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State filters in state residuated lattices

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Abstract. In this paper, we introduce the notions of prime state filters, obstinate state filters, and primary state filters in state residuated lattices and study some properties of them. Several characterizations of these state filters are given and the prime state filter theorem is proved. In addition, we investigate the relations between them.

1 Introduction and Preliminaries

The algebraic counterparts of some non-classical logics satisfy residuation and those logics can be considered in a frame of residuated lattices. In fact, residuated lattices were introduced by M. Ward et.al. [23]. The theory of filters and ideals plays an important role in studying these algebras. From logical point of view, filters correspond to sets of provable formulas. Some types of filters of a residuated lattice such as Boolean filters, implicative filters, positive implicative filters, and obstinate filters were introduced, for example, in [2, 12, 21, 22].

Also, F. Forouzesh and et.al. introduced obstinate ideals in MV-algebras and investigated some properties of them in [5]. States on MV-algebras

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were introduced by Mundici [16] with the intent of measuring the average truth-value of propositions in Łukasiewicz logic, which are a generalization of probability measures on Boolean algebras. States on MV-algebras have been deeply investigated [4, 11]. Consequently, the notion of states has been extended to other logical algebras such as BL-algebras [19], MTL-algebras [13], [15], R_0-algebras [14], and residuated lattices [3], [20] and their non-commutative cases. Different approaches to the generalization mainly gave rise to two different notions, namely, Rieˇcan states [19] and Bosbach states [6]. It was proved by Ciungu [3] that in any good non-commutative residuated lattice Bosbach states and Rieˇcan states coincide, while the converse is not always true. Thus, the notion of Rieˇcan states is more general than that of Bosbach states. What the two have in common is that they both have as codomain the closed unit interval [0, 1].

State residuated lattices were introduced by Pengfei He, et.al. in 2015 [10]. They introduced the notion of state operators on residuated lattices and investigated some related properties of such operators. Also, they gave characterizations of Rl-monoids and Heyting algebras, and discussed the relations between state operators and states on residuated lattices.

In this paper, we introduce the concept of the prime state filters in the state residuated lattices. We give a characterization of the prime state filters and prove the prime state filter theorem. In addition, we define the notion of obstinate state filters in a state residuated lattice and investigate the relations between the obstinate state filters and some other state filters of a residuated lattice. We obtain extension theorem of obstinate state filters and give several characterizations of these state filters. Finally, we introduce the notion of the primary state filters in state residuated lattices and give a characterization of the primary state filters.

In this section, we summarize some definitions and results about residuated lattices and lattices, which will be used in the following sections of the paper.

Definition 1.1. [9, 23] An algebraic structure $(L, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a *residuated lattice* if it satisfies the following conditions

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(2) $(L, \odot, 1)$ is a commutative monoid,
(3) $x \odot y \leq z$ if and only if $x \leq y \to z$,

for all $x, y, z \in L$, where $\leq$ is the partial order of the lattice $(L, \wedge, \vee, 0, 1)$.

In what follows, by L we denote the universe of a residuated lattice $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$. For any $x \in L$ and a natural number n, we define $x^* = x \rightarrow 0$, $x^{**} = (x^*)^*$, $x^0 = 1$, and $x^n = x^{n-1} \odot x$.

Proposition 1.2. [9, 18] *In a residuated lattice* $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$*, the following properties hold, for every* $x, y, z, t \in L$*:*

(1) $1 \rightarrow x = x, x \rightarrow 1 = 1$,
(2) $x \leq y$ *if and only if* $x \rightarrow y = 1$,
(3) $x \odot x^* = 0$, *and* $x \odot y = 0$ *if and only if* $x \leq y^*$,
(4) *if* $x \leq y$, *then* $y \rightarrow z \leq x \rightarrow z$, $z \rightarrow x \leq z \rightarrow y$,
(5) $x \odot (x \rightarrow y) \leq y$, $x \odot y \leq x, y, x \vee y \geq x, y$,
(6) $x \odot y \leq x \wedge y$, $x \leq y \rightarrow x$,
(7) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$,
(8) $0^* = 1$, $1^* = 0$, $x \leq x^{**}$, $x^{***} = x^*$, $(x \vee y)^* = x^* \wedge y^*$,
(9) $x \odot (y \rightarrow z) \leq y \rightarrow (x \odot z) \leq (x \odot y) \rightarrow (x \odot z)$,
(10) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
(11) $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$, *hence* $x \vee y^n \geq (x \vee y)^n$ *and* $x^m \vee y^n \geq (x \vee y)^{mn}$ *for every natural numbers m,n,*
(12) $x \rightarrow (x \wedge y) = x \rightarrow y$,
(13) $x \odot y = x \odot (x \rightarrow x \odot y)$,
(14) *if* $x \leq y$, $z \leq t$, *then* $x \odot z \leq y \odot t$,
(15) $x \leq y$ *if and only if* $x \wedge y = x$, $x \vee y = y$,
(16) $x \vee y = 1$ *implies* $x \wedge y = x \odot y$,
(17) $x^* \leq x \rightarrow y$.

Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a residuated lattice. A nonempty set F of L is called a *filter* if it satisfies: (1) For $x, y \in F$, $x \odot y \in F$; (2) For $x \in F$, $y \in L$, and $x \leq y$, we have $y \in F$. We denote by $F[L]$ the set of all filters of L. A proper filter F of L is called a *maximal* filter if it is not strictly contained in any proper filter of L. Note that, a proper filter F of L is maximal if and only if, for all $x \in L$, the following holds: $x \notin F$ if and only if there exists an integer $n \geq 1$ such that $(x^n)^* \in F$. If X is a nonempty subset of L, then we denote the filter generated by X by $\langle X \rangle$. Clearly, we have $\langle X \rangle = \{x \in L : x \geq x_1 \odot x_2 \odot \ldots \odot x_n, x_i \in X\}$, see [18].

Definition 1.3. [18] A *deductive system* of a residuated lattice L is a subset F containing 1 such that if $x \to y \in F$ and $x \in F$, then $y \in F$. Note that a subset F of a residuated lattice L is a deductive system of L if and only if F is a filter of L .

Definition 1.4. [17] Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a residuated lattice.
(1) L is called *simple* if it has exactly two filters: $\{1\}$ and L.
(2) L is called *local* if it has exactly one maximal filter.

Definition 1.5. [7, page 13] A complete lattice L is called a *frame* if it satisfies the infinite distributive law $x \wedge \bigvee S = \bigvee\{x \wedge s : s \in S\}$, for all $x \in L$ and $S \subseteq L$.

We denote by L the bounded distributive lattice $(L, \wedge, \vee, 0, 1)$.

