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Lattice of compactifications of a topological group

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Abstract. We show that the lattice of compactifications of a topological group G is a complete lattice which is isomorphic to the lattice of all closed normal subgroups of the Bohr compactification bG of G. The correspondence defines a contravariant functor from the category of topological groups to the category of complete lattices. Some properties of the compactification lattice of a topological group are obtained.

1 Introduction

Let X be a Tychonoff space. It is well known that the collection $K(X)$ of all Hausdorff compactifications of X forms a complete upper semi-lattice under the order relation defined by $c_1X \leq c_2X$ if and only if there is a continuous map $f : c_2X \to c_1X$, which leaves X pointwise fixed. In general $K(X)$ is not a complete lattice and it is a complete lattice if and only if X is locally compact. There are many results studying the relationship between topological properties of X and the order structure of $K(X)$ (see [4], [9]-[13], [16]).

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Here we consider the Hausdorff compactfications of a Hausdorff topological group G.

Let G be a topological group. Recall that a compactification of G is a pair (K, φ), where K is a compact Hausdorff group and $\varphi : G \to K$ is a dense continuous homomorphism. For two compactifications (K, φ) and (H, ψ) of G, we say that (K, φ) and (H, ψ) are equivalent if there is a topological isomorphism $h : K \to H$ such that $\psi = h \circ \varphi$. This defines an equivalence relation on the collection of all Hausdorff compactifications of G. We use the same symbol $K(G)$ (like in the case of topological spaces) to denote the set of all equivalence classes of the Hausdorff compactifications of G. There exists a natural order relation on $K(G)$ defined as follows:

$(K, \varphi) \leq (H, \psi)$ if there exists a continuous homomorphism $\mu : H \to K$ such that $\varphi = \mu \circ \psi$.

The following result is well known. For convenience of the readers we give a proof here.

Lemma 1.1. *Let (K, φ) and (H, ψ) be two compactifications of G. Then (K, φ) and (H, ψ) are equivalent if and only if $(K, \varphi) \leq (H, \psi)$ and $(H, \psi) \leq (K, \varphi)$ both hold.*

Proof. The necessity is clear. We only need to show the sufficiency. Suppose that $(K, \varphi) \leq (H, \psi)$ and $(H, \psi) \leq (K, \varphi)$ both hold. We have continuous homomorphisms $f : H \to K$ and $g : K \to H$ such that $\varphi = f \circ \psi$, $\psi = g \circ \varphi$. It follows that $g \circ f \circ \varphi = g \circ \psi = \varphi$. This implies that $g \circ f = id_H$ since φ is dense. Similarly we have $f \circ g = id_K$. Hence $f : H \to K$ is a topological isomorphism. □

By this lemma, $(K(G), \leq)$ is a partially ordered set. Clearly $K(G)$ has the least element, that is the trivial group and trivial homomorphism, and also has the largest element, the Bohr compactification of G.

In this note we investigate the order algebraic structure of the compactification lattice $K(G)$ of a given topological group G and its relationship with the properties of G.

2 Preliminaries

Throughout this paper we consider the category **TopGrp** of Hausdorff topological groups and continuous homomorphisms. **TopAb** and **Ab** respectively denote the category of topological abelian groups and the category of discrete abelian groups. The categories of complete lattice and sup-preserving maps and the category of complete lattice and inf-preserving maps are denoted by **CLat** and $\mathbf{CLat^{op}}$, respectively.

We denote by $\mathbb{N}$ the set of positive natural numbers; by $\mathbb{Z}$ the integers, by $\mathbb{Q}$ the rationals, by $\mathbb{R}$ the reals, and by $\mathbb{T}$ the unit circle group which is identified with $\mathbb{R}/\mathbb{Z}$. We write $\mathbb{U} = \prod_{n=1}^{\infty} U(n)$, where $U(n)$ is the group of all $n \times n$ unitary matrices in $GL_n(\mathbb{C})$. According to Peter-Weyl's theorem, for every compact group K, the continuous homomorphisms $K \to \mathbb{U}$ separate the points of K [7]. It follows that every compact group can be topologically embedded into some power of $\mathbb{U}$. Hence we can take $\mathbb{U}$ in **TopGrp** in the place of the circle group $\mathbb{T}$ in the category **TopAb** of topological Abelian groups. The cyclic group of order $n > 1$ is denoted by $\mathbb{Z}(n)$.

