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Mappings to Realcompactifications

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This article is dedicated to Professor A.R. Aliabad

Abstract. In this paper, we introduce and study a mapping from the collection of all intermediate rings of $C(X)$ to the collection of all realcompactifications of X contained in βX. By establishing the relations between this mapping and its converse, we give a different approach to the main statements of De et. al. Using these, we provide different answers to the four basic questions raised in Acharyya et.al. Finally, we give some notes on the realcompactifications generated by ideals.

1 Introduction and Preliminaries

Throughout this article all topological spaces are assumed to be Tychonoff. For a given topological space X, $C(X)$ denotes the algebra of all real-valued continuous functions on X and $C^*(X)$ denotes the subalgebra of $C(X)$ consisting of all bounded elements. The reader is referred to [11] for undefined terms and notations concerning $C(X)$. A subring $A(X)$ of $C(X)$ is called an intermediate ring, if $C^*(X) \subseteq A(X)$. A subring of $C(X)$ is called a C-ring, if it is isomorphic with $C(Y)$ for some Tychonoff space Y. Those C-rings which are also intermediate rings of $C(X)$ are

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called intermediate C-rings. It is well-known that every $f \in C(X)$ has a continuous extension f^* from βX to $\mathbb{R}^*$ (the one-point compactification of $\mathbb{R}$). For each element f of an intermediate ring $A(X)$, we set $S_A(f) = \{p \in \beta X : (fg)^*(p) = 0, \forall g \in A(X)\}$. We can easily observe that $\mathrm{cl}_{\beta X} Z(f) \subseteq S_A(f) \subseteq Z(f^*)$ and thus $S_A(f) \cap X = Z(f)$ for each $f \in A(X)$. We use M_A^p to denote the set $\{f \in A(X) : p \in S_A(f)\}$ for each $p \in \beta X$. Evidently, $M_C^p = M^p$ and $M_{C^*}^p = M^{*p}$. From [15, Theeorem 2.8 and Theorem 2.9] it follows that the collection of all the maximal ideals of an intermediate ring $A(X)$ is $\{M_A^p : p \in \beta X\}$. An ideal I of a commutative ring R is called a z-ideal, if whenever $f \in I$, then $M_f(R) \subseteq I$ in which $M_f(R)$ denotes the intersection of all the maximal ideals of R containing f. We use M_f instead of $M_f(C(X))$ for each $f \in C(X)$. It is well-known that $M_f = \{g \in C(X) : Z(f) \subseteq Z(g)\}$ for each $f \in C(X)$. Also, from [12, Proposition 2.7], it follows that $M_f(A(X)) = \{g \in A(X) : S_A(f) \subseteq S_A(g)\}$ for each element f of an intermediate ring $A(X)$. We denote by $\upsilon_f X$ the set $\{p \in \beta X : f^*(p) < \infty\}$ for each $f \in C(X)$. Also, we use $\upsilon_A X$ to denote the set $\bigcap_{f \in A} \upsilon_f X$ for each $A \subseteq C(X)$. Evidently, $\upsilon_C X = \upsilon X$ and $\upsilon_{C^*} X = \beta X$. Also, $\upsilon X \subseteq \upsilon_A X$ for each subset A of $C(X)$. By a realcompactification of X we mean a realcompact space containing X as a dense subspace. It is easy to see that $\upsilon_A X$ is a realcompactification of X for each subset A of $C(X)$. Moreover, each realcomapctification of X which is contained in βX is of the form $\upsilon_A X$ for some $A \subseteq C(X)$, see [11, 8B,3]. The aim of this paper is to investigate a new approach to the results of [8] concerning characterization of intermediate C-rings of $C(X)$ and to answer the four questions raised in [1] by a different way. It should be noted that these questions have previously been answered in [4] and [9]. To these aims, we consider "υ" as a mapping from the collection of all intermediate rings of $C(X)$ to the collection of all realcompactifications of X contained in βX which is called the mapping of realcompactification. Using this, we give new short proofs to the main results of [8]. This paper consists of three sections. Section 1, which is already noticed, is the introduction and preliminaries in which some necessary notations and terminologies are introduced. In Section 2, we define the mapping "υ" which assigns to each intermediate ring $A(X)$ the realcompactification $\upsilon_A X$ of X. The converse of this mapping, namely, "υ^{-1}", is also defined. By stating the relations between these two mappings, we offer a different approach to the main results of [8]. Moreover, we answer four questions raised in [1]. In Section 3, we give some notes on the realcompact spaces which ideals of $C(X)$ and of intermediate rings generate. These notes extend some results in [6] concerning the realcompactifications which ideals generate.

