Categories and
General Algebraic Structures
with Applications
Volume 10, Number 1, January 2019, 1-15.

An equivalence functor between local vector lattices and vector lattices

Karim Boulabiar

Abstract. We call a local vector lattice any vector lattice with a distinguished positive strong unit and having exactly one maximal ideal (its radical). We provide a short study of local vector lattices. In this regards, some characterizations of local vector lattices are given. For instance, we prove that a vector lattice with a distinguished strong unit is local if and only if it is clean with non no-trivial components. Nevertheless, our main purpose is to prove, via what we call the radical functor, that the category of all vector lattices and lattice homomorphisms is equivalent to the category of local vectors lattices and unital (i.e., unit preserving) lattice homomorphisms.

1 Introduction

All vector lattices we consider in this paper are assumed to be real. Let L and M be vector lattices. Recall that a linear map $L \overset{\omega}{\to} M$ is called a *lattice homomorphism* if

$$\omega\left(|x|\right) = \left|\omega\left(x\right)\right|, \quad \text{for all } x \in L.$$

Keywords: Category, equivalence functor, maximal ideal, local vector lattice, radical, lattice homomorphism, strong unit.
Mathematics Subject Classification [2010]: 06F20, 18D35.
Received: 13 November 2017, Accepted: 10 February 2018
ISSN Print: 2345-5853 Online: 2345-5861

The category whose objects are all vector lattices and morphisms are lattice homomorphisms is denoted by **VL**. Thus, by a **VL**-*object* we mean a vector lattice and any lattice homomorphism between two **VL**-objects is referred to as a **VL**-*morphism*.

Now, a vector subspace I of a **VL**-object E is called an (*order*) *ideal* in E if I contains with any x all y such that $|y| \leq |x|$. Clearly, if $x \in E$ then the set

$$I_x = \{y \in E : |y| \leq n\,|x|\,, \text{ for some } n = 1, 2, ...\}$$

is an ideal in E. Actually, I_x is the smallest ideal in E containing x. An element $x \in E$ for which $I_x = E$ is called a *strong* (*order*) *unit* in E. As usual, an ideal I in E is said to be *proper* if $I \neq E$. By the way, a proper ideal in E contains no strong units. A proper ideal M in E is said to be *maximal* if M is not contained in any other proper ideal in E. We call a *unital vector lattice* any **VL**-object along with a distinguished positive strong unit. Moreover, if E and F are unital vector lattices with distinguished strong units e and f, respectively, then any **VL**-morphism $E \overset{\omega}{\to} F$ for which the equality $\omega(e) = f$ holds is said to be *unital*. It is well-known that any unital vector lattice E has maximal ideals (see [7, Theorem 27.4]). It could happen that E has exactly one maximal ideal. In this situation, we call E a *local vector lattice* (see [8] for the corresponding concept in Ring Theory). The subcategory of **VL** with objects all local vector lattices and morphisms unital lattice homomorphisms is denoted by **LVL**. In other words, an **LVL**-object is a local vector lattice and an **LVL**-morphism is a unital lattice homomorphism between two local vector lattices. The main purpose of this note is to prove that the categories **VL** and **LVL** turn out to be equivalent. The equivalence functor $\mathcal{R} : \mathbf{VL} \to \mathbf{LVL}$ in question is defined as follows.

As proved in [7, Theorem 27.4], any **LVL**-object E has maximal ideals. Then, the intersection of all maximal ideals in E is the *radical* of E, which we denote by $\operatorname{Rad}(E)$. For every **LVL**-morphism $E \overset{\omega}{\to} F$, we define a **VL**-morphism $\operatorname{Rad}(E) \overset{\omega_0}{\to} \operatorname{Rad}(F)$ by putting

$$\omega_0(x) = \omega(x) \quad \text{for all } x \in \operatorname{Rad}(E)\,.$$

Actually, ω_0 is well-defined because, as we shall see later, ω sends $\operatorname{Rad}(E)$ to $\operatorname{Rad}(F)$. Roughly speaking, ω_0 is the restriction of ω to $\operatorname{Rad}(E)$. Then, $\mathcal{R}$ is the functor that takes each **LVL**-object E to the **VL**-object $\operatorname{Rad}(E)$, and each **LVL**-morphism $E \overset{\omega}{\to} F$ to the **VL**-morphism $\operatorname{Rad}(E) \overset{\omega_0}{\to} \operatorname{Rad}(F)$. It will turn out that $\mathcal{R}$ is the functor we are looking for.

Finally, we point out that the standard monograph [7] is used as a unique source of unexplained terminology and notations on Vector Lattices. But for Category Theory, the reader can consult the text [6].