The subgroup generated by a subset X of a group G is denoted by $\langle X \rangle$, and $\langle x \rangle$ is the cyclic subgroup of G generated by an element $x \in G$. The abbreviation $K \leq G$ means that K is a subgroup of G, and $N \lhd G$ means that N is a normal subgroup of G.

For an arbitrary topological group G, we denote the family of all continuous homomorphisms from G to $\mathbb{U}$ by the symbol $C^*(G)$. The Bohr compactification of G is denoted by $b : G \to bG$. The group G endowed with the Bohr topology, that is, the topology induced by the family of all continuous homomorphisms from G to $\mathbb{U}$, is denoted by G^+. The von Neumann's kernel of G is denoted by $N(G)$, that is, $N(G) = ker(b) = \bigcap\{ker(f) \mid f \in C^*(G)\}$.

Throughout the paper all topological groups are assumed to be Hausdorff. All unexplained topological terms can be found in [3].

3 Compactification lattice of a topological group

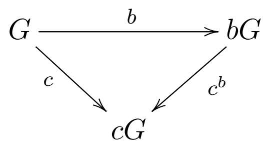
Lemma 3.1. *For every topological group G, the partially ordered set $K(G)$ is a complete lattice.*

Proof. We only need to show that every family of compactifications of G has supremum.

Let $\{c_i : G \to c_iG \mid i \in I\}$ be a family of compactifications of G. Consider the diagonal map $\langle c_i \rangle : G \to \prod_{i \in I} c_iG$. Denote by $cG = \overline{\langle c_i \rangle(G)}$ the closure of the image of G under $\langle c_i \rangle$, and $c : G \to cG$ the restriction of the mapping $\langle c_i \rangle$. Then $c : G \to cG$ is a compactification of G, and the equalities $p_i \circ c = c_i, i \in I$, imply that $c : G \to cG$ is an upper bound of $\{c_i : G \to c_iG \mid i \in I\}$, where $p_i : cG \to c_iG$ is the i^{th} projection. We now show that the compactification $c : G \to cG$ is the supremum of $\{c_i : G \to c_iG \mid i \in I\}$.

Indeed, suppose that $k : G \to K$ is a compactification of G such that for each $i \in I$, there exists a continuous homomorphism $h_i : K \to c_iG$ satisfying $c_i = h_i \circ k$. Then the diagonal map $\langle h_i \rangle : K \to \prod_{i \in I} c_iG$ satisfies $\langle h_i \rangle \circ k = \langle h_i \circ k \rangle = \langle c_i \rangle$. Also we have $\langle h_i \rangle(K) = \langle h_i \rangle(\overline{k(G)}) \subseteq \overline{\langle h_i \rangle(k(G))} = \overline{\langle c_i \rangle(G)}$. Let $h : K \to cG$ be the restriction of $\langle h_i \rangle$, then we have $h \circ k = c$. $\square$

Let G be a topological group. For every compactification $c : G \to cG$ of G, c can be uniquely factored through $b : G \to bG$ as $c = c^b \circ b$:



Clearly, c^b is surjective since c is dense and c^b is closed. Denote by $L(cG) = ker(c^b)$ the kernel of c^b. Then $L(cG)$ is a closed normal subgroup of bG.

Let also $CN(bG)$ be the set of all closed normal subgroups of bG ordered by inverse inclusion. We have the following result.