2 Mappings of realcompactification

Let $\Sigma(X)$ denote the collection of all intermediate rings of $C(X)$ and $\mathcal{R}(\beta X)$ denote the collection of all realcompactifications of X contained in βX. We consider the relation $\sim$ on $\Sigma(X)$ as follows, for $A(X), B(X) \in \Sigma(X)$, $A(X) \sim B(X)$, if $\upsilon_A X = \upsilon_B X$. Clearly, $\sim$ is an equivalence relation on $\Sigma(X)$. The equivalence class of $A(X) \in \Sigma(X)$ is denoted by $[A(X)]$. Also, the collection of all the largest members of the equivalence classes $[A(X)]$ for $A(X) \in \Sigma(X)$ is denoted by $\Sigma_1(X)$. For more details about the collections $\Sigma(X)$ and $\Sigma_1(X)$ refer to [1]. Let $\upsilon : \Sigma(X) \longrightarrow \mathcal{R}(\beta X)$ and $\upsilon^{-1} : \mathcal{R}(\beta X) \longrightarrow \Sigma(X)$ be defined by $\upsilon(A(X)) = \upsilon_A X$ and $\upsilon^{-1}(T) = \{f \in C(X) : f^*(p) < \infty, \forall p \in T\}$ for each $A(X) \in \Sigma(X)$ and each $T \in \mathcal{R}(\beta X)$, respectively. The next statement shows that the mappings υ and υ^{-1} are Galois conjugate (in the sense of lattice theory) which has a straightforward proof.

Lemma 2.1. *For each $A(X) \in \Sigma(X)$ and each $T \in \mathcal{R}(X)$, we have*

$$\upsilon\upsilon^{-1}\upsilon(A(X)) = \upsilon(A(X)), \quad \upsilon^{-1}\upsilon\upsilon^{-1}(T) = \upsilon^{-1}(T)$$

It follows from [1, Theorem 2.2] that the equivalence class $[A(X)]$ has the largest member for each $A(X) \in \Sigma(X)$. The next statement gives a different approach to this fact.

Theorem 2.2. *For each $A(X) \in \Sigma(X)$, the largest member of the equivalence class $[A(X)]$ exists and is equal to $\upsilon^{-1}\upsilon(A(X))$.*

Proof. It is evident that

$$[A(X)] = \{B(X) \in \Sigma(X) : \upsilon^{-1}\upsilon(A(X)) = \upsilon^{-1}\upsilon(B(X))\}.$$

It follows that $B(X) \subseteq \upsilon^{-1}\upsilon(B(X)) = \upsilon^{-1}\upsilon(A(X))$ for each $B(X) \in \Sigma(X)$. Also, clearly, $\upsilon^{-1}\upsilon(A(X)) \in [A(X)]$. Thus, $\upsilon^{-1}\upsilon(A(X))$ is the largest member of $[A(X)]$. $\square$

The next statement follows from Theorem 2.2 and Lemma 2.1 which provides an alternative method to [8, Theorem 1.2].

Corollary 2.3. [8, Theorem 1.2] *$B(X)$ is the largest member of $[A(X)]$ if and only if $B(X) = \upsilon^{-1}(T)$ for some $T \in \mathcal{R}(X)$.*

Proof. ($\Rightarrow$) This is clear by Theorem 2.2.

($\Leftarrow$) Let $B(X) \in [A(X)]$ and $B(X) = \upsilon^{-1}(T)$ for some $T \in \mathcal{R}(X)$. By Theorem 2.2, we have $B(X) \subseteq \upsilon^{-1}\upsilon(A(X))$. As $B(X) \in [A(X)]$, we have $\upsilon(A(X)) = \upsilon(B(X))$. Therefore, by Lemma 2.1, $\upsilon^{-1}\upsilon(A(X)) = \upsilon^{-1}\upsilon(B(X)) = \upsilon^{-1}\upsilon\upsilon^{-1}(T) = \upsilon^{-1}(T) = B(X)$ which completes the proof. $\square$

Theorem 2.4. [8, Theorem 1.1] *For each $A(X) \in \Sigma(X)$, $\upsilon^{-1}\upsilon(A(X)) = \{g|_X : g \in C(\upsilon_A X)\}$.*

