Categories and
General Algebraic Structures
with Applications
Volume 9, Number 1, July 2018, 139-161.

Convex L-lattice subgroups in L-ordered groups

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Abstract. In this paper, we have focused to study convex L-subgroups of an L-ordered group. First, we introduce the concept of a convex L-subgroup and a convex L-lattice subgroup of an L-ordered group and give some examples. Then we find some properties and use them to construct convex L-subgroup generated by a subset S of an L-ordered group G . Also, we generalize a well known result about the set of all convex subgroups of a lattice ordered group and prove that $C(G)$, the set of all convex L-lattice subgroups of an L-ordered group G, is an L-complete lattice on height one. Then we use these objects to construct the quotient L-ordered groups and state some related results.

1 Introduction

Zhang and Liu in [20] defined a kind of an L-frame by a pair $(A; i_A)$, where A is a classical frame and $i_A : L \to A$ is a frame morphism. For a stratified L-topological space $(X; \delta)$, the pair $(\delta; i_X)$ is one of this kind of L-frames, where $i_X : L \to \delta$, is a map which sends $a \in L$ to the constant map with the value a. Conversely, a point of an L-frame $(A; i_A)$ is a frame morphism

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Keywords: L-ordered group, convex L-subgroup, (normal) convex L-lattice subgroup.
Mathematics Subject Classification[2010]: 06D72, 08A72, 03E72.
Received: 26 March 2017, Accepted: 1 June 2017
ISSN Print: 2345-5853 Online: 2345-5861

$p : (A; i_A) \to (L; id_L)$ satisfying $p \circ i_A = id_L$ and $Lpt(A)$ denotes the set of all points of $(A; i_A)$. Then $\{\Phi_x : Lpt(A) \to L \mid \forall p \in Lpt(A); \Phi_x(p) = p(x)\}$ is a stratified L-topology on $Lpt(A)$. By these two assignments, Zhang and Liu constructed an adjunction between $SL-Top$ and $L-Loc$ and consequently they established the Stone representation theorem for distributive lattices by means of this adjunction. They pointed out that, from the viewpoint of lattice theory, Rodabaugh's fuzzy version of the Stone representation theory is just one and it has nothing different from the classical one. Recently, based on complete Heyting algebras, Fan and Zhang [7, 19] studied quantitative domains through fuzzy set theory. Their approach uses a fuzzy partial order, specifically a degree function, on a non-empty set. Yao [16] introduced the notion of L-frames. It is an L-complete ordered set with the meet operation having a right fuzzy adjoint. Indeed, the category of L-frames introduced by Yao is isomorphic to the category of L-frames defined by Zhang and Liu in [20] (for more details see [17]). He established an adjunction between the category of stratified L-topological spaces and the category of L-locales, the opposite category of this kind of L-frames. Borzooei et al. in [4] defined the notions of L-ordered and L-lattice ordered groups and found a relation between positive cones and L-ordered relations of a group and verified a quotient L-ordered group constructed by a convex normal L-subgroup. They also stated a general form for Riesz decomposition property in L-lattice ordered groups. In [5], they continued the study of this structure and defined the notion of totally L-ordered group.

The content of this paper is organized as follows. In Section 2, some notions and results about L-ordered groups and L-lattice ordered groups are recalled. In Section 3, the concepts of positive cone, convex L-subgroup and convex L-lattice subgroup in L-ordered groups, where L is a frame, are defined, and it is proved that the set of all convex L-lattice subgroups is an L-complete lattice of height one, and using a normal convex L-subgroup, an L-ordered group is constructed and some related results are investigated.

2 Preliminaries

In this section, we gather some definitions and results which will be used in the paper.

We recall that a frame is a complete lattice $(L, \vee, \wedge, 0, 1)$ satisfying the (infinite) distributive law

$$a \wedge (\bigvee_{b\in B} b) = \bigvee_{b\in B} (a \wedge b)$$

for any $a \in L$ and $B \subseteq L$. If $(L, \vee, \wedge, 0, 1)$ is a frame, then we have a binary operation $\rightarrow: L \times L \to L$ defined by $x \to y = \bigvee\{z \in L | x \wedge z \leq y\}$, for every $x, y \in L$. The following equations hold in all frames, for each $x, y, z \in L$ and $Y \subseteq L$:

(i) $(x \wedge y) \to z = x \to (y \to z)$;

(ii) $x \to (\bigwedge Y) = \bigwedge_{y\in Y}(x \to y)$;

(iii) $(\bigvee Y) \to z = \bigwedge_{y\in Y}(y \to z)$.

From now on, in this paper, $(L, \vee, \wedge, 0, 1)$ or simply L is a frame.

Let P be a set and $e : P \times P \to L$ be a map. The pair (P, e) is called an *L-ordered set* if for all $x, y, z \in P$,

(E1): $e(x, x) = 1$,

(E2): $e(x, y) \wedge e(y, z) \leq e(x, z)$,

(E3): $e(x, y) = e(y, x) = 1$ implies $x = y$.

In an L-ordered set (P, e), the map e is called an *L-order relation* on P. Note that, if $(P, \leq)$ is a poset, then $(P, \chi_{\leq})$ is an L-ordered set, where $\chi_{\leq}$ is the characteristic function of $\leq$. Moreover, for each L-ordered set (P, e), the set $\leq_e = \{(x, y) \in P \times P \mid e(x, y) = 1\}$ is a partial order on P and so $(P, \leq_e)$ is a poset. We denote L^P for the set of all L-subsets of P, that is $L^P = \{f \mid f : P \to L\}$. A map $f : (P, e_P) \to (Q, e_Q)$ between two L-ordered sets is called *monotone* if for all $x, y \in P$, $e_P(x, y) \leq e_Q(f(x), f(y))$. Let (P, e) be an L-ordered set and S be an L-subset of P. Then the support of S is defined by $Supp(S) = \{x \in P | \ 0 < S(x)\}$. For an L-ordered set (P, e) and $S \in L^P$, an element $x_0 \in P$ is called a *join* (*meet*) of S, in symbol $x_0 = \sqcup S$ $(x_0 = \sqcap S)$, if for all $x \in P$,

(J1) $S(x) \leq e(x; x_0)$ ((M1) $S(x) \leq e(x_0; x)$),

(J2) $\bigwedge_{y\in P}(S(y) \to e(y, x)) \leq e(x_0, x)$ ((M2) $\bigwedge_{y\in P}(S(y) \to e(x, y)) \leq e(x, x_0)$).

If the join or meet of S exists, then they are unique. An L-ordered set (P, e) is called a (*weak*) *L-lattice* if for every $(x, y \in G)$ $S \in L^P$ where $Supp(S)$ is finite, ($\sqcap_{\chi\{x,y\}}$ and $\sqcup_{\chi\{x,y\}}$) $\sqcup S$ and $\sqcap S$ exist. It can be easily seen that if

(P, e) is an L-lattice, then $(P, \leq_e)$ is a lattice and $\sqcap_{\chi\{x,y\}}$ and $\sqcup_{\chi\{x,y\}}$ are $x \wedge y$ and $x \vee y$, respectively. An L-ordered set (P, e) is called an L-complete lattice if for any $S \in L^P$, $\sqcup S$ and $\sqcap S$ exist (see [18, 19, 21]).