Definition 1.6. [1, page 174] A *Heyting algebra* is a lattice $(L, \wedge, \vee)$ with 0 such that for every $x, y \in L$ there is the element $x \to y = sup\{z \in L : x \wedge z \leq y\} \in L$. Consequently, $x \wedge z \leq y$ if and only if $z \leq x \to y$.

Definition 1.7. [18] An element a in a residuated lattice L is called *nilpotent* if and only if there exists a natural number n such that $a^n = 0$. The minimum n such that $a^n = 0$ is called the *nilpotenece order* of a and will be denoted by $\mathrm{ord}(a)$.

Definition 1.8. [8] We recall that an element $a \in L$ is called *complemented* if there is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$; if such an element b exists it is called a *complement* of a. A complement of an element in a distributive lattice, if exists, is unique, and is denoted by $b = a'$; and the set of all complemented elements of L is denoted by $B(L)$.

Notation 1. [18] If $e \in B(L)$ and $x, y \in L$, then $e \wedge (x \vee y) = (e \wedge x) \vee (e \wedge y)$.

Definition 1.9. [8] A lattice $(L, \wedge, \vee)$ is called *Brouwerian* if it satisfies the identity $a \wedge (\bigvee_i b_i) = \bigvee_i (a \wedge b_i)$ (whenever the arbitrary joins exists).

Proposition 1.10. [18, Proposition 1.33] *Let L be a residuated lattice. Then, $(F[L], \cap, \vee, \{1\}, L)$ is a Brouwerian lattice.*

Proposition 1.11. [18] *Let L be a residuated lattice.*

(i) *If $F \in F[L]$ and $a \in L$, then*

$$\langle F, a\rangle = \{x \in L : x \geq (f_1 \odot a^{n_1}) \odot \ldots \odot (f_k \odot a^{n_k})$$

with $f_i \in F$, $n_i \in \mathbb{N}$, $i \in \{1, \ldots, k\}, k \geq 1\}$;

(ii) *If $x, y \in L$, then $\langle x \vee y\rangle = \langle x\rangle \cap \langle y\rangle$;*

(iii) *If $F \in F[L]$ and $a \in L \setminus F$, then $\langle F, a\rangle = F \vee \langle a\rangle$;*

(iv) *$\langle F, a\rangle \cap \langle F, b\rangle = \langle F, a \vee b\rangle$, for every $a, b \in L$.*

Proposition 1.12. *Let L be a residuated lattice. For $P \in F[L]$ the following assertions are equivalent:*

(i) *if $P_1, P_2 \in F[L]$ and $P_1 \cap P_2 \subseteq P$, then $P_1 \subseteq P$ or $P_2 \subseteq P$;*

(ii) *if $P_1, P_2 \in F[L]$ and $P_1 \cap P_2 = P$, then $P_1 = P$ or $P_2 = P$;*

(iii) *if $x, y \in L$ and $x \vee y \in P$, then $x \in P$ or $y \in P$.*

Proof. (i)⇒(ii) Let $P_1, P_2 \in F[L]$ and $P_1 \cap P_2 = P$. Then, $P_1 \cap P_2 \subseteq P$, by (i), we get $P_1 \subseteq P$ or $P_2 \subseteq P$. If $P_1 \subseteq P$, then $P = P_1 \cap P_2 \subseteq P_1 \subseteq P$. Hence $P_1 = P$. If $P_2 \subseteq P$, then, by similar way, we obtain $P_2 = P$.

(ii)⇒(i) Let $P_1, P_2 \in F[L]$ and $P_1 \cap P_2 \subseteq P$. Then, $(P_1 \cap P_2) \vee P = P$. It follows from Proposition 1.10, that $(P_1 \vee P) \cap (P_2 \vee P) = P$. Now, by (ii), $P_1 \vee P = P$ or $P_2 \vee P = P$. Thus $P_1 \subseteq P$ or $P_2 \subseteq P$.

(ii)⇒(iii) Let $x \vee y \in P$. It follows, from Proposition 1.11 (ii), that $\langle x\rangle \cap \langle y\rangle = \langle x \vee y\rangle \subseteq P$. Hence we have $(\langle x\rangle \cap \langle y\rangle) \vee P = P$. By Proposition 1.10, we conclude that $(\langle x\rangle \vee P) \cap (\langle y\rangle \vee P) = P$. By hypothesis, we obtain $\langle x\rangle \vee P = P$ or $\langle y\rangle \vee P = P$. So, we get $\langle x\rangle \subseteq P$ or $\langle y\rangle \subseteq P$. This results $x \in P$ or $y \in P$.

(iii)⇒(ii) Let $P_1, P_2 \in F[L]$ and $P_1 \cap P_2 = P$. If we suppose that $P_1 \neq P$ and $P_2 \neq P$, then, there are $x \in P_1 - P$ and $y \in P_2 - P$. Since P_1 and P_2 are filters and $x, y \leq x \vee y$, so $x \vee y \in P_1 \cap P_2 = P$ and, by hypothesis, we get $x \in P$ or $y \in P$, which is a contradiction. Thus $P_1 = P$ or $P_2 = P$. □

A filter F which satisfies one of the equivalent conditions of Proposition 1.12, is called *prime*. We denote the set of all prime filters of L by $Spec(L)$.

Suppose F is a filter of a residuated lattice L. Define $x \equiv_F y$ if and only if $x \to y \in F$ and $y \to x \in F$. Then, $\equiv_F$ is a congruence relation on L. The set of all congruence classes is denoted by L/F, and so $L/F := \{[x] : x \in L\}$, where $[x] = \{y \in L : x \equiv_F y\}$. Define $\bullet, \rightharpoonup, \sqcap, \sqcup$ on L/F by

$[x] \bullet [y] = [x \odot y], [x] \rightharpoonup [y] = [x \to y]$,
$[x] \sqcap [y] = [x \wedge y], [x] \sqcup [y] = [x \vee y]$.
Therefore $(L/F, \sqcap, \sqcup, \bullet, \rightharpoonup, [1], [0])$ is a residuated lattice with respect to F, see [9].

Definition 1.13. [10] Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a residuated lattice. A mapping $\tau : L \to L$ is called a *state operator* on L if it satisfies the following conditions:
$(L1)$ $\tau(0) = 0$;
$(L2)$ $x \to y = 1$ implies $\tau(x) \to \tau(y) = 1$;
$(L3)$ $\tau(x \to y) = \tau(x) \to \tau(x \wedge y)$;
$(L4)$ $\tau(x \odot y) = \tau(x) \odot \tau(x \to (x \odot y))$;
$(L5)$ $\tau(\tau(x) \odot \tau(y)) = \tau(x) \odot \tau(y)$;
$(L6)$ $\tau(\tau(x) \to \tau(y)) = \tau(x) \to \tau(y)$;
$(L7)$ $\tau(\tau(x) \vee \tau(y)) = \tau(x) \vee \tau(y)$;
$(L8)$ $\tau(\tau(x) \wedge \tau(y)) = \tau(x) \wedge \tau(y)$,
for any $x, y \in L$.