2 Preliminaries on local vector lattices

As usual, the symbol $\mathbb{R}$ is used to denote the totally-ordered field of all real numbers. As mentioned before in the case of local vector lattices, any unital vector lattice E has maximal ideals (see Theorem 27.4 in [7]) and the radical of E (i.e., the intersection of all maximal ideals in E) is denoted by $\operatorname{Rad}(E)$. It turns out that $\operatorname{Rad}(E)$ coincides with the ideal E_0 of all infinitely small elements of E (see [7, Theorem 27.5]). Recall here that an element $x \in E$ is said to be *infinitely small* if there exists $y \in E$ such that

$$n\,|x| \leq |y|\,, \quad \text{for all } n \in \{1, 2, ...\}\,.$$

Furthermore, from [7, Theorem 27.3] it follows that if M is a maximal ideal in E then the direct sum

$$E = \mathbb{R}e \oplus M$$

holds, where $\mathbb{R}e$ is the line generated by the vector e. Consequently, if E has a unique maximal ideal, then $E = \mathbb{R}e \oplus \operatorname{Rad}(E)$. Conversely, if $E = \mathbb{R}e \oplus \operatorname{Rad}(E)$ then $\operatorname{Rad}(E)$ is the only maximal ideal in E. Indeed, let M be a maximal ideal in E and pick $x \in M$. Then there exists $r \in \mathbb{R}$ and $x_0 \in \operatorname{Rad}(E)$ such that

$$re = x - x_0 \in M.$$

It follows that $r = 0$ (otherwise, M would contains a strong unit which is impossible). Therefore,

$$x = x_0 \in \operatorname{Rad}(E)$$

and so

$$M \subset \operatorname{Rad}(E) \subset M.$$

We derive directly that $\operatorname{Rad}(E)$ is the unique maximal ideal in E. We call E a *local vector lattice* provided one the following equivalent conditions hold.

(1) E has a unique maximal ideal.

(2) The direct sum $E=\mathbb{R}e\oplus\operatorname{Rad}(E)$ holds.

Thus, for any element x in the local vector lattice E there exist a unique real number $r(x)$ and a unique element $\rho(x)\in\operatorname{Rad}(E)$ such that

$$x=r(x)\,e+\rho(x)\in\mathbb{R}e\oplus\operatorname{Rad}(E).$$

We call $r(x)\,e$ the *real component* of x and $\rho(x)$ its *infinitely small component*. By the way, the maps r and ρ that take each $x\in E$ to $r(x)$ and $\rho(x)$, respectively, are linear. But one might still wonder what a local vector lattice looks like. Some elements are clarified next.

Lemma 2.1. *Let E be a local vector lattice and $x\in E$. Then the following hold.*

(i) *$x\in E^{+}$ if and only if, either $r(x)>0$, or $0\leq x\in\operatorname{Rad}(E)$.*

(ii) *If $x\notin\operatorname{Rad}(E)$ then $|x|=\operatorname{sign}(r(x))\,x$.*

(iii) *$r(|x|)=|r(x)|$ and, if $r(x)\neq 0$ then $\rho(|x|)=\operatorname{sign}(r(x))\,\rho(x)$.*

Proof. (i) Let us prove the 'if' part. There is nothing to prove if $0\leq x\in\operatorname{Rad}(E)$. Hence, assume that $r(x)>0$. Since $\rho(x)$ is an infinitely small element of E, there exists $y\in E$ such that

$$n\,|\rho(x)|\leq|y|\,,\quad\text{for all } n\in\{1,2,...\}\,.$$

Choose $m\in\{1,2,...\}$ such that $|y|\leq me$. Thus,

$$\frac{n}{m}\,|\rho(x)|\leq e,\quad\text{for all } n\in\{1,2,...\}\,.$$

Therefore, we can take $n\in\{1,2,...\}$ large enough for the inequality $m\leq nr(x)$ to hold. Whence,

$$\frac{-1}{r(x)}\rho(x)\leq\frac{1}{r(x)}\,|\rho(x)|\leq\frac{n}{m}\,|\rho(x)|\leq e.$$

It follows that $-\rho(x)\leq r(x)\,e$, which yields that

$$x=r(x)\,e+\rho(x)\geq 0.$$

Now, we establish the 'only if' part. Assume that $x \in E^+$. Clearly, it suffices to prove that $r(x) \geq 0$. Arguing by contradiction, suppose that $r(x) < 0$. Hence,

$$r(-x) = -r(x) > 0$$

and so, by the 'if' part, $-x \in E^+$. We derive that $x = 0$ and so $r(x) = 0$, which contradicts the inequality $r(x) < 0$. The proof of (i) is complete.

(ii) Suppose that $x \notin \mathrm{Rad}(E)$ or, equivalently, $r(x) \neq 0$. Using (i) and the equality $r(-x) = -r(x)$, we infer that

$$x \geq 0 \text{ if } r(x) > 0 \quad \text{and} \quad x \leq 0 \text{ if } r(x) < 0.$$

Summarizing, we get the equality $|x| = \mathrm{sign}(r(x))\, x$ and (ii) follows.