Proposition 3.2. *$L : K(G) \to CN(bG)$ is an isomorphism.*

Proof. Notice that L has an inverse map which sends each closed normal subgroup $N \lhd G$ to the compactification $q \circ b : G \to bG/N$, where $q : bG \to bG/N$ is the quotient map. Also for two compactifications $c_1 : G \to c_1G$ and $c_2 : G \to c_2G$ of G, it is clear that $c_1 \leq c_2$ if and only if $ker(c_2^b) \subseteq ker(c_1^b)$, that is, $L(c_2G) \subseteq L(c_1G)$. Hence $L : K(G) \to CN(bG)$ is an isomorphism. $\square$

Corollary 3.3. *Let G and H be topological groups. If bG is topologically isomorphic to bH, then $K(G)$ and $K(H)$ are isomorphic.*

A classical result in [11] showed that if X and Y are locally compact spaces, then their lattices of compactifications $K(X)$ and $K(Y)$ are isomorphic if and only if $\beta X \setminus X$ and $\beta Y \setminus Y$ are homeomorphic. For a topological group G, the remainder $bG \setminus b(G)$ in general does not determine the order structure of $K(G)$. Indeed, if we take $S(X)$ the permutation group of an infinite set X, we know that $S(X)$ is a minimally almost periodic groups, that is, $N(S(X)) = S(X)$. Put $H = \mathbb{T}$ the unit circle group. Then $bS(X)$ is the trivial group and $bH = H$. Hence the remainder $bS(X) \setminus b(S(X)) = bH \setminus b(H) = \emptyset$. But it is clear that $K(bS(X))$ is not isomorphic to $K(H)$.

Corollary 3.4. *Let L be a complete lattice. L is isomorphic to the compactification lattice $K(G)$ of a topological group G if and only if L is isomorphic to the lattice of all closed normal subgroups of a compact group.*

Let G be a topological group and let $H \leq G$ be a dense subgroup of G. Then for every compactification $c : G \to cG$ of G the restriction of c to H is a compactification of H. Conversely every compactification $c : H \to cH$ of H admits a unique extension $\tilde{c} : G \to cH$. Hence the following result is clear.

Proposition 3.5. *If H is a dense subgroup of a topological group G, then $K(G) = K(H)$.*

If G is a topological Abelian group, we can give an alternative description of $K(G)$. Let $C^*(G)$ denote the group of all continuous characters on G, that is, the group of all continuous homomorphisms $f : G \to \mathbb{T}$. Suppose that $N \leq C^*(G)$. Then we have a compactification $c_N : G \to c_N G$ corresponding to N such that $c_N G$ is the closure of the image of G in $\mathbb{T}^N$ under the diagonal map c_N. For two subgroups $N_1 \leq C^*(G), N_2 \leq C^*(G)$, we write $N_1 \sim N_2$ if the compactifications $c_{N_1} : G \to c_{N_1} G$ and $c_{N_2} : G \to c_{N_2} G$ corresponding to N_1 and N_2, respectively, are equivalent. Denote by $Subgp(C^*(G))/\sim$ the quotient set of the set $Subgp(C^*(G))$ of all subgroups of $C^*(G)$. Let $[N], [M] \in Subgp(C^*(G))/\sim$. Define $[N] \leq [M]$ if and only if $c_N G \leq c_M G$, where $c_N G$ and $c_M G$ are the compactifications corresponding to N and M, respectively.

Proposition 3.6. *The partially ordered set $(Subgp(C^*(G))/\sim, \leq)$ is isomorphic to $K(G)$.*

Corollary 3.7. *Let G be a topological abelian group. Then $\mid K(G) \mid \leq 2^{|C^*(G)|}$.*

Note that the above corollary is true for an arbitrary topological group G, though then $C^*(G)$ is not a group.