Theorem 2.1. [19] *Let (P, e) be an L-ordered set and $S \in L^P$. Then*
(i) $x_0 = \sqcup S$ *if and only if* $e(x_0, x) = \bigwedge_{y\in P}(S(y) \to e(y, x))$, *for all* $x \in P$;
(ii) $x_0 = \sqcap S$ *if and only if* $e(x; x_0) = \bigwedge_{y\in P}(S(y) \to e(x, y))$, *for all* $x \in P$.

Proposition 2.2. [4] *Let (P, e) be a weak L-lattice. Then for all $x, y, a \in P$, the following conditions hold:*
(i) $e(a, x \wedge y) = e(a, x) \wedge e(a, y)$;
(ii) $e(x \vee y, a) = e(x, a) \wedge e(y, a)$.

Definition 2.3. [4] $S \in L^P$ is called a *convex L-subset* of L-ordered set (P, e) if for every $x, y, a \in P$,

$$S(x) \wedge S(y) \wedge e(x, a) \wedge e(a, y) \leq S(a).$$

An *L-ordered group* (or an *L-fuzzy ordered group*) $(G, e, ., 1)$, is a group $(G, ., 1)$ together with an L-order relation $e : G \times G \to L$ such that for any $a \in G$, the translation maps $(\ ).a : G \to G$ and $a.(\ ) : G \to G$ are monotone or, equivalently, (FOG): $e(x, y) \leq e(bxa, bya)$, for every $x, y, a, b \in G$.

An *L-lattice ordered group* is an L-ordered group in which for every $x, y \in G$, $\sqcup_{\chi\{x,y\}}$ and $\sqcap_{\chi\{x,y\}}$ exist. In each L-lattice ordered group $(G, e, ., 1)$, the ordered set $(G, \leq_e)$ is a lattice. Let $(G; e, \cdot, 1)$ be an L-ordered group, $a \in G$ and $S \in L^G$. We define $a \wedge S$, $a \vee S$ and aS by

$$(a \wedge S)(y) = \bigvee\{S(x) | x \in G, a \wedge x = y\},$$

$$(a \vee S)(y) = \bigvee\{S(x) | x \in G, a \vee x = y\}, \quad aS(y) = S(a^{-1}y)$$

for any $y \in G$. A map $f : (G, e_G, \cdot, 1_G) \to (H, e_H, \cdot, 1_H)$ between two L-ordered groups is called an *L-ordered group homomorphism* if it is monotone and group homomorphism or, equivalently, f preserves the operations $\cdot$, 1 and for each $x, y \in G$, $e(x, y) \leq e(f(x), f(y))$. If an L-ordered group homomorphism is one to one and onto, then it is called an L-ordered group isomorphism (for more details see [4]).

Proposition 2.4. [4] *Let $(G; e, ., 1)$ be an L-ordered group. Then forever $x, y, a, b \in G$ and $S \in L^G$, the following conditions hold:*
(i) $e(x, y) = e(bxa, bya)$;
(ii) $e(x, y) = e(y^{-1}, x^{-1})$;
(iii) *If* $x \leq y$, *then* $e(y, a) \leq e(x, a)$ *and* $e(a, x) \leq e(a, y)$;
(iv) $a \sqcup S = \sqcup(aS)$, $a \sqcap S = \sqcap(aS)$ *and* $(\sqcap S)^{-1} = \sqcap S^{-1}$;
(v) $(G, \leq_e)$ *is an ordered group.*

Definition 2.5. [4] Let $(G; e, \cdot, 1)$ be an L-ordered group. Then

(i) $S \in L^G$ is called an *L-subgroup* of G if $S(1_G) = 1$, $S(x) = S(x^{-1})$ and $S(x) \wedge S(y) \leq S(xy)$, for every $x, y \in G$.

(ii) L-subgroup S of G is called *normal* if $S(y) \leq S(xyx^{-1})$ for all $x, y \in G$. Clearly, if S is normal, then $S(y) = S(xyx^{-1})$, for all $x, y \in G$.

(iii) The *positive cone* of $S \in L^G$ is the map $S^+ : G \to L$, which is defined by $S^+(x) = S(x) \wedge e(1, x)$, for all $x \in G$.

(iv) The positive (negative) cone of G is defined by

$$G^+(x) = e(1, x)(G^-(x) = e(x, 1))\forall x \in G.$$

Theorem 2.6. [4] *Let $(G; e, \cdot, 1)$ be an L-ordered group. Then $S \in L^G$ is a convex L-subset of G if and only if for every $x, a \in G$,*

$$S(x) \wedge e(1, a) \wedge e(a, x) \leq S(a).$$

3 Convex L-subgroups and convex L-lattice subgroups

In this section, we study some properties of convex L-subgroups and convex L-lattice subgroups in L-ordered groups. Throughout this section $(G; e, \cdot, 1)$ (simply denoted by G) is an L-lattice ordered group, unless otherwise stated.

Definition 3.1. An L-subset S of G is called

(i) an *L-Lattice subgroup* if S is an L-subgroup such that $S(x) \wedge S(y) \leq S(x \wedge y)$ and $S(x) \wedge S(y) \leq S(x \vee y)$, for all $x, y \in G$;

(ii) a *convex L-subgroup* of G, if it is both an L-subgroup and a convex L-subset of G. A convex L-subgroup $C \in L^G$ is called a *convex L-lattice subgroup* of $(G; e, \cdot, 1)$ if it is an L-Lattice subgroup of G.

Example 3.2. Let $G = \mathbb{Z} \times \mathbb{Z}$ and $(L = \{0, a, b, 1\}, \leq)$ be a poset such that $a \vee b = 1$ and $a \wedge b = 0$. Then L is a frame. Now, let $e : G \times G \to L$ and $C \in L^G$ be defined by

$$e((x_1, y_1), (x_2, y_2)) = \begin{cases} 1 & \text{if } x_1 \leq x_2, y_1 \leq y_2 \\ a & \text{if } x_1 \leq x_2, y_2 < y_1 \\ b & \text{if } x_2 < x_1, y_1 \leq y_2 \\ 0 & \text{if } x_2 < x_1, y_2 < y_1 \end{cases}, \qquad C(x, y) = \begin{cases} 1 & \text{if } y = 0 \\ a & \text{if } y \neq 0. \end{cases}$$

Then C is a convex L-lattice subgroup of G.