The pair (L, τ) is said to be a *state residuated lattice*, or more precisely, a residuated lattice with internal state.

Proposition 1.14. [10] *Let (L, τ) be a state residuated lattice. Then, for any $x, y \in L$, we have*
(1) $\tau(1) = 1$,
(2) $x \leq y$ *implies* $\tau(x) \leq \tau(y)$,
(3) $\tau(x^*) = (\tau(x))^*$,
(4) $\tau(x \odot y) \geq \tau(x) \odot \tau(y)$ *and if* $x \odot y = 0$, *then* $\tau(x \odot y) = \tau(x) \odot \tau(y)$,
(5) $\tau(x \odot y^*) \geq \tau(x) \odot (\tau(y))^*$ *and if* $x \leq y$, *then* $\tau(x \odot y^*) = \tau(x) \odot (\tau(y))^*$,
(6) $\tau(x \to y) \leq \tau(x) \to \tau(y)$. *In particular, if* x, y *are comparable, then* $\tau(x \to y) = \tau(x) \to \tau(y)$,
(7) *if* τ *is faithful, then* $x < y$ *implies* $\tau(x) < \tau(y)$,
(8) $\tau^2(x) = \tau(x)$,
(9) $\tau(L) = Fix(\tau)$, *where* $Fix(\tau) = \{x \in L : \tau(x) = x\}$,
(10) $\tau(L)$ *is a subalgebra of* L,
(11) *Ker(τ) is a filter of* L.

Definition 1.15. [10] Let (L, τ) be a state residuated lattice. A nonempty subset F of L is called a *state filter* of (L, τ) if F is a filter of L such that if $x \in F$, then $\tau(x) \in F$ for all $x \in L$. A proper state filter of (L, τ) is called *maximal* if it not strictly contained in any proper state filter of (L, τ).

We denote the set of all state filters of (L, τ) by $SF[L]$, and the set of all maximal state filters of (L, τ) by $Max_\tau(L)$.

Definition 1.16. [10] A state residuated lattice (L, τ) is called *state simple* if it has exactly two state filters: $\{1\}$ and L.

Let (L, τ) be a state residuated lattice. For any nonempty set X of L, we denote by $\langle X \rangle_\tau$ the state filter of (L, τ) generated by X, that is, $\langle X \rangle_\tau$ is the smallest state filter of (L, τ) containing X.

Proposition 1.17. [10] *Let F, F_1, F_2 be state filters of (L, τ) and $a \notin F$. Then*

(1) $\langle a \rangle_\tau = \{x \in L : x \geq (a \odot \tau(a))^n, n \geq 1\}$, *which is called the principal state filter of* (L, τ),

(2) $\langle F, a \rangle_\tau = \{x \in L : x \geq f \odot (a \odot \tau(a))^n, f \in F, n \geq 1\}$,

(3) $\langle F_1 \cup F_2 \rangle_\tau = \{x \in L : x \geq f_1 \odot f_2, f_1 \in F_1, f_2 \in F_2\}$.

Lemma 1.18. [10] *Let (L, τ) be a state residuated lattice. A proper state filter F of (L, τ) is maximal if and only if, for any $a \notin F$, there exists an integer $n \geq 1$ such that $(\tau(a)^n)^* \in F$.*

Definition 1.19. [10] A state residuated lattice (L, τ) is called *state local* if it has exactly one maximal state filter.

Proposition 1.20. [10] *Let (L, τ) be a state residuated lattice and $a, b \in L$. Then, the following hold:*

(1) *if* $a \leq b$, *then* $\langle b \rangle_\tau \subseteq \langle a \rangle_\tau$,

(2) $\langle \tau(a) \rangle_\tau \subseteq \langle a \rangle_\tau$,

(3) $\langle a \odot \tau(a) \rangle_\tau = \langle a \rangle_\tau$,

(4) $\langle a \rangle_\tau \cap \langle b \rangle_\tau = \langle (a \odot \tau(a)) \vee (b \odot \tau(b)) \rangle_\tau$,

(5) $\langle a \rangle_\tau \vee \langle b \rangle_\tau = \langle a \wedge b \rangle_\tau = \langle a \odot b \rangle_\tau$.

For any $F_1, F_2 \in SF[L]$, we put $F_1 \hookrightarrow F_2 = \{x \in L : F_1 \cap \langle x \rangle_\tau \subseteq F_2\}$. For $F \in SF[L]$, we define $F^* = F \hookrightarrow \{1\} = \{x \in L : F \cap \langle x \rangle_\tau = 1\}$, see [10].

Theorem 1.21. [10] *In the frame* $(SF[L], \subseteq)$, *for any* $F, F_1, F_2 \in SF[L]$, *we have:*

(1) $F_1 \cap F \subseteq F_2 \Rightarrow F \subseteq F_1 \hookrightarrow F_2$, *that is,* $F_1 \hookrightarrow \_ : SF[L] \to SF[L]$ *is the right adjoint of* $F_1 \cap \_ : SF[L] \to SF[L]$.

(2) $F_1 \hookrightarrow F_2 = \{x \in L : f \vee (x \odot \tau(x))^n \in F_2,$ *for all* $f \in F_1$ *and* $n \geq 1\}$.

2 Prime state filters

In this section, we introduce the notion of the prime state filters in a state residuated lattice. We prove the prime state filter theorem and investigate some properties of them.

Proposition 2.1. *Let* (L, τ) *be a state residuated lattice and* P *be a proper state filter of* (L, τ). *Then, the following are equivalent:*

(i) *If* $P_1, P_2 \in SF[L]$ *and* $P = P_1 \cap P_2$, *then* $P = P_1$ *or* $P = P_2$;

(ii) *If* $P_1, P_2 \in SF[L]$ *and* $P_1 \cap P_2 \subseteq P$, *then* $P_1 \subseteq P$ *or* $P_2 \subseteq P$;

(iii) *If* $a, b \in L$ *so that* $(a \odot \tau(a)) \vee (b \odot \tau(b)) \in P$, *then* $a \in P$ *or* $b \in P$.

Proof. (i)⇒(ii) It is similar to the proof of Proposition 1.12.