Let L be a complete lattice. An element $a \in L$ is called compact if for every family $\{a_i \mid i \in I\} \subset L$ with $a \leq \bigvee a_i$, there exists finite set $J \subset I$ such that $a \leq \bigvee_{i \in J} a_i$. Let $k(L)$ be the set of all compact elements of L. A complete lattice L is said to be an algebraic lattice if $a = \bigvee\{c \in k(L) \mid c \leq a\}$ for every $a \in L$. L is said to be a dual algebraic lattice if the dual lattice L^{op} of L is algebraic.

Theorem 3.8. *If L is a compactification lattice of a topological Abelian group, then there is a dual algebraic lattice S such that L is a retraction of S.*

Proof. Suppose that G is a topological Abelian group and L is order isomorphic to the compactifiction lattice $K(G)$ of G. Let S be the lattice of all subgroups of bG. We first show that S is a dual algebraic lattice (with respect to the inverse inclusion order).

Indeed, S is a complete lattice and it contains L as a sub-upper semilattice. It is clear that the family of all finitely generated subgroups of bG and the family of all compact element of S^{op} coincide, and also every subgroup $H \leq bG$ can be represented as a supremum of a family of finitely generated subgroups of bG.

Let $r : S \to L$ be such that for every $a \in S$, $r(a)$ is in the closure of a in bG. Then r is a retraction. □

4 The contravariant lattice-valued functor

Let $f : G \to H$ be a continuous homomorphism. We define $K(f) : K(G) \to K(H)$ as the mapping which sends every compactification $c : H \to cH$ to the compactification $c \circ f : G \to \overline{c(f(G))}$ of G, where $\overline{c(f(G))}$ means the closure of $c(f(G))$ in cH. Equivalently, if we regard $K(G)$ as $CN(bG)$, and $K(H)$ as $CN(bH)$, then $K(f) : CN(bH) \to CN(bG)$ sends every closed normal subgroup $N \lhd bH$ to the closed normal subgroup $bf^{-}(N) \lhd bG$, where $bf : bG \to bH$ is the extension of f.

Proposition 4.1. *The assignment $G \mapsto K(G)$ is a contravariant functor from the category* $\mathbf{TopGp^{op}}$ *to the category* **CLat** *of complete lattices.*

Proof. For two continuous homomorphisms $f : G \to H$ and $g : H \to M$ of topological groups, it is clear that $K(g \circ f) = K(g) \circ K(f)$. We only need to show that $K(f) : CN(bH) \to CN(bG)$ preserves arbitrary joins, for any continuous homomorphism $f : G \to H$.

$K(f)$ maps H to G, that is, it preserves the bottom element. Let $\{N_i \mid i \in I\}$ be a family of closed normal subgroups of bH. We have $\bigvee_{i\in I} N_i = \bigcap_{i\in I} N_i$, and hence

$$bf^{-}(\bigvee_{i\in I} N_i) = bf^{-}(\bigcap_{i\in I} N_i) = \bigcap_{i\in I} bf^{-}(N_i) = \bigvee_{i\in I} bf^{-}(N_i).$$

This shows that $K(f)$ preserves arbitrary joins. □

Lemma 4.2. *Let G and H be topological groups and let $f : G \to H$ be a dense continuous homomorphism. Then $K(f) : K(H) \to K(G)$ is injective.*

Proof. The composition $G \to bG \to bH = G \to H \to bH$ is dense, since $b_H : H \to bH$ is dense. It follows that $bf : bG \to bH$ is dense and, hence, it is surjective since it is closed:

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \downarrow & & \downarrow \\ bG & \xrightarrow{bf} & bH \end{array}$$

Thus $K(f) : K(H) \to K(G)$ is injective. □

We know that in the category **TopAb** of topological abelian groups, monomorphisms are precisely one-to-one continuous homomorphisms and epimorphisms are precisely dense continuous homomorphisms. Hence we have the following result.