Example 3.3. Let $0 < 1$ and $L = \prod_{n=1}^{\infty} \{0, 1\}$ with the pointwise order relation and $G = \prod_{n=1}^{\infty} \mathbb{Z}$. For any $(x_i)_{\mathbb{N}}, (y_i)_{\mathbb{N}} \in G$, define $e((x_i)_{\mathbb{N}}, (y_i)_{\mathbb{N}}) = (\chi_{\leq}(x_i, y_i))_{\mathbb{N}}$ where $\leq$ is the natural order on $\mathbb{Z}$. Define $S \in L^G$ by

$$\pi_j(S((x_i)_{i \in \mathbb{N}})) = \begin{cases} 1 & \text{if } x_j \text{ is even} \\ 0 & \text{if } x_j \text{ is odd,} \end{cases}$$

where π_j is the j-th canonical projection map, for all $j \in \mathbb{N}$. Then clearly, $(G, e, +, (0)_{\mathbb{N}})$ is an L-ordered group. Moreover, S is an L-subgroup which is not convex. Indeed, since 0 is even, $S((0)_{\mathbb{N}}) = (1)_{\mathbb{N}}$. For any $x \in \mathbb{Z}$, x is even if and only if $-x$ is even. So, $S((x_i)_{\mathbb{N}}) = S((-x_i)_{\mathbb{N}})$. For each $x, y \in \mathbb{Z}$, $x + y$ is odd if and only if one of x, y is odd. It follows that, for all $(x_i)_{\mathbb{N}}, (y_i)_{\mathbb{N}} \in G$, and all $j \in \mathbb{N}$,

$$\pi_j(S((x_i)_{\mathbb{N}})) \wedge \pi_j(S((y_i)_{\mathbb{N}})) \leq \pi_j(S((x_i)_{\mathbb{N}} + (y_i)_{\mathbb{N}})).$$

Hence, $S((x_i)_{\mathbb{N}}) \wedge S((y_i)_{\mathbb{N}}) \leq S((x_i)_{\mathbb{N}} + (y_i)_{\mathbb{N}})$. That is, S is an L-subgroup. Also, we have

$$\begin{aligned} S((2)_{\mathbb{N}}) \wedge e((0_{\mathbb{Z}})_{\mathbb{N}}, (1)_{\mathbb{N}}) \wedge e((1)_{\mathbb{N}}, (2)_{\mathbb{N}}) &= (1)_{\mathbb{N}} \wedge (1)_{\mathbb{N}} \wedge (1)_{\mathbb{N}} \\ &\not\leq (0)_{\mathbb{N}} = S((1)_{\mathbb{N}}). \end{aligned}$$

So, S is not convex.

Lemma 3.4. *Let $C \in L^G$ be a convex L-subgroup of G. Then for any $x \in G$,*

$$C(x \vee x^{-1}) = C(x \wedge x^{-1}) = C(x).$$

Proof. Let $x \in G$. Then $C(x \vee x^{-1}) = C((x \vee x^{-1})^{-1}) = C(x \wedge x^{-1})$ and $e(x \wedge x^{-1}, x) = e(x, x \vee x^{-1}) = 1$. Since C is convex,

$$C(x \vee x^{-1}) = C(x \vee x^{-1}) \wedge C(x \wedge x^{-1}) \wedge e(x \wedge x^{-1}, x) \wedge e(x, x \vee x^{-1}) \leq C(x).$$

On the other hand, since C is an L-lattice, $C(x) \leq C(x \wedge x^{-1}) = C(x \vee x^{-1})$. Therefore,

$$C(x \vee x^{-1}) = C(x \wedge x^{-1}) = C(x).$$

□

Theorem 3.5. *Let $C \in L^G$ be an L-subgroup of G. Then C is a convex L-lattice subgroup of G if and only if for every $x, g \in G$, $C(x) \wedge e(g \vee g^{-1}, x \vee x^{-1}) \leq C(g)$.*

Proof. ($\Rightarrow$) Let C be a convex L-lattice subgroup of G and $x, g \in G$. Then

$$\begin{aligned} C(x) \wedge e(g \vee g^{-1}, x \vee x^{-1}) &= C(x \vee x^{-1}) \wedge e(g \vee g^{-1}, x \vee x^{-1}), \text{by Lemma 3.4} \\ &= C(x \vee x^{-1}) \wedge e(g \vee g^{-1}, x \vee x^{-1}) \wedge e(1, g \vee g^{-1}) \\ &\leq C(g \vee g^{-1}), \text{ by the definition of convexity} \\ &= C(g), \quad \text{by Lemma 3.4.} \end{aligned}$$

($\Leftarrow$) Let $C \in L^G$. Since for any $x \in G$, we have

$$e(1, x) = e(1, x) \wedge e(x^{-1}, 1) \leq e(x^{-1}, x) = e(x^{-1}, x) \wedge e(x, x) = e(x^{-1} \vee x, x).$$

Hence, for every $x, a \in G$,

$$\begin{aligned} C(a) \wedge e(1, x) \wedge e(x, a) &\leq C(a) \wedge e(1, x) \wedge e(x, a \vee a^{-1}), \quad \text{by Lemma 2.4(iii)} \\ &\leq C(a) \wedge e(x \vee x^{-1}, x) \wedge e(x, a \vee a^{-1}) \\ &\leq C(a) \wedge e(x \vee x^{-1}, a \vee a^{-1}), \text{ by (E2)} \\ &\leq C(x). \end{aligned}$$

Then by Theorem 2.6, C is convex. □

Proposition 3.6. *The intersection of any convex L-lattice subgroups of G is a convex L-lattice subgroup of G, too.*

Proof. Let $\{C_i \in L^G | i \in I\}$ be a family of convex L-subgroups of $(G; e, \cdot, 1)$. It is easy to see that $\bigwedge_i C_i$ is an L-lattice subgroup of $(G; e, \cdot, 1)$. It is enough to show that $\bigwedge_i C_i$ is convex. For ever $a, x \in G$, we have

$$\begin{aligned}(\bigwedge_{i\in I} C_i)(a) &= \bigwedge_{i\in I}(C_i(a)) \geq \bigwedge_{i\in I}(C_i(x) \wedge e(1,a) \wedge e(a,x)) \\ &= \bigwedge_{i\in I}(C_i(x)) \wedge e(1,a) \wedge e(a,x).\end{aligned}$$

Hence $\bigwedge_{i\in I} C_i$ is convex. □

Lemma 3.7. *For every $x, a \in G$*

$$G^+(x) = G^+(axa^{-1}) \ , \ G^-(x) = G^-(axa^{-1}).$$

Proof. For every $x, a \in G$, we have

$$G^+(x) = e(1, x) = e(a1a^{-1}, axa^{-1}) = e(1, axa^{-1}) = G^+(axa^{-1}).$$

For G^-, the proof is similar. □

If S is an L-subgroup of an L-ordered group $(G; e, \cdot, 1)$, then by Proposition 3.6, the intersection of any convex L-subgroups of $(G; e, \cdot, 1)$ that contains S is a convex L-subgroup of $(G; e, \cdot, 1)$ which contains S, and is denoted by $\langle S \rangle_C$. We used $\langle S \rangle_C$ to denote the *convex L-subgroup of G generated by S.*

Let $(G; e, \cdot, 1)$ be an L-ordered group and $S, T \in L^G$. Then for any $x \in G$, we define $S \cdot T \in L^G$ by

$$S \cdot T(x) = \bigvee_{ab=x} (S(a) \wedge T(b)),$$

for any $x \in G$.