(iii) If $r(x) = 0$ then $x \in \mathrm{Rad}(E)$ and so $|x| \in \mathrm{Rad}(E)$ because $\mathrm{Rad}(E)$ is an ideal in E. Hence, $r(|x|) = 0 = |r(x)|$. Let's suppose that $r(x) \neq 0$ or, equivalently, $x \notin \mathrm{Rad}(E)$. By (ii), we have

$$\begin{aligned} |x| &= \mathrm{sign}(r(x))\, x = \mathrm{sign}(r(x))\,(r(x)\, e + \rho(x)) \\ &= \mathrm{sign}(r(x))\, r(x)\, e + \mathrm{sign}(r(x))\, \rho(x) \\ &= |r(x)|\, e + \mathrm{sign}(r(x))\, \rho(x). \end{aligned}$$

Accordingly,

$$r(|x|) = |r(x)|, \quad \text{for all } x \in E$$

and

$$\rho(|x|) = \mathrm{sign}(r(x))\, \rho(x), \quad \text{for all } x \in E \text{ with } r(x) \neq 0.$$

This completes the proof of the lemma. □

The following characterization of local vector lattices may well not have been quite on the agenda, but we think that it could be of interest to some. Let E be vector lattice with a distinguished strong unit $e > 0$. We call a *component* of E any element $p \in E$ such that

$$p \wedge (e - p) = 0.$$

Notice that 0 and e are components in E. The set of all components in E is denoted by $C(E)$. Clearly,

$$0 \leq p \leq e \text{ and } e - p \in C(E), \quad \text{for all } p \in C(E).$$

More about components can be found in [1, Section 1.4]. Following the recent work [5] (see also [4]), we say that E is *clean* if each element of E can be written as a sum of a strong unit and a component of E. In other words, E is clean if and only if, for every $x \in E$ there exist a strong unit u of E and $p \in C(E)$ such that $x = u + p$. We are in position now to give the characterization we were talking about.

Theorem 2.2. *A vector lattice E with with a strong unit $e > 0$ is local if and only if E is clean and $C(E) = \{0, e\}$.*

Proof. Necessity. Choose $p \in C(E)$ and assume that $0 < p < e$. This assumption together with the equality

$$p \wedge (1 - p) = 0$$

yields that p is not a strong unit in E (otherwise, we would get $e - p = 0$ and so $p = e$). Since $\mathrm{Rad}(E)$ is the unique maximal ideal in E, Theorem 27.4 in [7] shows that $p \in \mathrm{Rad}(E)$. Analogously, $e - p \in \mathrm{Rad}(E)$ and so

$$e = p + (e - p) \in \mathrm{Rad}(E).$$

But $\mathrm{Rad}(E)$ is a maximal ideal in E and thus cannot contain strong units. This contradiction allows us to conclude that $p \in \{0, e\}$ and so $C(E) = \{0, e\}$. Now, we claim that E is clean. To this end, we prove that if $x \in E$ then $x = u + p$ for some strong unit u in E and $p \in \{0, e\}$. If x itself is a strong unit in E, then take $u = x$ and $p = 0$. So, suppose that x is not a strong unit in E. Again by Theorem 27.4 in [7], we infer that $x \in \mathrm{Rad}(E)$. Assume that $e - x$ is not a strong unit in E. Similarly, we would obtain that $e - x \in \mathrm{Rad}(E)$ which leads to the contradiction

$$e = e - x + x \in \mathrm{Rad}(E).$$

Therefore, $e - x$ is a strong unit in E. It suffices thus to put $u = e - x$ and $p = e$. This means that E is clean and proves Necessity.

Sufficiency. Assume that E is clean and $C(E) = \{0, e\}$. Let M and N be two maximal ideals in E. We claim that $M = N$. To this end, we shall argue by contradiction. Without loss of generality, suppose that there exists $x \in M$ such that $x \notin N$. So, there is $r \in \mathbb{R}$ for which

$$r \neq 0 \quad \text{and} \quad x - re \in N$$

(see [7, Theorem 27.3]). By (iii), we can find a strong unit u in E and $p \in C(E) = \{0, e\}$ such that

$$\frac{1}{r}x = u + p.$$

If $p = 0$ then $ru = x \in M$, which contradicts the maximality of M. Analogously, if $p = e$ then $ru = x - re \in N$, contradicting here the maximality of N. We deduce that $M = N$ and then E has only one maximal ideal. We can conclude that E is local, as desired. □

The following (and last) result in this section will be of much greater use in the next section, in which categories are involved. For this reason, we are once again using the Category language. Recall from the introduction that **VL** denotes the category of all vector lattices and lattice homomorphisms, while **LVL** is the used to denote the subcategory of **VL** whose objects are local vector lattices and morphisms are unital lattice homomorphisms. For convenience, recall that if E and F are **LVL**-objects with distinguished positive strong units e and f, respectively, then a **LVL**-morphism $E \overset{\omega}{\to} F$ is a **VL**-morphism for which $\omega(e) = f$. For instance, it follows from the third assertion in Lemma 2.1 that the map $r : E \to \mathbb{R}$ that takes each element $x \in E$ to the real number $r(x)$ is an **LVL**-morphism. Notice by the way that $\mathbb{R}$ is an **LVL**-object with $\operatorname{Rad}(\mathbb{R}) = \{0\}$. As previously pointed out, we end this section with a lemma that will often come in handy in the next section.