Proof. If $f \in \upsilon^{-1}\upsilon(A(X))$, then $f^*(p) < \infty$ for each $p \in \upsilon_A X$. This means that f could be extended to $\upsilon_A X$. Therefore, $f^{\upsilon_A} \in C(\upsilon_A X)$ and $f^{\upsilon_A}|_X = f \in A(X)$. Conversely, if $f \in \{g|_X : g \in C(\upsilon_A X)\}$, then $f \in C(X)$ and $f^*(p) < \infty$ for each $p \in \upsilon_A X$. This clearly implies that $f \in \upsilon^{-1}\upsilon(A(X))$. $\square$

The next statement easily follows from Theorem 2.4.

Corollary 2.5. *The extension mapping $f \longmapsto f^{\upsilon_A}$ is an isomorphism from $\upsilon^{-1}\upsilon(A(X))$ onto $C(\upsilon_A X)$ for each $A(X) \in \Sigma(X)$.*

We need the following lemma which is first stated in [16, Theorem 4.5] and is proved in [13, Lemma 2.1] by a different way.

Lemma 2.6. *Let $A(X)$ and $B(Y)$ be two isomorphic intermediate rings of $C(X)$ and $C(Y)$, respectively. Then $\upsilon_A X$ is homeomorphic with $\upsilon_B Y$.*

Theorem 2.7. [8, Theorem 1.4] *An intermediate ring $A(X)$ of $C(X)$ is a C-ring if and only if $A(X)$ is the largest member of $[A(X)]$.*

Proof. ($\Rightarrow$) Let $A(X) \cong C(Y)$ for some Tychonoff space Y. By Lemma 2.6, we have $\upsilon_A X \simeq \upsilon Y$ and thus $A(X) \cong C(Y) \cong C(\upsilon Y) \cong C(\upsilon_A X)$. Using this and by a routine reasoning, we can prove that the extension mapping is an isomorphism from $A(X)$ to $C(\upsilon_A X)$. Also, by Corollary 2.5, $\upsilon^{-1}\upsilon(A(X))$ is isomorphic with $C(\upsilon_A X)$ under the extension mapping. It easily inferred that $A(X) = \upsilon^{-1}\upsilon(A(X))$.

($\Leftarrow$) An easy consequence of Corollary 2.5. $\square$

In the final part of this section we consider four basic questions raised in [1] concerning relations between $\Sigma(X)$ and $\Sigma_1(X)$. We answer these questions using the notion of singly generated intermediate rings over a given intermediate ring. Following [10], for an intermediate ring $A(X)$ and $f \in C(X)$, we use $A(X)[f]$ to denote the singly generated intermediate ring of $C(X)$ over $A(X)$ generated by f, which is the smallest intermediate ring of $C(X)$ containing both $A(X)$ and f. It is easy to see that $A(X)[f] = \{\sum_{i=0}^{n} f^i g_i : g_i \in A(X), n \in \mathbb{N} \cup \{0\}\}$ Also, as stated in [9], whenever $f \in C(X)$ such that $Z(f) = \emptyset$ and $\frac{1}{f} \in C^*(X)$, then $A(X)[f] = \{gf^n : g \in A(X), n \in \mathbb{N} \cup \{0\}\}$.

The following questions raised in [1].

Question 1. Does there exist at all any Tychonoff space X for which $\Sigma_1(X) = \Sigma(X)$?

Question 2. For any σ-compact locally compact space X, does there exist an $H(X) \in \Sigma(X) \setminus \Sigma_1(X)$?

Question 3. If for any X, there are two different $A, B \in \Sigma(X)$ with $\upsilon_A X = \upsilon_B X$, does there necessarily exist an infinite family $\{A_\alpha : \alpha \in \Lambda\}$ for which $\upsilon_{A_\alpha} X = \upsilon_{A_\beta} X$ for all $\alpha, \beta \in \Lambda$?

Question 4. Is $C^*(X)$ the only lone equivalence class of $\Sigma(X)$ generated by the equvalence relation $\sim$ of $\Sigma(X)$, for any realcompact non-compact space X?

We answer the first two questions by the following statement.

Theorem 2.8. *A topoogical space X is pseudocompact if and only if $\Sigma_1(X) = \Sigma(X)$.*

Proof. ($\Rightarrow$) Evident.