Theorem 3.8. *Let S be an L-subgroup of an L-ordered group $(G; e, \cdot, 1)$. Then*

$$\langle S \rangle_C = S \cdot G^+ \wedge S \cdot G^-.$$

Proof. First we show that $S\cdot G^+ \wedge S\cdot G^-$ is a convex L-subgroup of $(G;e,\cdot,1)$. Since S is an L-subgroup of $(G;e,\cdot,1)$, $S(1)=1$. From $G^+(1)=e(1,1)=1$ and $G^-(1)=e(1,1)=1$, we get

$$\begin{aligned}(S.G^+ \wedge S.G^-)(1) &= (S\cdot G^+)(1) \wedge (S\cdot G^-)(1)\\ &= \bigvee_{xy=1}(S(x)\wedge G^+(y)) \wedge \bigvee_{xy=1}(S(x)\wedge G^-(y))\\ &\geq S(1)\wedge G^+(1)\wedge S(1)\wedge G^-(1)=1.\end{aligned}$$

Now for every $x,y\in G$, by Definition 2.5 and Lemma 3.7, we have:

$$\begin{aligned}&(S\cdot G^+ \wedge S\cdot G^-)(x) \wedge (S\cdot G^+ \wedge S\cdot G^-)(y)\\ &\quad = (\bigvee_{ab=x}(S(a)\wedge G^+(b)) \wedge \bigvee_{a'b'=x}(S(a')\wedge G^-(b')))\\ &\quad\quad \wedge(\bigvee_{cd=y}(S(c)\wedge G^+(d)) \wedge \bigvee_{c'd'=y}(S(c')\wedge G^-(d')))\\ &\quad = \bigvee_{ab=x}\bigvee_{cd=y}((S(a)\wedge S(c))\wedge(G^+(b)\wedge G^+(d)))\\ &\quad\quad \wedge \bigvee_{ab=x}\bigvee_{cd=y}((S(a')\wedge S(c'))\wedge(G^-(b')\wedge G^-(d')))\\ &\quad \leq \bigvee_{abcd=xy}((S(ac))\wedge(G^+(c^{-1}bc)\wedge G^+(d)))\\ &\quad\quad \wedge \bigvee_{a'b'c'd'=xy}((S(a'c'))\wedge(G^-(c'^{-1}b'c')\wedge G^-(d')))\\ &\quad \leq \bigvee_{acc^{-1}bcd=xy}((S(ac))\wedge(G^+(c^{-1}bcd)))\\ &\quad\quad \wedge \bigvee_{a'c'c'^{-1}b'c'd'=xy}((S(a'c'))\wedge(G^+(c^{-1}bcd)))\\ &\quad \leq \bigvee_{uv=xy}((S(u))\wedge(G^+(v)))\wedge \bigvee_{u'v'=xy}((S(u'))\wedge(G^-(v')))\\ &\quad = (S\cdot G^+ \wedge S\cdot G^-)(xy).\end{aligned}$$

Let $x \in G$. Then

$$\begin{aligned}(S\cdot G^+ \wedge S\cdot G^-)(x^{-1}) &= \bigvee_{ab=x^{-1}} (S(a)\wedge G^+(b)) \wedge \bigvee_{a'b'=x^{-1}} (S(a')\wedge G^-(b'))\\ &= \bigvee_{ab^{-1}=x} (S(a^{-1})\wedge G^+(b)) \wedge \bigvee_{a'^{-1}b'=x} (S(a'^{-1})\wedge G^-(b'))\\ &= (S\cdot G^+ \wedge S\cdot G^-)(x).\end{aligned}$$

So $S\cdot G^+ \wedge S\cdot G^-$ is an L-subgroup of $(G; e, \cdot, 1)$. Now, for every $x, y \in G$ we have

$$\begin{aligned}& (S\cdot G^+ \wedge S\cdot G^-)(x)\wedge e(1,y)\wedge e(y,x)\\ = & \bigvee_{ab=x} (S(a)\wedge G^+(b)) \wedge \bigvee_{a'b'=x} (S(a')\wedge G^-(b'))\wedge e(1,y)\wedge e(y,x)\\ \le & \bigvee_{a'b'=x} (S(a')\wedge e(b',1))\wedge e(1,y)\wedge e(y,x)\\ = & \bigvee_{a'b'=x} (S(a')\wedge e(b',1)\wedge e(1,y)\wedge e(y,x))\\ & \wedge \bigvee_{a'b'=x} (S(a')\wedge e(b',1)\wedge e(1,y)\wedge e(y,x))\\ \le & \bigvee_{a'b'=x} (S(a')\wedge e(1,(b')^{-1})\wedge e(1,y)\wedge e(y,a'b'))\\ & \wedge \bigvee_{a'b'=x} (S(a')\wedge e(b',1)\wedge e(x^{-1}y,1))\\ \le & \bigvee_{a'b'=x} (S(a')\wedge e(1,(b')^{-1})\wedge e((b')^{-1},y(b')^{-1})\wedge e(y(b')^{-1},a'))\\ & \wedge \bigvee_{a'b'=x} (S(a')\wedge e(b'x^{-1}y,1))\\ \le & \bigvee_{a'b'=x} (S(a')\wedge e(1,y)\wedge e(1,a'))\wedge \bigvee_{a'b'=x} (S(a')\wedge e(b'x^{-1}y,1))\\ \le & \bigvee_{a'b'=x} (S(a'^{-1})\wedge e(1,a'y))\wedge \bigvee_{a'b'=x} (S(a')\wedge e(b'x^{-1}y,1))\\ = & \bigvee_{a'b'=x} (S(a'^{-1})\wedge G^+(a'y))\wedge \bigvee_{a'b'=x} (S(a')\wedge G^-(b'x^{-1}y))\end{aligned}$$

$$\leq \bigvee_{st=y} (S(s) \wedge G^{+}(t)) \wedge \bigvee_{s't'=y} (S(s') \wedge G^{-}(t')), \text{ since } y = a'^{-1}a'y, y = a'b'x^{-1}y$$
$$= \quad S \cdot G^{+} \wedge S \cdot G^{-}(y).$$

Hence $S \cdot G^{+} \wedge S \cdot G^{-}$ is convex. Moreover, for any $x \in G$, we have

$$\begin{aligned} S(x) &= (S(x) \wedge e(1,1)) \wedge (S(x) \wedge e(1,1)) \\ &\leq \bigvee_{ab=x} (S(a) \wedge G^{+}(b)) \wedge \bigvee_{ab=x} (S(a) \wedge G^{-}(b)) = (S \cdot G^{+} \wedge S \cdot G^{-})(x). \end{aligned}$$

It follows that $S \cdot G^{+} \wedge S \cdot G^{-}$ contains S. Now, let C be a convex L-subgroup of $(G; e, \cdot, 1)$ such that $S(a) \leq C(a)$ for any $a \in G$. Then

$$\begin{aligned} (S.G^{+} \wedge S.G^{-})(x) &= \bigvee_{ab=x} (S(a) \wedge G^{+}(b)) \wedge \bigvee_{a'b'=x} (S(a') \wedge G^{-}(b')) \\ &= \bigvee_{ab=x} (S(a) \wedge G^{+}(a^{-1}x)) \wedge \bigvee_{a'b'=x} (S(a') \wedge G^{-}(a'^{-1}x)) \\ &= \bigvee_{ab=x} (S(a) \wedge e(1, a^{-1}x)) \wedge \bigvee_{a'b'=x} (S(a') \wedge e(a'^{-1}x, 1)) \\ &= \bigvee_{ab=x} (S(a) \wedge e(a, x)) \wedge \bigvee_{a'b'=x} (S(a') \wedge e(x, a')) \\ &= \bigvee_{ab=x} \bigvee_{a'b'=x} (S(a) \wedge e(a, x) \wedge S(a') \wedge e(x, a')) \\ &\leq \bigvee_{ab=x} \bigvee_{a'b'=x} (C(a) \wedge e(a, x) \wedge C(a') \wedge e(x, a')) \\ &\leq C(x). \end{aligned}$$

Therefore, $\langle S \rangle_C = S \cdot G^{+} \wedge S \cdot G^{-}$. $\square$

It is well known that the set of all convex subgroups of a lattice ordered group H is a complete lattice (see [16]). In the next theorem, we want to generalize this result for the set of all convex L-lattice subgroups of G. First, we recall that if X is a set, then for each $S \in L^X$, the height of S is defined by $ht(S) = \bigvee_{x \in X} S(x)$.