Lemma 2.3. *Let $E \overset{\omega}{\to} F$ be an* **LVL***-morphism. Then*

$$\omega(x) \in \operatorname{Rad}(F) \quad \textit{for all } x \in \operatorname{Rad}(E).$$

Proof. Since the radical of a unital vector lattice coincides with the ideal of all infinitely small elements, it suffices to prove that if an element $x \in E$ is infinitely small, then so is $\omega(x) \in F$. So, let x be an infinitely small element of E. Therefore, there exists $y \in E$ such that

$$n|x| \leq |y|, \quad \text{for all } n \in \{1, 2, ...\}.$$

Whence, if $n \in \{1, 2, ...\}$ then

$$n|\omega(x)| = \omega(n|x|) \leq \omega(|y|) = |\omega(y)|.$$

It follows that $\omega(x)$ is an infinitely small element of F, leading to the desired result. $\square$

3 The radical functor

Notwithstanding its simpleness, Lemma 2.3 allow us to introduce a functor from **LVL** to **VL**. Indeed, for every **LVL**-morphism $E \overset{\omega}{\to} F$, we define a **VL**-morphism $\mathrm{Rad}(E) \overset{\omega_0}{\to} \mathrm{Rad}(F)$ by putting

$$\omega_0(x) = \omega(x), \quad \text{for all } x \in \mathrm{Rad}(E).$$

Roughly speaking, ω_0 is the restriction of ω to $\mathrm{Rad}(E)$. Then, let $\mathcal{R}$ be the function that takes each **LVL**-object E to the **VL**-object $\mathrm{Rad}(E)$, and each **LVL**-morphism $E \overset{\omega}{\to} F$ to the **VL**-morphism $\mathrm{Rad}(E) \overset{\omega_0}{\to} \mathrm{Rad}(F)$. We can readily check that $\mathcal{R}$ is a functor from **LVL** to **VL**. We call $\mathcal{R}$: **LVL** $\to$ **VL** the *radical functor*. We begin our investigation with the following first property of the radical functor. We emphasize that we shall keep the same notations as previously used in the preceding sections.

Lemma 3.1. *The radical functor* $\mathcal{R}$: **LVL** $\to$ **VL** *is faithful.*

Proof. Let E, F be two **LVL**-objects and let e, f denote the distinguished positive strong units of E and F, respectively. Pick two **LVL**-morphisms $E \overset{\omega}{\to} F$ and $E \overset{\psi}{\to} F$ such that

$$\omega_0 = \mathcal{R}(\omega) = \mathcal{R}(\psi) = \psi_0.$$

We claim that $\omega = \psi$. To this end, pick $x \in E$ and observe that

$$\begin{aligned} \omega(x) &= \omega(r(x)e + \rho(x)) = r(x)\omega(e) + \omega(\rho(x)) \\ &= r(x)f + \omega_0(\rho(x)) = r(x)\psi(e) + \psi_0(\rho(x)) \\ &= r(x)\psi(e) + \psi(\rho(x)) = \psi(r(x)e + \rho(x)) = \psi(x). \end{aligned}$$

This completes the proof of the lemma. $\square$

Next, we give the second fundamental property of the radical functor.

Lemma 3.2. *The radical functor* $\mathcal{R}$: **LVL** $\to$ **VL** *is full.*

Proof. Let E, F be two **LVL**-objects, and choose a **VL**-morphism $\mathrm{Rad}\,(E) \overset{\varphi}{\to} \mathrm{Rad}\,(F)$. We claim that there exists an **LVL**-morphism $E \overset{\omega}{\to} F$ such that

$$\omega_0 = \mathcal{R}\,(\omega) = \varphi.$$

To this end, let e and f denote the distinguished positive strong units of E and F, respectively. Define $\omega : E \to F$ by putting

$$\omega\,(x) = r\,(x)\,f + \varphi\,(\rho\,(x))\,, \quad \text{for all } x \in E.$$

Observe that $\varphi\,(\rho\,(x)) \in \mathrm{Rad}\,(F)$ (where we use Lemma 2.3). Furthermore, it follows readily from the direct sum

$$E = \mathbb{R}e \oplus \mathrm{Rad}\,(E)$$

that ω is linear. Also, $\omega\,(e) = f$ because $r\,(e) = 1$ and $\rho\,(e) = 0$. Moreover, if $x \in \mathrm{Rad}\,(E)$ then $r\,(x) = 0$ and thus

$$\omega_0\,(x) = \omega\,(x) = \varphi\,(\rho\,(x)) = \varphi\,(x)\,.$$

It remains to show that ω is a lattice homomorphism. Let $x \in E$ and observe that

$$\begin{aligned} \omega\,(x) &= \omega\,(r\,(x)\,e + \rho\,(x)) \\ &= r\,(x)\,\omega\,(e) + \omega\,(\rho\,(x)) \\ &= r\,(x)\,f + \varphi\,(\rho\,(x))\,. \end{aligned}$$

