Categories and
General Algebraic Structures
with Applications
Volume 9, Number 1, July 2018, 15-27.

Total graph of a 0-distributive lattice

S. Ebrahimi Atani, S. Dolati, M. Khoramdel*, and M. Sedghi

Abstract. Let $\pounds$ be a 0-distributive lattice with the least element 0, the greatest element 1, and $\mathrm{Z}(\pounds)$ its set of zero-divisors. In this paper, we introduce the total graph of $\pounds$, denoted by $\mathrm{T}(G(\pounds))$. It is the graph with all elements of $\pounds$ as vertices, and for distinct $x, y \in \pounds$, the vertices x and y are adjacent if and only if $x \vee y \in \mathrm{Z}(\pounds)$. The basic properties of the graph $\mathrm{T}(G(\pounds))$ and its subgraphs are studied. We investigate the properties of the total graph of 0-distributive lattices as diameter, girth, clique number, radius, and the independence number.

1 Introduction

There has been a lot of activity over the past several years in associating a graph to an algebraic system such as a ring or semiring [1, 3, 5, 8, 9, 11, 14]. Recently, the study of the total graph property in the rings and modules has become quite popular. In many ways this program began with the paper in 2008, by D.F. Anderson and A. Badawi [2]. They introduced the total graph of a commutative ring R. Let $\mathrm{Z}(R)$ be the set of zero-divisors of R. The total graph of R, denoted by $\mathrm{T}(\Gamma(R))$, is the graph with all elements

* Corresponding author
Keywords: Lattice, minimal prime ideal, zero-divisor graph, total graph.
Mathematics Subject Classification [2010]: 06B35, 05C25.
Received: 27 January 2017, Accepted: 20 June 2017
ISSN Print: 2345-5853 Online: 2345-5861

of R as vertices, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in \mathrm{Z}(R)$. In [2], they studied the three (induced) subgraphs $\mathrm{Nil}(\Gamma(R))$, $\mathrm{Z}(\Gamma(R))$, and $\mathrm{Reg}(\Gamma(R))$ of $\mathrm{T}(\Gamma(R))$ with vertices $\mathrm{Nil}(R)$, $\mathrm{Z}(R)$, and $\mathrm{Reg}(R)$, respectively.

The study of zero-divisor graph of a poset was initiated by Halaš and Jukl [13]. There are many papers which study the zero-divisor graph of lattices [11, 14, 16, 17]. Let $\pounds$ be a lattice with the least element 0 and the greatest element 1. An element a of $\pounds$ is said to be *zero-divisor* if there exists $0 \neq b \in \pounds$ such that $a \wedge b = 0$. Let $\mathrm{Z}(\pounds)$ be its set of zero-divisors. The *zero-divisor graph*, $G(\pounds)$, is the graph with vertices $\mathrm{Z}(\pounds)^* = \mathrm{Z}(\pounds) \setminus \{0\}$, the set of non-zero zero-divisors of $\pounds$, and for distinct $x, y \in \mathrm{Z}(\pounds)^*$, the vertices x and y are adjacent if and only if $x \wedge y = 0$.

In this paper, we introduce the *total graph* of a lattice $\pounds$ with respect to zero-divisor elements of $\pounds$, denoted by $\mathrm{T}(G(\pounds))$. It is the graph with all elements of $\pounds$ as vertices, and for distinct $x, y \in \pounds$, the vertices x and y are adjacent if and only if $x \vee y \in \mathrm{Z}(\pounds)$. We are interested in investigating the total graph of a lattice to use other notions of total graph and associate which exist in the literature as laid forth in [2].

Now, we summarize the content of the paper. In the present paper, we study three (induced) subgraphs $\mathrm{Z}(G(\pounds))$, $\mathrm{Reg}(G(\pounds))$, and $\mathrm{Z}^*(G(\pounds))$ of $\mathrm{T}(G(\pounds))$, with vertices $\mathrm{Z}(\pounds)$, $\pounds \setminus \mathrm{Z}(\pounds)$, and $\mathrm{Z}(\pounds)^*$, respectively. In Section 2, we show that $\mathrm{T}(G(\pounds))$ is not connected, but its subgraph $\mathrm{Z}(G(\pounds))$ is always connected with $\mathrm{diam}(\mathrm{Z}(G(\pounds))) \in \{1, 2\}$ and $\mathrm{gr}(\mathrm{Z}(G(\pounds))) \in \{3, \infty\}$. It is shown that $\mathrm{diam}(\mathrm{Z}(G(\pounds))) = 1$ if and only if $\mathrm{Z}(\pounds)$ is an ideal of $\pounds$ and $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = 3$ if and only if $|\mathrm{Z}(\pounds)| \geq 4$. Moreover, we give a description of a lower bound for the clique number of $\mathrm{Z}(G(\pounds))$. In Section 3, it is proved that $\mathrm{Z}^*(G(\pounds))$ is connected if and only if $|\min(\pounds)| \neq 2$. Also, if $\mathrm{Z}^*(G(\pounds))$ is connected, then $\mathrm{diam}(\mathrm{Z}^*(G(\pounds))) \in \{1, 2\}$. Moreover, $\mathrm{gr}(\mathrm{Z}^*(G(\pounds))) \in \{3, \infty\}$. It is investigated when $\mathrm{diam}(\mathrm{Z}^*(G(\pounds))) = \mathrm{diam}(\mathrm{Z}(G(\pounds)))$ or $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = \mathrm{gr}(\mathrm{Z}^*(G(\pounds)))$. Further, we prove that if $\pounds$ is a lattice with $\min(\pounds)$ is finite, then there is no vertex of $\mathrm{Z}^*(G(\pounds))$ which is adjacent to every other vertex of $\mathrm{Z}^*(G(\pounds))$ and the radius of $\mathrm{Z}^*(G(\pounds))$ is 2, provided that $\mathrm{Z}^*(G(\pounds))$ is connected. It is shown that if $\pounds$ is a lattice with $\min(\pounds)$ is finite, then the independence number of $\mathrm{Z}^*(G(\pounds))$ is equal to $|\min(\pounds)|$.