Definition 3.9. Let (P, e) be an L-ordered set. If for $S \in L^P$ with $ht(S) = 1$, $\sqcap S$ and $\sqcup S$ exist then (P, e) is called an *L-complete lattice of hight one.*

In the next theorem, we will show that $C(G)$ is an L-complete lattice of hight one.

Theorem 3.10. *Let $C(G)$ be the set of all convex L-lattice subgroups of G. Then for each φ belonging to $\mathcal{C} = \{\varphi \in L^{C(G)} \mid \bigvee_{f\in C(G)} \varphi(f) = 1\}$, $\sqcup\varphi$ and $\sqcap\varphi$ exist and belong to $C(G)$.*

Proof. By [2] and [11, Exa. 3.7], we know that (L^G, e') is an L-complete lattice where $e'(A, B) = \bigwedge_{x\in G}(A(x) \to B(x))$, for all $A, B \in L^G$. Consider the L-ordered set $(C(G), e')$. For any $\varphi \in \mathcal{C}$, we claim that if $S_0(x) = \bigvee_{f\in C(G)}(\varphi(f) \wedge f(x))$ and $S_1(x) = \bigwedge_{f\in C(G)}(\varphi(f) \to f(x))$, for all $x \in G$, then $\sqcup\varphi = S_0$ and $\sqcap\varphi = S_1$ in $(C(G), e')$. First, we show that $S_0, S_1 \in C$.
(1) For all $x, y, a \in G$ we have

$$\begin{aligned}
&S_0(x) \wedge S_0(y) \wedge e(x, a) \wedge e(a, y) \\
&\quad= \Big(\bigvee_{f\in C(G)} (\varphi(f) \wedge f(x))\Big) \wedge \Big(\bigvee_{g\in C(G)} (\varphi(g) \wedge g(y))\Big) \wedge e(x, a) \wedge e(a, y) \\
&\quad= \bigvee_{g\in C(G)} \bigvee_{f\in C(G)} \Big(\varphi(f) \wedge \varphi(g) \wedge f(x) \wedge g(y) \wedge e(x, a) \wedge e(a, y)\Big) \\
&\quad\leq \bigvee_{f\in C(G)} \Big(\varphi(f) \wedge f(x) \wedge f(y) \wedge e(x, a) \wedge e(a, y)\Big) \\
&\qquad\quad \wedge \bigvee_{g\in C(G)} \Big(\varphi(g) \wedge g(x) \wedge g(y) \wedge e(x, a) \wedge e(a, y)\Big) \\
&\quad\leq \bigvee_{f\in C(G)} (\varphi(f) \wedge f(a)) \wedge \bigvee_{g\in C(G)} (\varphi(g) \wedge g(a)) = S_0(a).
\end{aligned}$$

(2) Since $\varphi \in \mathcal{C}$, $S_0(1) = \bigvee_{f\in C(G)}(\varphi(f) \wedge f(1)) = \bigvee_{f\in C(G)} \varphi(f) = 1$.
(3) Let $x, y \in G$. Then

$$\begin{aligned}
S_0(xy) &= \bigvee_{f\in C(G)} (\varphi(f) \wedge f(xy)) \geq \bigvee_{f\in C(G)} (\varphi(f) \wedge f(x) \wedge f(y)) \\
&\geq (\bigvee_{f\in C(G)} \varphi(f) \wedge f(x)) \wedge (\bigvee_{f\in C(G)} \varphi(f) \wedge f(y)) \\
&= S_0(x) \wedge S_0(y).
\end{aligned}$$

In a similar way, we can show that $S_0(x \wedge y), S_0(x \vee y) \geq S_0(x) \wedge S_0(y)$.

From (1)-(3), it follows that $S_0 \in C(G)$. Now, we show that $S_0 = \sqcup\varphi$. Since S_0 is $\sqcup\varphi$ in (L^G, e') (see the proof of [21, Theorem 2.20]), then for each $f \in C(G)$, by (J1), we have $\varphi(f) \le e'(f, S_0)$. Let $f \in C(G)$. Then

$$\begin{aligned}
\bigwedge_{g\in C(G)} (\varphi(g) \to e'(g,f)) &= \bigwedge_{g\in C(G)} \big(\varphi(g) \to \bigwedge_{x\in G} (g(x) \to f(x))\big) \\
&= \bigwedge_{g\in C(G)} \bigwedge_{x\in G} \big(\varphi(g) \to (g(x) \to f(x))\big) \\
&= \bigwedge_{g\in C(G)} \bigwedge_{x\in G} \big((\varphi(g) \wedge g(x)) \to f(x)\big) \\
&= \bigwedge_{x\in G} \bigwedge_{g\in C(G)} \big((\varphi(g) \wedge g(x)) \to f(x)\big) \\
&= \bigwedge_{x\in G} \big((\bigvee_{g\in C(G)} (\varphi(g) \wedge g(x))) \to f(x)\big) \\
&= \bigwedge_{x\in G} (S_0(x) \to f(x)) = e'(S_0 \to f).
\end{aligned}$$

Therefore, $S_0 = \sqcup\varphi$. Also,

(4) $S_1(1) = \bigwedge_{g\in C(G)}(\varphi(g) \to g(1)) = \bigwedge_{g\in C(G)}(\varphi(g) \to 1) = 1$.

(5) For each $x, y \in G$,

$$\begin{aligned}
S_1(x.y) &= \bigwedge_{g\in C(G)} (\varphi(g) \to g(xy)) \ge \bigwedge_{g\in C(G)} (\varphi(g) \to (g(x) \wedge g(y))) \\
&= \bigwedge_{g\in C(G)} \big((\varphi(g) \to g(x)) \wedge (\varphi(g) \to g(y))\big) = S_1(x) \wedge S_1(y).
\end{aligned}$$

In a similar way, it can be easily seen that $S_1(x\wedge y), S_1(x\vee y) \ge S(x)\wedge S(y)$.