(i)⇒(iii) Suppose that $((a \odot \tau(a)) \vee (b \odot \tau(b)) \in P$, $a, b \in L$. Let $P_1 = \langle P, a\rangle_\tau$ and $P_2 = \langle P, b\rangle_\tau$. Obviously, $P \subseteq P_1 \cap P_2$. Let $x \in P_1 \cap P_2$. Then, by Proposition 1.17 (2), there are $l, k \in P$ and $m, n \geq 1$ such that $x \geq k \odot (a \odot \tau(a))^n$ and $x \geq l \odot (b \odot \tau(b))^m$. Then, by the property of joins, we have

$$
\begin{aligned}
x &\geq (k \odot (a \odot \tau(a))^n) \vee (l \odot (b \odot \tau(b))^m) \\
&\geq ((k \odot (a \odot \tau(a))^n) \vee l) \odot ((k \odot (a \odot \tau(a))^n) \vee (b \odot \tau(b))^m)) \\
&\geq ((k \vee l) \odot ((a \odot \tau(a))^n \vee l)) \odot ((k \vee (b \odot \tau(b))^m) \\
&\odot ((a \odot \tau(a))^n \vee (b \odot \tau(b))^m)) \\
&\geq (k \vee l) \odot ((a \odot \tau(a))^n \vee l) \odot (k \vee (b \odot \tau(b))^m) \\
&\odot ((a \odot \tau(a)) \vee (b \odot \tau(b)))^{nm} \text{ (by Proposition 1.2 (11)).}
\end{aligned}
$$

But $(k \vee l), (a \odot \tau(a))^n \vee l, k \vee (b \odot \tau(b))^m, ((a \odot \tau(a)) \vee (b \odot \tau(b)))^{nm} \in P$, by the property of filters, $x \in P$. Thus $P = P_1 \cap P_2$. Therefore by (i), $P = P_1$ or $P = P_2$, that is, $a \in P$ or $b \in P$.

(iii)⇒(i) Let $P_1, P_2 \in SF[L]$ such that $P = P_1 \cap P_2$. Suppose that $P \neq P_1$ and $P \neq P_2$ and let $a \in P_1 \setminus P$ and $b \in P_2 \setminus P$. Then, $a \odot \tau(a) \in$

$P_1, b \odot \tau(b) \in P_2$. So, $(a \odot \tau(a)) \vee (b \odot \tau(b)) \in P_1 \cap P_2 = P$, that is, by (iii), $a \in P$ or $b \in P$, which is a contradiction. Thus $P = P_1$ or $P = P_2$. □

Definition 2.2. Let (L, τ) be a state residuated lattice. A proper state filter P of (L, τ) is called *prime* if it satisfies one of the equivalent conditions of Proposition 2.1.

We denote the set of all prime state filters of (L, τ) by $Spec_\tau(L)$.

Theorem 2.3. (Prime state filter theorem) *Let (L, τ) be a state residuated lattice, I be an ideal in the lattice L, and F be a state filter of (L, τ) such that $F \cap I = \emptyset$. Then, there is a prime state filter P such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof. Let us consider the set

$$\kappa(F) = \{F' : F' \in SF[L], F \subseteq F' \text{ and } F' \cap I = \emptyset\}.$$

Since $F \in \kappa(F)$, it follows that $\kappa(F)$ is nonvoid. One can easily prove that the set $\kappa(F)$ is inductively ordered by inclusion and, by Zorn's lemma, it has a maximal element P. We will prove that P is a prime state filter. Since $P \in \kappa(F)$, we deduce that P is a proper state filter and $P \cap I = \emptyset$. Let $a, b \in L$ such that $(a \odot \tau(a)) \vee (b \odot \tau(b)) \in P$. Suppose that $a \notin P$ and $b \notin P$ and consider the sets $\langle P, a\rangle_\tau$ and $\langle P, b\rangle_\tau$. Then, P is strictly contained in $\langle P, a\rangle_\tau$ and $\langle P, b\rangle_\tau$, and the maximality of P implies that $\langle P, a\rangle_\tau \notin \kappa(F)$ and $\langle P, b\rangle_\tau \notin \kappa(F)$. Thus $\langle P, a\rangle_\tau \cap I \neq \emptyset$ and $\langle P, b\rangle_\tau \cap I \neq \emptyset$. Let $x \in \langle P, a\rangle_\tau \cap I$ and $y \in \langle P, b\rangle_\tau \cap I$. According to Proposition 1.17 (2), there are $k, l \in P$ and $n, m \geq 1$. So that $x \geq k \odot (a \odot \tau(a))^n$ and $y \geq l \odot (b \odot \tau(b))^m$. Then, $x \vee y \geq (k \odot (a \odot \tau(a))^n) \vee (l \odot (b \odot \tau(b))^m) \geq (k \vee l) \odot ((a \odot \tau(a))^n \vee l) \odot (k \vee (b \odot \tau(b))^m) \odot ((a \odot \tau(a)) \vee (b \odot \tau(b)))^{nm}$ (similar to Proposition 2.1). But $k \vee l, (a \odot \tau(a))^n \vee l, k \vee (b \odot \tau(b))^m, ((a \odot \tau(a)) \vee (b \odot \tau(b)))^{nm} \in P$, so, by the property of filters, $x \vee y \in P$. On the other hand, since I is an ideal of the lattice L, we deduce that $x \vee y \in I$, and therefore $P \cap I \neq \emptyset$, which is a contradiction. Thus P is a prime state filter. □

Proposition 2.4. *Let (L, τ) be a state residuated lattice and F be a proper state filter of (L, τ). Then, there is a maximal state filter F_0 of (L, τ) such that $F \subseteq F_0$.*

Proof. One can easily prove that

$$L_F = \{F' : F' \text{ is a proper state filter containing } F\}$$

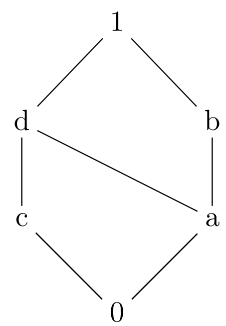
is nonvoid and inductively ordered by inclusion so, by Zorn's lemma, L_F has a maximal element F_0. We are going to prove that F_0 is a maximal state filter of (L, τ). Indeed, if F_1 is a proper state filter of (L, τ) such that $F_0 \subseteq F_1$, then $F_1 \in L_F$ and the maximality of F_0 implies that $F_1 = F_0$. □

Proposition 2.5. *Let (L, τ) be a state residuated lattice and $a \in L, a < 1$. Then, there is a prime state filter P of (L, τ) so that $a \notin P$.*

Proof. Since $\{1\}$ is a state filter and $\{1\} \cap (a] = \emptyset$, by Theorem 2.3, there exists a prime state filter P such that $P \cap (a] = \emptyset$. Thus, $a \notin P$. □

The following example shows that not every prime state filter of (L, τ) is a prime filter of L.