($\Leftarrow$) Let X be non-pseudocompact, thus there exists some $f \in C(X) \setminus C^*(X)$. Clearly, we could choose f to be a positive function. Let $p \in \beta X \setminus \upsilon_f X$. Set $A(X) = M^p + C^*(X)$ and $B(X) = A(X)[\ln f]$. From [13, Proposition 2.3] it follows that $\upsilon_A X = \upsilon X \cup \{p\}$. It is clear that $B(X)$ is an intermediate ring of $C(X)$. Also, $B(X)$ is not a C-ring (see [9, Lemma 2.3]). Therefore, $\upsilon_B X = \upsilon_A X \cap \upsilon_{\ln_f} X = (\upsilon X \cup \{p\}) \cap \upsilon_f X = \upsilon X$. It follows that the largest member of $[B(X)]$ which is $C(X)$ does not equals to $B(X)$. This means that $B(X) \in \Sigma(X)$, however, $B(X) \notin \Sigma_1(X)$. $\square$

Answer of Question 1. By Theorem 2.8, the only topological spaces X for which $\Sigma(X) = \Sigma_1(X)$ are pseudocompact spaces.

Answer of Question 2. By Theorem 2.8, for any σ-compact locally comapct and non-pseudocompact space X, $\Sigma(X) \neq \Sigma_1(X)$ and thus there exists some $H(X) \in \Sigma(X) \setminus \Sigma_1(X)$.

It is stated in [11, 9D.2] that whenever X is a non-pseudocompact space, then $\beta X \setminus \upsilon X$ contains at least 2^c points. We generalize this fact to $\beta X \setminus \upsilon_f X$ for each unbounded element $f \in C(X)$ by the next statement.

Lemma 2.9. *Let X be a non-pseudocompact topological space and $f \in C(X) \setminus C^*(X)$. Then $\beta X \setminus \upsilon_f X$ has at least 2^c points.*

Proof. Since f is unbounded on X, we get that $\upsilon_f X \subset \beta X$. Also the fact that $\upsilon_f X$ is realcompact ensures that $\upsilon(\upsilon_f X) = \upsilon_f X$. On writing $Y = \upsilon_f X$, the set $\beta X \setminus \upsilon_f X$ reduces to $\beta Y \setminus \upsilon Y = \beta Y \setminus Y$ which is nonempty and therefore contains at least 2^c elements. $\square$

It follows from Lemma 2.9 that $\beta X \setminus \upsilon_A X$ has at least 2^c points for each non-pseudocompact space X and each intermediate ring $A(X)$ of $C(X)$ different from $C^*(X)$. By using Lemma 2.9, we provide an answer to the third question.

Answer of Question 3. Since $C^*(X)$ is the only member of its equivalence class modulo the relation $\sim$ defined on $\Sigma(X)$, the hypothesis implies that X can not be pseudocompact. Let X be a non-pseudocompact space. Thus, there exists $f \in C(X) \setminus C^*(X)$. Clearly, $g = 1 + |f| \in C(X) \setminus C^*(X)$. For each $p \in \beta X \setminus \upsilon_g X$, let $A_p = M^p + C^*(X)$ and $B_p = A_p[\ln g]$. As $Z(g) = \emptyset$ and $\frac{1}{g} \in C^*(X)$, we have $B_p = \{h(\ln g)^n : h \in A_p, n \in \mathbb{N} \cup \{0\}\}$. As noted earlier, $\upsilon_{A_p} X = \upsilon X \cup \{p\}$ and B_p is an intermediate ring of $C(X)$ which is not a C-ring. It follows that $\upsilon_{B_p} X = (\upsilon X \cup \{p\}) \cap \upsilon_g X = \upsilon X$. Now, we show that $B_p \neq B_q$ for every two distinct points $p, q \in \beta X \setminus \upsilon_g X$. Let

$p \neq q \in \beta X \backslash \upsilon_g X$. Thus, there exists $h \in C^*(X)$ such that $p \in \mathrm{cl}_{\beta X} Z(h)$ and $q \notin Z(h^\beta)$. Hence, $h \in M^p \subseteq B_p$ and thus $gh \in M^p \subseteq B_p$. We claim that $gh \notin B_q$. Assume on the contrary that $gh \in B_q$. Thus, there exist $l \in A_q$ and $n \in \mathbb{N}$ such that $gh = l(\ln g)^n$. It follows that $l = \frac{gh}{(\ln g)^n}$ and hence $l^*(q) = \frac{(gh)^*(q)}{((\ln g)^n)^*(q)} = \frac{g^*(q)}{(\ln g)^*(q)} = \infty$ which is a contradiction, since, $l \in A_q$ which implies that $l^*(q) < \infty$. Moreover, By Lemma 2.9, $\beta X \setminus \upsilon_g X$ has at least 2^c points. Therefore, $\{B_p : p \in \beta X \setminus \upsilon_f X\}$ is an uncountable family of different intermediate rings of $C(X)$ such that $\upsilon_{B_p} X = \upsilon_{B_q} X = \upsilon X$ for each $p, q \in \beta X \setminus \upsilon_f X$. $\square$