(6) For all $x, y, a \in G$ we have

$$\begin{aligned}
S_1(x) \wedge S_1(y) \wedge e(x,a) \wedge e(a,y)& \\
&= (\bigwedge_{g\in C(G)} (\varphi(g) \to g(x))) \wedge (\bigwedge_{f\in C(G)} (\varphi(f) \to f(y))) \wedge e(x,a) \wedge e(a,y) \\
&\leq \bigwedge_{f,g\in C(G)} \Big((\varphi(g) \to g(y)) \wedge (\varphi(f) \to f(x))\Big) \wedge e(x,a) \wedge e(a,y) \\
&\leq \bigwedge_{f\in C(G)} \Big((\varphi(f) \to f(y)) \wedge (\varphi(f) \to f(x))\Big) \wedge e(x,a) \wedge e(a,y) \\
&\leq \bigwedge_{f\in C(G)} \Big(\varphi(f) \to (f(y) \wedge f(x))\Big) \wedge e(x,a) \wedge e(a,y) \\
&= \bigwedge_{f\in C(G)} \Big((\varphi(f) \to (f(y) \wedge f(x))) \wedge e(x,a) \wedge e(a,y)\Big) \\
&\leq \bigwedge_{f\in C(G)} \Big((\varphi(f) \to (f(y) \wedge f(x))) \wedge \big(\varphi(f) \to (e(x,a) \wedge e(a,y))\big)\Big) \\
&\leq \bigwedge_{f\in C(G)} \Big(\varphi(f) \to (f(y) \wedge f(x) \wedge e(x,a) \wedge e(a,y))\Big) \\
&\leq \bigwedge_{f\in C(G)} (\varphi(f) \to f(a)) = S_1(a).
\end{aligned}$$

From (4)-(6), it follows that $S_1 \in C(G)$.
(7) Let $f \in C(G)$. Then

$$\begin{aligned}
e'(f, S_1) &= \bigwedge_{x\in G} (f(x) \to S_1(x)) = \bigwedge_{x\in G} \Big(f(x) \to (\bigwedge_{g\in C(G)} (\varphi(g) \to g(x)))\Big) \\
&= \bigwedge_{x\in G} \bigwedge_{g\in C(G)} (f(x) \to (\varphi(g) \to g(x))) \\
&= \bigwedge_{g\in C(G)} \bigwedge_{x\in G} (\varphi(g) \to (f(x) \to g(x))) \\
&= \bigwedge_{g\in C(G)} \Big(\varphi(g) \to (\bigwedge_{x\in G} (f(x) \to g(x)))\Big) = \bigwedge_{g\in C(G)} (\varphi(g) \to e'(f,g)).
\end{aligned}$$

That is, $\sqcap\varphi = S_1$. Summing up the above results, we get that for each $\varphi \in \mathcal{C}$, $\sqcup\varphi$ and $\sqcap\varphi$ exist and belong to $C(G)$. $\square$

Proposition 3.11. *Let $(G; e, \cdot, 1)$ be an L-ordered group, S be a normal convex L-subgroup of G and let*

$$G/S = \{aS | a \in G\}\ ,\ \bar{e}(aS, bS) = \bigvee_{x \in G} (e(ax, b) \wedge S(x)),$$

$$(aS)(bS) = (ab)S,\ \textit{and}\ (aS)^{-1} = a^{-1}S.$$

Then $(G/S, \bar{e}, \cdot, S)$ is an L-ordered group.

Proof. Let $(G; e, \cdot, 1)$ be an L-ordered group and S be a normal L-subgroup of G. For every $a, y \in G$,

$$\begin{aligned} aS(y) &= S(a^{-1}y) = S(y^{-1}a) = S(a^{-1}ay^{-1}a) \\ &= S(ay^{-1}) = a^{-1}S(y^{-1}) = a^{-1}S(y). \end{aligned}$$

So $aS = a^{-1}S$. Let $a, a', y \in G$. If $aS(y) = a'S(y)$ then $S(a^{-1}y) = S(a'^{-1}y)$. Also, for every $a, b, a', b' \in G$, if $aS = a'S$ and $bS = b'S$, then

$$\begin{aligned} \bar{e}(aS, bS) = &\bigvee_{x \in G} (e(ax, b) \wedge S(x)) = \bigvee_{x \in G} (e(b^{-1}, x^{-1}a^{-1}) \wedge S(x)) \\ = &\bigvee_{x \in G} (e(xb^{-1}, a^{-1}) \wedge S(x)) = \bigvee_{x \in G} (e(b^{-1}bxb^{-1}, a^{-1}) \wedge S(bxb^{-1})) \\ = &\bigvee_{y \in G} (e(b^{-1}y, a^{-1}) \wedge S(y)) = \bar{e}(b^{-1}S, a^{-1}S) \\ = &\ \bar{e}(bS, aS) \end{aligned}$$

and

$$\begin{aligned} \bar{e}(aS, bS) &= \bigvee_x (e(ax, b) \wedge S(x)) = \bigvee_x (e(a'a'^{-1}ax, b) \wedge S(a^{-1}ax)) \\ &= \bigvee_x (e(a'a'^{-1}ax, b) \wedge S(a'^{-1}ax)), \quad \text{since } aS(ax) = a'S(ax) \\ &= \bigvee_x (e(a'y, b) \wedge S(y)), \quad \text{let } y = a'^{-1}ax \\ &= \bar{e}(a'S, bS). \end{aligned}$$

Hence

$$\bar{e}(aS, bS) = \bar{e}(a'S, bS) = \bar{e}(bS, a'S) = \bar{e}(b'S, a'S) = \bar{e}(a'S, b'S),$$

and so $\bar{e}$ is well-defined. Now, we show that $(G/S, \bar{e})$ is an L-ordered set.
(E1): Let $a \in G$. Then $\bar{e}(aS, aS) = \bigvee_x (e(ax, a) \wedge S(x)) \geq e(a1_G, a) \wedge S(1_G) = 1 \wedge 1 = 1$.
(E2): Let $a, b, c \in G$. Then

$$\begin{aligned}
\bar{e}(aS, bS) \wedge \bar{e}(bS, cS) &= \bigvee_{x \in G} (e(ax, b) \wedge S(x)) \wedge \bigvee_{y \in G} (e(by, c) \wedge S(y)) \\
&= \bigvee_{x \in G} (e(ax, b) \wedge S(x)) \wedge \bigvee_{y \in G} (e(b, cy^{-1}) \wedge S(y)) \\
&= \bigvee_{x \in G} \bigvee_{y \in G} (e(ax, b) \wedge S(x) \wedge e(b, cy^{-1}) \wedge S(y)) \\
&\leq \bigvee_{x,y \in G} (e(ax, cy^{-1}) \wedge S(xy)), \quad \text{by Definition 2.5} \\
&= \bigvee_{xy \in G} (e(axy, c) \wedge S(xy)) \\
&= \bigvee_{z \in G} (e(az, c) \wedge S(z)) = \bar{e}(aS, cS).
\end{aligned}$$

(E3): Let $a, b, c \in G$ such that $\bar{e}(aS, bS) = \bar{e}(bS, aS) = 1$. Then

$$\bigvee_x (e(ax, b) \wedge S(x)) = \bigvee_x (e(bx, a) \wedge S(x)) = 1.$$

Since S is convex, $S(ab^{-1}) \geq S(s) \wedge S(t) \wedge e(x, ab^{-1}) \wedge e(ab^{-1}, t)$ for every $s, t \in G$. Thus,

$$\begin{aligned}
S(ab^{-1}) &\geq \bigvee_{s \in G} \bigvee_{t \in G} (S(s) \wedge S(t) \wedge e(s, ab^{-1}) \wedge e(ab^{-1}, t)) \\
&= \bigvee_{s \in G} (S(s) \wedge e(s, ab^{-1})) \wedge \bigvee_{t \in G} (S(t) \wedge e(ab^{-1}, t)) \\
&= \bigvee_{s \in G} (S(s) \wedge e(a^{-1}s, b^{-1})) \wedge \bigvee_{-t \in G} (S(t^{-1}) \wedge e(b^{-1}t^{-1}, a^{-1})), \text{ by (FOG)} \\
&= \bar{e}(a^{-1}S, b^{-1}S) \wedge \bar{e}(b^{-1}S, a^{-1}S) = \bar{e}(aS, bS) \wedge \bar{e}(bS, aS) = 1.
\end{aligned}$$