In order to make this paper easier to follow, we recall in this section various notions which will be used in the sequel. For a graph G by $E(G)$ and $V(G)$ we denote the set of all edges and vertices, respectively. We recall that a graph is *connected* if every two distinct vertices is connected by a path. A graph G is said to be *totally disconnected* if it has no edge. The *distance* between two distinct vertices a and b, denoted by $\mathrm{d}(a,b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a,a) = 0$ and $d(a,b) = \infty$). The *diameter* of a graph G, denoted by $\mathrm{diam}(G)$, is equal to $\sup\{d(a,b) : a,b \in V(G)\}$. If a and b are two adjacent vertices of G, then we write $a - b$. The *eccentricity* of a vertex a is defined as $\mathrm{e}(a) = \max\{d(a,b) : \ b \in V(G)\}$ and the *radius* of G is given by $\mathrm{rad}(G) = \min\{\mathrm{e}(x) : \ x \in V(G)\}$. A graph is *complete* if it is connected with diameter less than or equal to one. We denote the complete graph on n vertices by K_n. The *girth* of a graph G, denoted by $\mathrm{gr}(G)$, is the length of the shortest cycle in G, provided G contains a cycle; otherwise, $\mathrm{gr}(G) = \infty$. A *complete bipartite* graph with part sizes m and n is denoted by $K_{m,n}$. A *clique* of a graph is its maximal complete subgraph and the number of vertices in the largest clique of a graph G, denoted by $w(G)$, is called the *clique number* of G. An *induced subgraph* of a graph G by the set $S \subseteq V(G)$ is a subgraph H of G where vertices are adjacent in H precisely when adjacent in G. In a graph $G = (V,E)$, a set $S \subseteq V$ is an *independent set* if the subgraph induced by S is totally disconnected. The *independence number* $\alpha(G)$ is the maximum size of an independent set in G [7].

A *lattice* is a poset $(\pounds, \leq)$ in which every pair of elements x, y has a g.l.b. (called the meet of x and y, and written $x \wedge y$) and a l.u.b. (called the join of x and y, and written $x \vee y$). A lattice $\pounds$ is *complete* when each of its subsets X has a l.u.b. and a g.l.b. in $\pounds$. Setting $X = \pounds$, we see that any complete lattice contains the least element 0 and the greatest element 1 (in this case, we say that $\pounds$ is a lattice with 0 and 1). A non-empty subset I of a lattice $\pounds$ is called an *ideal*, if for $a \in I$, $b \in \pounds$, $b \leq a$ implies $b \in I$ (then I is called a *down-directed set*) and for every $a, b \in I$ we have $a \vee b \in I$. A non-empty subset F of a lattice $\pounds$ is called a *filter*, if for $a \in F$, $b \in \pounds$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\pounds$ is a lattice with 1, then $\{1\}$ is a filter of $\pounds$). A lattice $\pounds$ with 0 is called *$\pounds$-domain* if $a \wedge b = 0$ $(a, b \in \pounds)$, then $a = 0$ or $b = 0$. We say that a subset $D \subseteq \pounds$ is *meet closed* if $1 \in D$ and $a \wedge b \in D$ for all $a, b \in D$. A proper ideal P of $\pounds$ is called *prime* if $x \wedge y \in P$, then $x \in P$ or $y \in P$. A prime ideal P of a lattice $\pounds$ is said to be a *minimal prime ideal*, if it is minimal among all prime ideals containing $\{0\}$ (note that in an $\pounds$-domain, the only minimal prime ideal is $\{0\}$). The set of all minimal prime ideals of $\pounds$ is denoted by $\min(\pounds)$. An element a of a lattice $\pounds$ is called an *atom*, if there is no $y \in \pounds$ such that $0 < y < a$ [6]. A 0-distributive lattice $\pounds$ is a lattice with 0 in which $a \wedge b = 0 = a \wedge c$ implies $a \wedge (b \vee c) = 0$ [4]. Note that every distributive lattice with 0 is a 0-distributive lattice. Let I be an ideal of a lattice $\pounds$. A prime ideal P containing an ideal I of a lattice $\pounds$ is said to be a *maximal N-prime* of I, if P is maximal with respect to the property of being contained in $\mathrm{Z}_I(\pounds) = \{x \in \pounds : \ x \wedge y \in I \text{ for some } y \in \pounds \setminus I\}$ [15].

Proposition 1.1. (1) *Let $\pounds$ be a lattice with the least element* 0 *and the greatest element* 1*. A non-empty subset I of $\pounds$ is an ideal of $\pounds$ if and only if $a \wedge c \in I$ and $a \vee b \in I$ for all $a, b \in I$ and $c \in \pounds$. Moreover, if I is an ideal and $x \vee y \in I$, then $x, y \in I$ for every $x, y \in \pounds$ and $0 \in I$.*

(2) *Let $\pounds$ be a lattice with* 0 *and $a \in \pounds$. Then the set $(0 : a) = \{b \in \pounds : b \wedge a = 0\}$ is a down-directed set. Moreover, $\pounds$ is a 0-distributive lattice if and only if $(0 : a)$ is an ideal for every $a \in \pounds$.*

Proposition 1.2. [4, Theorem 3.1] *Let $\pounds$ be a* 0*-distributive lattice and $\{P_\alpha\}_{\alpha \in \Lambda}$ the set of all prime ideals of $\pounds$. Then $\bigcap_{\alpha \in \Lambda} P_\alpha = \{0\}$. Moreover, if $P_1, \cdots, P_n$ are the only distinct minimal prime ideals of $\pounds$, then $\bigcap_{i=1}^{n} P_i = \{0\}$.*

2 Basic structures of $\mathrm{T}(G(\pounds))$ and the subgraph $\mathrm{Z}(G(\pounds))$

In this section, we study two (induced) subgraphs $\mathrm{Z}(G(\pounds))$ and $\mathrm{Reg}(G(\pounds))$ of $\mathrm{T}(G(\pounds))$, with vertices $\mathrm{Z}(\pounds)$ and $\pounds \setminus \mathrm{Z}(\pounds)$, respectively. It is clear that $V(\mathrm{T}(G(\pounds))) = \mathrm{Z}(\pounds) \cup (\pounds \setminus \mathrm{Z}(\pounds))$.