Proposition 4.3. *The functor K : $\mathbf{TopAb}^{\mathbf{op}} \to \mathbf{CLat}$ preserves monomorphisms.*

Let G be a topological group. We know that the Bohr compactification bG of G can be obtained in two steps:

first taking $pG = (G/N(G))^{+}$ the quotient group $G/N(G)$ endowed with the Bohr topology. Let $q : G \to pG$ be the quotient map, which is in fact the precompact reflection of G. Next we put $bG = \varrho pG$ to be the Raĭkov completion of pG. Then bG is the Bohr compactification of G. Suppose that $f : G \to H$ is a continuous homomorphism. Then f has a continuous homomorphism extension $pf : pG \to pH$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \downarrow & & \downarrow \\ pG & \xrightarrow{pf} & pH \end{array}$$

It is clear that $Ker(pf) = \{e_{pG}\}$ if and only if $N(G) = f^{-}(N(H))$. By the construction of the Raĭkov completion, we know that $Ker(pf) = \{e_{pG}\}$ if and only if $Ker(bf) = \{e_{bG}\}$:

$$\begin{array}{ccc} pG & \xrightarrow{pf} & pH \\ \downarrow & & \downarrow \\ bG & \xrightarrow{bf} & bH \end{array}$$

Lemma 4.4. *Let $f : G \to H$ be a continuous homomorphism. Then $K(f) : K(H) \to K(G)$ is surjective if and only if $N(G) = f^{-}(N(H))$.*

Proof. Suppose that $N(G) = f^{-}(N(H))$. By the above argument, $ker(bf)$ is trivial, hence bf is one-to-one. It follows that $bf : bG \to bH$ is a closed embedding, thus $K(f) : K(H) \to K(G)$ is surjective.

Conversely, if $K(f) : K(H) \to K(G)$ is surjective, then there exists a closed normal subgroup $N \lhd bH$ such that $bf^{-}(N) = \{e_{bG}\}$. It follows that $ker(bf) = \{e_{bG}\}$. This implies that $N(G) = f^{-}(N(H))$. □

Let G and H be two discrete abelian groups and let $f : G \to H$ be a one-to-one homomorphism. Then each homomorphism $g : G \to \mathbb{T}$ has an extension $\hat{g} : H \to \mathbb{T}$ such that $g = \hat{g} \circ f$. It follows that $N(G) = f^{-}(N(H))$.

Corollary 4.5. *The functor $K : \mathbf{Ab}^{\mathbf{op}} \to \mathbf{CLat}$ preserves epimorphisms.*

In general the functor $K : \mathbf{TopAb}^{\mathbf{op}} \to \mathbf{CLat}$ does not preserve epimorphisms. Indeed, if we take a topological Abelian group H such that H is not maximally almost periodic and denote by G the group H endowed with discrete topology, then $N(G)$ is the trivial group and $N(H) \neq \{e_H\}$. By Lemma 3.4, $K(id) : K(G) \to K(H)$ is not surjective.

Proposition 4.6. *If $f : G \to H$ is a dense continuous homomorphism such that $N(G) = f^{-}(N(H))$, then bG and bH are topologically isomorphic. So $K(G)$ is order isomorphic to $K(H)$.*

Proof. Clearly we have $ker(bf) = \{e_G\}$ and hence $bf : bG \to bH$ is injective, since $N(G) = f^{-}(N(H))$. Also the composition $G \to bG \to bH = G \to H \to bH$ is dense, since $f : G \to H$ is dense. It follows that $bf : bG \to bH$ is dense and surjective, since it is closed. □

Furthermore, we have the following result.

