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Uniformities and covering properties for partial frames (I)

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Abstract. Partial frames provide a rich context in which to do pointfree structured and unstructured topology. A small collection of axioms of an elementary nature allows one to do much traditional pointfree topology, both on the level of frames or locales, and that of uniform or metric frames. These axioms are sufficiently general to include as examples bounded distributive lattices, σ-frames, κ-frames and frames.

Reflective subcategories of uniform and nearness spaces and lately coreflective subcategories of uniform and nearness frames have been a topic of considerable interest. In [9] an easily implementable criterion for establishing certain coreflections in nearness frames was presented. Although the primary application in that paper was in the setting of nearness frames, it was observed there that similar techniques apply in many categories; we establish here, in this more general setting of structured partial frames, a technique that unifies these.

We make use of the notion of a partial frame, which is a meet-semilattice in which certain designated subsets are required to have joins, and finite meets distribute over these. After presenting our axiomatization of partial frames, which we call $\mathcal{S}$-frames, we add structure, in the form of $\mathcal{S}$-covers and nearness, and provide the promised method of constructing certain coreflections. We illustrate the method with the examples of uniform, strong and totally bounded nearness $\mathcal{S}$-frames.

In Part (II) of this paper ([10]) we consider regularity, normality and compactness for partial frames.

Keywords: Frame, $\mathcal{S}$-frame, Z-frame, partial frame, σ-frame, κ-frame, meet-semilattice, nearness, uniformity, strong inclusion, uniform map, coreflection, P-approximation, strong, totally bounded, regular, normal, compact.
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1 Introduction

Partial frames provide a fertile context in which to do structured and unstructured topology in a pointfree way. This paper serves to lay the foundations for a long-term investigation of this setting, the aims of which are many and varied.

It will be seen that a small collection of axioms of an elementary nature allows one to do much traditional pointfree topology, both on the level of frames or locales, and that of uniform or metric frames. These axioms are sufficiently general to include as examples bounded distributive lattices, σ-frames, κ-frames and frames. Although not the main motivation of this work, this paper can also act as a self-contained and readable source for topological material common to these contexts. The smooth flow of this presentation owes much to the careful choice of the axioms; indeed, at first, it was not at all obvious to us which axioms would be needed.

Our initial involvement in this topic came from a different source, namely, reflective subcategories of uniform and nearness spaces. These have been a topic of abiding interest; latterly a similar interest in coreflective subcategories of uniform and nearness frames has arisen. In [9] an easily implementable criterion for establishing certain coreflections in nearness frames was presented. Although the primary application in that paper was in the setting of nearness frames, it was observed there that similar techniques apply in other categories, for example, uniform frames, uniform σ-frames, nearness σ-frames and pre-nearness frames. We will establish here, in this more general setting of structured partial frames, a technique that unifies all the above.

An essential aspect of our work is the use of covers: not surprisingly, they are used in the presentation of uniform and nearness structures, as well as compactness. Perhaps more unexpectedly, the notions of regularity and normality are also very naturally defined using covers. This theme of the richness of covers will continue in our subsequent papers, on complete regularity, coherence and the Lindelöf property. At this stage, the distinguished rôle of σ-frames will become apparent. Further rewards from pursuing this point of view will include interesting new results concerning completions.

In [12] Madden states:

> "It will be possible, I believe, to formulate a useful notion of a

> partial frame. This would be a meet-semilattice in which certain distinguished subsets would all have suprema and in which meets would distribute over joins of such subsets...My hope is that a theory of partial frames could provide substantial insight into large classes of epireflective properties and covering properties in locale theory and topology..."

It is in this spirit that we introduce our notion of a partial frame, which we call an $\mathcal{S}$-frame. We make use of so-called *selection functions* to specify the distinguished subsets that are required to have joins. The idea of using such selection functions is not new; see, for instance, [14], [20] and [19]. A selection function must satisfy certain axioms to produce a tractable theory, and each of the authors cited uses different but overlapping collections of such axioms, as do we.

After introducing the basics of $\mathcal{S}$-frames and their maps, we consider sub $\mathcal{S}$-frames and show how to generate sub $\mathcal{S}$-frames from a given set of elements. Then we add structure, in the form of $\mathcal{S}$-covers and nearness. Our chosen axioms allow for the formation of uniform images of nearness structures in a simple way. They also allow us to conclude that the collection of subobjects of a nearness $\mathcal{S}$-frame forms a complete lattice. With these ingredients in place, we provide the promised method of constructing certain coreflections, which we illustrate with the three examples of uniform, strong and totally bounded nearness $\mathcal{S}$-frames. At this stage it is clear that we have indeed found a general setting in which our coreflection construction applies.

In Part (II) we return to the unstructured setting to consider regularity and normality. Both properties are naturally expressed in terms of $\mathcal{S}$-covers, but can also be characterized in other familiar ways. We see that an $\mathcal{S}$-frame is normal and regular if and only if the collection of all finite $\mathcal{S}$-covers forms a basis for an $\mathcal{S}$-uniformity on it.

Next we consider the most fundamental covering property of topology: compactness. Various results about strong inclusions culminate in the proposition that every compact, regular $\mathcal{S}$-frame has a unique compatible $\mathcal{S}$-uniformity.

For the convenience of the reader, we conclude by considering how our axioms are related to the formalisms appearing elsewhere in the literature.

The concrete examples of nearness $\mathcal{S}$-frames which occur in this paper are all instances of what could be termed structured κ-frames. It is an open question in the literature (see [19]) whether κ-frames can be characterized by any of the existing axiomatizations of partial frames; the same applies to our axiomatization.

2 Background

See [15], [11], [17] and [16] as references for frame theory; see [2], [3], [7] and [5] for nearness frames; see [4], [13] and [18] for σ-frames.

Frames

1. A *meet-semilattice* L is a partially ordered set in which all finite subsets have a meet. In particular, we regard the empty set as finite, so a meet-semilattice comes equipped with a top element, which we denote by 1. We also insist that a meet-semilattice should have a bottom element, which we denote by 0. (Technically, one might wish to refer to these as *bounded* meet-semilattices.) A function $f : L \to M$ is a *meet-semilattice map* if it preserves finite meets, as well as the top and the bottom element.

2. A *bounded distributive lattice* L is a meet-semilattice in which $\bigvee B$ exists for any finite $B \subseteq L$ and for such B, $a \wedge \bigvee B = \bigvee\{a \wedge b : b \in B\}$ for all $a \in L$. A *bounded distributive lattice map* is a meet-semilattice map between bounded distributive lattices that preserves finite joins.

3. A *σ-frame* L is a meet-semilattice in which $\bigvee B$ exists for any countable $B \subseteq L$ and for such B, $a \wedge \bigvee B = \bigvee\{a \wedge b : b \in B\}$ for all $a \in L$. A *σ-frame map* is a meet-semilattice map between σ-frames that preserves countable joins.

4. A *κ-frame*, where κ is a regular cardinal, has the same definition as a σ-frame, but the countable subsets are replaced by subsets with cardinality less than κ.

5. A *frame* L is a meet-semilattice in which $\bigvee B$ exists for any $B \subseteq L$ and for such B, $a \wedge \bigvee B = \bigvee\{a \wedge b : b \in B\}$ for all $a \in L$. A *frame map* is a meet-semilattice map between frames which preserves all joins.