Proposition 2.1. $\mathrm{T}(G(\pounds)) = \emptyset$ *if and only if $\pounds$ is an $\pounds$-domain.*

Proof. Suppose that $\mathrm{T}(G(\pounds)) = \emptyset$ and let $x \wedge y = 0$ with $x \neq 0$. If $y \neq 0$, then $x \in \mathrm{Z}(\pounds)$; hence $0 - x$ is a path in $\mathrm{T}(G(\pounds))$, a contradiction. Thus $\pounds$ is an $\pounds$-domain. Conversely, assume on the contrary, $\mathrm{T}(G(\pounds)) \neq \emptyset$. Then there exist $a, b \in \pounds$ such that $a \vee b \in \mathrm{Z}(\pounds)$ with $a \neq b$, so $(a \vee b) \wedge c = 0$ for some $0 \neq c \in \pounds$. It is clear that $(a \wedge c) \vee (b \wedge c) \leq (a \vee b) \wedge c$. Therefore

$a \wedge c = 0 = b \wedge c$. Thus $a = b = 0$, which is a contradiction. Hence $\mathrm{T}(G(\pounds)) = \emptyset$. $\square$

Theorem 2.2. $\mathrm{Reg}(G(\pounds))$ *is totally disconnected. Thus, in particular,* $\mathrm{T}(G(\pounds))$ *is always disconnected.*

Proof. First, we show that no element of $\pounds \setminus \mathrm{Z}(\pounds)$ is adjacent to any element of $\mathrm{Z}(\pounds)$. Let $x \in \pounds \setminus \mathrm{Z}(\pounds)$ (so $x \neq 0$). If $x \vee y \in \mathrm{Z}(\pounds)$ for some $y \in \mathrm{Z}(\pounds)$, then there exists $0 \neq r \in \mathrm{Z}(\pounds)$ such that $(x \vee y) \wedge r = 0$, so $x \vee y \in (0 : r)$. As $(x \wedge r) \vee (y \wedge r) \leq (x \vee y) \wedge r = 0$, $x \in (0 : r) \subseteq \mathrm{Z}(\pounds)$, a contradiction. Similarly, if $x, y \in \pounds \setminus \mathrm{Z}(\pounds)$, then $x \vee y \notin \mathrm{Z}(\pounds)$. Finally, $\mathrm{Reg}(G(\pounds))$ being totally disconnected gives $\mathrm{T}(G(\pounds))$ is always disconnected. $\square$

From now on, unless otherwise stated, we assume that $\pounds$ is a 0-distributive lattice with the least element 0 and the greatest element 1.

Remark 2.3. It is well-known that $\mathrm{diam}(G(\pounds)) \leq 3$ and if $G(\pounds)$ contains a cycle, then $\mathrm{gr}(G(\pounds)) \leq 4$ ([11] and [17]). Assume that $G(\pounds)$ is complete and let a, b, and c be distinct elements of $\mathrm{Z}(\pounds)^*$. Then $a \wedge b = 0$, $a \wedge c = 0$, and $b \wedge c = 0$. Thus $b \vee c \in (0 : a)$, since $(0 : a)$ is an ideal of $\pounds$. Clearly, $b \vee c \neq 0$. Hence $b \vee c \in \mathrm{Z}(\pounds)^*$. If $b \vee c = c$, then $b \leq c$, so $0 = b \wedge c$ implies $b = 0$, which is a contradiction. So $b \vee c \neq c$. Since $G(\pounds)$ is complete, $(b \vee c) \wedge c = 0$; hence $c = 0$ which is impossible. Therefore $G(\pounds)$ is not complete. Also, $G(\pounds)$ contains no loop. Thus if $|\mathrm{Z}(\pounds)^*| \geq 3$, then $\mathrm{diam}(G(\pounds)) \in \{2, 3\}$.

The next theorem gives a more explicit description of the diameter of $\mathrm{Z}(G(\pounds))$.

Theorem 2.4. *The following hold*:

(1) $\mathrm{Z}(G(\pounds))$ *is connected with* $\mathrm{diam}(\mathrm{Z}(G(\pounds))) \leq 2$.

(2) *If* $\mathrm{T}(G(\pounds)) \neq \emptyset$, *then* $\mathrm{Z}(G(\pounds))$ *is a complete graph if and only if* $\mathrm{Z}^*(G(\pounds))$ *is a complete graph if and only if* $\mathrm{Z}(\pounds)$ *is an ideal of* $\pounds$ *if and only if* $\mathrm{Z}(\pounds)$ *is a join sub-semilattice of* $\pounds$.

(3) *If* $\mathrm{T}(G(\pounds)) \neq \emptyset$, *then* $\mathrm{diam}(\mathrm{Z}(G(\pounds))) = 2$ *if and only if* $\mathrm{Z}(\pounds)$ *is not an ideal of* $\pounds$.

Proof. (1) Since $0 \wedge a = 0$ for every $a \in \pounds$, $0 \in \mathrm{Z}(\pounds)$. Thus $x - 0 - y$ is a path in $\mathrm{Z}(G(\pounds))$ for distinct $x, y \in \mathrm{Z}(\pounds)$. Thus $\mathrm{Z}(G(\pounds))$ is connected with $\mathrm{diam}(\mathrm{Z}(G(\pounds))) \in \{1, 2\}$.

(2) Let $\mathrm{Z}(G(\pounds))$ be a complete graph and $x, y \in \mathrm{Z}(\pounds)$, $z \in \pounds$. Then $x \vee y \in \mathrm{Z}(\pounds)$ and $x \wedge c = 0$ for some $0 \neq c \in \pounds$. Now $(x \wedge z) \wedge c = 0 \wedge z = 0$ gives $x \wedge z \in \mathrm{Z}(\pounds)$. Thus $\mathrm{Z}(\pounds)$ is an ideal of $\pounds$. The other implications are clear.

(3) It is clear from (2). $\square$

Our next theorem characterizes the girth of subgraph $\mathrm{Z}(G(\pounds))$ of $\mathrm{T}(G(\pounds))$.

Theorem 2.5. *The following hold*:

(1) $\mathrm{gr}(\mathrm{Z}(G(\pounds))) \in \{3, \infty\}$.

(2) *If* $\mathrm{T}(G(\pounds)) \neq \emptyset$, *then* $|\mathrm{Z}(\pounds)| = 3$ *if and only if* $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = \infty$.

(3) *If* $\mathrm{T}(G(\pounds)) \neq \emptyset$, *then* $|\mathrm{Z}(\pounds)| \geq 4$ *if and only if* $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = 3$.