Example 2.6. Let $L_2 = \{0, a, b, c, d, 1\}$ be a set with Hasse diagram and cayley tables as follows:



$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then, $(L_2, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice. Now, we define a map τ_2 on L_2 by

$$\tau_2(0) = \tau_2(a) = \tau_2(b) = 0, \tau_2(c) = \tau_2(d) = \tau_2(1) = 1.$$

One can easily check that (L_2, τ_2) is a state residuated lattice. The filters of L_2 are $\{1\}, \{b, 1\}, \{c, d, 1\}$, and L_2, and the state filters of (L_2, τ_2) are $\{1\}, \{c, d, 1\}$, and L_2. The filter $\{1\}$ is not a prime filter of L_2, because $\{1\} = \{b, 1\} \cap \{c, d, 1\}$ but $\{1\} \neq \{b, 1\}$ and $\{1\} \neq \{c, d, 1\}$ (according to Proposition 1.12 (ii)). Still, as a state filter of (L_2, τ_2), $\{1\}$ is a prime state filter (according to Proposition 2.1 (ii)).

Proposition 2.7. *Let (L, τ) be a state residuated lattice and $a, b \in L$. Then,*
(i) $(\langle a \rangle_\tau)^* = \{x \in L : (a \odot \tau(a)) \vee (x \odot \tau(x)) = 1\}$;
(ii) $(\langle a \rangle_\tau)^* \cap (\langle b \rangle_\tau)^* = (\langle a \odot b \rangle_\tau)^*$.

Proof. (i)

$$\begin{aligned}
(\langle a \rangle_\tau)^* &= \{x \in L : \langle a \rangle_\tau \cap \langle x \rangle_\tau = \{1\}\} \\
&= \{x \in L : \langle (a \odot \tau(a)) \vee (x \odot \tau(x)) \rangle_\tau = \{1\}\} \\
&\qquad\qquad\qquad \text{(by Proposition 1.20 (4))} \\
&= \{x \in L : (a \odot \tau(a)) \vee (x \odot \tau(x)) = 1\}.
\end{aligned}$$

(ii) Let $x \in (\langle a \rangle_\tau)^* \cap (\langle b \rangle_\tau)^*$. Then, by (i), it follows that

$$(a \odot \tau(a)) \vee (x \odot \tau(x)) = (b \odot \tau(b)) \vee (x \odot \tau(x)) = 1.$$

Also, by Proposition 1.2 (11),

$$((a \odot \tau(a)) \odot (b \odot \tau(b)) \vee (x \odot \tau(x)) \geq ((a \odot \tau(a)) \vee (x \odot \tau(x))) \odot ((b \odot \tau(b) \vee (x \odot \tau(x)),$$

and so $((a \odot \tau(a)) \odot (b \odot \tau(b))) \vee (x \odot \tau(x)) = 1$. Then, by Proposition 1.2 (11),

$$((a \odot \tau(a)) \odot (b \odot \tau(b)))^2 \vee (x \odot \tau(x)) \geq (((a \odot \tau(a)) \odot (b \odot \tau(b))) \vee (x \odot \tau(x)))^2 = 1.$$

On the other hand, by Propositions 1.2 (5) and 1.14 (4),

$$\begin{aligned}
&((a \odot \tau(a)) \odot (b \odot \tau(b)))^2 \\
&= (a \odot \tau(a)) \odot (b \odot \tau(b)) \odot (a \odot \tau(a)) \odot (b \odot \tau(b)) \\
&\leq a \odot b \odot \tau(a) \odot \tau(b) \\
&\leq (a \odot b) \odot \tau(a \odot b).
\end{aligned}$$

So, $((a \odot b) \odot \tau(a \odot b)) \vee (x \odot \tau(x)) = 1$, that is, by (i), we have that $x \in (\langle a \odot b\rangle_\tau)^*$. Conversely, let $x \in (\langle a \odot b\rangle)^*$, and therefore $(x \odot \tau(x)) \vee ((a \odot b) \odot \tau(a \odot b)) = 1$. Since $a \odot \tau(a) \geq (a \odot b) \odot \tau(a \odot b)$ (by Propositions 1.2 (5) and Definition 1.13 (L4)), we deduce that

$$(a \odot \tau(a)) \vee (x \odot \tau(x)) \geq ((a \odot b) \odot \tau(a \odot b)) \vee (x \odot \tau(x)) = 1,$$

and so $(a \odot \tau(a)) \vee (x \odot \tau(x)) = 1$. Analogously, $(b \odot \tau(b)) \vee (x \odot \tau(x)) = 1$, and therefore $x \in (\langle a\rangle_\tau)^* \cap (\langle b\rangle_\tau)^*$. □

Theorem 2.8. *The following are equivalent:*

(i) $(SF[L], \cap, \vee, \rightarrow, \{1\}, L)$ *is a Boolean algebra.*

(ii) *Every state filter of* (L, τ) *is a principal state filter and for every* $a \in L$ *there exists* $n \geq 1$ *so that*

$$(a \odot \tau(a)) \vee (((a \odot \tau(a))^n)^* \odot \tau(((a \odot \tau(a))^n)^*) = 1.$$

Proof. (i)⇒(ii) If $F \in SF[L]$, then $F \vee F^* = L$. Thus $0 \in F \vee F^*$. But, according to Proposition 1.17 (3),

$$F \vee F^* = \langle F \cup F^*\rangle_\tau = \{x \in L : x \geq f \odot g, f \in F, g \in F^*\}.$$

So, there are $f \in F, g \in F^*$ such that $f \odot g = 0$. According to Theorem 1.21 (2),

$$F^* = \{x \in L : a \vee (x \odot \tau(x))^n = 1, \text{ for every } n \geq 1 \text{ and } a \in F\}.$$

Thus $a \vee (g \odot \tau(g))^n = 1$, for every $a \in F$ and $n \geq 1$, that is, $a \vee g = 1$, for every $a \in F$. Therefore $f \vee g = 1$. By Proposition 1.2 (16), it follows that $f \wedge g = f \odot g$. We will prove that $F = \langle f\rangle_\tau$. Since $f \in F$, we deduce that $\langle f\rangle_\tau \subseteq F$. Let $x \in F$. Then, $x \vee g = 1$, by Notation 1, since $f \in B(L)$, we have

$$\begin{aligned} f &= f \wedge 1 \\ &= f \wedge (x \vee g) \\ &= (f \wedge x) \vee (f \wedge g) \\ &= (f \wedge x) \vee (f \odot g) \\ &= (f \wedge x) \vee 0 \\ &= f \wedge x. \end{aligned}$$