Nearness and uniformity

The following are well-known:

1. For a frame L, $C \subseteq L$ is a *cover of* L if $\bigvee C = 1$. For covers C and D of L, $C \wedge D = \{c \wedge d : c \in C, d \in D\}$ is again a cover of L. We say that C *refines* D if for any $c \in C$ there exists $d \in D$ with $c \leq d$; we then write $C \leq D$.

2. For $a, b \in L$ and C a cover of L we write $a \lhd_C b$ if $Ca = \bigvee\{c \in C : c \wedge a \neq 0\} \leq b$ and say *a is uniformly below b with respect to C*. If $CC = \{Cc : c \in C\} \leq D$, we write $C <^* D$.

3. A non-empty collection of covers, $\mathcal{N}L$, of L is a *nearness* on L if it is filtered by meet and refinement and satisfies the following *compatibility condition*: For any $x \in L, x = \bigvee\{t \in L : t \lhd_C x$ for some $C \in \mathcal{N}L\}$.

4. The members of $\mathcal{N}L$ are called *uniform* covers. The pair $(L, \mathcal{N}L)$ is a *nearness frame*. We also say that if $t \lhd_C x$ for some $C \in \mathcal{N}L$, then t is *uniformly below x in* $(L, \mathcal{N}L)$.

5. For a nearness frame $(L, \mathcal{N}L)$ if for any uniform cover D there is a uniform cover C such that $C <^* D$, then $(L, \mathcal{N}L)$ is a *uniform frame* and $\mathcal{N}L$ is a *uniformity* on L.

6. For nearness frames $(L, \mathcal{N}L)$ and $(M, \mathcal{N}M)$, a frame map $f : L \to M$ is a *uniform* map if for every $C \in \mathcal{N}L$, $f[C] = \{f(c) : c \in C\} \in \mathcal{N}M$. The category of nearness frames and uniform maps is denoted by **NearFrm**. The category of uniform frames and uniform maps is denoted by **UniFrm**.

Remark 2.1. Nearness and uniform structures for σ-frames occur in the literature (see [13]). Essentially two modifications are needed: (1) A *cover* C of a σ-frame L must be a *countable* subset of L whose join is the top. (2) The compatibility condition requires any element of L to be a *countable* join of elements uniformly below it.

3 $\mathcal{S}$-frames

An $\mathcal{S}$-frame is a meet-semilattice in which certain joins are required to exist and finite meets are required to distribute over these joins. We specify these joins by means of a selection function as defined below:

Definition 3.1. A *selection function* is a rule, which we usually denote by $\mathcal{S}$, which assigns to each meet-semilattice A a collection $\mathcal{S}A$ of subsets of A, such that the following conditions hold (for all meet-semilattices A and B):

(S1) For all $x \in A$, $\{x\} \in \mathcal{S}A$.

(S2) If $G, H \in \mathcal{S}A$ then $\{x \wedge y : x \in G, y \in H\} \in \mathcal{S}A$.

(S3) If $G \in \mathcal{S}A$ and, for all $x \in G$, $x = \bigvee H_x$ for some $H_x \in \mathcal{S}A$, then

$$\bigcup_{x \in G} H_x \in \mathcal{S}A.$$

(S4) For any meet-semilattice map $f : A \to B$,

$$\mathcal{S}(f[A]) = \{f[G] : G \in \mathcal{S}A\} \subseteq \mathcal{S}B.$$

Remark 3.2.

1. Once a selection function, $\mathcal{S}$, has been fixed, we speak informally of the members of $\mathcal{S}A$ as the *designated* subsets of A.
2. We note that, in Axiom (S4) above, for $\mathcal{S}(f[A])$ to be defined, $f[A]$ needs to be a meet-semilattice. This is of course so, since meet-semilattice maps preserve the top element, the bottom element and finite meets.
3. A consequence of Axiom (S4) is the fact that, for any meet-semilattice map $f : A \to B$, $G \in \mathcal{S}A$ implies $f[G] \in \mathcal{S}B$. So any meet-semilattice map sends designated subsets of the domain to designated subsets of the codomain.
4. Axioms (S1) to (S4) appear as (A1) to (A4) in Paseka's [14]. In his list of axioms a further axiom, (A5), appears; we have no need at present for this, but it will be important in later work.

We can now present our particular approach to partial frames, which we call $\mathcal{S}$-frames.

Definition 3.3. Let $\mathcal{S}$ be a selection function.

1. An $\mathcal{S}$*-frame*, L, is a meet-semilattice that satisfies the following two conditions:

   (a) For all $G \in \mathcal{S}L$, G has a join in L (i.e. $\bigvee G$ exists).

   (b) For all $x \in L$, for all $G \in \mathcal{S}L$, $x \wedge \bigvee G = \bigvee_{y \in G} x \wedge y$.

2. Let L and M be $\mathcal{S}$-frames. An $\mathcal{S}$*-frame map* $f : L \to M$ is a meet-semilattice map such that, for all $G \in \mathcal{S}L$, $f(\bigvee G) = \bigvee_{y \in G} f(y)$.

3. $\mathcal{S}$**Frm** is the category of $\mathcal{S}$-frames as objects and $\mathcal{S}$-frame maps as morphisms.

Remark 3.4. 1. With regard to conditions (a) and (b) above, we note that, since $\{x\} \in \mathcal{S}L$, for any $G \in \mathcal{S}L$ we have that $\{x \wedge y : y \in G\} \in \mathcal{S}L$ by Axiom (S2) so $\bigvee_{y \in G} x \wedge y$ exists.

2. With regard to condition 2. above, we note that for $G \in \mathcal{S}L$, Axiom (S4) guarantees that $f[G] \in \mathcal{S}M$, so $\bigvee_{y \in G} f(y)$ exists.

Example 3.5. We give several selection functions, together with their corresponding categories of $\mathcal{S}$-frames. Throughout, A is an arbitrary meet-semilattice.

1. $\mathcal{S}A = \{\{x\} : x \in A\}$. $\mathcal{S}$**Frm** is just the category of meet-semilattices.

2. $\mathcal{S}A = \{G \subseteq A : G \text{ is finite}\}$. $\mathcal{S}$**Frm** is the category of bounded distributive lattices.

3. $\mathcal{S}A = \{G \subseteq A : G \text{ is countable}\}$. $\mathcal{S}$**Frm** is the category of σ-frames.

4. $\mathcal{S}A = \{G \subseteq A : \operatorname{card}(G) < \kappa\}$, where $\operatorname{card}(G)$ denotes the cardinality of G and κ is a regular cardinal. $\mathcal{S}$**Frm** is the category of κ-frames.

5. $\mathcal{S}A = \mathcal{P}A$, the power set of A. $\mathcal{S}$**Frm** is the category of frames.

The above examples appear in the literature; see, for example, [14], [12], [19].

We now introduce an appropriate notion of subobject for $\mathcal{S}$-frames; this will play a crucial rôle in our construction of coreflections in Section 5.