It follows quickly, again from Lemma 2.3, that

$$r\,(\omega\,(x)) = r\,(x) \quad \text{and} \quad \rho\,(\omega\,(x)) = \varphi\,(\rho\,(x))\,.$$

Moreover, if $r\,(x) = 0$ then

$$\omega\,(|x|) = \varphi\,(|x|) = |\varphi\,(x)| = |\omega\,(x)|$$

On the other hand, if $r\,(x) \neq 0$ then Lemma 2.1 (iii) yields that

$$|x| = r\,(|x|)\,e + \rho\,(|x|) = |r\,(x)|\,e + \mathrm{sign}\,(r\,(x))\,\rho\,(x)\,.$$

Hence, by linearity and Lemma 2.1 (iii), we derive that

$$\begin{aligned}\omega\left(|x|\right) &= \omega\left(|r\left(x\right)|\,e + \operatorname{sign}\left(r\left(x\right)\right)\rho\left(x\right)\right)\\ &= |r\left(x\right)|\,\omega\left(e\right) + \omega\left(\operatorname{sign}\left(r\left(x\right)\right)\rho\left(x\right)\right)\\ &= |r\left(\omega\left(x\right)\right)|\,f + \varphi\left(\operatorname{sign}\left(r\left(\omega\left(x\right)\right)\right)\rho\left(x\right)\right)\\ &= |r\left(\omega\left(x\right)\right)|\,f + \operatorname{sign}\left(r\left(\omega\left(x\right)\right)\right)\varphi\left(\rho\left(x\right)\right)\\ &= |r\left(\omega\left(x\right)\right)|\,f + \operatorname{sign}\left(r\left(\omega\left(x\right)\right)\right)\rho\left(\omega\left(x\right)\right) = |\omega\left(x\right)|\,.\end{aligned}$$

Thus, ω is a lattice homomorphism the proof is complete. □

To conclude that the radical functor is an equivalence, it remains to show that it is isomorphism-dense. More precisely, we have to show that for every **VL**-morphism $L \overset{\varphi}{\to} M$, there exists an **LVL**-morphism $E \overset{\omega}{\to} F$ such that

$$\operatorname{Rad}\left(E\right) \overset{\omega_0}{\to} \operatorname{Rad}\left(F\right) = L \overset{\varphi}{\to} M.$$

It turns out that this property is quite involved and requires much more work than the properties of being faithful and full. In this prospect, further backgrounds are needed. Let L be a **VL**-object. The Cartesian product $\mathbb{R} \times L$ is a real vector space with respect to the coordinatewise addition and scalar multiplication. Moreover, it is clear that L can be identified with the vector subspace $\{0\} \times L$ of $\mathbb{R} \times L$. Henceforth, we shall consider L as a vector subspace of $\mathbb{R} \times L$. Analogously, identifying $\mathbb{R}$ with $\mathbb{R} \times \{0\}$, we can assume that $\mathbb{R}$ is a vector subspace of $\mathbb{R} \times L$. Thus, we have the direct sum

$$\mathbb{R} \times L = \mathbb{R} \oplus L = \left\{r + u : r \in \mathbb{R} \text{ and } u \in L\right\}.$$

Consequently, each vector $x \in \mathbb{R} \oplus L$ can be written uniquely as a sum of a scalar $\operatorname{Re}\left(x\right) \in \mathbb{R}$ and a vector $L\left(x\right) \in L$. Then, we have

$$x - \operatorname{Re}\left(x\right) = L\left(x\right) \in L, \quad \text{for all } x \in \mathbb{R} \oplus L.$$

Also, we can perform a simple calculation to see that

$$\operatorname{Re}\left(rx + y\right) = r\operatorname{Re}\left(x\right) + \operatorname{Re}\left(y\right), \quad \text{for all } r \in \mathbb{R} \text{ and } x, y \in \mathbb{R} \oplus L \tag{3.1}$$

and

$$L\left(rx + y\right) = rL\left(x\right) + L\left(y\right), \quad \text{for all } r \in \mathbb{R} \text{ and } x, y \in \mathbb{R} \oplus L. \tag{3.2}$$

We have gathered now all the ingredient we need to prove the following theorem.

Theorem 3.3. *Let L be a* **VL**-*object. Then $\mathbb{R} \oplus L$ is an* **LVL**-*object with radical L and positive strong unit* 1.

