

$\rightarrow$	0	a	1
0	1	1	a
a	0	1	1
1	0	a	1

$\mathbf{L}_7$ (lattice 1–a–0):

$\rightarrow$	0	a	1
0	1	a	a
a	0	1	a
1	0	a	1

$\mathbf{L}_8$ (lattice 1–a–0):

$\rightarrow$	0	a	1
0	1	1	a
a	0	1	a
1	0	a	1

$\mathbf{L}_9$ (lattice 1–a–0):

$\rightarrow$	0	a	1
0	1	0	0
a	0	1	1
1	0	a	1

$\mathbf{L}_{10}$ (lattice 1–a–0):

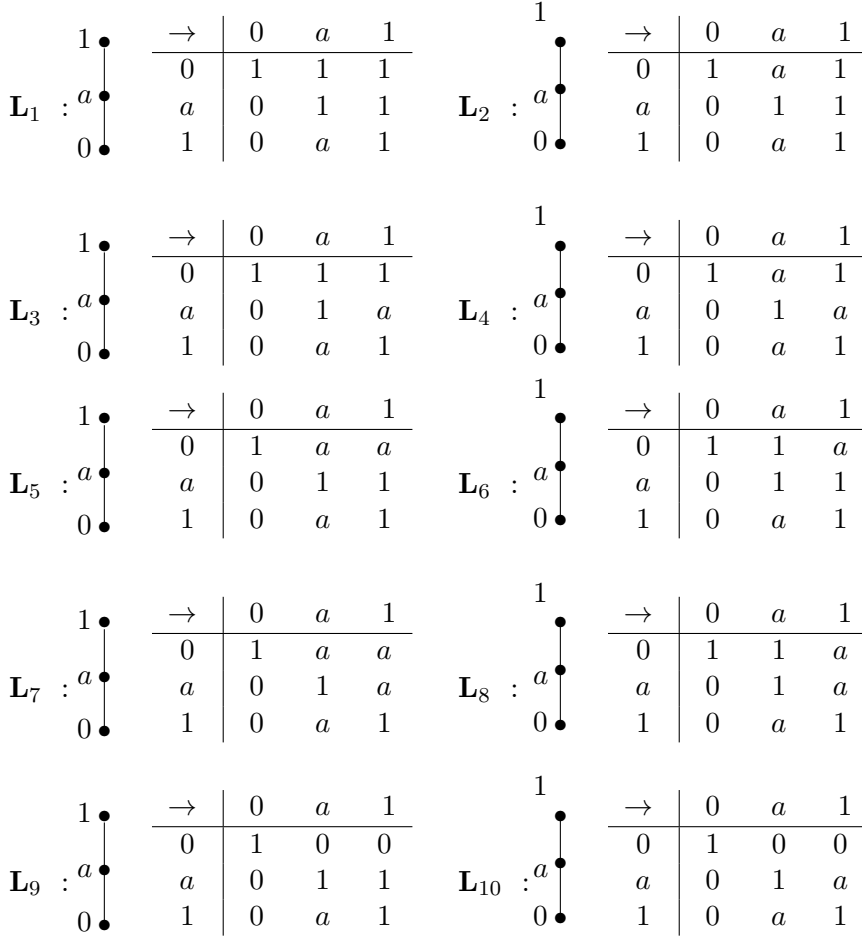| $\rightarrow$	0	a	1
0	1	0	0
a	0	1	a
1	0	a	1

Figure 2

Let $\mathbf{L}_i^{dp}$, $i = 1, \ldots, 10$, denote the expansion of $\mathbf{L}_i$ (shown in Figure 2) by adding the unary operation $'$ such that $0' = 1$, $1' = 0$, and $a' = 1$; and let $\mathbf{L}_i^{dm}$, $i = 1, \ldots, 10$, denote the expansion of $\mathbf{L}_i$ by adding the unary operation $'$ such that $0' = 1$, $1' = 0$, and $a' = a$. We Let $\mathbf{C}_{10}^{\mathbf{dp}} := \{\mathbf{L}_i^{dp} : i = 1, \ldots, 10\}$ and $\mathbf{C}_{10}^{\mathbf{dm}} := \{\mathbf{L}_i^{dm} : i = 1, \ldots, 10\}$. We also let $\mathbf{C}_{\mathbf{20}} := \mathbf{C}_{10}^{\mathbf{dm}} \cup \mathbf{C}_{10}^{\mathbf{dp}}$.