Definition 3.6. If L is a meet-semilattice and $M \subseteq L$, we call M a *sub meet-semilattice of* L if M is a meet-semilattice with the same finite meet, bottom and top as L; this is equivalent to $0 \in M, 1 \in M$ and $x, y \in M \Rightarrow x \wedge y \in M$.

Remark 3.7. We note that M is then a sub meet-semilattice of L if and only if the identical embedding $i : M \to L$ is a meet-semilattice map. Further, in such a case, Axiom (S4) guarantees that, for any selection function $\mathcal{S}$, $G \in \mathcal{S}M$ implies that $G \in \mathcal{S}L$.

Lemma 3.8. *For any $\mathcal{S}$-frame L and $M \subseteq L$, the following conditions are equivalent:*

1. *M is an $\mathcal{S}$-frame and the identical embedding $i : M \to L$ is an $\mathcal{S}$-frame map.*
2. *M is a sub meet-semilattice of L and $G \in \mathcal{S}M$ implies that $\bigvee\limits_M G = \bigvee\limits_L G$.*
3. *M satisfies the conditions:*

$$\text{(i) } 0 \in M \quad \text{(ii) } 1 \in M \quad \text{(iii) } x, y \in M \Rightarrow x \wedge y \in M$$

$$\text{(iv) } G \in \mathcal{S}M \Rightarrow \bigvee\limits_L G \in M.$$

Proof. (1) $\Rightarrow$ (2): Since, in particular, the identical embedding is a meet-semilattice map, M is a sub meet-semilattice of L. (See Remark 3.7.) Since M and L are $\mathcal{S}$-frames, for $G \in \mathcal{S}M$ we have $G \in \mathcal{S}L$, so $\bigvee\limits_M G$ and $\bigvee\limits_L G$ exist. By assumption $i(\bigvee\limits_M G) = \bigvee\limits_L i[G]$, that is $\bigvee\limits_M G = \bigvee\limits_L G$.

(2) $\Rightarrow$ (3): Clear, since $\bigvee\limits_M G \in M$ of course.

(3) $\Rightarrow$ (1): Clearly M is a meet-semilattice. For $G \in \mathcal{S}M$, $\bigvee\limits_L G \in M$ and so this join is also the join of G in M, that is, $\bigvee\limits_L G = \bigvee\limits_M G$. Further, for

any $a \in M$, $a \wedge \bigvee_M G = a \wedge \bigvee_L G = \bigvee_L \{a \wedge y : y \in G\} = \bigvee_M \{a \wedge y : y \in G\}$. (Here we use Axioms (S1) and (S2) to obtain $\{a \wedge y : y \in G\} \in \mathcal{S}M$.) So M is an $\mathcal{S}$-frame. That the identical embedding is an $\mathcal{S}$-frame map is then clear. □

Definition 3.9. Let $\mathcal{S}$ be a selection function and L an $\mathcal{S}$-frame. We shall call a subset M of L a *sub $\mathcal{S}$-frame of L* if it satisfies any of the equivalent conditions of Lemma 3.8.

The following intuitively appealing axiom allows one to obtain information about designated subsets of a sub $\mathcal{S}$-frame. We note that all the selection functions mentioned in Examples 3.5 are easily seen to satisfy it. We have not encountered this axiom elsewhere in the literature.
Axiom (S5): For any $\mathcal{S}$-frame L, if M is a sub $\mathcal{S}$-frame of L, $G \subseteq M$ and $G \in \mathcal{S}L$, then $G \in \mathcal{S}M$.

From now on, all selection functions in this paper will be assumed to satisfy Axiom (S5).

Axiom (S5) is used in the result below, but a more vital use will occur in Proposition 5.2.

Lemma 3.10. *For any $\mathcal{S}$-frame L and $J \subseteq L$ define*

$$\langle J \rangle = \{x \in L : x = \bigvee H_x \text{ for some } H_x \in \mathcal{S}L\} \cup \{0\}$$

where any such H_x consists of finite meets of elements of J.

(a) *$\langle J \rangle$ is a sub $\mathcal{S}$-frame of L.*

(b) *If J is a sub $\mathcal{S}$-frame of L, then $\langle J \rangle = J$.*

Proof. (a) $0 \in \langle J \rangle$ by definition.
$1 \in \langle J \rangle$ since $\{1\} \in \mathcal{S}L$ and 1 is the meet of the empty set.
Suppose that $x, y \in \langle J \rangle$. We need only consider the case where $x \neq 0, y \neq 0$. Take $H_x, H_y \in \mathcal{S}L$ such that all their elements are finite meets of members of J and $x = \bigvee H_x, y = \bigvee H_y$. Then Axiom (S2) guarantees that $\{a \wedge b : a \in H_x, b \in H_y\} \in \mathcal{S}L$ and clearly all elements of this set are finite meets of members of J. Then $x \wedge y = \bigvee\{a \wedge b : a \in H_x, b \in H_y\}$. So $x \wedge y \in \langle J \rangle$.

Thus far, it has been established that $\langle J\rangle$ is a sub meet-semilattice of L.
Take $G \in \mathcal{S}\langle J\rangle$. If $G = \emptyset$ then $\bigvee G = 0 \in \langle J\rangle$. Otherwise, for each $x \in G$, $x = \bigvee H_x$ for some $H_x \in \mathcal{S}L$ such that all elements of H_x are finite meets of elements of J. Since $\langle J\rangle$ is a sub meet-semilattice of L, $G \in \mathcal{S}L$. Apply Axiom (S3) to get $\bigcup\limits_{x\in G} H_x \in \mathcal{S}L$. Then $\bigvee(\bigcup\limits_{x\in G} H_x) \in \langle J\rangle$. A straightforward check shows that $\bigvee G = \bigvee(\bigcup\limits_{x\in G} H_x)$, so $\bigvee G \in \langle J\rangle$, as required.

(b) It is easy to see that $J \subseteq \langle J\rangle$. For the converse, one uses Axiom (S5). □

Definition 3.11. With the notation of Lemma 3.10, we call $\langle J\rangle$ the *sub $\mathcal{S}$-frame of L generated by J.*

Remark 3.12. We note that our definition of selection function (see Definition 3.1) does not require the empty set to be selected. Had that condition been imposed, the definition of $\langle J\rangle$ would automatically result in $0 \in \langle J\rangle$. We chose our formalism to keep the number of axioms to a minimum.

4 Structured $\mathcal{S}$-frames

The definition below provides an appropriate notion of cover for an $\mathcal{S}$-frame, which we then use to construct nearness and uniform structures.

Definition 4.1. Let $\mathcal{S}$ be a selection function and L an $\mathcal{S}$-frame.

1. We call C an *$\mathcal{S}$-cover* of L if $C \in \mathcal{S}L$ and $\bigvee C = 1$.
2. If C and D are $\mathcal{S}$-covers of L, then $C \wedge D = \{c \wedge d : c \in C, d \in D\}$ is an $\mathcal{S}$-cover of L.
3. If C and D are $\mathcal{S}$-covers of L, we say that C *refines* D and write $C \leq D$ if, for all $c \in C$, there exists $d \in D$ such that $c \leq d$.
4. If $a \in L$ and C is an $\mathcal{S}$-cover of L, we set $C_a = \{c \in C : c \wedge a \neq 0\}$. In general, one cannot expect $\bigvee C_a$ to exist, but when it does, we write $Ca = \bigvee C_a$, as usual.