Proposition 4.7. *Let $f : G \to H$ be a continuous homomorphism. Then $K(f) : K(H) \to K(G)$ is an isomorphism if and only if $bf : bG \to bH$ is an embedding and for any two closed subgroups N_1 and N_2 of bH, $N_1 \cap bG = N_2 \cap bG$ implies that $N_1 = N_2$.*

Proof. If $K(f) : K(H) \to K(G)$ is surjective, then there exists a closed normal subgroup $N \lhd bH$ such that $bf^{-1}(N) = \{e_{bG}\}$, which implies that $ker(bf) = \{e_{bG}\}$. Hence $K(f) : K(H) \to K(G)$ is surjective if and only if $bf : bG \to bH$ is an embedding. It is clear that $K(f) : K(H) \to K(G)$ is injective if and only if for any two closed subgroups N_1 and N_2 of bH, $N_1 \cap bG = N_2 \cap bG$ implies that $N_1 = N_2$. □

The functor $K : \mathbf{TopGp^{op}} \to \mathbf{CLat}$ does not preserve coproducts. Indeed, if we take $G = H = \mathbb{Z}(2)$, then $bG = bH = \mathbb{Z}(2)$, and $b(G \times H) = \mathbb{Z}(2) \times \mathbb{Z}(2)$. As $\mid K(G \times H) \mid= 5$ and $\mid K(G) \times K(H) \mid= 4$, so $K(G \times H)$ is not isomorphic to $K(G) \times K(H)$. Since in the category **CLat** of complete lattices, coproducts of objects are equivalent to products of objects, hence $K(G \times H)$ is not the coproduct of $K(G)$ and $K(H)$.

The following result was proved in [6].

Lemma 4.8. *Let $\{G_i \mid i \in I\}$ be a family of topological groups. Then $b\prod_{i\in I} G_i$ is topologically isomorphic to $\prod_{i\in I} bG_i$.*

Proposition 4.9. *Let $\{G_i \mid i \in I\}$ be a family of topological groups. Then $\prod K(G_i)$ is a sub-complete lattice of the complete lattice $K(\prod G_i)$, and there exists an inf-preserving mapping $r : K(\prod G_i) \to \prod K(G_i)$ which leaves $\prod K(G_i)$ pointwise fixed.*

Proof. First $\prod K(G_i)$ is isomorphic to $\prod CN(bG_i)$. By Lemma 4.8, $K(\prod G_i)$ is isomorphic to $CN(\prod bG_i)$. So we have a natural embedding $f : \prod CN(bG_i) \to CN(\prod bG_i)$ of complete lattices which sends every element $(N_i) \in \prod CN(bG_i)$ to the element $\prod N_i \in CN(\prod bG_i)$.

Let $r : CN(\prod bG_i) \to \prod CN(bG_i)$ such that for each closed normal subgroup $N \lhd \prod bG_i$, $r(N) = (p_i(N))$, where $p_i : \prod bG_i \to bG_i$ is the projection to i^{th} coordinate. Then r is a mapping satisfying the required condition. □

Let $\{G_i \mid i \in I\}$ be a family of topological groups. We write $\bigotimes G_i$ the coproduct of $\{G_i \mid i \in I\}$. It is clear that $C^*(\bigotimes G_i)$ is a one-to-one correspondence to $\prod C^*(G_i)$. It is natural to ask the following question.

Open Question 4.10. Does the functor $K : \mathbf{TopGp^{op}} \to \mathbf{CLat}$ preserve products?

When focused on topological abelian groups, we have a functor from the category **TopAb** of topological abelian groups to the category $\mathbf{CLat^{op}}$ of complete lattices.

Indeed, let G, H be topological abelian groups and let $f : G \to H$ be a continuous homomorphism. Denote by $\hat{K}(G)$ and $\hat{K}(H)$ the closed subgroup lattice (with respect to the inverse inclusion order) of bG and bH, respectively. Then $\hat{K}(G)$ and $\hat{K}(H)$ are isomorphic to the compactification lattices of G and H, respectively, and $bf : bG \to bH$ maps each closed subgroup $N \leq bG$ onto a closed subgroup $bf(N) \leq bH$. Suppose that $\{N_j \mid j \in J\}$ is a family of closed subgroups of bG. Then $bf(\bigwedge N_j) = bf(\overline{\langle \bigcup N_j \rangle}) = \overline{bf(\langle \bigcup N_j \rangle)} = \overline{\langle \bigcup bf(N_j) \rangle} = \bigwedge bf(N_j)$. This implies that we have a functor $\hat{K} : \mathbf{TopAb} \to \mathbf{CLat^{op}}$.