Proof. We shall keep the above notations. Let $\mathcal{K}$ denote the set of all $x \in \mathbb{R} \oplus L$ such that, either $\mathrm{Re}\,(x) > 0$, or $\mathrm{Re}\,(x) = 0$ and $x \in L^+$. First, we claim that $\mathcal{K}$ is a cone in $\mathbb{R} \oplus L$ (see Definition 1.2 in [3]). To this end, notice that

$$0 \leq \mathrm{Re}\,(x)\,, \quad \text{for all } x \in \mathcal{K}.$$

Choose $x, y \in \mathcal{K}$ and assume that $\mathrm{Re}\,(x) > 0$ or $\mathrm{Re}\,(y) > 0$. Since

$$\mathrm{Re}\,(x + y) = \mathrm{Re}\,(x) + \mathrm{Re}\,(y)$$

(see (3.1)), we derive that $\mathrm{Re}\,(x + y) > 0$ and thus $x + y \in \mathcal{K}$. Otherwise, $\mathrm{Re}\,(x) = \mathrm{Re}\,(y) = 0$ and so $x, y \in L^+$. We get again $x + y \in \mathcal{K}$. This means that $\mathcal{K}$ is closed under addition. Furthermore, take $r \in [0, \infty)$ and observe that $rx \in L^+$ if $\mathrm{Re}\,(x) = 0$ or $r = 0$, and $\mathrm{Re}\,(rx) = r\,\mathrm{Re}\,(x) > 0$ if $\mathrm{Re}\,(x) > 0$ and $r > 0$ (where we use (3.1)). We derive that $rx \in \mathcal{K}$. Now, pick $x \in \mathbb{R} \oplus L$ such that $\pm x \in \mathcal{K}$. This together with (3.1) leads to

$$\pm\,\mathrm{Re}\,(x) = \mathrm{Re}\,(\pm x) \geq 0.$$

Accordingly, $\mathrm{Re}\,(x) = 0$ and so $\pm x \in L^+$. This yields that $x = 0$ and finally that $\mathcal{K}$ is a cone in $\mathbb{R} \oplus L$, as required. In other words, $\mathbb{R} \oplus L$ is an ordered vector space with $\mathcal{K}$ as positive cone (see Theorem 11.4 in [7]). Henceforth, we give up the symbol $\mathcal{K}$ and we use instead the classical notation $(\mathbb{R} \oplus L)^+$. In the next step, we prove that the order vector space $\mathbb{R} \oplus L$ is a vector lattice. Actually, we shall see that any element in $\mathbb{R} \oplus L$ has an absolute value in $\mathbb{R} \oplus L$.

Hence, let $x \in \mathbb{R} \oplus L$ and define $y \in \mathbb{R} \oplus L$ by

$$y = \begin{cases} |x|\,, & \text{if } \mathrm{Re}\,(x) = 0 \\ \\ \mathrm{sign}\,(\mathrm{Re}\,(x))\,x, & \text{if } \mathrm{Re}\,(x) \neq 0. \end{cases}$$

Observe that if $\mathrm{Re}\,(x) \neq 0$ then

$$y = |\mathrm{Re}\,(x)| + \mathrm{sign}\,(\mathrm{Re}\,(x))\,L\,(x)\,.$$

We infer that $y \in (\mathbb{R} \oplus L)^{+}$. Moreover,

$$y - x = \begin{cases} |x| - x, & \text{if } \operatorname{Re}(x) = 0 \\ \\ [|\operatorname{Re}(x)| - \operatorname{Re}(x)] + (\operatorname{sign}(\operatorname{Re}(x)) - 1)\, L(x), & \text{if } \operatorname{Re}(x) \neq 0. \end{cases}$$

Clearly, if $\operatorname{Re}(x) = 0$ then $y - x \in (\mathbb{R} \oplus L)^{+}$. Furthermore, if $\operatorname{Re}(x) > 0$ then

$$y - x = 0 \in (\mathbb{R} \oplus L)^{+},$$

and if $\operatorname{Re}(x) < 0$ then

$$y - x = 2\, |\operatorname{Re}(x)| - 2L(x) \in (\mathbb{R} \oplus L)^{+}.$$

Analogously, we may show that $y + x \in (\mathbb{R} \oplus L)^{+}$. Consequently, y is an upper bounded in $\mathbb{R} \oplus L$ of the pair $\{-x, x\}$. Choose another upper bound z of the pair $\{-x, x\}$ in $\mathbb{R} \oplus L$. We derive that $\operatorname{Re}(z + x) \geq 0$ and $\operatorname{Re}(x - z) \geq 0$. It follows from (3.1) that $\operatorname{Re}(z) \geq |\operatorname{Re}(x)|$. If $\operatorname{Re}(x) \neq 0$ then

$$y = \operatorname{sign}(\operatorname{Re}(x))\, x = \pm x \leq z.$$

Also, if $\operatorname{Re}(z) = 0$ then $\operatorname{Re}(x) = 0$ and so $x, z \in L$. Accordingly, the inequalities $z \geq \pm x$ holds in L which gives $z \geq |x| = y$. Finally, if $\operatorname{Re}(x) = 0$ and $\operatorname{Re}(z) > 0$ then, by (3.1),

$$\operatorname{Re}(z - y) = \operatorname{Re}(z - |x|) = \operatorname{Re}(z) > 0$$