5. If $a, b \in L$ and C is an $\mathcal{S}$-cover of L, we write $a \lhd_C b$ if $C_a \subseteq \downarrow b$. Here, $\downarrow b = \{t \in L : t \leq b\}$ as usual. We say that a is *uniformly below* b *with respect to* C.

6. If C and D are $\mathcal{S}$-covers of L, we say that C *star-refines* D, and write $C <^* D$, if, for all $c \in C$, there is $d \in D$ such that $c \lhd_C d$.

Remark 4.2. (1) The definition of $\mathcal{S}$-cover adopted above appears in [14] and naturally generalizes the notion of cover of a σ-frame. (See, for instance, [13].)
(2) We note that the definitions of C_a and $\lhd_C$ above appear in [6] but these authors use a different notion of cover in their paper.

Lemma 4.3. *If $f : L \to M$ is an $\mathcal{S}$-frame map between $\mathcal{S}$-frames and C is an $\mathcal{S}$-cover of L, then $f[C]$ is an $\mathcal{S}$-cover of M.*

Proof. Since f is, in particular, a meet-semilattice map, Axiom (S4) applies, so $C \in \mathcal{S}L$ implies that $f[C] \in \mathcal{S}M$. Since f is an $\mathcal{S}$-frame map, it preserves joins of designated sets, so $f(\bigvee C) = \bigvee f[C]$, giving $\bigvee f[C] = 1$. □

Definition 4.4. Let $\mathcal{S}$ be a selection function and L an $\mathcal{S}$-frame. We call $\mathcal{K}L$ an *$\mathcal{S}$-nearness on L* if

1. $\mathcal{K}L$ is a non-empty collection of $\mathcal{S}$-covers of L.

2. For $C, D \in \mathcal{K}L$, $C \wedge D \in \mathcal{K}L$.

3. If $C \in \mathcal{K}L$ and D is an $\mathcal{S}$-cover of L such that $C \leq D$, then $D \in \mathcal{K}L$.

4. For each $a \in L$, there exists $T \in \mathcal{S}L$ such that $a = \bigvee T$ and, for each $t \in T$, $t \lhd_C a$ for some $C \in \mathcal{K}L$. (This is the compatibility condition.)

- If $\mathcal{K}L$ is an $\mathcal{S}$-nearness on L, we call $(L, \mathcal{K}L)$ a *nearness $\mathcal{S}$-frame.*

- We write $a \lhd b$ *in* $(L, \mathcal{K}L)$ if there exists $C \in \mathcal{K}L$ such that $a \lhd_C b$.

- A nearness $\mathcal{S}$-frame $(L, \mathcal{K}L)$ is *strong* if for all $D \in \mathcal{K}L$ there exists $C \in \mathcal{K}L$ such that, for all $c \in C$ there exists $d \in D$ and $E \in \mathcal{K}L$ such that $c \lhd_E d$. In this case, we write $C \lhd D$.

- An $\mathcal{S}$-nearness $\mathcal{K}L$ on L is *totally bounded* if for each $C \in \mathcal{K}L$, there exists $D \in \mathcal{K}L$ such that D is finite and $D \leq C$.

- A nearness $\mathcal{S}$-frame $(L, \mathcal{K}L)$ is a *uniform* $\mathcal{S}$-frame if for all $D \in \mathcal{K}L$ there exists $C \in \mathcal{K}L$ with $C <^* D$. We then call $\mathcal{K}L$ an *$\mathcal{S}$-uniformity on L*.

Definition 4.5. Let $(L, \mathcal{K}L)$, $(M, \mathcal{K}M)$ be nearness $\mathcal{S}$-frames. Then $f : (L, \mathcal{K}L) \to (M, \mathcal{K}M)$ is said to be *uniform* if $f : L \to M$ is an $\mathcal{S}$-frame map and, for each $C \in \mathcal{K}L$, $f[C] \in \mathcal{K}M$.

Definition 4.6. **Near$\mathcal{S}$Frm** is the category having nearness $\mathcal{S}$-frames as objects and uniform maps as morphisms. **Uni$\mathcal{S}$Frm** is the category having uniform $\mathcal{S}$-frames as objects and uniform maps as morphisms.

Example 4.7. For the following two examples of selection functions, already mentioned in Example 3.5, the categories **Near$\mathcal{S}$Frm** are well-known:

1. For $\mathcal{S}A = \mathcal{P}A$, **Near$\mathcal{S}$Frm** is the category of nearness frames and **Uni$\mathcal{S}$Frm** is the category of uniform frames.
2. For $\mathcal{S}A = \{G \subseteq A : G \text{ is countable}\}$, **Near$\mathcal{S}$Frm** is the category of nearness σ-frames and **Uni$\mathcal{S}$Frm** is the category of uniform σ-frames.

Remark 4.8. For any selection function $\mathcal{S}$, the two-element chain $\{0, 1\}$, denoted by $\mathbf{2}$, is an $\mathcal{S}$-frame. It carries a unique $\mathcal{S}$-nearness structure, but what the structure is depends on $\mathcal{S}$. For $\mathcal{S}A = \{\{x\} : x \in A\}$, $\mathcal{K}\mathbf{2} = \{\{1\}\}$. For all the other cases of Example 3.5, $\mathcal{K}\mathbf{2} = \{\{0, 1\}, \{1\}\}$.

The following straightforward lemma summarizes the expected elementary properties of star-refinements.

Lemma 4.9. 1. *Let E_1, E_2, D_1 and D_2 be $\mathcal{S}$-covers of an $\mathcal{S}$-frame. Then*
(i) $E_1 <^* D_1 \Rightarrow E_1 \leq D_1$ *and*
(ii) $E_1 <^* D_1, E_2 <^* D_2 \Rightarrow E_1 \wedge E_2 <^* D_1 \wedge D_2$.

2. *If f is a uniform map from $(L, \mathcal{K}L)$ to $(M, \mathcal{K}M)$ and $C <^* D$ in $\mathcal{K}L$ then $f[C] <^* f[D]$ in $\mathcal{K}M$.*

3. *Every uniform $\mathcal{S}$-frame is strong.*

Proof. All these proofs are routine. We illustrate with a proof of 2. Suppose that $f : (L, \mathcal{K}L) \to (M, \mathcal{K}M)$ is a uniform map between nearness $\mathcal{S}$-frames, and $E <^* D$, for $E, D \in \mathcal{K}L$. Then, for each $e \in E$, there exists $d \in D$ with $E_e \subseteq \downarrow d$. Then $f[E]_{f(e)} \subseteq \downarrow f(d)$, showing that $f[E] <^* f[D]$. $\square$

As in the case of frames and σ-frames, we have:

Proposition 4.10. *A strong, totally bounded nearness $\mathcal{S}$-frame is uniform.*

Proof. Let $(L, \mathcal{K}L)$ be a strong totally bounded nearness $\mathcal{S}$-frame. Take $C \in \mathcal{K}L$. There exists a finite $D \in \mathcal{K}L$ such that $D \leq C$. Further, there exists $E \in \mathcal{K}L$ such that $E \lhd D$. Lastly, take a finite $F = \{x_1, x_2, \ldots, x_n\} \in \mathcal{K}L$ such that $F \leq E$. Now take $A \in \mathcal{K}L$ such that, for all $j = 1, 2, \ldots, n$, $A_{x_j} \subseteq \downarrow c_j$ for some $c_j \in C$. Such A exists, since F is finite. One concludes the proof by showing in a routine way that $A \wedge F <^* C$. $\square$

Constructing uniform images is an important ingredient in the theory of uniform and nearness frames; we show now that Axioms (S1) to (S5) allow us to do the same for nearness $\mathcal{S}$-frames.