Answer of Question 4. Let X be a realcompact non-compact space. Also, let $C^*(X) \neq A(X) \in \Sigma(X)$ be given. We show that $[A(X)]$ has a memeber other than $A(X)$. As $A(X) \neq C^*(X)$, we can choose some non-negative element $f \in A(X) \setminus C^*(X)$. Thus, there exists some $p \in \beta X \setminus \upsilon_f X$. Let $B(X) = M^p + C^*(X)$ and $D(X) = A(X) \cap B(X)$. Since, $\upsilon_B X = \upsilon X \cup \{p\}$. We claim that $\upsilon_D X = \upsilon_A X \cup \{p\}$. Evidently, $\upsilon_A X \cup \{p\} \subseteq \upsilon_D X$. Let $q \notin \upsilon_A X \cup \{p\}$. As $p \neq q$, there exists some $g \in C^*(X)$ such that $p \in \mathrm{cl}_{\beta X} Z(g)$ and $q \notin Z(g^\beta)$. Also, as $q \notin \upsilon_A X$, there exists some $h \in A(X)$ such that $h^*(q) = \infty$. It follows that $g \in M^p$ and thus $hg \in M^p \subseteq B(X)$. Moreover, $hg \in A(X)$. Therefore, $hg \in D(X)$ and $(hg)^*(q) = \infty$; that is, $q \notin \upsilon_D X$. This completes the proof of our claim. Now, set $E(X) = D(X)[\ln f]$. As noted earlier, $E(X)$ is an intermediate ring of $C(X)$ which is not a C-ring. Also, using [9, Lemma 2.3], $f = \mathrm{e}^{\ln f} \notin E(X)$ and $\upsilon_E X = \upsilon_D X \cap \upsilon_f X = \upsilon_A X$. It follows that $E(X) \neq A(X)$ and $E(X) \in [A(X)]$. $\square$

3 Notes on realcompactifications induced by ideals

It is stated in [6] that for each ideal I in $C(X)$, we have $\upsilon_I X = \upsilon_{I_z} X = \upsilon X \cup \theta(I)$ in which I_z denotes the smallest z-ideal in $C(X)$ containing I. Also the largest z-ideal in $C(X)$ contained in I is denoted by I^z. For an intermediate ring $A(X)$ of $C(X)$, we use $I_z(A)$ (respectively, $I^z(A)$) the smallest z-ideal in $A(X)$ contained in I (respectively, the largest z-ideal in $A(X)$ containing in I). In [7, Proposition 1.2], it is proved that $I^z = \{f \in C(X) : M_f \subseteq I\} = \sum_{M_f \subseteq I} M_f$ and $I_z = \{g \in C(X) : \exists f \in I; g \in M_f\} = \sum_{f \in I} M_f$. Using the following lemma, we show that for each ideal I in $C(X)$, the realcompactifications induced by I and I^z coincide.

Lemma 3.1. (a) *For each ideal I of $C(X)$, $\theta(I) = \bigcap_{M_f \subseteq I} cl_{\beta X} Z(f)$.*

(b) *For each family $\{I_\lambda\}_{\lambda\in\Lambda}$ of ideals of $C(X)$, we have $\theta(\sum_{\lambda\in\Lambda} I_\lambda) = \bigcap_{\lambda\in\Lambda} \theta(I_\lambda)$.*