So $S(ab^{-1}) = 1$. Since S is normal, for any $x \in G$,

$$\begin{aligned}
bS(x) &= S(b^{-1}x) = S(ab^{-1}xa^{-1}) \geq S(ab^{-1}) \wedge S(xa^{-1}) \\
&= 1 \wedge S(xa^{-1}) = S(x^{-1}xa^{-1}x) = S(a^{-1}x) = aS(x).
\end{aligned}$$

Hence $aS = bS$. Therefore, $(G/S, e')$ is an L-ordered set. Finally, since for every $a, b, c \in G$,

$$\begin{aligned}\bar{e}((aS).(cS),(bS).(cS)) &= \bar{e}(acS, bcS) = \bigwedge_{x\in G}(e(acx, bc)\wedge S(x))\\ &= \bigwedge_{x\in G}(e(acxc^{-1}, b)\wedge S(cxc^{-1})), \quad \text{by Definition 2.5}\\ &= \bigwedge_{x'\in G}(e(ax', b)\wedge S(x')), \quad (\text{Let } x' = cxc^{-1})\\ &= \bar{e}(aS, bS)\end{aligned}$$

and

$$\begin{aligned}\bar{e}((cS).(aS),(cS).(bS)) &= \bar{e}(caS, cbS) = \bigwedge_{x\in G}(e(cax, cb)\wedge S(x))\\ &= \bigwedge_{x\in G}(e(ax, b)\wedge S(x)) = \bar{e}(aS, bS),\end{aligned}$$

we get that $(G/S, e', ., S)$ is an L-ordered group. □

Lemma 3.12. *Let $(G; e_1, \cdot, 1_G)$ and $(H; e_2, \cdot, 1_H)$ be two L-ordered groups, $f : G \to H$ be an L-ordered group homomorphism, and $Kerf \in L^{G\times G}$ be defined by*

$$(Kerf)(x, y) = e_2(f(x), f(y)) \wedge e_2(f(y), f(x)).$$

If we define $N_f \in L^G$, for any $x \in G$, by

$$\begin{aligned}N_f(x) &= (Kerf)(1_G, x) = e_2(f(1_G), f(x)) \wedge e_2(f(x), f(1_G))\\ &= e_2(1_H, f(x)) \wedge e_2(f(x), 1_H),\end{aligned}$$

then for all $x \in G$,

$$N_f(x) = e_2(1_H, f(x) \wedge f(x^{-1})).$$

Proof. Let $x \in G$. Then

$$\begin{aligned}N_f(x) &= e_2(1_H, f(x)) \wedge e_2(f(x), 1_H) = e_2(1_H, f(x)) \wedge e_2(1_H, f(x)^{-1})\\ &= e_2(1_H, f(x)) \wedge e_2(1_H, f(x^{-1}))\\ &= e_2(1_H, f(x) \wedge f(x^{-1})).\end{aligned}$$

□

Proposition 3.13. *Let $(G; e_1, \cdot, 1_G)$ and $(H; e_2, \cdot, 1_H)$ be two L-ordered groups and $f : G \to H$ be an L-ordered group homomorphism. Then N_f is a normal convex L-subgroup of G and there is a one to one and onto L-ordered homomorphism from G/N_f to $Im(f)$.*

Proof. Clearly $N_f(1_G) = e_2(1_H, f(1_G)) \wedge e_2(f(1_G), 1_H) = 1 \wedge 1 = 1$.
Now, let $x, y \in G$. Then, by Proposition 2.4 (i),

$$\begin{aligned}
&N_f(x) \wedge N_f(y) \\
&\quad = e_2(1_H, f(x)) \wedge e_2(f(x), 1_H) \wedge e_2(1_H, f(y)) \wedge e_2(f(y), 1_H) \\
&\quad = e_2(1_H, f(x)) \wedge e_2(f(x)f(y), f(y)) \wedge e_2(f(x), f(x)f(y)) \wedge e_2(f(y), 1_H) \\
&\quad = e_2(1_H, f(x)) \wedge e_2(f(xy), f(y)) \wedge e_2(f(x), f(xy)) \wedge e_2(f(y), 1_H) \\
&\quad \leq e_2(1_H, f(xy)) \wedge e_2(f(xy), 1_H), \quad \text{by (E3)} \\
&\quad = N_f(xy).
\end{aligned}$$

Moreover, for any $x \in G$, by Proposition 2.4 (ii),

$$\begin{aligned}
N_f(x) &= e_2(1_H, f(x)) \wedge e_2(f(x), 1_H) = e_2((f(x))^{-1}, 1_H) \wedge e_2(1_H, (f(x))^{-1}) \\
&= e_2(f(x^{-1}), 1_H) \wedge e_2(1_H, f(x^{-1})) \\
&= N_f(x^{-1}).
\end{aligned}$$

Hence N_f is an L-subgroup of G. Also N_f is normal, because for every $x, y \in G$:

$$\begin{aligned}
N_f(x) &= e_2(1_H, f(x)) \wedge e_2(f(x), 1_H) \\
&= e_2(f(y^{-1}y), f(x)) \wedge e_2(f(x), f(y^{-1}y)) \\
&= e_2(f(y)^{-1}f(y), f(x)) \wedge e_2(f(x), f(y)^{-1}f(y)) \\
&= e_2(1_H, f(y)f(x)f(y)^{-1}) \wedge e_2(f(y)f(x)f(y)^{-1}, 1_H), \quad \text{by (FOG)} \\
&= e_2(1_H, f(yxy^{-1})) \wedge e_2(f(yxy^{-1}), 1_H) \\
&= N_f(yxy^{-1}).
\end{aligned}$$

Now, we prove that N_f is convex. Let $a, x \in G$. Since f is monotone, we get that

$$\begin{aligned}
&N_f(x) \wedge e_1(1_G, a) \wedge e_1(a, x)\\
&\qquad = e_2(1_H, f(x)) \wedge e_2(f(x), 1_H) \wedge e_1(1_G, a) \wedge e_1(a, x)\\
&\qquad \leq e_2(1_H, f(x)) \wedge e_2(f(x), 1_H) \wedge e_2(1_H, f(a)) \wedge e_2(f(a), f(x))\\
&\qquad \leq e_2(1_H, f(x)) \wedge e_2(f(a), 1_H) \wedge e_2(1_H, f(a)), \quad \text{by (E2)}\\
&\qquad \leq e_2(1_H, f(a)) \wedge e_2(f(a), 1_H)\\
&\qquad = \quad N_f(a).
\end{aligned}$$