It is easy to verify that $\mathbf{C}_{10}^{\mathbf{dp}} \subseteq \mathbf{DPCStSH} \subseteq \mathbf{DQDStSH}$ and $\mathbf{C}_{10}^{\mathbf{dm}} \subseteq \mathbf{DMStSH} \subseteq \mathbf{DQDStSH}$. It is also easy to see that these 20 algebras are the only 3-element $\mathbf{DQDStSH}$-algebras.

We are also interested in the three 4-element algebras $\mathbf{D_1}$, $\mathbf{D_2}$ and $\mathbf{D_3}$ described below. Each of the three algebras has a Boolean lattice reduct with the universe $\{0, a, b, 1\}$, in which b is the complement of a, and the unary operation $'$ is defined as follows: $a' = a$, $b' = b$, $0' = 1$, $1' = 0$, and $\rightarrow$ is defined in Figure 3:

$\mathbf{D_1}$:

$\rightarrow$	0	1	a	b
0	1	0	b	a
1	0	1	a	b
a	b	a	1	0
b	a	b	0	1

$\mathbf{D_2}$:

$\rightarrow$	0	1	a	b
0	1	1	1	1
1	0	1	a	b
a	b	1	1	b
b	a	1	a	1

$\mathbf{D_3}$:

$\rightarrow$	0	1	a	b
0	1	a	1	a
1	0	1	a	b
a	b	a	1	0
b	a	1	a	1

Figure 3

It was shown in [20] that the algebras $\mathbf{D_1}$, $\mathbf{D_2}$, and $\mathbf{D_3}$ are simple and the variety generated by the three algebras is precisely the variety $\mathbf{DQDBSH}$ which is the subvariety of $\mathbf{DQDSH}$ defined by (Bo). The semi-Heyting reducts of $\mathbf{D_1}$, $\mathbf{D_2}$ and $\mathbf{D_3}$ are, up to isomorphism, $\bar{\mathbf{2}} \times \bar{\mathbf{2}}$, $\mathbf{2} \times \mathbf{2}$, and $\mathbf{2} \times \bar{\mathbf{2}}$ respectively. Notice also that $\{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\} \models x'' \approx x$, and $\mathbf{D2}$ has a Heyting algebra reduct, while the other two do not. Furthermore, $\mathbf{D_1}$ satisfies the commutative law: $x \rightarrow y \approx y \rightarrow x$.

The following definition is from [20].

Definition 2.3. Let $x \in \mathbf{L} \in \mathbf{DQDSH}$. For $n \in \omega$, we define $t_n(x)$ recursively as follows:

$$x^{0('*)} = x,$$
$$x^{(n+1)('*)} = (x^{n('*)})'^*, \text{ for } n \geq 0.$$
$$t_0(x) = x, \; t_{n+1}(x) = t_n(x) \wedge x^{(n+1)('*)}, \text{ for } n \geq 0.$$

For $n \in \omega$, the *nth level* (or *level n*) subvariety $\mathbf{DQDSH_n}$ of $\mathbf{DQDSH}$

is defined by the identity:

$$t_n(x) \approx t_{n+1}(x).$$

For $n \in \omega$, we let $\mathbf{BDQDSH_n} := \mathbf{BDQDSH} \cap \mathbf{DQDSH_n}$; and similarly $\mathbf{SBDQDSH_n}$, etc. are defined. It was shown in [20] that, for $n \in \omega$, the variety $\mathbf{BDQDSH_n}$, and hence the subvariety $\mathbf{SBDQDSH_n}$, is a discriminator variety.

Note that the level 0 (or 0th level) variety $\mathbf{DQDSH_0}$ is defined by the identity: $x \approx x \wedge x'^*$, (equivalently, $x'^* \approx x$) and it was shown in [20] that this variety is generated by $\{\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}}\}$.

An interesting open problem is the problem of investigation of the (complex) structure of the lattice of subvarieties of the variety $\mathbf{DQDSH_1}$ which is defined by: $x \wedge x'^* \wedge x'^{*'*} \approx x \wedge x'^*$.