Proof. (1) If $x \vee y \in \mathrm{Z}(\pounds)$ for some $x, y \in \mathrm{Z}(\pounds)^*$ with $x \neq y$, then $0-x-y-0$ is a cycle in $\mathrm{Z}(G(\pounds))$; hence $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = 3$. Let $x \vee y \notin \mathrm{Z}(\pounds)$ for each $x, y \in \mathrm{Z}(\pounds)^*$. So $\mathrm{Z}(G(\pounds))$ does not contain any cycle; hence $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = \infty$.

(2) Let $\mathrm{Z}(\pounds) = \{0, a, b\}$. If $\mathrm{Z}(\pounds)$ is an ideal of $\pounds$, then either $a \vee b = b$ or $a \vee b = a$. We can assume that $a \vee b = b$. Then $a \wedge b = 0$ gives $a = a \wedge (a \vee b) = a \wedge b = 0$, which is impossible. Thus $\mathrm{Z}(\pounds)$ is not an ideal of $\pounds$, and so $a \vee b \notin \mathrm{Z}(\pounds)$; hence we have the path $a - 0 - b$, which gives $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = \infty$. Conversely, let $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = \infty$. By Proposition 2.1, it is not hard to see that $|\mathrm{Z}(\pounds)| \neq 1, 2$. We show that $|\mathrm{Z}(\pounds)| = 3$. Suppose on the contrary that $|\mathrm{Z}(\pounds)| \geq 4$. Since $\mathrm{diam}(G(\pounds)) \in \{2, 3\}$, by Remark 2.3, there exist $a, b \in \mathrm{Z}(\pounds)^*$ such that $d(a, b) = 2$. Thus there is $c \in \mathrm{Z}(\pounds)^*$ such that $a - c - b$ is a path in $G(\pounds)$; hence $a \wedge c = 0$ and $b \wedge c = 0$. So $a, b \in (0 : c)$ which gives $a \vee b \in (0 : c) \subseteq \mathrm{Z}(\pounds)$, because $(0 : c)$ is an ideal of $\pounds$. So $0 - a - b - 0$ is a cycle in $\mathrm{Z}(G(\pounds))$ and $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = 3$, a contradiction.

(3) Let $|\mathrm{Z}(\pounds)| \geq 4$. Since $G(\pounds)$ is not complete and $\mathrm{diam}(G(\pounds)) \in \{2, 3\}$, by Remark 2.3, there exist $a, b \in \mathrm{Z}(\pounds)^*$ such that $d(a, b) = 2$ in $G(\pounds)$, so there exists $c \in \mathrm{Z}(\pounds)^*$ such that $a-c-b$ is a path in $G(\pounds)$; hence $a, b \in (0 : c)$, which gives $a \vee b \in (0 : c) \subseteq \mathrm{Z}(\pounds)$. This implies $0 - a - b - 0$ is a cycle in $\mathrm{Z}(G(\pounds))$ and $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = 3$. Conversely, assume that $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = 3$; we show that $|\mathrm{Z}(\pounds)| \geq 4$. Clearly, $|\mathrm{Z}(\pounds)| \neq 1, 2$. Suppose, on the contrary, that $\mathrm{Z}(\pounds) = \{0, a, b\}$. Since $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = 3$, a and b are adjacent in $\mathrm{Z}(G(\pounds))$, which gives $a \vee b \in \mathrm{Z}(\pounds)$. Hence $\mathrm{Z}(\pounds)$ is an ideal of $\pounds$, which is a contradiction. Therefore $|\mathrm{Z}(\pounds)| \geq 4$. $\square$

The next theorem gives a description of a lower bound for the clique number of $\mathrm{Z}(G(\pounds))$ in terms of the number of atoms in $\pounds$.

Theorem 2.6. *Let $A(\pounds)$ be the set of all atoms in $\pounds$. Then we have $\omega(\mathrm{Z}(G(\pounds))) \geq |A(\pounds)|$.*

Proof. If $|A(\pounds)| \in \{1, 2\}$, then there is nothing to prove, because $\mathrm{Z}(G(\pounds))$ is connected. Let $|A(\pounds)| > 2$ and $x, y \in A(\pounds)$. As $|A(\pounds)| \geq 3$, there exists $z \in A(\pounds)$ such that $x \neq z$ and $y \neq z$. Clearly, $x \wedge z = 0$ and $y \wedge z = 0$. Thus $(x \vee y) \wedge z = 0$; and $x \vee y \in \mathrm{Z}(\pounds)$. Therefore $A(\pounds)$ is a clique in $\mathrm{Z}(G(\pounds))$, and so $\omega(\mathrm{Z}(G(\pounds))) \geq |A(\pounds)|$. □

Proposition 2.7. $\omega(\mathrm{Z}(G(\pounds))) \geq \ |\{|\mathcal{C}| : \ \mathcal{C}$ is a chain in $Z(\pounds)\}|$.

Proof. It is clear. □

3 Properties of the subgraph $\mathrm{Z}^*(G(\pounds))$

We continue to use the notations already established, so $\pounds$ is a 0-distributive lattice with 0 and 1. In this section, we refine our results on $\mathrm{diam}(\mathrm{Z}^*(G(\pounds)))$, $\mathrm{gr}(\mathrm{Z}^*(G(\pounds)))$, and the relation between $\mathrm{Z}^*(G(\pounds))$ and $\mathrm{Z}(G(\pounds))$.

Remark 3.1. It is not hard to see that (using mathematical induction on n): If I is an ideal and $P_1, P_2, \cdots, P_n$ are prime ideals of $\pounds$ with $I \subseteq \bigcup_{i=1}^{n} P_i$, then $I \subseteq P_r$ for some $1 \leq r \leq n$.

Theorem 3.2. *The following hold*:

(1) *If* $\min(\pounds) = \{P_\alpha\}_{\alpha\in\Lambda}$, *then* $\mathrm{Z}(\pounds) = \cup_{\alpha\in\Lambda} P_\alpha$.

(2) *If* $\mathrm{T}(G(\pounds)) \neq \emptyset$, *then* $\mathrm{Z}^*(G(\pounds))$ *is connected if and only if* $|\min(\pounds)| \neq 2$. *Moreover, if* $\mathrm{Z}^*(G(\pounds))$ *is connected, then we have* $\mathrm{diam}(\mathrm{Z}^*(G(\pounds))) \in \{1, 2\}$.