Proposition 4.11. *Suppose that L and M are $\mathcal{S}$-frames, $(L, \mathcal{K}L)$ is a nearness $\mathcal{S}$-frame and $f : L \to M$ is an $\mathcal{S}$-frame map. Define $f(L, \mathcal{K}L) = (f[L], \langle f[\mathcal{K}L]\rangle)$, where $f[L] = \{f(x) : x \in L\}$ and $\langle f[\mathcal{K}L]\rangle = \{D \in \mathcal{S}M : D \geq f[C]$, for some $C \in \mathcal{K}L\}$. Then $f(L, \mathcal{K}L)$ is a nearness $\mathcal{S}$-frame, and $f : (L, \mathcal{K}L) \to f(L, \mathcal{K}L)$ is a uniform map.*

Proof. First we show that $f[L]$ is an $\mathcal{S}$-frame. As mentioned in Remark 3.2 (2), $f[L]$ is a meet-semilattice. Now take $H \in \mathcal{S}(f[L])$. By Axiom (S4) $H = f[G]$, for some $G \in \mathcal{S}L$. Since f is an $\mathcal{S}$-frame map, $f(\bigvee G) = \bigvee f[G] = \bigvee H$, so $\bigvee H$ does exist. To check the required distributivity, take $y \in f[L]$. Then $y = f(x)$, for some $x \in L$.
Then

$$\begin{aligned} y \wedge \bigvee H &= f(x) \wedge \bigvee f[G] = f(x) \wedge f(\bigvee G) = f(x \wedge \bigvee G) \\ &= f(\bigvee_{a \in G} x \wedge a) = \bigvee_{a \in G} f(x \wedge a) = \bigvee_{a \in G} f(x) \wedge f(a) = \bigvee_{b \in H} y \wedge b. \end{aligned}$$

We note that the fifth equality uses the fact that $\{x \wedge a : a \in G\} \in \mathcal{S}L$ which follows from the facts that $\{x\}, G \in \mathcal{S}L$ and Axiom (S2).

Next we show that $\langle f[\mathcal{K}L]\rangle$ is an $\mathcal{S}$-nearness on $f[L]$. By Lemma 4.3, if $C \in \mathcal{K}L$, then $f[C]$ is an $\mathcal{S}$-cover of M, and also of $f[L]$, by (S5). So $\langle f[\mathcal{K}L]\rangle$ does consist of $\mathcal{S}$-covers of $f[L]$.

If $D_1, D_2 \in \langle f[\mathcal{K}L] \rangle$, then $D_1 \geq f[C_1]$ and $D_2 \geq f[C_2]$ for some $C_1, C_2 \in \mathcal{K}L$. Then $D_1 \wedge D_2 \geq f[C_1] \wedge f[C_2] = f[C_1 \wedge C_2]$ and $C_1 \wedge C_2 \in \mathcal{K}L$. Further, $D_1 \wedge D_2 \in \mathcal{S}M$, by Axiom (S2). So $\langle f[\mathcal{K}L] \rangle$ is closed under finite meets.

It is clear that, if $D \in \langle f[\mathcal{K}L] \rangle$ and E is an $\mathcal{S}$-cover of $f[L]$ such that $E \geq D$, then $E \in \langle f[\mathcal{K}L] \rangle$.

For the compatibility condition, begin with $b \in f[L]$. Then $b = f(a)$ for some $a \in L$. Now $a = \bigvee T$ for some $T \in \mathcal{S}L$ such that $t \in T$ implies that $t \triangleleft_C a$ for some $C \in \mathcal{K}L$. Then $f[C] \in \langle f[\mathcal{K}L] \rangle$ and $f(a) = f(\bigvee T) = \bigvee f[T]$, since f is an $\mathcal{S}$-frame map. We conclude the proof by showing that $t \triangleleft_C a$ implies that $f(t) \triangleleft_{f[C]} f(a)$:

$$\begin{aligned} &f[C]_{f(t)} = \{f(c) : c \in C, f(c) \wedge f(t) \neq 0\} \subseteq \{f(c) : c \in C, c \wedge t \neq 0\} \\ &= f[C_t] \subseteq f[{\downarrow} a] \subseteq {\downarrow} f(a) \end{aligned}$$

The fact that $f : (L, \mathcal{K}L) \to f(L, \mathcal{K}L)$ is then uniform is clear, since $C \in \mathcal{K}L$ implies that $f[C] \in \langle f[\mathcal{K}L] \rangle$, as remarked above. □

5 Constructing coreflections

We now investigate the structure of the sub nearness $\mathcal{S}$-frames of a given nearness $\mathcal{S}$-frame, showing that they form a complete lattice.

Definition 5.1. Let $(L, \mathcal{K}L)$ and $(M, \mathcal{K}M)$ be nearness $\mathcal{S}$-frames. We call $(L, \mathcal{K}L)$ a *sub nearness $\mathcal{S}$-frame* of $(M, \mathcal{K}M)$ if L is a sub $\mathcal{S}$-frame of M and $\mathcal{K}L \subseteq \mathcal{K}M$. We note that this is equivalent to the identical embedding from $(L, \mathcal{K}L)$ to $(M, \mathcal{K}M)$ being a uniform map. We then write $(L, \mathcal{K}L) \leq (M, \mathcal{K}M)$.

Proposition 5.2. *Let $(L, \mathcal{K}L)$ be a nearness $\mathcal{S}$-frame. The collection of all sub nearness $\mathcal{S}$-frames of $(L, \mathcal{K}L)$ forms a complete lattice.*

Proof. The relation $\leq$ given in Definition 5.1 is a partial order. The bottom element is clearly the two element frame with its unique $\mathcal{S}$-nearness (except in the case where L is degenerate, in which case it is L itself).

Let $\{(L_\alpha, \mathcal{K}L_\alpha) : \alpha \in I\}$ be a non-empty collection of sub nearness $\mathcal{S}$-frames of $(L, \mathcal{K}L)$.

• Let $\widetilde{L}$ be the sub $\mathcal{S}$-frame of L generated by $\bigcup\limits_{\alpha \in I} L_\alpha$.