In this paper we are interested in a special case of this problem, namely that of describing the subvariety lattice of the variety $\mathbf{RDQDStSH_1}$ of regular dually quasi-De Morgan Stone semi-Heyting algebras of level 1 (see Section 4 for definition).

The variety $\mathbf{V(C_{20})}$ generated by the twenty 3-element algebras (Figure 2) and the variety $\mathbf{DQDBSH}$ generated by the three 4-element algebras (Figure 3) are subvarieties of $\mathbf{DQDSH_1}$ that were axiomatized in [20]. In this paper, we are also interested in the problem of finding an equational base (an equational axiomatization) for the join of the varieties $\mathbf{V(C_{20})}$ and $\mathbf{DQDBSH}$. It is somewhat surprising that these two problems are closely related (see Part II, Theroem 3.4).

It is easy to see that all the twenty five algebras described above in Figures 1-3 are actually in $\mathbf{SBDQDStSH_1}$.

The following lemma is useful in the sequel. In particular, it aids us in giving an alternative definition of $\mathbf{DQDStSH_n}$.

Lemma 2.4. *Let* $\mathbf{L} \in \mathbf{DQDStSH}$ *with* $|L| \geq 2$. *Then*
(1) $x^{*''} = x^*$
(2) $x'^{**'} \leq x$
(3) $x'^{**} \vee x^{*'} = 1$
(4) $x'^{*'*} \leq x$
(5) $x \wedge x'^* \wedge x'^{*'*} \approx (x \wedge x'^*)'^*$.

Proof. From $x^* \vee x^{**} = 1$, we have $x^{*''} \vee x^{**''} = 1$, implying $x^{**} \vee x^{*''} = 1$. It follows that $x^* \leq x^{*''}$, from which we can conclude (1). Next, $x' \leq x'^{**}$

implies $x'^{**'} \leq x'' \leq x$, proving (2). From (2) we get $x^* \wedge x'^{**'} = 0$, which implies $x^{*'} \vee x'^{**} = 1$, in view of (1)– this proves (3). Now, from (3) we have $x^{*'**} \vee x^{**'} = 1$, yielding $x^{*'*} \leq x^{**'}$. Replacing x by x' and using (2), we get $x'^{*'*} \leq x'^{**'} \leq x$, thus proving (4). Finally, $(x \wedge x'^*)'^* = x'^* \wedge x'^{*'*} = x \wedge x'^* \wedge x'^{*'*}$ in view of (4), which proves (5). $\qquad\square$

The following theorem, which is now immediate from Lemma 2.4(5) and the definition of t_n, gives an alternative definition of **DQDStSH$_\mathbf{n}$**.

Theorem 2.5. *For $n \in \omega$,* **DQDStSH$_\mathbf{n}$** *is defined by the identity:* $(x \wedge x'^*)^{n('^*)} \approx (x \wedge x'^*)^{(n+1)('^*)}$, *modulo* **DQDStSH**.

In particular, the variety **DQDStSH$_\mathbf{1}$** *is defined by the identity:* $x \wedge x'^* \approx (x \wedge x'^*)'^*$, *relative to* **DQDStSH**.

3 Dually quasi-De Morgan Stone semi-Heyting algebras of level 1

We now focus our attention on the (sub)variety **DQDStSH$_\mathbf{1}$** which, in view of Theorem 2.5, consists of dually quasi-De Morgan Stone semi-Heyting algebras satisfying the identity:

(L1) $x \wedge x'^* \approx (x \wedge x'^*)'^*$ (Level 1).

If the underlying semi-Heyting algebra of a **DQDStSH**-algebra is a Heyting algebra we denote the algebra by **DQDStH**-algebra. The corresponding varieties are denoted by **DQDStSH** and **DQDStH** respectively.

In this section our goal is to prove **DQDStSH$_\mathbf{1}$ = SBDQDStSH$_\mathbf{1}$**. To achieve this, we need the following lemmas.

Lemma 3.1. *Let $x \in \mathbf{L} \in$ **DQDStSH$_\mathbf{1}$**. Then*
(1) $(x^* \vee x^{**'})^* \approx x^{**} \wedge x^{*'}$
(2) $x^{**} \wedge x^{*'} \wedge x^{**'} = 0$
(3) $x^{*'} \wedge x^{**'} = 0$.