(3) *If* $\mathrm{T}(G(\pounds)) \neq \emptyset$ *and* $\mathrm{Z}^*(G(\pounds))$ *is a complete graph, then both* $\mathrm{Z}(\pounds)$ *and* $\min(\pounds)$ *are infinite.*

Proof. (1) It follows on the lines of Remark 1.2 of [15].

(2) Assume that $\mathrm{Z}^*(G(\pounds))$ is connected and let $\min(\pounds) = \{P_1, P_2\}$. Then $\mathrm{Z}(\pounds) = P_1 \cup P_2$, by (1). If $0 \neq x \in P_1$ and $0 \neq y \in P_2$, then $x \vee y \notin \mathrm{Z}(\pounds)$ (for if $x \vee y \in \mathrm{Z}(\pounds)$, then $x \vee y \in P_1$ or $x \vee y \in P_2$ and so $x \in P_2 \cap P_1 = \{0\}$

or $y \in P_1 \cap P_2 = \{0\}$, which is a contradiction, by Proposition 1.2). Then $x \vee y \notin \mathrm{Z}(\pounds)$ gives none of elements of P_1 and P_2 are adjacent in $\mathrm{Z}(G(\pounds))$; so $\mathrm{Z}^*(G(\pounds))$ is not connected, a contradiction. Conversely, suppose that $|\min(\pounds)| \neq 2$. If $|\min(\pounds)| = 1$, then $\mathrm{Z}(\pounds) = P = \{0\}$ for some minimal prime ideal P of $\pounds$, by (1) and Proposition 1.2. Let $a \wedge b = 0$ for some $a, b \in \pounds$ with $a \neq 0$; so $b \in \mathrm{Z}(\pounds) = \{0\}$. Hence $\pounds$ is an $\pounds$-domain, which is impossible by Proposition 2.1. Therefore $|\min(\pounds)| \geq 3$. We claim that $P_i \cap P_j \neq \{0\}$ for each $P_i, P_j \in \min(\pounds)$. Suppose, on the contrary, that $P_i \cap P_j = \{0\}$ for some $P_i, P_j \in \min(\pounds)$. We show that $\mathrm{Z}(\pounds) = P_i \cup P_j$. If $x \in \mathrm{Z}(\pounds)^* \setminus P_i \cup P_j$, then there exists $y \in \mathrm{Z}(\pounds)^*$ such that $x \wedge y = 0 \in P_i \cap P_j$. Since $x \notin P_i, P_j$, we have $y \in P_i \cap P_j = \{0\}$, a contradiction. Thus $\mathrm{Z}(\pounds) \subseteq P_i \cup P_j$; hence $\mathrm{Z}(\pounds) = P_i \cup P_j$, which implies $\min(\pounds) = \{P_i, P_j\}$ (because, if there exists $P_k \in \min(\pounds) \setminus \{P_i, P_j\}$, then $P_k \subseteq \mathrm{Z}(\pounds) = P_i \cup P_j$, by (1), which implies $P_k \subseteq P_i$ or $P_k \subseteq P_j$, by Remark 3.1, a contradiction). Thus $P_i \cap P_j \neq \{0\}$ for each minimal prime ideals P_i, P_j of $\pounds$. Now, let $x, y \in \mathrm{Z}(\pounds)^*$. If $x \vee y \in \mathrm{Z}(\pounds)^*$, then $d(x, y) = 1$. Let $x \vee y \notin \mathrm{Z}(\pounds)^*$. So $x \in P_i$ and $y \in P_j$, where P_i, P_j are distinct minimal prime ideals of $\pounds$. Choose $0 \neq z \in P_i \cap P_j$. Then $x - z - y$ is a path in $\mathrm{Z}^*(G(\pounds))$ and $d(x, y) = 2$.

(3) Since $\mathrm{Z}^*(G(\pounds))$ is a complete graph, $\mathrm{Z}(\pounds)$ is an ideal of $\pounds$. If $\mathrm{Z}(\pounds) = \{x_1, x_2, \cdots, x_n\}$, then $\bigvee_{i=1}^{n} x_i \in \mathrm{Z}(\pounds)$; hence $\bigvee_{i=1}^{n} x_i = x_j \neq 0$ for some $1 \leq j \leq n$. Since $x_j \in \mathrm{Z}(\pounds)$, there exists $0 \neq x_k \in \mathrm{Z}(\pounds)$ such that $x_j \wedge x_k = 0$. So $x_k = (x_1 \vee x_2 \vee \cdots \vee x_n) \wedge x_k = x_j \wedge x_k = 0$, a contradiction. Hence $\mathrm{Z}(\pounds)$ is infinite. We show that $\min(\pounds)$ is infinite. Suppose $\min(\pounds)$ is finite. So $\mathrm{Z}(\pounds) = \bigcup_{i=1}^{n} P_i$, where P_i's are minimal prime ideals of $\pounds$, by (1). Since $\mathrm{Z}(\pounds)$ is an ideal of $\pounds$, $\mathrm{Z}(\pounds) = P_i$ for some $1 \leq i \leq n$, by Remark 3.1. Thus $\mathrm{Z}(\pounds) = P_i = \{0\}$, by Proposition 1.2, which gives $\pounds$ is an $\pounds$-domain, which is a contradiction, by Proposition 2.1. □

Proposition 3.3. *Let* $\mathrm{T}(G(\pounds)) \neq \emptyset$. *If* $\mathrm{diam}(G(\pounds)) = 2$ *and* $\min(\pounds)$ *is a finite set, then* $|\min(\pounds)| = 2$.

Proof. Since $\mathrm{diam}(G(\pounds)) = 2$, $|\mathrm{Z}(\pounds)| \geq 3$; hence $|\min(\pounds)| \neq 1$. Let $\min(\pounds) = \{P_1, P_2, \cdots, P_n\}$. By Remark 3.1, $P_1 \not\subseteq \bigcup_{i=2}^{n} P_i$, so there exists an element in $\mathrm{Z}(\pounds)$ which is contained in a unique minimal prime ideal P_1 of $\pounds$. Let $a \in P_1 \setminus \bigcup_{i=2}^{n} P_i$. Suppose, on the contrary, that there are at least two other minimal prime ideals P_2 and P_3. If $P_2 \setminus P_1 \cup P_3 = \emptyset$, then