• Define $\mathcal{K}\widetilde{L}$ as follows: $C \in \mathcal{K}\widetilde{L}$ if and only if $C \in \mathcal{S}\widetilde{L}$ and there exists a natural number n and $D_{\alpha_j} \in \mathcal{K}L_{\alpha_j}$ for $j = 1, \ldots, n$ such that $D_{\alpha_1} \wedge \ldots \wedge D_{\alpha_n} \leq C$.
We now show that $(\widetilde{L}, \mathcal{K}\widetilde{L})$ is the join of $\{(L_\alpha, \mathcal{K}L_\alpha) : \alpha \in I\}$, by noting the following points:

1. For each $\alpha \in I$, $\mathcal{K}L_\alpha \subseteq \mathcal{K}\widetilde{L}$.
2. For $a, b \in L_\alpha$, $a \lhd b$ in $(L_\alpha, \mathcal{K}L_\alpha)$ implies that $a \lhd b$ in $(\widetilde{L}, \mathcal{K}\widetilde{L})$, since $C_a \subseteq{\downarrow}b$ for some $C \in \mathcal{K}L_\alpha$ gives $C_a \subseteq{\downarrow}b$ for that same $C \in \mathcal{K}\widetilde{L}$.
3. $\mathcal{K}\widetilde{L}$ is closed under finite meets.
4. If $C \in \mathcal{K}\widetilde{L}$, $D \in \mathcal{S}\widetilde{L}$ and $C \leq D$, then $D \in \mathcal{K}\widetilde{L}$.
5. Take $a \in \widetilde{L}$. The case $a = 0$ presents no difficulties. Write $a = \bigvee H$ for some $H \in \mathcal{S}L$ such that all the elements of H are finite meets of elements of $B = \bigcup\limits_{\alpha \in I} L_\alpha$.
   Fix $x \in H$. Write $x = b_{\alpha_1} \wedge \ldots \wedge b_{\alpha_n}$ for some $b_{\alpha_j} \in L_{\alpha_j}$. For each $j = 1, \ldots, n$, $b_{\alpha_j} = \bigvee G_{\alpha_j}$ for some $G_{\alpha_j} \in \mathcal{S}L_{\alpha_j}$ such that $u \in G_{\alpha_j} \Rightarrow u \lhd b_{\alpha_j}$ in $(L_{\alpha_j}, \mathcal{K}L_{\alpha_j})$, and hence in $(\widetilde{L}, \mathcal{K}\widetilde{L})$.
   Let $Z_x = G_{\alpha_1} \wedge \ldots \wedge G_{\alpha_n}$. Then $Z_x \in \mathcal{S}\widetilde{L}$ and $w \in Z_x \Rightarrow w \lhd x$ in $(\widetilde{L}, \mathcal{K}\widetilde{L})$. Further, $x = \bigvee Z_x$.
   Now let $Z = \bigcup\limits_{x \in H} Z_x$. Then $a = \bigvee Z$ and $w \in Z \Rightarrow w \lhd a$ in $(\widetilde{L}, \mathcal{K}\widetilde{L})$.
   What remains to be shown is that $Z \in \mathcal{S}\widetilde{L}$.
   Since $a = \bigvee H$, $H \in \mathcal{S}L$ and for all $x \in H$, $x = \bigvee Z_x$ for $Z_x \in \mathcal{S}L$, Axiom (S3) guarantees that $Z \in \mathcal{S}L$. Since $\widetilde{L}$ is a sub $\mathcal{S}$-frame of L, $Z \subseteq \widetilde{L}$ and $Z \in \mathcal{S}L$, Axiom (S5) guarantees that $Z \in \mathcal{S}\widetilde{L}$.
6. $(\widetilde{L}, \mathcal{K}\widetilde{L})$ is a nearness $\mathcal{S}$-frame from 3,4 and 5 above.
7. $(\widetilde{L}, \mathcal{K}\widetilde{L})$ is a sub nearness $\mathcal{S}$-frame of $(L, \mathcal{K}L)$, since $\widetilde{L}$ is a sub $\mathcal{S}$-frame of L and $\mathcal{K}\widetilde{L} \subseteq \mathcal{K}L$. The latter follows since $\mathcal{K}L_\alpha \subseteq \mathcal{K}L$ and if $C \in \mathcal{S}\widetilde{L}$ with $D_{\alpha_1} \wedge \ldots \wedge D_{\alpha_n} \leq C$, for some $D_{\alpha_j} \in \mathcal{K}L_{\alpha_j}$, then $D_{\alpha_1} \wedge \ldots \wedge D_{\alpha_n} \in \mathcal{K}L$ and so $C \in \mathcal{K}L$.
8. If $(M, \mathcal{K}M)$ is a sub nearness $\mathcal{S}$-frame of $(L, \mathcal{K}L)$ such that $(L_\alpha, \mathcal{K}L_\alpha) \leq (M, \mathcal{K}M)$ for all $\alpha \in I$, then L_α is a sub $\mathcal{S}$-frame of M and $\mathcal{K}L_\alpha \subseteq \mathcal{K}M$

for all $\alpha \in I$. So $\widetilde{L}$ is a sub $\mathcal{S}$-frame of M and $\mathcal{K}\widetilde{L} \subseteq \mathcal{K}M$. We see that $(\widetilde{L}, \mathcal{K}\widetilde{L})$ is indeed the join of $\{(L_\alpha, \mathcal{K}L_\alpha) : \alpha \in I\}$, as required.

□

In this paper, we will use P to denote an arbitrary property that a nearness $\mathcal{S}$-frame might have. We introduce the idea of a P-approximation of a nearness $\mathcal{S}$-frame and use it to construct a functor from **Near$\mathcal{S}$Frm** to itself, in the case that the property P is preserved by uniform images.

Definition 5.3. Let $(L, \mathcal{K}L)$ be a nearness $\mathcal{S}$-frame.

1. We call those sub nearness $\mathcal{S}$-frames of $(L, \mathcal{K}L)$ that have property P, the *P-approximations* of $(L, \mathcal{K}L)$.
2. Define $\Gamma_P(L, \mathcal{K}L)$ to be the join of all the P-approximations of $(L, \mathcal{K}L)$ (as provided in Proposition 5.2 of course).

By Proposition 5.2, $\Gamma_P(L, \mathcal{K}L)$ is a nearness $\mathcal{S}$-frame. We make no claim that $\Gamma_P(L, \mathcal{K}L)$ necessarily satisfies property P, but will, of course, be most interested in those properties P where it does.

We note that $\Gamma_P(L, \mathcal{K}L)$ is defined in the case where a given nearness $\mathcal{S}$-frame has no P-approximations. It is the empty join.

Remark 5.4. Let $h : (L, \mathcal{K}L) \to (M, \mathcal{K}M)$ be a uniform map between nearness $\mathcal{S}$-frames. If we define $h(L, \mathcal{K}L)$ as in Proposition 4.11 it is straightforward to check that $h(L, \mathcal{K}L)$ is a sub nearness $\mathcal{S}$-frame of $(M, \mathcal{K}M)$ and that $h : (L, \mathcal{K}L) \to h(L, \mathcal{K}L)$ is a uniform map.