So, by Proposition 1.2 (15), $f \leq x$ and, since $f \in \langle f\rangle_\tau$, it follows that $x \in \langle f\rangle_\tau$. Thus $F = \langle f\rangle_\tau$. Let now $a \in L$. Then, $\langle a\rangle_\tau \vee (\langle a\rangle_\tau)^* = L$, so, there are $b \in \langle a\rangle_\tau, c \in (\langle a\rangle_\tau)^*$ such that $b \odot c = 0$. Since $b \in \langle a\rangle_\tau$, by Proposition 1.17 (1), we deduce that there is $n \geq 1$ such that $b \geq (a \odot \tau(a))^n$. According to Proposition 1.2 (4), we have $b \odot c \geq (a \odot \tau(a))^n \odot c$. Therefore $(a \odot \tau(a))^n \odot c = 0$. Since $c \in (\langle a\rangle_\tau)^*$, by Proposition 2.7 (i), we get $(a \odot \tau(a)) \vee (c \odot \tau(c)) = 1$. Then,

$$\begin{aligned} c \vee (a \odot \tau(a))^n &\geq (c \odot \tau(c)) \vee (a \odot \tau(a))^n \\ &\geq ((c \odot \tau(c)) \vee (a \odot \tau(a)))^n \\ &= 1 \text{ (by Proposition 1.2 (5,11)}. \end{aligned}$$

So, $c \vee (a \odot \tau(a))^n = 1$. By Proposition 1.2 (16), it follows that $(a \odot \tau(a))^n \wedge c = (a \odot \tau(a))^n \odot c$. Since $c \odot (a \odot \tau(a))^n = 0$, by Proposition 1.2 (3), we have $c \leq ((a \odot \tau(a))^n)^*$. Then, by Proposition 1.14 (2), $\tau(c) \leq \tau(((a \odot \tau(a))^n)^*)$. It follows that

$$\begin{aligned} & (a \odot \tau(a)) \vee (((a \odot \tau(a))^n)^* \odot \tau(((a \odot \tau(a))^n)^*)) \\ \geq\ & ((a \odot \tau(a)) \vee ((a \odot \tau(a))^n)^*) \\ \odot\ & ((a \odot \tau(a)) \vee \tau(((a \odot \tau(a))^n)^*)) \\ \geq\ & ((a \odot \tau(a)) \vee c) \odot ((a \odot \tau(a)) \vee \tau(c) \\ \geq\ & ((a \odot \tau(a)) \vee (c \odot \tau(c)))^2 \\ =\ & 1. \end{aligned}$$

(ii)$\Rightarrow$(i) Since $(SF[L], \cap, \vee, \rightarrow, \{1\}, L)$ is a Heyting algebra. In order to prove that it is a Boolean algebra it is enough to prove that for every $F \in SF[L]$, we have $F^* = \{1\}$ if only if $F = L$ (according to [18, Proposition 1.8]). Let $F \in SF[L]$ with $F^* = \{1\}$. By the hypothesis, there is $a \in L$ so that $F = \langle a\rangle_\tau$ therefore $(\langle a\rangle_\tau)^* = \{1\}$. There is $n \geq 1$ such that

$$(a \odot \tau(a)) \vee (((a \odot \tau(a))^n)^* \odot \tau(((a \odot \tau(a))^n)^*) = 1$$

and, by Proposition 2.7 (i), it follows that $((a \odot \tau(a))^n)^* \in (\langle a\rangle_\tau)^*$. So, $((a \odot \tau(a))^n)^* = 1$, that is, $(a \odot \tau(a))^n = 0$. Since $(a \odot \tau(a))^n \in \langle a\rangle_\tau$, we deduce that $0 \in \langle a\rangle_\tau$. In conclusion, $F = \langle a\rangle_\tau = L$. $\square$

3 Obstinate state filters

In this section, we introduce the notion of obstinate state filters in a state residuated lattice and give some characterizations of obstinate state filters. Also, we introduce Boolean and primary state filters in a state residuated lattice and investigate some properties of them.

Definition 3.1. Let (L, τ) be a state residuated lattice. A state filter F is an *obstinate state filter* of (L, τ) if it satisfies $0 \notin F$ (that is, F is a proper state filter) and $x, y \notin F$ imply $\tau(x) \to \tau(y) \in F$ and $\tau(y) \to \tau(x) \in F$.

The following proposition is an equivalent condition for obstinate state filters.

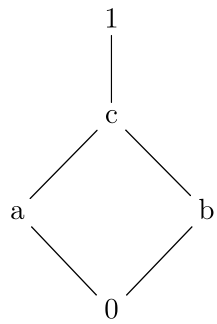
Proposition 3.2. *Let (L, τ) be a state residuated lattice. A proper state filter F of (L, τ) is an obstinate state filter if and only if it satisfies the following condition:*

$$\forall x \in L,\ x \notin F \Rightarrow \exists n \geq 1,\ ((\tau(x))^*)^n \in F.$$

Proof. Suppose that F is an obstinate proper state filter and $0, x \notin F$. Then, $1 = \tau(0) \to \tau(x) \in F$ and $(\tau(x))^* = \tau(x) \to \tau(0) \in F$. So, $((\tau(x))^*)^n \in F$, for $n = 1$, and obtain the result. Conversely, let $x, y \notin F$. We show that $\tau(x) \to \tau(y) \in F$ and $\tau(y) \to \tau(x) \in F$. By hypothesis, $((\tau(x))^*)^n \in F$ and $((\tau(y))^*)^m \in F$, for some $n, m \geq 1$. We know that, $((\tau(x))^*)^n \leq (\tau(x))^*$ and $((\tau(y))^*)^m \leq (\tau(y))^*$. By the property of filters, $(\tau(x))^* \in F$ and $(\tau(y))^* \in F$. By Proposition 1.2 (17), we have $(\tau(x))^* \leq \tau(x) \to \tau(y)$ and $(\tau(y))^* \leq \tau(y) \to \tau(x)$, for all $x, y \in L$. By the property of filters, we get $\tau(x) \to \tau(y) \in F$ and $\tau(y) \to \tau(x) \in F$. $\square$

Example 3.3. In Example 2.6, we can check that $F_2 = \{c, d, 1\}$ is a state filter. By Proposition 3.2, it is easy to check that F_2 is an obstinate state filter of (L_2, τ_2).