Accordingly, $z - y \in (\mathbb{R} \oplus L)^{+}$ and finally y is precisely the absolute value of x in $\mathbb{R} \oplus L$. Summarizing, $\mathbb{R} \oplus L$ is a vector lattice and the absolute value in $\mathbb{R} \oplus L$ is given by

$$|x| = \begin{cases} |L(x)| & \text{if } \operatorname{Re}(x) = 0 \\ \\ \operatorname{sign}(\operatorname{Re}(x))\, x = |\operatorname{Re}(x)| + \operatorname{sign}(\operatorname{Re}(x))\, L(x) & \text{if } \operatorname{Re}(x) \neq 0. \end{cases}$$

In particular,

$$\operatorname{Re}(|x|) = |\operatorname{Re}(x)|, \quad \text{for all } x \in \mathbb{R} \oplus L \tag{3.3}$$

and

$$L(|x|) = \operatorname{sign}(\operatorname{Re}(x))\, L(x), \quad \text{for all } x \in \mathbb{R} \oplus L \text{ with } \operatorname{Re}(x) \neq 0. \tag{3.4}$$

At this point, observe that $1 \in (\mathbb{R} \oplus L)^+$. We claim that 1 is a strong unit in $\mathbb{R} \oplus L$. To do this, pick $x \in \mathbb{R} \oplus L$ and choose $n \in \{1, 2, ...\}$ such that $n > |\mathrm{Re}\,(x)|$. Hence, by (3.1),

$$\mathrm{Re}\,(n - x) = n - \mathrm{Re}\,(x) > 0 \quad \text{and} \quad \mathrm{Re}\,(n + x) = n + \mathrm{Re}\,(x) > 0.$$

We derive that

$$n - x \in (\mathbb{R} \oplus L)^+ \quad \text{and} \quad n + x \in (\mathbb{R} \oplus L)^+.$$

This means that the inequality $|x| \leq n$ holds in $\mathbb{R} \oplus L$. It follows that 1 is a strong unit in $\mathbb{R} \oplus L$, as required.

The last step in the proof is to show that L is the unique maximal ideal in $\mathbb{R} \oplus L$. Let $x \in L$ and $y \in \mathbb{R} \oplus L$ such that $|y| \leq |x|$ in $\mathbb{R} \oplus L$. Whence, $|x| - |x| \in (\mathbb{R} \oplus L)^+$ from which we derive that

$$0 \leq \mathrm{Re}\,(|x| - |y|) = \mathrm{Re}\,(|x|) - \mathrm{Re}\,(|y|) = -\,\mathrm{Re}\,(|y|) = -\,|\mathrm{Re}\,(y)| \leq 0$$

(where we use (3.1) and (3.3)). This yields that $\mathrm{Re}\,(y) = 0$ and so $y \in L$. We infer that L is an ideal in $\mathbb{R} \oplus L$, as desired. On the other hand, choose a maximal ideal M and pick $x \in M$. We claim that $\mathrm{Re}\,(x) = 0$. To this end, we shall argue by contradiction assuming that $\mathrm{Re}\,(x) > 0$ (if $\mathrm{Re}\,(x) < 0$ then we can work with $-x$). Observe that

$$\frac{2}{\mathrm{Re}\,(x)} x = 1 + \left(1 + \frac{2}{\mathrm{Re}\,(x)} L\,(x)\right) \geq 1 \text{ in } \mathbb{R} \oplus L$$

because

$$1 + \frac{2}{\mathrm{Re}\,(x)} L\,(x) \geq 0 \text{ in } \mathbb{R} \oplus L.$$

But then x would be a strong unit in $\mathbb{R} \oplus L$, which contradicts the fact that M is a maximal ideal. Therefore, $\mathrm{Re}\,(x) = 0$ and thus M is contained in L. By maximality, we obtain that $L = M$. This yields that L is the unique maximal ideal in $\mathbb{R} \oplus L$ and completes the proof of the theorem. □

As an application of Theorem 3.3, we get the following property of the radical functor $\mathcal{R} : \mathbf{LVL} \to \mathbf{VL}$.