Definition 5.5. If a property P satisfies the condition that, whenever a nearness $\mathcal{S}$-frame $(L, \mathcal{K}L)$ has property P, then $h(L, \mathcal{K}L)$ has property P for any uniform h, we say that *P is preserved by uniform images.*

Proposition 5.6. *Let P be a property that is preserved by uniform images. Then Γ_P : **Near$\mathcal{S}$Frm** $\to$ **Near$\mathcal{S}$Frm** is a functor.*

Proof. Γ_P was defined on objects in Definition 5.3. We define Γ_P on morphisms as follows. Let $h : (L, \mathcal{K}L) \to (M, \mathcal{K}M)$ be a uniform map between nearness $\mathcal{S}$-frames. We show below that $\Gamma_P\, h : \Gamma_P(L, \mathcal{K}L) \to \Gamma_P(M, \mathcal{K}M)$ given by restricting the domain and codomain of h is again a uniform map. For brevity we write $\Gamma_P(L, \mathcal{K}L) = (\widetilde{L}, \mathcal{K}\widetilde{L})$ and $\Gamma_P(M, \mathcal{K}M) = (\widetilde{M}, \mathcal{K}\widetilde{M})$.

Let $\{(L_\alpha, \mathcal{K}L_\alpha) : \alpha \in I\}$ be the collection of all P-approximations of $(L, \mathcal{K}L)$. For any $\alpha \in I$, by assumption, $h(L_\alpha, \mathcal{K}L_\alpha)$ is a P-approximation of $(M, \mathcal{K}M)$. So $h(L_\alpha, \mathcal{K}L_\alpha) \leq (\widetilde{M}, \mathcal{K}\widetilde{M})$. Then $h[L_\alpha] \subseteq \widetilde{M}$ for all $\alpha \in I$, giving $h[\widetilde{L}] \subseteq \widetilde{M}$. Further, $h[\mathcal{K}L_\alpha] \subseteq \mathcal{K}\widetilde{M}$ for all $\alpha \in I$, so $\langle h[\mathcal{K}\widetilde{L}]\rangle \subseteq \mathcal{K}\widetilde{M}$.

That Γ_P preserves identities and composition is clear. □

We are now in a position to provide the promised construction of certain coreflections.

Theorem 5.7. *Let P be a property satisfying the conditions:*

1. *P is preserved by uniform images, and*
2. *for any nearness $\mathcal{S}$-frame $(L, \mathcal{K}L)$, the join $\Gamma_P(L, \mathcal{K}L)$ of all P-approximations of $(L, \mathcal{K}L)$ has property P.*

Then the nearness $\mathcal{S}$-frames with property P form a full monocoreflective subcategory of all nearness $\mathcal{S}$-frames.

Proof. Let P be a property described as above and let $(L, \mathcal{K}L)$ be a nearness $\mathcal{S}$-frame. We show that the identical embedding $\eta_P : \Gamma_P(L, \mathcal{K}L) \to (L, \mathcal{K}L)$ is the desired coreflection map.
Let $(M, \mathcal{K}M)$ be a nearness $\mathcal{S}$-frame with property P and $f : (M, \mathcal{K}M) \to (L, \mathcal{K}L)$ a uniform map. Now $f(M, \mathcal{K}M)$ is a sub nearness $\mathcal{S}$-frame of $(L, \mathcal{K}L)$ and has property P, so is a P-approximation of $(L, \mathcal{K}L)$. This makes the identical embedding $i : f(M, \mathcal{K}M) \to \Gamma_P(L, \mathcal{K}L)$ a uniform map and we have the following obvious commuting diagram:

$$\begin{tikzcd}
\Gamma_P(L, \mathcal{K}L) \arrow[r, "\eta_P"] & (L, \mathcal{K}L) \\
f(M, \mathcal{K}M) \arrow[u, "i"'] & (M, \mathcal{K}M) \arrow[l, "f"] \arrow[u, "f"]
\end{tikzcd}$$

The factorization of f is unique, because η_P is $1-1$, and hence a monomorphism. □

Definition 5.8. We call the coreflection constructed in Theorem 5.7 the *P-coreflection of nearness $\mathcal{S}$-frames.*

We note that a morphism $h : (L, \mathcal{K}L) \to (M, \mathcal{K}M)$ in **Near$\mathcal{S}$Frm** is an isomorphism if and only if $h : L \to M$ is an $\mathcal{S}$-frame isomorphism and $\langle h[\mathcal{K}L]\rangle = \mathcal{K}M$. Further, if $f : (L, \mathcal{K}L) \to (M, \mathcal{K}M)$ is a morphism in **Near$\mathcal{S}$Frm** and f is $1-1$, then $(L, \mathcal{K}L)$ is isomorphic to $f(L, \mathcal{K}L)$ which is a sub nearness $\mathcal{S}$-frame of $(M, \mathcal{K}M)$.

In the next result, we show that any full, isomorphism-closed coreflective subcategory, **H**, of **Near$\mathcal{S}$Frm** for which the coreflection maps are all $1-1$, can in fact be obtained by the construction of Theorem 5.7. We simply define $(L, \mathcal{K}L)$ to have property P whenever $(L, \mathcal{K}L)$ is an object of **H**; the details are as in the corresponding result in [9].

Proposition 5.9. *Let* **H** *be a full, isomorphism-closed, coreflective subcategory of* **Near$\mathcal{S}$Frm** *for which the* **H***-coreflection maps are all* $1-1$. *Define* P *by stating that a nearness* $\mathcal{S}$*-frame* $(L, \mathcal{K}L)$ *satisfies* P *if and only if* $(L, \mathcal{K}L)$ *is an object of* **H**. *Then the* P*-coreflection and the* **H***-coreflection of any nearness* $\mathcal{S}$*-frame are isomorphic.*

The results in this section can of course be used to obtain the corresponding results for nearness frames in [9].

6 Three applications

It is well known that uniform frames are coreflective in nearness frames and that totally bounded nearness frames are coreflective in nearness frames [8], [1]; the fact that strong nearness frames are coreflective in nearness frames was established only recently, in [9]. These three results can now be generalized elegantly to the nearness $\mathcal{S}$-frame setting, using the techniques of the previous section.

Proposition 6.1. *The strong nearness* $\mathcal{S}$*-frames form a coreflective subcategory of all nearness* $\mathcal{S}$*-frames.*

Proof. We apply Theorem 5.7.
We begin by showing that the uniform image of a strong nearness $\mathcal{S}$-frame is again strong. Let $(L, \mathcal{K}L)$ be strong and $f : (L, \mathcal{K}L) \to f(L, \mathcal{K}L)$ provide a uniform image, as defined in Proposition 4.11. For any $F \in \langle f[\mathcal{K}L]\rangle$, $F \geq f[E]$ for some $E \in \mathcal{K}L$. Since $\mathcal{K}L$ is strong, there exists $D \in \mathcal{K}L$ such that $D \lhd E$. So, for all $d \in D$, there exists $e \in E$ and $C \in \mathcal{K}L$ such that

$d \lhd_C e$, that is, $C_d = \{c \in C : c \wedge d \neq 0\} \subseteq\downarrow e$. Then $f(d) \lhd_{f[C]} f(e)$ as shown in the proof of Proposition 4.11. So $f[D] \lhd f[E]$. So we have $f[D] \in \langle f[\mathcal{K}L] \rangle$ such that $f[D] \lhd F$, as required.