Proof. $(x^* \vee x^{**'})^* = (x^{*''} \vee x^{*''*'})^* = (x^{*'} \wedge x^{*''*})'^* = x^{*'} \wedge x^{*''*} = x^{*'} \wedge x^{**}$, in view of Lemma 2.4(1) and the axiom (L1), proving (1). For (2), we have
$x^{**'} \wedge x^{**} \wedge x^{*'}$
$= x^{**'} \wedge (x^* \vee x^{**'})^*$ by (1)

$= x^{**\prime} \wedge [(x^* \vee x^{**\prime}) \to 0]$
$= x^{**\prime} \wedge [x^{**\prime} \to 0]$ by (SH2)
$= 0.$
For (3), first, we note that $x^* \wedge x^{**\prime} \wedge x^{*\prime} = 0$ by (2), since $x^{***} = x^*$. Now,
$x^{*\prime} \wedge x^{**\prime}$
$= (x^{*\prime} \wedge x^{**\prime}) \wedge (x^* \vee x^{**})$ by (St)
$= 0$ by what was noted earlier, which proves (3). $\qquad\square$

Lemma 3.2. *Let $x \in \mathbf{L} \in \mathbf{DQDStSH_1}$. Then*
(1) $x^{**\prime} = x^{*\prime*}$
(2) $x^{*\prime} \vee x^{*\prime*} = 1$
(3) $x^{*\prime**} = x^{*\prime}$.

Proof. $x^{**\prime} \leq x^{*\prime*}$ is immediate from Lemma 3.1(3), while the other in-
equality follows from Lemma 2.2(iv), so (1) is proved. Next, $x^{*\prime} \vee x^{*\prime*} = x^{*\prime} \vee x^{**\prime} = (x^* \wedge x^{**})^\prime = 1$ by (1), proving (2). Finally, using (2), we have
$x^{*\prime**} \wedge (x^{*\prime} \vee x^{*\prime*}) = x^{*\prime**}$, which, by distributivity, implies that $x^{*\prime**} \leq x^{*\prime}$.
Since the other inequality is well known, proof of (3) is complete. $\qquad\square$

Lemma 3.3. *Let $x, y \in \mathbf{L} \in \mathbf{DQDStSH}$. Then*
(1) $(x^\prime \vee y^{*\prime})^\prime \vee (x^\prime \vee y^{**\prime})^\prime = x^{\prime\prime}$
(2) $(x^\prime \vee y^{*\prime})^\prime = x^{\prime\prime} \wedge y^*$
(3) $(x \vee y^{*\prime})^\prime = x^\prime \wedge y^*$.

Proof. We have

$$
\begin{aligned}
(x^\prime \vee y^{*\prime})^\prime \vee (x^\prime \vee y^{**\prime})^\prime &= (x \wedge y^*)^{\prime\prime} \vee (x \wedge y^{**})^{\prime\prime} \\
&= [x \wedge (y^* \vee y^{**})]^{\prime\prime} \\
&= x^{\prime\prime} \quad \text{by (St),}
\end{aligned}
$$

proving (1). Next, we have

$$
\begin{aligned}
x^{\prime\prime} \wedge y^* &= y^* \wedge [(x^\prime \vee y^{*\prime})^\prime \vee (x^\prime \vee y^{**\prime})^\prime] \quad \text{by (1)} \\
&= [y^* \wedge (x^\prime \vee y^{*\prime})^\prime] \vee [y^* \wedge (x^\prime \vee y^{**\prime})^\prime] \\
&= (x^\prime \vee y^{*\prime})^\prime,
\end{aligned}
$$

as $y^* \geq y^{*\prime\prime} \geq (x' \vee y^{*\prime})'$ and $y^{**} \geq y^{**\prime\prime} \geq (x' \vee y^{**\prime})'$. Thus (2) is proved. Finally,

$$
\begin{aligned}
(x \vee y^{*\prime})' &= (x'' \vee y^{*\prime})' \\
&= x''' \wedge y^* \quad \text{by (2)} \\
&= x' \wedge y^*.
\end{aligned}
$$

Thus (3) is proved. □

We are now ready to prove our main theorem of this section.