Corollary 3.4. *The radical functor* $\mathcal{R} : \mathbf{LVL} \to \mathbf{VL}$ *is isomorphic-dense.*

Proof. Let $L \overset{\varphi}{\to} M$ be a **VL**-morphism. Put

$$E = \mathbb{R} \oplus L \quad \text{and} \quad F = \mathbb{R} \oplus L.$$

From Theorem 3.3 it follows that E is an **LVL**-object with 1 as a positive strong unit and $\operatorname{Rad}(E) = L$. Similarly, F is an **LVL**-object with 1 as a positive strong unit and $\operatorname{Rad}(F) = M$. Now, since

$$\varphi(L(x)) \in M, \quad \text{for all } x \in L,$$

we can define a map $E \overset{\omega}{\to} F$ by putting

$$\omega(x) = \operatorname{Re}(x) + \varphi(L(x)), \quad \text{for all } x \in E.$$

It turns out that ω is an **LVL**-morphism. Indeed, it is clear that $\omega(1) = 1$. Moreover, the linearity of ω follows straightforwardly from (3.1) and (3.2). Also, if $x \in L$ then $\operatorname{Re}(x) = 0$ and so

$$\omega(x) = \varphi(L(x)) = \varphi(x).$$

On the other hand, pick $x \in E$ and observe that if $\operatorname{Re}(x) = 0$ then

$$|\omega(x)| = |\varphi(L(x))| = \varphi(|L(x)|) = \varphi(|x|) = \omega(|x|).$$

Furthermore, if $\operatorname{Re}(x) \neq 0$ then, using (3.3) and (3.4), we get

$$\begin{aligned} |\omega(x)| &= |\operatorname{Re}(\omega(x))| + \operatorname{sign}(\operatorname{Re}(\omega(x)))\, M(\omega(x)) \\ &= |\operatorname{Re}(x)| + \varphi(\operatorname{sign}(\operatorname{Re}(x))(L(x))) \\ &= \operatorname{Re}(|x|) + \varphi(L(|x|)) = \omega(|x|). \end{aligned}$$

Summarizing, $E \overset{\omega}{\to} F$ is an **LVL**-morphism and we have

$$\operatorname{Rad}(E) \overset{\omega_0}{\to} \operatorname{Rad}(F) = L \overset{\varphi}{\to} M.$$

We conclude that the radical functor $\mathcal{R} : \mathbf{LVL} \to \mathbf{VL}$ is full. $\square$

Combining Lemma 3.1, Lemma 3.2, and Corollary 3.4, we get straightforwardly the following theorem, which is the main (and the last) result of the paper.

Theorem 3.5. *The radical functor $\mathcal{R} : \mathbf{LVL} \to \mathbf{VL}$ is an equivalence.*

We end this paper with the following comment. By Theorem 14.15 in [6], there must exist an equivalence functor $\mathcal{L} : \mathbf{VL} \to \mathbf{LVL}$. Such a functor can be constructed as follows. For any **VL**-object L, we denote by $\mathcal{L}(L)$ the **LVL**-object $\mathbb{R} \oplus L$ (see Theorem 3.3). Then, any **VL**-morphism $L \xrightarrow{\varphi} M$ extends uniquely to an **LVL**-morphism $\mathcal{L}(L) \xrightarrow{\mathcal{L}(\varphi)} \mathcal{L}(M)$ by putting

$$\mathcal{L}(\varphi)(x) = \operatorname{Re}(x) + \varphi(L(x)) \quad \text{for all } x \in \mathcal{L}(L).$$

The proof of this is similar to the proof of Corollary 3.4. Finally, it is routine to show that the function $\mathcal{L}$ that assigns to each **VL**-object L the **LVL**-object $\mathcal{L}(L)$, and to each **VL**-morphism $L \xrightarrow{\varphi} M$ the **LVL**-morphism $\mathcal{L}(L) \xrightarrow{\mathcal{L}(\varphi)} \mathcal{L}(M)$ is a functor from **VL** to **LVL**.

References

[1] Aliprantis, C.D. and Burkinshaw, O., "Locally Solid Riesz Spaces with Applications to Economics", Mathematical Surverys and Monographs 105, Amer. Math. Soc., 2003.

[2] Aliprantis, C.D. and Burkinshaw, O., "Positive Operators", Springer, 2006.

[3] Aliprantis, C.D. and R. Tourkey, R., "Cones and Duality", Graduate Studies in Mathematics 84, Amer. Math. Soc., 2007.

[4] Boulabiar, K. and Smiti, S., *Lifting components in clean abelian ℓ-groups*, Math. Slovaca, 68 (2018), 299-310.

[5] Hager, A.W., Kimber, C.M., and McGovern, W.W., *Clean unital ℓ-groups*, Math. Slovaca 63 (2013), 979-992.

[6] Herrlich, H. and Strecker, G.E., "Category Theory: an Introduction", Allyn and Bacon Inc., 1973.

[7] Luxemburg, W.A.J. and Zaanen, A.C., "Riesz spaces I", North-Holland Research Monographs, Elsevier, 1971.

[8] Matsumura, H., "Commutative Ring Theory", Cambridge University Press, 1989.

Karim Boulabiar *University of Tunis-El Manar, Faculty of Sciences of Tunis, Department of Mathematics 2092-El Manar, Tunisia.*

Email: karim.boulabiar@ipest.rnu.tn