Proof. (a) It is clear that whenever $f \in C(X)$ and $M_f \subseteq I$, then $\theta(I) \subseteq \theta(M_f) = \mathrm{cl}_{\beta X} Z(f)$. Thus, $\theta(I) \subseteq \bigcap_{M_f\subseteq I} \mathrm{cl}_{\beta X} Z(f)$. Now, let $p \notin \theta(I)$. Hence, there exists $f \in I$ such that $p \notin \mathrm{cl}_{\beta X} Z(f)$. Therefore, there exists $g \in C(X)$ such that $p \notin \mathrm{cl}_{\beta X} Z(g)$ and $\mathrm{cl}_{\beta X} Z(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(g)$. We claim that $M_g \subseteq I$. Let $h \in M_g$. It follows that $Z(g) \subseteq Z(h)$ and thus $\mathrm{cl}_{\beta X} Z(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(g) \subseteq \mathrm{cl}_{\beta X} Z(h)$. This implies that $Z(f) \subseteq \mathrm{int}_X Z(h)$, and hence by [11, 1D], there exists $k \in C(X)$ such that $h = kf$. Therefore, $h \in I$ which means that $M_g \subseteq I$ and the claim is proved. Hence, $p \notin \bigcap_{M_f\subseteq I} \mathrm{cl}_{\beta X} Z(f)$ and the required equality follows.

(b) It is evident that $\theta(\sum_{\lambda\in\Lambda} I_\lambda) \subseteq \bigcap_{\lambda\in\Lambda} \theta(I_\lambda)$. Now, let $p \notin \theta(\sum_{\lambda\in\Lambda} I_\lambda)$. Thus, there exists some $f \in \sum_{\lambda\in\Lambda} I_\lambda$ such that $p \notin \mathrm{cl}_{\beta X} Z(f)$. As $f \in \sum_{\lambda\in\Lambda} I_\lambda$, there exist $\lambda_1, ..., \lambda_n \in \Lambda$ such that $f = f_{\lambda_1} + ... + f_{\lambda_n}$. It follows that $\bigcap_{i=1}^n \mathrm{cl}_{\beta X} Z(f_{\lambda_i}) \subseteq \mathrm{cl}_{\beta X} Z(f)$ which implies $p \notin \mathrm{cl}_{\beta X} Z(f_{\lambda_k})$ for some $k \in \{1, ..., n\}$. Therefore, $f_{\lambda_k} \in I_{\lambda_k}$ and $p \notin \mathrm{cl}_{\beta X} Z(f_{\lambda_k})$. Hence, $p \notin \theta(I_{\lambda_k})$. Thus, $\bigcap_{\lambda\in\Lambda} \theta(I_\lambda) \subseteq \theta(\sum_{\lambda\in\Lambda} I_\lambda)$ and the equality follows. □

Proposition 3.2. *For each ideal I of $C(X)$, $\upsilon_I X = \upsilon_{I_z} X = \upsilon_{I^z} X = \upsilon X \cup \theta(I)$.*

Proof. By Lemma 3.1, we have $\theta(I^z) = \theta(\sum_{M_f\subseteq I} M_f) = \bigcap_{M_f\subseteq I} \theta(M_f) = \bigcap_{f\in I} \mathrm{cl}_{\beta X} Z(f) = \theta(I)$. Thus, using [6, Proposition 2.3], the proof is complete. □

As mentioned in the introduction, $M_f(A) = \{g \in A(X) : S_A(f) \subseteq S_A(g)\}$ for each element f of an intermediate ring $A(X)$. It follows that, for an ideal I in $A(X)$, the intersection of all the maximal ideals of $A(X)$ containing I is $M_A^{\theta_A(I)} = \{f \in A(X) : \theta_A(I) \subseteq S_A(f)\}$. The next lemma easily follows from Corollary 2.5 and Lemma 3.1.

Lemma 3.3. (a) *For each ideal I of an intermediate C-ring $C(X)$, $\theta_A(I) = \bigcap_{M_f(A)\subseteq I} S_A(f)$.*

(b) *For each family $\{I_\lambda\}_{\lambda\in\Lambda}$ of ideals of an intermediate ring $A(X)$, $\theta_A(\sum_{\lambda\in\Lambda} I_\lambda) = \bigcap_{\lambda\in\Lambda} \theta_A(I_\lambda)$.*

Proposition 3.4. *Let I be an ideal of an intermediate ring $A(X)$, then $\upsilon_I X = \upsilon_{I_z(A)} X = \upsilon_A X \cup \theta_A(I)$. Moreover, if $A(X)$ is an intermediate C-ring, then $\upsilon_I X = \upsilon_{I_z(A)} X = \upsilon_{I^z(A)} X$.*

Proof. For the proof of the first series of equalities see [6, Proposition 2.3]. The proof of the second series of equalities follows from Lemma 3.3 and [6, Proposition 2.3]. □

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