Theorem 3.4. DQDStSH$_1$ = SBDQDStSH$_1$

Proof.

$$
\begin{aligned}
(x \vee y^*)' &= (x \vee y^{*\prime\prime})' \quad \text{by Lemma 2.4(1)} \\
&= (x \vee y^{*\prime**\prime})' \quad \text{by Lemma 3.2(3)} \\
&= x' \wedge y^{*\prime**} \quad \text{by Lemma 3.3(3)} \\
&= x' \wedge y^{*\prime} \quad \text{by Lemma 3.2(3)}.
\end{aligned}
$$

Thus **DQDStSH$_1$** $\subseteq$ **SBDQDStSH$_1$**. Since the other inequality is trivial, the proof is complete. □

We conclude this section with the remark that there are algebras in **SBDQDSH$_1$** in which (St) fails; thus, **DQDStSH$_1$** is a proper subvariety of **SBDQDSH$_1$**. We also observe that there are algebras to show that **DQDStSH$_1$** and **DmsSH$_1$** are incomparable elements in the lattice of subvarieties of **DQDSH$_1$**.

4 Simple algebras in RDQDStSH$_1$

We now introduce the (sub)variety **RDQDSH** of regular dually quasi-De Morgan semi-Heyting algebras. (Recall that $x^+ = x'^{*\prime}$.)

Definition 4.1. Let **L** $\in$ **DQDSH**. Then **L** *is regular* if **L** satisfies:

(M) $x \wedge x^+ \leq y \vee y^*$.

The variety of regular **DQDSH**-algebras will be denoted by **RDQDSH**.

Remark 4.2. The reader is cautioned here not to confuse this notion of regularity with another notion of regularity defined by the identity: $x^{**} \approx x^{*\prime}$ (see [9], [20]). In general, the two notions are independent of each other. In this paper we do not have the occasion to use the other notion, so there is no confusion.

The purpose of this section is to give an explicit description of simple algebras in the variety **RDQDStSH₁**. Since **RDQDStSH₁** $\subseteq$ **SBDQDSH₁** by Theorem 3.4, we are going to obtain such a description as an application of the following theorem which is immediate from Corollaries 7.6 and 7.7 of [20].

Theorem 4.3. *Let* **L** $\in$ **BDQDSH₁** *with* $|L| \geq 2$. *Then the following are equivalent:*

(1) **L** *is simple*

(2) **L** *is subdirectly irreducible*

(3) (a) Cen L = $\{0, 1\}$ *or* Cen L = $\{0, a, a^*, 1\}$ *with* $a = a'$, *and*

   (b) *For every* $x \in L$, *if* $x \neq 1$, *then* $t_1(x) = 0$.

We first show that the condition (3)(a) in the above theorem is redundant. For this we need the following lemma.

Lemma 4.4. *Suppose* **L** $\in$ **BDQDSH₁** *satisfies the condition* (3)(b) *of the preceding theorem. Let* $a \in$ **L** *such that* $a \vee a^* = 1$ *and* $a \notin \{0, 1\}$. *Then*

(1) $a'' = a$

(2) $a' \leq a$

(3) $a' = a^{**\prime}$

(4) $a^* = 0$ *or* $a \leq a'$

(5) $a' = a$.

Proof. From $(a \vee a^*)'' = 1$, we get $a'' \vee a^{*''} = 1$, implying $a \wedge (a'' \vee a^{*''}) = a$, and hence $a \leq a''$, which leads us to conclude (1). To prove (2), if $a' = 1$, then $a = a'' = 0$ in view of (1), contrary to the hypothesis. Hence, $a' \neq 1$. Then from hypothesis (3)(b) we have $a' \wedge (a \vee a''^*) = a' \wedge a$. Therefore, $a \wedge a' = a' \wedge (a \vee a^*) = a'$ by (1) and the hypothesis. So, we get $a' \leq a$, proving (2). For (3), from $a^* \wedge a^{**''} = 0$ we get $a \vee a^{**''} = a$, since $a \vee a^* = 1$, implying $a^{**''} \leq a$; hence $a^{**'} \geq a'$. On the other hand, we have $a' \vee [(a \to 0) \to 0]' = a' \vee (0 \to 0)'$ by Lemma 2.2(x), which simplifies to $a' \geq a^{**'}$, proving (3). Suppose $a^{*'} \neq 1$. Then $a^{*'} \wedge a^{*''*} = 0$ by hypothesis. Hence $a^{*''} \vee a^{*''*'} = 1$, which simplifies to $a^* \vee a' = 1$, in view of Lemma 2.4(1) and item (3) above, from which we conclude $a \leq a'$. Thus we have $a^{*'} = 1$ or $a \leq a'$, which implies $a^{*''} = 0$ or $a \leq a'$, which, in view of Lemma 2.4(1), proves (4). Suppose $a \not\leq a'$. Then $a^* = 0$ by (4), whence $a^{**'} = 0$. Then by (3) $a' = 0$, implying $a \geq a'' = 1$, which is contrary to the hypothesis. Hence $a \leq a'$, which combined with (2) gives $a' = a$, proving (5). $\qquad\square$