Hence N_f is convex. Define $\phi : G/N_f \to Imf$ by $\phi(aN_f) = f(a)$, for any $a \in G$. It is obvious that ϕ is a group homomorphism, one to one and onto. It is enough to prove that φ is monotone. For every $a, b \in G$

$$\begin{aligned}
&\bar{e}(aN_f, bN_f) = \bigvee_x (e_1(ax, b) \wedge N_f(x))\\
&\quad = \bigvee_x (e_1(ax, b) \wedge e_2(1_H, f(x)) \wedge e_2(f(x), 1_H))\\
&\quad = \bigvee_x (e_1(x, a^{-1}b) \wedge e_2(1_H, f(x)) \wedge e_2(f(x), 1_H)), \quad \text{by Proposition 2.4}\\
&\quad \leq \bigvee_x (e_2(f(x), f(a^{-1}b)) \wedge e_2(1_H, f(x)) \wedge e_2(f(x), 1_H)), \text{ since } f \text{ is monotone}\\
&\quad \leq \bigvee_x e_2(1_H, f(a^{-1}b)) \wedge e_2(f(x), 1_H), \quad \text{by (E2)}\\
&\quad \leq e_2(1_H, f(a^{-1}b)) = e_2(1_H, f(a^{-1})f(b)) = e_2(1_H, f(a)^{-1}f(b))\\
&\quad = e_2(f(a), f(b)) = e_2(\phi(aN_f), \phi(bN_f))).
\end{aligned}$$

Hence, $\bar{e}(aN_f, bN_f) \leq e_2(\phi(aN_f), \phi(bN_f))$ and so ϕ is monotone. Therefore, ϕ is an L-ordered group isomorphism. □

Theorem 3.14. *Let S be a convex L-lattice subgroups of G. Then G/S is a distributive weak L-lattice ordered group.*

Proof. By Proposition 3.11, G/S is an L-ordered group. Let S be a convex L-lattice subgroups of G and $x, y \in G$. First we show that $xS \vee yS = (x \vee y)S$. It is clear that $\bar{e}(xS, (x \vee y)S) = \bar{e}(yS, (x \vee y)S) = 1$. Now, let

$\bar{e}(xS, dS) \wedge \bar{e}(yS, dS) = 1$. Then

$$\begin{aligned}
1 &= \bigvee_{a\in G} (S(a)\wedge e(xa, d))\wedge \bigvee_{b\in G} (S(b)\wedge e(yb, d)) \\
&\leq \bigvee_{a\in G} \bigvee_{b\in G} (S(a)\wedge S(b)\wedge e(xa, d)\wedge e(yb, d)) \\
&= \bigvee_{a\in G} \bigvee_{b\in G} (S(a)\wedge S(b)\wedge e(x(a\wedge b), d)\wedge e(y(a\wedge b), d)), \text{ by Proposition 2.4(iii)} \\
&= \bigvee_{a\in G} \bigvee_{b\in G} (S(a)\wedge S(b)\wedge e(x(a \wedge b)\vee y(a\wedge b), d)), \quad \text{by Proposition 2.2(ii)} \\
&= \bigvee_{a\in G} \bigvee_{b\in G} (S(a\wedge b)\wedge e((x\vee y)(a\wedge b), d)), \text{ since } S \text{ is } L\text{-lattice subgroup} \\
&\leq \bigvee_{c\in G} (S(c)\wedge e((x\vee y)(c), d)) = \bar{e}((x\vee y)S, dS).
\end{aligned}$$

So $\bar{e}((x \vee y)S, dS) = 1$. Therefore, $(x \vee y)S = xS \vee yS$. By a similar way, we can show that $(x \wedge y)S = xS \wedge yS$. So G/S is a weak L-lattice ordered group. Now, for every $x, y, z \in G$ we have

$$\begin{aligned}
xS \wedge (yS \vee zS) &= xS \wedge (y \vee z)S = (x \wedge (y \vee z))S \\
&= ((x \wedge y) \vee (x \wedge z))S = (xS \wedge yS) \vee (xS \wedge zS).
\end{aligned}$$

Therefore, G/S is a distributive weak L-lattice ordered group. □

Theorem 3.15. *Let $NC(G)$ be the set of all normal convex L-lattice subgroups of G and $\mathcal{N} = \{\varphi \in NC(G) \mid \bigvee_{f\in NC(G)} \varphi(f) = 1\}$. Consider the L-ordered relation e' in Theorem 3.10. Then for each $\varphi \in \mathcal{N}$, $\sqcap\varphi$ and $\sqcap\varphi$ exist and belong to $NC(G)$.*

Proof. Let $\varphi \in \mathcal{N}$. By Theorem 3.10, $\sqcap\varphi, \sqcup\varphi \in C(G)$. It suffices to show that $\sqcap\varphi, \sqcup\varphi$ are normal. Let $x, y \in G$. Then

$$S_0(xyx^{-1}) = \bigvee_{f\in NC(G)} (\varphi(f) \wedge f(xyx^{-1})) = \bigvee_{f\in NC(G)} (\varphi(f) \wedge f(y)) = S_0(y).$$

In a similar way, we can show that $S_1(xyx^{-1}) = S_1(y)$. □

Theorem 3.16. *Let C be a normal convex L-lattice subgroups of G and A be an L-subgroup of G, where $C \leq A$. Suppose that $A/C \in L^{G/C}$ is defined by*

$$(A/C)(xC) = \bigvee_{a \in G, aC = xC} A(a).$$

Then A/C is an L-subgroup of G/C.

Proof. By Proposition 3.11, G/C is an L-ordered group. Since $C = 1_G C$, $(A/C)(1_G C) = \bigvee_{a \in G, aC = 1_G C} A(a) \geq A(1_G) = 1$. For any $x \in G$,

$$\begin{aligned} (A/C)(xC) &= \bigvee_{a \in G, aC = xC} A(a) = \bigvee_{a \in G, a^{-1}C = x^{-1}C} A(a^{-1}) \\ &= \bigvee_{a' \in G, a'C = x^{-1}C} A(a') = (A/C)(x^{-1}C). \end{aligned}$$

Let $x, y \in G$. Then

$$\begin{aligned} (A/C)(xC) \wedge (A/C)(yC) &= \bigvee_{a \in G, aC = xC} A(a) \wedge \bigvee_{b \in G, bC = yC} A(b) \\ &= \bigvee_{a \in G, aC = xC} \bigvee_{b \in G, bC = yC} (A(a) \wedge A(b)) \\ &\leq \bigvee_{a \in G, aC = xC} \bigvee_{b \in G, bC = yC} (A(ab)), \quad \text{since } A \text{ is an } L\text{-subgroup of } G. \\ &\leq \bigvee_{ab \in G, aCbC = xCyC} (A(ab)) \\ &= \bigvee_{ab \in G, abC = xyC} (A(ab)) = (A/C)(xyC) \\ &= (A/C)(xCyC). \end{aligned}$$

So A/C is an L-subgroup of G/C. □

4 Conclusion

In this paper, the concepts of a convex L-subgroup and a convex L-lattice subgroup in L-ordered groups, where L is a frame, are defined and some properties are investigated. The convex L-subgroup generated by an L-subgroup is characterized. It is proved that the set of all convex L-lattice subgroups is an L-complete lattice on height one. Finally, using a normal convex L-subgroup, an L-ordered group constructed and some related results are investigated.

Acknowledgement

The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the editor and referees.

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