Proposition 4.11. *Let $f : G \to H$ be a continuous homomorphism of topological abelian groups. Then $\hat{K}(f) : \hat{K}(G) \to \hat{K}(K)$ is an order isomorphism if and only if $bf : bG \to bH$ is a topological isomorphism.*

Proof. Suppose that $\hat{K}(f) : \hat{K}(G) \to \hat{K}(H)$ is an order isomorphism. Then bf is surjective, since $bf(bG) = bH$, and $bf(\{e_{bG}\}) = bf(ker(bf)) = \{e_{bH}\}$ implies that $ker(bf) = \{e_{bG}\}$, since $\hat{K}(f)$ is injective. It follows that bf is a topological isomorphism, since bf is closed. The converse is clear. □

It is a natural question whether for two topological abelian groups G and H, $K(G)$ isomorphic to $K(H)$ implies that bG and bH are topological isomorphic. But in general this is not true. For example, we take $G = \mathbb{Z}(2)$ and $H = \mathbb{Z}(3)$, then $K(G) = K(H)$ are two-elements lattices, but $bG = G$, $bH = H$.

Open Question 4.12. Let G and H be topological groups. Can one give sufficient and necessary conditions on G and H for $K(G)$ to be isomorphic to $K(H)$?

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References

[1] Arhangel'skii, A.V. and Tkachenko, M., "Topological Groups and Related Structures", World Scientific, 2008.

[2] Dikranjan, D.N., *Closure operators in topological groups related to von Neumann's kernel*, Topology Appl. 153 (2006) 1930-1955.

[3] Engelking, R., "General Topology", Heldermann, 1989.

[4] Firby, P.A., *Lattices and Compactifications* I, II*, and* III, Proc. London Math. Soc. 27 (1973), 22-68.

[5] Hart, J.E. and Kunen, K., *Bohr compactifications of non-abelian groups*, Topology Proc. 26 (2001-2002), 593-626.

[6] He, W. and Xiao, Z., *The τ-precompact Hausdorff Group Reflection of Topological Groups*, Bull. Belg. Math. Soc. Simon Stevin, 25 (2018), 107–120.

[7] Hewitt, E. and Ross, K., "Abstract Harmonic Analysis", Vol. 1, Springer, 1963.

[8] Hušek, M. and de Vries, J., *Preservation of products by functors close to reflectors*, Topology Appl. 27 (1987), 171-189.

[9] Kannan, V. and Thrivikraman, T., *Lattices of Hausdorff compactifications of a locally compact space*, Pacific J. Math. 57 (1975), 441-444.

[10] Magill, K.D., Jr., *A note on compactifications*, Math. Z. 94 (1966), 322-325.

[11] Magill, K.D., Jr., *The lattice of compactifications of a locally compact space*, Proc. London Math. Soc. 18 (1968), 231-244.

[12] Mendivil, Franklin, *Function algebras and the lattic of compactifications*, Proc. Amer. Math. Soc. 127 (1999), 1863-1871.

[13] Rayburn, Marlon C., *On Hausdorff Compactifications*, Pacific J. Math. 44 (1973), 707-714.

[14] Tholen, W., *A categorical guide to separation, compactness and perfectness*, Homology Homotopy Appl. 1 (1999), 147-161.

[15] Thomas, B.V. Smith , *Categories of topological groups*, Quaest. Math. 2 (1977-1978), 355-377.

[16] Thrivikraman, T., *On the lattices of compactifications*, Proc. London Math. Soc. 4 (1972), 711-717.

[17] Ursul, M., "Topological Rings Satisfying Compactness Conditions", Springer, 2002.

[18] Uspenskij, V.V., *A universal topological group with a countable base*, Funct. Anal. Appl. 20 (1986), 160-161.

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