The following corollary is now immediate from Theorem 4.3 and (5) of Lemma 4.4.

Corollary 4.1. *Let* $\mathbf{L} \in \mathbf{BDQDSH_1}$ *with* $|L| \geq 2$. *Then the following are equivalent:*

(1) $\mathbf{L}$ *is simple*

(2) $\mathbf{L}$ *is subdirectly irreducible*

(3) *For every* $x \in L$, *if* $x \neq 1$, *then* $x \wedge x'^* = 0$.

Unless otherwise stated, in the rest of this section we assume that $\mathbf{L} \in \mathbf{RDQDStSH_1}$ and satisfies the following simplicity condition (which is the condition (3) of the preceding corollary):

(SC) For every $x \in L$, if $x \neq 1$ then $x \wedge x'^* = 0$.

Lemma 4.5. *Let* $x, y \in \mathbf{L}$. *Then*

(1) $x \wedge (x^+ \vee y \vee y^*) = x \wedge (y \vee y^*)$

(2) $x' \vee x'^* \vee x^+ = 1$.

Proof. We have

$$
\begin{aligned}
x \wedge (y \vee y^*) &= x \wedge [(x \wedge x^+) \vee (y \vee y^*)] \quad \text{by (M)} \\
&= (x \wedge x^+) \vee [x \wedge (y \vee y^*)] \\
&= x \wedge (x^+ \vee y \vee y^*),
\end{aligned}
$$

proving (1). Next,

$$
\begin{aligned}
x' \vee x'^* \vee x'^{*\prime} &\geq x' \vee (x \wedge x'^*) \vee x'^{*\prime} \\
&\geq (x \wedge x'^*)' \vee (x \wedge x'^*)'' \\
&= [(x \wedge x'^*) \wedge (x \wedge x'^*)']' \\
&= [(x \wedge x'^*)'^* \wedge (x' \vee x'^{*\prime})]' \quad \text{by (L1)} \\
&= [(x' \vee x'^{*\prime})^* \wedge (x' \vee x'^{*\prime})]' \\
&= 0', \text{since }^* \text{ is the pseudocomplement} \\
&= 1,
\end{aligned}
$$

which proves (2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It should perhaps be remarked here that the condition (SC) is not used in the above lemma.

Lemma 4.6. *Let $x, y \in L$. Then*

(1) $x = 1$ *or* $x \leq x'$

(2) $x \vee y = 1$ *or* $x \leq (x \vee y)'$

(3) $x \leq y$ *or* $x \vee y' = 1$.

Proof. Suppose $x \neq 1$. Then

$$
\begin{aligned}
x \wedge x' &= (x \wedge x') \vee (x \wedge x'^*) \quad \text{by (SC)} \\
&= x \wedge (x' \vee x'^*) \\
&= x \wedge (x' \vee x'^* \vee x'^{*\prime}) \quad \text{by Lemma 4.5(1)} \\
&= x \wedge 1 \quad \text{by Lemma 4.5(2)} \\
&= x,
\end{aligned}
$$

which proves (1). From (1) we have $x \vee y = 1$ or $x \vee y \leq (x \vee y)'$. The latter clearly implies $x \leq (x \vee y)'$, proving (2). From (2) we get $x \vee y' = 1$ or $x \leq (x \vee y')'$. But $(x \vee y')' \leq y'' \leq y$. Hence (3) holds. $\qquad\square$

Lemma 4.7. *Let $a, b \in \mathbf{L}$ such that $0 < a < b < 1$. Then*

(1) $b \leq a \vee a^*$

(2) $a' = 1$ *or* $a' = b$

(3) $a' = 1$

(4) $a^{*\prime} = 1$

(5) $a^{**\prime} = 0$

(6) $a^* = 0$.