$P_2 \subseteq P_1 \cup P_3$, and so $P_2 \subseteq P_1$ or $P_2 \subseteq P_3$, by Remark 3.1, a contradiction. Thus $P_2 \setminus P_1 \cup P_3 \neq \emptyset$. Let $b \in P_2 \setminus P_1 \cup P_3$. We show that $a \vee b \in \pounds \setminus \mathrm{Z}(\pounds)$. Since $a \notin \bigcup_{i=2}^{n} P_i$, $a \in \bigcap_{i=2}^{n}(\pounds \setminus P_i)$. It is not hard to see that $\pounds \setminus P_i$ is a filter for each i. It follows that $a \vee b \in \bigcap_{i=2}^{n}(\pounds \setminus P_i)$. Since $b \notin P_1$, $b \in \pounds \setminus P_1$, which gives $a \vee b \in \pounds \setminus P_1$, since it is a filter. Thus $a \vee b \in \bigcap_{i=1}^{n}(\pounds \setminus P_i) = \pounds \setminus \mathrm{Z}(\pounds)$. If $d(a,b) = 1$ in $G(\pounds)$, then $a \wedge b = 0 \in P_3$, which gives $a \in P_3$ or $b \in P_3$, since P_3 is a prime ideal of $\pounds$, a contradiction. If $d(a,b) = 2$ in $G(\pounds)$, then $a, b \in (0 : c)$ for some $c \in \mathrm{Z}(\pounds)^*$, which gives $a \vee b \in (0 : c) \subseteq \mathrm{Z}(\pounds)$, a contradiction with $a \vee b \in \pounds \setminus \mathrm{Z}(\pounds)$. Hence $|\min(\pounds)| = 2$. $\square$

Theorem 3.4. *If* $\mathrm{T}(G(\pounds)) \neq \emptyset$ *and* $\min(\pounds)$ *is a finite set, then the following hold*:

(1) $\mathrm{diam}(G(\pounds)) = 2$ *if and only if* $\mathrm{Z}^*(G(\pounds))$ *is not connected and* $|\mathrm{Z}(\pounds)^*| \geq 3$.

(2) $\mathrm{diam}(G(\pounds)) = 3$ *if and only if* $\mathrm{Z}^*(G(\pounds))$ *is connected.*

Proof. (1) Let $\mathrm{diam}(G(\pounds)) = 2$ (so $|\mathrm{Z}(\pounds)^*| \geq 3$). Thus $|\min(\pounds)| = 2$, by Proposition 3.3; hence $\mathrm{Z}^*(G(\pounds))$ is not connected, by Theorem 3.2(2). Conversely, assume that $\mathrm{Z}^*(G(\pounds))$ is not connected and $|\mathrm{Z}(\pounds)^*| \geq 3$. So $|\min(\pounds)| = 2$, by Theorem 3.2(2), say $\min(\pounds) = \{P_1, P_2\}$. Then $P_1 \cap P_2 = \{0\}$, by Proposition 1.2. Moreover, $\mathrm{Z}(\pounds) = P_1 \cup P_2$, by Theorem 3.2(1). Set $V_1 = P_1 \setminus \{0\}$ and $V_2 = P_2 \setminus \{0\}$. Let $x, y \in V_1$. If $x \wedge y = 0 \in P_2$, then $x \in P_2$ or $y \in P_2$, which is a contradiction. Thus none of the elements of V_1 are adjacent together. Similarly, none of the elements of V_2 are adjacent. This means that $G(\pounds)$ is a bipartite graph with parts V_1 and V_2 (note that at least one of the parts has more than one vertex). Thus $\mathrm{diam}(G(\pounds)) = 2$.

(2) If $\mathrm{diam}(G(\pounds)) = 3$, then $G(\pounds)$ is not complete bipartite. If $|\min(\pounds)| = 2$, then by an argument like that in (1), $G(\pounds)$ is complete bipartite, which is impossible. So $\min(\pounds) \neq 2$, and hence $\mathrm{Z}^*(G(\pounds))$ is connected, by Theorem 3.2(2). Conversely, assume that $\mathrm{diam}(G(\pounds)) \neq 3$. By (1), $\mathrm{diam}(G(\pounds)) \neq 2$. Hence $\mathrm{diam}(G(\pounds)) = 1$ and $|\mathrm{Z}(\pounds)^*| = 2$, by Remark 2.3. Hence $\mathrm{Z}(\pounds)$ is not an ideal of $\pounds$ (see Theorem 2.5(2)). Now $\mathrm{Z}^*(G(\pounds))$ being connected gives $\mathrm{Z}(\pounds)$ is an ideal of $\pounds$, a contradiction. Thus $\mathrm{diam}(G(\pounds)) = 3$. $\square$

Theorem 3.5. *The following hold*:

(1) $\mathrm{gr}(\mathrm{Z}^*(G(\pounds))) \in \{3, \infty\}$.

(2) *If* $\mathrm{T}(G(\pounds)) \neq \emptyset$ *and* $|\min(\pounds)| \neq 2$*, then*

$$\mathrm{diam}(\mathrm{Z}^*(G(\pounds))) = \mathrm{diam}(\mathrm{Z}(G(\pounds)));$$

(3) *If* $\mathrm{T}(G(\pounds)) \neq \emptyset$ *and* $|\mathrm{Z}(\pounds)| \neq 4, 5$*, then we have* $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = \mathrm{gr}(\mathrm{Z}^*(G(\pounds)))$.