Example 3.4. Let $L_1 = \{0, a, b, c, 1\}$ be a bounded lattice as shown in the following diagram:



Now, let $\odot$ and $\rightarrow$ be defined as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then, $(L_1, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice. Now, we define a map τ_1 on L_1 as follows:

$$\tau_1(0) = \tau_1(a) = 0, \tau_1(b) = \tau_1(c) = \tau_1(1) = 1.$$

One can easily check that (L_1, τ_1) is a state residuated lattice and $F = \{c, 1\}$ is a state filter of (L_1, τ_1). For $b \notin F$ and $0 \notin F$, since $0 = \tau_1(b) \rightarrow \tau_1(0) \notin F$, F is not obstinate state filter of (L_1, τ_1).

Theorem 3.5. *Let F be an obstinate state filter of (L, τ). Then, F is a maximal state filter of (L, τ).*

Proof. Let us suppose that F is an obstinate state filter which is not maximal. So, there exists a proper state filter G strictly greater then F (with respect to set inclusion). Let $a \in G \setminus F$. By Proposition 3.2, we have $((\tau(a))^*)^n \in F$ for some $n \geq 1$. Then, by Proposition 1.2 (5), $((\tau(a))^*)^n \leq (\tau(a))^*$. By the property of filters, $(\tau(a))^* \in F$ and also $(\tau(a))^* \in G$. Since $\tau(a) \in G$, by the property of filters and Proposition 1.2 (3), we have $\tau(a) \odot (\tau(a))^* = 0 \in G$, which is a contradiction. $\square$

The next example shows that the converse of the above theorem is not true.

Example 3.6. Consider $L = \{0, a, b, 1\}$ where $0 < a < b < 1$. Define $\odot$ and $\rightarrow$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then, $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice. Now we define a map τ on L as follows:

$$\tau(0) = 0, \tau(a) = a, \tau(b) = \tau(1) = 1.$$

One can easily check that (L, τ) is a state residuated lattice and it is clear that $F = \{b, 1\}$ is a maximal state filter of (L, τ) but $0, a \notin F$ and $\tau(a) \rightarrow \tau(0) \notin F$. Thus, F is not an obstinate state filter of (L, τ).

Theorem 3.7. *Let F be a proper state filter of (L, τ). Then, F is an obstinate state filter if and only if $x \in F$ or $(\tau(x))^* \in F$, for all $x \in L$.*

Proof. The proof is straightforward, by Proposition 3.2. □

Definition 3.8. Let F be a state filter of a state residuated lattice (L, τ). F is called a *Boolean state filter* if $(x \odot \tau(x)) \vee (x^* \odot (\tau(x))^*) \in F$, for all $x \in L$.

Theorem 3.9. *Let F be a state filter of (L, τ) and F a prime state filter and a Boolean state filter. Then, F is an obstinate state filter.*

Proof. Suppose F is a Boolean state filter. Then, $\forall x \in L$, $(x \odot \tau(x)) \vee (x^* \odot (\tau(x))^*) \in F$. It follows, from F being a prime state filters, $x \in F$ or $x^* \in F$. So, by Proposition 1.14 (3), $x \in F$ or $(\tau(x))^* = \tau(x^*) \in F$. Then, by Theorem 3.7, F is an obstinate state filter. □

Example 3.10. In Example 3.4, we can check that $\{1\}, \{b, c, 1\}, \{c, 1\}$, and L are state filters of (L, τ). It is easy to check that $F = \{c, 1\}$ is a prime state filter but it is not an obstinate state filter, since $a, b \notin F$ and $\tau(b) \rightarrow \tau(a) = 1 \rightarrow 0 = 0 \notin F$.

Remark 3.11. A state residuated lattice $(L,\, \tau)$ is state simple if and only if, for every $a \in L, a \neq 1 \Rightarrow ord(\tau(a)) < \infty$.

Proof. Let (L, τ) be state simple. Then, the state filter $\{1\}$ is maximal and since $a \notin \{1\}$, by Lemma 1.18, there exists $n \in \mathbb{N}$ such that $(\tau(a)^n)^* \in \{1\}$, that is $(\tau(a)^n)^* = 1$. Hence $\tau(a)^n \leq (\tau(a)^n)^{**} = 0$. Thus $\tau(a)^n = 0$. Therefore $ord(\tau(a)) = n < \infty$.

Conversely, let for every $a \neq 1, ord(\tau(a)) < \infty$. Then, there exists $n \in \mathbb{N}$ such that $\tau(a)^n = 0$. Hence $(\tau(a)^n)^* \in \{1\}$. It follows from Lemma 1.18 that $\{1\}$ is a maximal state ideal of (L, τ). Thus (L, τ) is state simple. $\square$

Example 3.12. Example 3.6 is a state local residuated lattice, but does not have obstinate state filters.

Theorem 3.13. (Extension theorem of obstinate state filter) *Suppose that F and G are two proper state filters such that $F \subseteq G$. If F is an obstinate state filter, then G is also an obstinate state filter.*

Proof. Let F be an obstinate state filter and $F \subseteq G$. Then, by Theorem 3.5, F is a maximal state filter. Since G is a proper state filter, we get that $F = G$. Hence G is an obstinate state filter. $\square$

Proposition 3.14. *Let F be an obstinate state filter. Then, $D(F) = \{x \in L : x^{**} \in F\}$ is also an obstinate state filter.*

Proof. If F is an obstinate state filter, then by applying Theorem 3.13 and $F \subseteq D(F)$, we get that $D(F)$ is an obstinate state filter. $\square$

Remark 3.15. • Let F and G be state filters of (L, τ). By Proposition 1.17, we have $\langle F \cup G \rangle_\tau = \{x \in L : x \geq f \odot g,\ f \in F, g \in G\}$. If F or G is an obstinate state filter, then by Theorem 3.13 and $F, G \subseteq F \cup G \subseteq \langle F \cup G \rangle_\tau$, we get that $\langle F \cup G \rangle_\tau$ is an obstinate state filter.

• If $\bigcap_{\alpha \in I} F_\alpha$ is an obstinate state filter of (L, τ) then, by Theorem 3.13 and $\bigcap_{\alpha \in I} F_\alpha \subseteq F_\alpha$, we get that F_α, for all $\alpha \in I$, are obstinate state filters of (L, τ).

Proposition 3.16. *Let F, G and I be state filters of (L, τ). If I is an obstinate state filter and $F \cap G \subseteq I$, then $F \subseteq I$ or $G \subseteq I$.*

Proof. Let $F \cap G \subseteq I$, but F, G are not subsets of I. We take $a \in F \setminus I$ and $b \in G \setminus I$. Then, $a \in F, a \notin I$ and $b \in G, b \notin I$ (1).
Since F and G are state filters, we have $\tau(a) \in F, \tau(b) \in G$. By Proposition 1.2 (5), $\tau(a), \tau(b) \leq \tau(a) \vee \tau(b)$, and since F and G are filters, we get that $\tau(a) \vee \tau(b) \in F$ and $\tau(a) \vee \tau(b) \in G$. Therefore $\tau(a) \vee \tau(b) \in F \cap G \subseteq I$. Hence $\tau(a) \vee \tau(b) \in I$ (2).