Proof. **Claim 1**: $a \leq b^+$. We have, from Lemma 4.6(3), $a \leq b^+$ or $a \vee b^{+\prime} = 1$. Suppose the latter holds, then $b \wedge (a \vee b^{\prime *\prime\prime}) = b$, which simplifies to $b \leq a$, as $b \wedge b^{\prime *\prime\prime} \leq b \wedge b^{\prime *} = 0$ since $b \neq 1$. Then we have a contradiction to the hypothesis $a < b$, so **Claim 1** is proved.

From Lemma 4.6(3), $a \leq b^{\prime *}$ or $a \vee b^+ = 1$. The former would imply $b \wedge a \leq b \wedge b^{\prime *} = 0$ by (SC), contrary to the hypothesis. Hence the latter holds, from which, using Claim 1, we have $b^+ = 1$. Hence by (M) we get $b = b \wedge 1 = b \wedge b^+ \leq a \vee a^*$, proving (1). Suppose $a' \neq 1$. Then from Lemma 4.6(1), $a' \leq a'' \leq b'' \leq b$; thus $a' \leq b$. On the other hand, as $b \neq 1$, $b \leq b'$ by Lemma 4.6(1), and we already know $b' \leq a'$, hence $b \leq a'$, implying $a' = b$ and thus proving (2). Now if $a' \neq 1$, then $a' = b$ by (2) and $a' \leq a''$ by Lemma 4.6(1). So $b = a' \leq a'' \leq a$–a contradiction, so $a' = 1$, proving (3). For (4) we first need to prove the following

Claim 2: $a \leq a^{*\prime\prime}$ or $a^{*\prime} = 1$. For, suppose $a \nleq a^{*\prime\prime}$. Then, by Lemma 4.6(3), $a \vee a^{*\prime\prime\prime} = 1$, whence $(a \vee a^{*\prime})' = 0$, and so, $(a'' \vee a^{*\prime})' = 0$. Hence, it follows from (3) that $a^{*\prime\prime} = 0$ and so $a^{*\prime} = 1$, proving **Claim 2**.

Thus we have $a \leq a^{*\prime\prime}$ or $a^{*\prime} = 1$, the former of which would imply $a \leq a^*$, leading to a contradiction since $a \neq 0$. Therefore, $a^{*\prime} = 1$, which proves (4). Next, $0 = 1' = (a^* \vee a^{**})' = (a^{*\prime\prime} \vee a^{**})' = a^{**\prime}$ by (4), thus proving (5). Finally, observe that $a^{**} \geq a^{**\prime\prime} = 1$ by (5), implying $a^* = 0$, proving (6). $\qquad\square$

Lemma 4.8. *If $\mathbf{L} \in \mathbf{RDQDStSH_1}$ satisfies the simplicity condition* (SC), *then $\mathbf{L}$ is of height at most 2 (that is, no chain in L is of cardinality ≥ 4).*

Proof. Suppose there are a, $b \in L$ such that $0 < a < b < 1$. Using Lemma 4.7(1), we get $b \leq a \vee a^*$. But $a^* = 0$ by Lemma 4.7(6). Thus $b \leq a$, which is a contradiction, proving the lemma. □

We are ready to prove the main theorem of this paper.

Theorem 4.9. *Let* $\mathbf{L} \in \mathbf{RDQDStSH_1}$ *with* $|L| \geq 2$. *Then the following are equivalent:*

(1) $\mathbf{L}$ *is simple*

(2) $\mathbf{L}$ *is subdirectly irreducible*

(3) *For every* $x \in L$, *if* $x \neq 1$, *then* $x \wedge x'^* = 0$

(4) $\mathbf{L}$ *is of height at most* 2

(5) $\mathbf{L} \in \{\mathbf{2^e}, \mathbf{\bar{2}^e}\} \cup \mathbf{C_{20}} \cup \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$, *up to isomorphism.*

Proof. (3) $\Rightarrow$ (4) by Lemma 4.8. Suppose (4) holds. Then it is clear that the lattice reduct of $\mathbf{L}$ is a chain of size 2, 3, or is a 4 element Boolean lattice. Observe that all algebras listed in (5) are in $\mathbf{RDQDStSH_1}$ and are the only ones of height at most 2. Hence $\mathbf{L}$ is isomorphic to one of the algebras in (5). Thus (5) holds; whence (4) $\Rightarrow$ (5). As it can be easily verified that all algebras listed in (5) are simple, (5) $\Rightarrow$ (1). The rest of the implications follow from Corollary 4.1. □

In concluding this section we point out that $\mathbf{RDQDStSH_1}$ is a proper subvariety of $\mathbf{SBDQDStSH_1}$.