Proof. (1) By Theorem 3.2(1), $\mathrm{Z}(\pounds) = \bigcup P_i$, where $P_i^{'}$ s are minimal prime ideals of $\pounds$. If $|\min(\pounds)| = 1$, then there is nothing to prove. If $|\min(\pounds)| = 2$, then $\mathrm{Z}(\pounds) = P_1 \cup P_2$. If $|P_1| \geq 4$ or $|P_2| \geq 4$, then $P_1 \setminus \{0\}$ and $P_2 \setminus \{0\}$ are complete subgraphs of $\mathrm{Z}^*(G(\pounds))$; so $\mathrm{gr}(\mathrm{Z}^*(G(\pounds))) = 3$. If $|P_1|, |P_2| \leq 3$, then there is no cycle in P_1 and P_2. Also there is no cycle between the elements of P_1 and P_2, since none of the elements of P_1 and P_2 are adjacent, by the proof of Theorem 3.2(2). Hence there is no cycle in $\mathrm{Z}^*(G(\pounds))$, and so $\mathrm{gr}(\mathrm{Z}^*(G(\pounds))) = \infty$. Thus, suppose that $|\min(\pounds)| \geq 3$. We show that for each $P_i \in \min(\pounds)$, $|P_i| \geq 3$. If there exists a minimal prime ideal P_i of $\pounds$ such that $P_i = \{0, x\}$, then, by the proof of Theorem 3.2(2), $P_i \cap P_j \neq \{0\}$ for each $P_j \in \min(\pounds)$. Hence $P_i \cap P_j = \{0, x\}$, which implies $P_i \subseteq P_j$, a contradiction. So $|P_i| \geq 3$ for any minimal prime ideal P_i of $\pounds$. If $|P_i| \geq 4$ for some $P_i \in \min(\pounds)$, then $P_i \setminus \{0\}$ is a complete subgraph of $\mathrm{Z}^*(G(\pounds))$, and so $\mathrm{gr}(\mathrm{Z}^*(G(\pounds))) = 3$. Now suppose that $|P_i| = 3$ for each minimal prime ideal P_i of $\pounds$, and set $P_i = \{0, x_1, x_2\}$. Since $x_1 \neq 0$, there exists a minimal prime ideal P_j of $\pounds$ such that $x_1 \notin P_j$, by Proposition 1.2. Hence $P_i \cap P_j = \{0, x_2\}$, since $P_i \cap P_j \neq \{0\}$. As $x_2 \neq 0$, there exists $P_j \neq P_k \in \min(\pounds)$ such that $x_2 \notin P_k$; so $P_i \cap P_k = \{0, x_1\}$. On the other hand, $P_k \cap P_j \neq \{0\}$, $x_2 \in P_j \setminus P_k$, and $x_1 \in P_k \setminus P_j$, and so there exists $0 \neq x \in \mathrm{Z}(\pounds)$ such that $x \in P_j \cap P_k$. Thus $P_j = \{0, x_1, x\}$ and $P_k = \{0, x_2, x\}$. So $x_1 - x - x_2 - x_1$ is a cycle in $\mathrm{Z}^*(G(\pounds))$ and $\mathrm{gr}(\mathrm{Z}^*(\mathrm{G}(\pounds))) = 3$.

(2) If $\mathrm{diam}(\mathrm{Z}(\mathrm{G}(\pounds))) = 1$, then $\mathrm{Z}(\pounds)$ being an ideal of $\pounds$ gives $a \vee b \in Z(\pounds)^*$ for each $a, b \in \mathrm{Z}(\pounds)^*$, which implies $\mathrm{diam}(\mathrm{Z}^*(G(\pounds))) = 1$. If $\mathrm{diam}(\mathrm{Z}(G(\pounds)) = 2$, then there exist $c, d \in \mathrm{Z}(\pounds)^*$ such that $c \vee d \notin \mathrm{Z}(\pounds)^*$. By Theorem 3.2(2), there exists $e \in \mathrm{Z}(\pounds)^*$ such that $c - e - d$ is a path in $\mathrm{Z}^*(G(\pounds))$; hence $\mathrm{diam}(\mathrm{Z}^*(G(\pounds))) = 2$. Thus $\mathrm{diam}(\mathrm{Z}(G(\pounds))) = \mathrm{diam}(\mathrm{Z}^*(G(\pounds)))$.

(3) By Proposition 2.1, $\pounds$ is not an $\pounds$-domain, and so $|\mathrm{Z}(\pounds)| \neq 1, 2$. If $|\mathrm{Z}(\pounds)| = 3$, then $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = \infty$, by Theorem 2.5(2). Also $|\mathrm{Z}(\pounds)^*| = 2$ gives $\mathrm{Z}^*(G(\pounds))$ contains no cycle; so $\mathrm{gr}(\mathrm{Z}^*(G(\pounds))) = \infty$. Thus $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = \mathrm{gr}(\mathrm{Z}^*(G(\pounds)))$. If $|\mathrm{Z}(\pounds)| \geq 6$, then $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = 3$, by Theorem 2.5(3).

Now we show that $\mathrm{gr}(\mathrm{Z}^*(G(\pounds))) = 3$. If $|\min(\pounds)| \geq 3$, then $\mathrm{gr}(\mathrm{Z}^*(G(\pounds))) = 3$, by the proof of (1). If $|\min(\pounds)| = 2$ and P_1, P_2 are two minimal prime ideals of $\pounds$, then at least one of the P_i' s has more than 3 vertices, and thus each $P_i \setminus \{0\}$ is a complete subgraph of $\mathrm{Z}^*(G(\pounds))$; hence $\mathrm{gr}(\mathrm{Z}^*(G(\pounds))) = 3$. Therefore $\mathrm{gr}(\mathrm{Z}(G(\pounds))) = \mathrm{gr}(\mathrm{Z}^*(G(\pounds)))$. □

Proposition 3.6. *Let $\pounds$ be a lattice and* $\min(\pounds)$ *be finite. Then the following hold*:

(1) *There is no vertex of* $\mathrm{Z}^*(G(\pounds))$ *which is adjacent to every other vertex of* $\mathrm{Z}^*(G(\pounds))$.

(2) *If* $\mathrm{Z}^*(G(\pounds))$ *is connected, then* $\mathrm{rad}(\mathrm{Z}^*(G(\pounds))) = 2$.

Proof. (1) Let x be a vertex of $\mathrm{Z}^*(G(\pounds))$ which is adjacent to every other vertex of $\mathrm{Z}^*(G(\pounds))$. As $x \neq 0$, $x \notin P_i$ for some $P_i \in \min(\pounds)$. Let $y \in P_i \setminus \bigcup_{j\neq i, P_j \in \min(\pounds)} P_j$ (by Remark 3.1, $P_i \setminus \bigcup_{j\neq i, P_j \in \min(\pounds)} P_j \neq \emptyset$). As $x \vee y \in \mathrm{Z}(\pounds)$, we have either $x \vee y \in P_i$ or $x \vee y \in \bigcup_{j\neq i, P_j \in \min(\pounds)} P_j$. In two cases, we have a contradiction. Hence, there is no vertex of $\mathrm{Z}^*(G(\pounds))$ which is adjacent to every other vertex of $\mathrm{Z}^*(G(\pounds))$.