Applying the hypothesis (I is an obstinate state filter) and (1), we obtain $(\tau(a))^* \in I$ and $(\tau(b))^* \in I$. Since I is a filter, we get that $(\tau(a))^* \odot (\tau(b))^* \in I$. By Proposition 1.2 (6), we have $(\tau(a))^* \odot (\tau(b))^* \leq (\tau(a))^* \wedge (\tau(b))^*$, hence $(\tau(a))^* \wedge (\tau(b))^* \in I$. By Proposition 1.2 (8), we know that $(\tau(a))^* \wedge (\tau(b))^* = (\tau(a) \vee \tau(b))^*$, hence $(\tau(a) \vee \tau(b))^* \in I$. Therefore $(\tau(a) \vee \tau(b)) \to 0 = (\tau(a) \vee \tau(b))^* \in I$, by (2), we have $\tau(a) \vee \tau(b) \in I$. Since I is a filter, by Definition 1.3, we get that $0 \in I$. This is a contradiction, and therefore $F \subseteq I$ or $G \subseteq I$. $\square$

Remark 3.17. Let F, G, and I be state filters of (L, τ). If I is an obstinate state filter and $I = F \cap G$, then $F = I$ or $G = I$.

Proof. Suppose that $I = F \cap G$, and hence $F \cap G \subseteq I$. Applying the above proposition, we obtain $F \subseteq I$ or $G \subseteq I$. It is enough to show that $I \subseteq F$ or $I \subseteq G$. By hypothesis, we have $I = F \cap G \subseteq F, G$. Hence the proof is complete. $\square$

By Theorem 3.13, it is easy to prove the following remark.

Remark 3.18. $\{1\}$ is an obstinate state filter of (L, τ) if and only if every state filter F of (L, τ) is an obstinate state filter.

Definition 3.19. Let (L, τ) be a state residuated lattice. A proper state filter F of (L, τ) is called a *primary state filter* if for every $a, b \in L$, $(a \odot b)^* \in F$ implies $(\tau(a)^n)^* \in F$ for some $n \geq 1$, or $(\tau(b)^m)^* \in F$ for some $m \geq 1$.

Proposition 3.20. *A state residuated lattice (L, τ) is local if and only if $ord(\tau(x))$ or $ord(\tau(x)^*)$ is finite, for every $x \in L$.*

Proof. Suppose that (L, τ) is local, that is, it has only one maximal state filter F. Let $x \in L$. Suppose that $ord(\tau(x)) = ord(\tau(x^*)) = \infty$. If $\langle x \rangle_\tau = L$ then, according to Proposition 1.17, there is $n \geq 1$ such that $(x \odot \tau(x))^n = 0$, so, $\tau((x \odot \tau(x))^n) = 0$ and, since $\tau((x \odot \tau(x))^n) \geq \tau(x)^{2n}$, it follows that

$\tau(x)^{2n} = 0$, which is a contradiction. Thus $\langle x\rangle_\tau$ is proper. Analogously, $\langle x^*\rangle_\tau$ is proper. Then, $\langle x\rangle_\tau, \langle x^*\rangle_\tau \subseteq F$, and therefore $x, x^* \in F$, so, $x \odot x^* = 0 \in F$, which is a contradiction.

Conversely, suppose that there are $F_1, F_2 \in Max_\tau(L), F_1 \neq F_2$ and let $a \in F_1 \setminus F_2$, for example. Then, by Lemma 1.18, there is $n \geq 1$ such that $(\tau(a)^n)^* \in F_2$, so, $\tau((\tau(a)^n)^*) \in F_2$. Let $x = \tau(a)^n$. Since $\tau(x^*) \in F_2$, we deduce that $(\tau(x^*))^n \in F_2$, for all $n \in N$. $ord(\tau(x^*)) = ord(\tau(x)^*) = \infty$ and, by the hypothesis, it follows that $ord(\tau(x)) < \infty$, so, there is $m \geq 1$ such that $\tau(x)^m = 0$, that is, $\tau(\tau(a)^n)^m = 0$. According to Proposition 1.14 (4), we have $\tau(\tau(a)^n)^m \geq (\tau(\tau(a))^n)^m = \tau(a)^{mn}$, and infer that $\tau(a)^{mn} = 0$. But $a \in F_1$, and therefore $0 = \tau(a)^{mn} \in F_1$, which is a contradiction. Thus (L, τ) has only a maximal state filter, and so, (L, τ) is local. $\square$

Theorem 3.21. *Let (L, τ) be a state residuated lattice. Then, the following are equivalent:*

(i) (L, τ) *is local;*

(ii) *Every proper state filter of* (L, τ) *is a primary state filter.*

Proof. (i)⇒(ii) Suppose that (L, τ) is local and let F_0 be its only maximal state filter. Let F be a proper state filter of (L, τ) and $a, b \in L$ such that $(a \odot b)^* \in F$. Since $F \subseteq F_0$, it follows that $(a \odot b)^* \in F_0$, so $a \odot b \notin F_0$, therefore $a \notin F_0$ or $b \notin F_0$. Because, if $a \in F_0$ and $b \in F_0$, then $a \odot b \in F_0$, which is a contradiction. Suppose that $a \notin F_0$. Then, $\langle a\rangle_\tau$ is not a subset of F_0 since (L, τ) is local, so $\langle a\rangle_\tau = L$, and so, there is $n \geq 1$ such that $(a \odot \tau(a))^n = 0$. Since $\tau((a \odot \tau(a))^n) \geq \tau(a)^{2n}$, we deduce that $\tau(a)^{2n} = 0$, that is, $(\tau(a)^{2n})^* = 1 \in F$. Analogously, if $b \notin F_0$, then there is $m \geq 1$ such that $(\tau(b)^{2m})^* = 1 \in F$. Thus F is a primary state filter.

(ii)⇒(i) Let $F = \{1\}$ be a proper state filter of (L, τ) and $x \in L$. Then, $(x \odot x^*)^* = 1 \in F$, and so, there is $n, m \geq 1$ such that $(\tau(x)^n)^* \in F$ or $(\tau(x^*)^m)^* \in F$. That is, $(\tau(x)^n)^* = 1$ or $(\tau(x^*)^m)^* = 1$, and therefore $\tau(x)^n = 0$ or $\tau(x^*)^m = 0$. Thus $ord(\tau(x)) < \infty$ or $ord(\tau(x^*)) < \infty$ and, according to Proposition 3.20, it follows that (L, τ) is local. $\square$

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