References

[1] M. Abad, J.M. Cornejo and J.P. Díaz Varela, *The variety of semi-Heyting algebras satisfying the equation* $(0 \to 1)^* \vee (0 \to 1)^{**} \approx 1$, Reports on Mathematical Logic 46 (2011), 75-90.

[2] M. Abad, J.M. Cornejo and J.P. Díaz Varela, *The variety generated by semi-Heyting chains*, Soft Computing 15 (2011), 721-728.

[3] M. Abad and L. Monteiro, *Free symmetric Boolean algebras*, Revista de la U.M.A. 27 (1976), 207-215.

[4] R. Balbes and PH. Dwinger, "Distributive Lattices", Univ. of Missouri Press, Columbia, 1974.

[5] S. Burris and H.P. Sankappanavar, "A Course in Universal Algebra", Springer-Verlag, New York, 1981. The free, corrected version (2012) is available online as a PDF file at math.uwaterloo.ca/~snburris.

[6] G. Grätzer, "Lattice Theory", W.H.Freeman and Co., San Francisco, 1971.

[7] A. Horn, *Logic with truth values in a linearly ordered Heyting algebras*, J. Symbolic. Logic 34 (1969), 395-408.

[8] B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand. 21 (1967), 110-121.

[9] V.Yu. Meskhi, *A discriminator variety of Heyting algebras with involution*, Algebra i Logika 21 (1982), 537-552.

[10] A. Monteiro, *Sur les algebres de Heyting symetriques*, Portugaliae Mathemaica 39 (1980), 1-237.

[11] W. McCune, *Prover9 and Mace 4*, http://www.cs.unm.edu/mccune/prover9/.

[12] H. Rasiowa, "An Algebraic Approach to Non-Classical Logics", North–Holland Publ.Comp., Amsterdam, 1974.

[13] H. Rasiowa and R. Sikorski, "The Mathematics of Metamathematics", Warsazawa, 1970.

[14] H.P. Sankappanavar, *Heyting algebras with dual pseudocomplementation*, Pacific J. Math. 117 (1985), 405-415.

[15] H.P. Sankappanavar, *Pseudocomplemented Okham and De Morgan algebras*, Zeitschr. f. math. Logik und Grundlagen d. Math. 32 (1986), 385-394.

[16] H.P. Sankappanavar, *Heyting algebras with a dual lattice endomorphism*, Zeitschr. f. math. Logik und Grundlagen d. Math. 33 (1987), 565–573.

[17] H.P. Sankappanavar, *Semi-De Morgan algebras*, J. Symbolic. Logic 52 (1987), 712-724.

[18] H.P. Sankappanavar, *Semi-Heyting algebras: An abstraction from Heyting algebras*, Actas del IX Congreso Dr. A. Monteiro (2007), 33-66.

[19] H.P. Sankappanavar, *Semi-Heyting algebras* II. In Preparation.

[20] H.P. Sankappanavar, *Expansions of semi-Heyting algebras.* I: *Discriminator varieties*, Studia Logica 98 (1-2) (2011), 27-81.

[21] H.P. Sankappanavar, *Expansions of semi-Heyting algebras.* II. In Preparation.

[22] J, Varlet, *A regular variety of type* $\langle 2, 2, 1, 1, 0, 0 \rangle$, Algebra Universalis 2 (1972), 218-223.

[23] H. Werner, "Discriminator Algebras", Studien zur Algebra und ihre Anwendungen, Band 6, Academie–Verlag, Berlin, 1978.

Hanamantagouda P. Sankappanavar, *Department of Mathematics, State University of New York, New Paltz, NY 12561*

Email: sankapph@newpaltz.edu