For the second condition, let $(L, \mathcal{K}L)$ be a nearness $\mathcal{S}$-frame, $\{(L_\alpha, \mathcal{K}L_\alpha) : \alpha \in I\}$ the set of its strong approximations and $(\widetilde{L}, \mathcal{K}\widetilde{L})$ their join. (So this is $\Gamma_P(L, \mathcal{K}L)$ where P is the property of being strong.) We show that $(\widetilde{L}, \mathcal{K}\widetilde{L})$ is strong.
For $C \in \mathcal{K}\widetilde{L}$, $C \geq D_{\alpha_1} \wedge \ldots \wedge D_{\alpha_n}$ for some $D_{\alpha_j} \in \mathcal{K}L_{\alpha_j}$, $j = 1, \ldots, n$. For all j, take $E_{\alpha_j} \in \mathcal{K}L_{\alpha_j}$ such that $E_{\alpha_j} \lhd D_{\alpha_j}$ in $(L_{\alpha_j}, \mathcal{K}L_{\alpha_j})$, and let $E = E_{\alpha_1} \wedge \ldots \wedge E_{\alpha_n}$. Then $E \in \mathcal{K}\widetilde{L}$, since $\mathcal{K}L_{\alpha_j} \subseteq \mathcal{K}\widetilde{L}$ and $\mathcal{K}\widetilde{L}$ is closed under finite meets. We conclude the proof by showing that $E \lhd C$ in $(\widetilde{L}, \mathcal{K}\widetilde{L})$. For this, begin with $e = e_1 \wedge \ldots \wedge e_n \in E$, where $e_j \in E_{\alpha_j}$. For each $j = 1, \ldots, n$, there exists $d_j \in D_{\alpha_j}$ and $F^j \in \mathcal{K}L_{\alpha_j}$ such that $e_j \lhd_{F^j} d_j$ in $(L_{\alpha_j}, \mathcal{K}L_{\alpha_j})$. Write $F = F^1 \wedge \ldots \wedge F^n$ and $d = d_1 \wedge \ldots \wedge d_n$. A straightforward calculation shows that $F_e \subseteq F^j_{e_j} \subseteq\downarrow d_j$ for all $j = 1, \ldots, n$ giving $F_e \subseteq\downarrow d$ and so $e \lhd_F d$ as required. □

Proposition 6.2. *The totally bounded nearness $\mathcal{S}$-frames form a coreflective subcategory of all nearness $\mathcal{S}$-frames.*

Proof. One checks routinely that uniform images of totally bounded nearness $\mathcal{S}$-frames remain totally bounded, and that the join of all totally bounded approximations of a nearness $\mathcal{S}$-frame is again totally bounded. Theorem 5.7 then applies. □

Proposition 6.3. *The uniform $\mathcal{S}$-frames form a coreflective subcategory of all nearness $\mathcal{S}$-frames.*

Proof. We show that Theorem 5.7 applies: To check that uniform images of uniform nearness $\mathcal{S}$-frames are uniform, use Lemma 4.9(2) to see that if $A <^* B$ then $h[A] <^* h[B]$, where h is a uniform map.
Let $(L, \mathcal{K}L)$ be a nearness $\mathcal{S}$-frame, $\{(L_\alpha, \mathcal{K}L_\alpha) : \alpha \in I\}$ the set of its uniform approximations and $(\widetilde{L}, \mathcal{K}\widetilde{L})$ their join. We show that $(\widetilde{L}, \mathcal{K}\widetilde{L})$ is uniform.
For $C \in \mathcal{K}\widetilde{L}$, $C \geq D_{\alpha_1} \wedge \ldots \wedge D_{\alpha_n}$ for some $D_{\alpha_j} \in \mathcal{K}L_{\alpha_j}$, $j = 1, \ldots, n$. For all j, take $E_{\alpha_j} \in \mathcal{K}L_{\alpha_j}$ such that $E_{\alpha_j} <^* D_{\alpha_j}$ in $(L_{\alpha_j}, \mathcal{K}L_{\alpha_j})$. Let $E = E_{\alpha_1} \wedge \ldots \wedge E_{\alpha_n}$. Then $E <^* D_{\alpha_1} \wedge \ldots \wedge D_{\alpha_n}$ (see Lemma 4.9) and $E \in \mathcal{K}\widetilde{L}$, since $\mathcal{K}L_{\alpha_j} \subseteq \mathcal{K}\widetilde{L}$ and $\mathcal{K}\widetilde{L}$ is closed under finite meets. □

In Part (II) of this paper ([10]) we consider regularity, normality and compactness for partial frames.

References

[1] D. Baboolal and R.G. Ori, Samuel compactification and uniform coreflection of nearness frames, Proceedings Symposium on Categorical Topology (1994), University of Cape Town, 1999.

[2] B. Banaschewski, *Completion in pointfree topology*, Lecture Notes in Math. and Applied Math., University of Cape Town, No. 2 (1996).

[3] B. Banaschewski, *Uniform completion in pointfree topology*, chapter in Topological and Algebraic Structures in Fuzzy Sets, S.E. Rodabaugh and E.P. Klement (Ed.s), Kluwer Academic Publishers, (2003) 19-56.

[4] B. Banaschewski and C.R.A. Gilmour, *Realcompactness and the cozero part of a frame*, Appl. Categ. Structures 9 (2001), 395-417.

[5] B. Banaschewski, S.S. Hong, and A. Pultr, *On the completion of nearness frames*, Quaest. Math. 21 (1998), 19-37.

[6] B. Banaschewski and A. Pultr, *A general view of approximation*, Appl. Categ. Structures 14 (2006), 165-190.

[7] B. Banaschewski and A. Pultr, *Cauchy points of uniform and nearness frames*, Quaest. Math. 19 (1996), 101-127.

[8] T. Dube, *A note on complete regularity and normality*, Quaest. Math. 19 (1996), 467-478.

[9] J. Frith and A. Schauerte, *A method for constructing coreflections for nearness frames*, Appl. Categ. Structures (to appear).

[10] J. Frith and A. Schauerte, *Uniformities and covering properties for partial frames* (II), Categ. General Alg. Struct. Appl. 2(1) (2014), 23-35.

[11] P.T. Johnstone, "Stone Spaces", Cambridge University Press, Cambridge, 1982.

[12] J.J. Madden, *κ-frames*, J. Pure Appl. Algebra 70 (1991), 107-127.

[13] I. Naidoo, *Aspects of nearness in σ-frames*, Quaest. Math. 30 (2007), 133-145.

[14] J. Paseka, Covers in generalized frames, in: General Algebra and Ordered Sets (Horni Lipova 1994), Palacky Univ. Olomouc, Olomouc, 84-99.

[15] J. Picado and A. Pultr, "Frames and Locales", Springer, Basel, 2012.

[16] J. Picado, A. Pultr, and A. Tozzi, Locales, chapter in Categorical Foundations, MC Pedicchio and W Tholen (eds), Encyclopedia of Mathematics and its Applications 97, Cambridge University Press, Cambridge, (2004) 49-101.

[17] S. Vickers, "Topology via Logic", Cambridge Tracts in Theoretical Computer Science 5, Cambridge University Press, Cambridge, 1989.

[18] J. Walters, *Compactifications and uniformities on sigma frames*, Comment. Math. Univ. Carolinae 32(1) (1991), 189-198.

[19] E.R. Zenk, *Categories of partial frames*, Algebra Universalis 54 (2005), 213-235.

[20] D. Zhao, *On projective Z-frames*, Canad. Math. Bull. 40(1) (1997), 39-46.

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