(2) By (1), $\mathrm{e}(x) \neq 1$ for all $x \in V(\mathrm{Z}^*(G(\pounds)))$. Therefore, $\mathrm{rad}(\mathrm{Z}^*(G(\pounds)))$ is not equal to 1. As $\mathrm{rad}(\mathrm{Z}^*(G(\pounds))) \leq \mathrm{diam}(\mathrm{Z}^*(G(\pounds))) \leq 2$, we get $\mathrm{rad}(\mathrm{Z}^*(G(\pounds))) = 2$. □

Theorem 3.7. *Let $\pounds$ be a lattice and* $\min(\pounds)$ *be finite. Then we have* $\alpha(\mathrm{Z}^*(G(\pounds))) = |\min(\pounds)|$.

Proof. Let $\min(\pounds) = \{P_1, P_2, ..., P_n\}$. For each $1 \leq i \leq n$, let $x_i \in P_i \setminus \bigcup_{j\neq i, P_j \in \min(\pounds)} P_j$. Set $I = \{x_1, x_2, ..., x_n\}$. We will show that I is an independent set. If $x_s \vee x_t \in \mathrm{Z}(\pounds)$ for some $x_s, x_t \in I$, where $s \neq t$, then $x_s \vee x_t \in P_k$ for some $1 \leq k \leq n$. This implies that $x_s \in P_k$ and $x_t \in P_k$. Hence $s = t = k$, a contradiction. Thus I is an independent set and so $\alpha(\mathrm{Z}^*(G(\pounds))) \geq n$. Now, let $\alpha(\mathrm{Z}^*(G(\pounds))) = m$ and $S = \{y_1, y_2, ..., y_m\}$ be a maximal independent set in $\mathrm{Z}^*(G(\pounds))$. If $m > n$, then by Pigeon hole principle, there exist $1 \leq i, j \leq n$ and $P \in \min(\pounds)$ such that $y_i, y_j \in P$. Hence $y_i \vee y_j \in P \subseteq \mathrm{Z}(\pounds)$, a contradiction. Hence $\alpha(\mathrm{Z}^*(G(\pounds))) = |\min(\pounds)|$. □

The following example shows that the condition “$\min(\pounds)$ is finite" is not superficial in Proposition 3.6 and Theorem 3.7.

Example 3.8. Consider the set $\mathbb{N}$ of natural numbers. Let $\pounds = \mathbb{N} \cup \{0\}$ and $a, b \in \pounds$. We write $a \leq b$ if and only if $a|b$, that is, $b = ac$ for some $c \in \pounds$. Then $\pounds$ becomes a lattice with the smallest element 1, the greatest element 0, and $x \wedge y = gcd(x, y)$, $x \vee y = lcm(x, y)$. One can show that $\mathrm{Z}^*(\pounds) = \mathbb{N}$ and the number of minimal prime ideals is infinite. However $\alpha(\mathrm{Z}^*(G(\pounds))) = 0$ (because $\mathrm{Z}^*(G(\pounds))$ is complete). Moreover $\mathrm{rad}(\mathrm{Z}^*(G(\pounds))) = 1$.

Acknowledgement

The authors express their deep gratitude to the referee for her/his helpful suggestions for the improvement of this work.

References

[1] Anderson, D.D. and Naseer, M., *Beck's coloring of commutative rings*, J. Algebra 159(2) (1993), 500-514.

[2] Anderson, D.F. and Badawi, A., *The total graph of a commutative ring*, J. Algebra 320(7) (2008), 2706-2719.

[3] Anderson, D.F. and Livingston, P.S., *The zero-divisor graph of a commutative rings*, J. Algebra 217(2) (1999), 434-447.

[4] Balasubramani, P. and Venkatanarasimhan, P.V., *Characterizations of the* 0-*distributive lattice*, Indian J. Pure Appl. Math. 32(3) (2001), 315-324.

[5] Beck, I., *Coloring of commutative rings*, J. Algebra 116(1) (1988), 208-226.

[6] Birkhoff, G., "Lattice Theory", Amer. Math. Soc., 1973.

[7] Bondy, A. and Murty, U.S.R., "Graph Theory with Applications", American Elsevier, 1976.

[8] Coykendall, J. Sather-Wagstaff, S., Sheppardson, L., and Spiroff, S., *On zero-divisor graphs*, Prog. Comm. Algebra 2 (2012), 241-299.

[9] Ebrahimi Atani, S., Dolati Pish Hesari, S., and Khoramdel, M., *Total graph of a commutative semiring with respect to identity-summand elements*, J. Korean Math. Soc. 51(3) (2014), 593-607.

[10] Ebrahimi Atani, S. and Sedghi Shanbeh Bazari, M., *On* 2-*absorbing filters of lattices*, Discuss. Math. Gen. Algebra and Appl. 36(2) (2016), 157-168.

[11] Estaji, E. and Khashyarmanesh, K., *The zero-divisor graph of a lattice*, Results Math. 61 (2012), 1-11.

[12] Grätzer, G., "General Lattice Theory", Birkhauser, 1978.

[13] Halaš, R. and Jukl, M., *On Beck's coloring of posets*, Discrete Math. 309 (2009), 4584-4589.

[14] *Joshi, V. and Khiste, A.*, *On the zero-divisor graphs of pm-lattices*, Discrete Math. 312 (2012), 2076-2082.

[15] Joshi, V. and Khiste, A., *Complement of the zero-divisor graph of a lattice*, Bull. Aust. Math. Soc. 89 (2014), 177-190.

[16] Nimbhorkar, S.K. Wasadikar, M.P., and Pawar, M.M., *Coloring of lattices, Math. Slovaca* 60 (2010), 419-434.

[17] Tamizh Chelvam, T. and Nithya, S., *A note on the zero-divisor graph of a lattices*, Trans. Comb. 3(3) (2014), 51-59.

Shahabaddin Ebrahimi Atani *Faculty of Mathematical Sciences, University of Guilan, P.O. Box 1914, Rasht, Iran.*

Email: ebrahimi@guilan.ac.ir

Saboura Dolati Pish Hesari *Faculty of Mathematical Sciences, University of Guilan, P.O. Box 1914, Rasht, Iran.*

Email: saboura_dolati@yahoo.com

Mehdi Khoramdel *Faculty of Mathematical Sciences, University of Guilan, P.O. Box 1914, Rasht, Iran.*

Email: mehdikhoramdel@gmail.com

Maryam Sedghi Shanbeh Bazari *Faculty of Mathematical Sciences, University of Guilan, P.O. Box 1914, Rasht, Iran.*

Email: ms.maryamsedghi55@gmail.com