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On the property U-(G-PWP) of acts

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Abstract. In this paper first of all we introduce Property U-(G-PWP) of acts, which is an extension of Condition (G-PWP) and give some general properties. Then we give a characterization of monoids when this property of acts implies some others. Also we show that the strong faithfulness (P-cyclicity) and (P-)regularity of acts imply the property U-(G-PWP). Finally, we give a necessary and sufficient condition under which all cyclic (finitely generated) right acts or all (strongly, $\Re$-) torsion free cyclic (finitely generated) right acts satisfy Property U-(G-PWP).

1 Introduction

Throughout this paper S will denote a monoid and $\mathbb{N}$ will stand for the set of natural numbers. We refer the reader to [5] and [6] for basic definitions and terminology relating to semigroups and acts over monoids, to [7] and [9] for definitions and results on flatness which are used here.

We use the following abbreviations,

weak pullback flatness = (WPF).

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weak kernel flatness = WKF.

principal weak kernel flatness = PWKF.

translation kernel flatness = TKF.

weak homoflatness = (WP).

principal weak homoflatness = (PWP).

weak flatness = WF.

principal weak flatness = PWF.

2 General properties

In this section first of all we introduce Property U-(G-PWP) of acts and give some general properties.

We recall from [9] that a right S-act A_S satisfies *Condition* (PWP) if $as = a's$, for $a, a' \in A_S$, and $s \in S$, implies that there exist $a'' \in A_S$ and $u, v \in S$, such that $a = a''u$, $a' = a''v$, and $us = vs$. Also we recall from [1] that a right S-act A_S satisfies *Condition* (G-PWP) if $as = a's$, for $a, a' \in A_S$ and $s \in S$, implies that there exist $a'' \in A_S$, $u, v \in S$ and $n \in \mathbb{N}$ such that $a = a''u$, $a' = a''v$, and $us^n = vs^n$. It is obvious that Condition PWP implies Condition $G - PWP$ but not the converse (see [1, Example 2.2]).

Definition 2.1. Let S be a monoid. A right S-act A_S satisfies Property U-(G-PWP) if there exists a family $\{B_i \mid i \in I\}$ of subacts of A_S such that $A = \bigcup_{i \in I} B_i$ and B_i, $i \in I$ satisfies Condition (G-PWP).

If I is a proper right ideal of S, then

$$A_S = S \coprod^{I} S = \{(\alpha, x) \mid \alpha \in S \setminus I\} \mathbin{\dot\cup} I \mathbin{\dot\cup} \{(\beta, y) \mid \beta \in S \setminus I\}$$

with

$$(\alpha, z)s = \begin{cases} (\alpha s, z) & \alpha s \notin I \\ \alpha s & \text{otherwise} \end{cases}$$

for every $\alpha \in S \setminus I, s \in S$ and $z \in \{x, y\}$ is a right S-act.

Recall from [6] that a right S-act A_S is said to be *decomposable* if there exist two subacts $B_S, C_S \subseteq A_S$ such that $A_S = B_S \cup C_S$ and $B_S \cap C_S = \emptyset$. Otherwise, A_S is called *indecomposable.*

Also recall that a right S-act A_S is *locally cyclic* if every finitely generated subact of A_S is contained within a cyclic subact of A_S.

Theorem 2.2. *Let S be a monoid. Then*

(1) *Θ_S and S_S satisfy U-(G-PWP).*

(2) *Every right S-act satisfying Condition (G-PWP) satisfies U-(G-PWP).*

(3) *If $\{B_i \mid i \in I\}$ is a family of subacts of a right S-act A_S such that for every $i \in I$, B_i satisfies U-(G-PWP), then $\bigcup_{i \in I} B_i$ satisfies U-(G-PWP).*

(4) *A right S-act A_S satisfies U-(G-PWP) if and only if for every $a \in A_S$ there exists a subact B of A_S such that $a \in B$ and B satisfies Condition (G-PWP).*

(5) *If A_S is a right S-act and I is a non-empty set such that B_i is a subact of A_S and satisfies Condition (G-PWP) for every $i \in I$, then the right S-act $\bigcup_{i \in I} B_i$ satisfies U-(G-PWP).*

(6) *For every proper right ideal I of S, $A_S = S \coprod^I S$ satisfies U-(G-PWP), where it is indecomposable and is generated exactly by two elements, but it is not locally cyclic.*

(7) *Every cyclic right S-act A_S satisfies Condition (G-PWP) if and only if A_S satisfies U-(G-PWP).*

Proof. The proofs of (1)-(5) and (7) are straightforward.

(6) Let I be a proper right ideal of S and

$$A_S = S \coprod^{I} S = \{(l, x) |\; l \in S \setminus I\} \,\dot\cup\, I \,\dot\cup\, \{(t, y) |\; t \in S \setminus I\},$$

$$B = \{(l, x) |\; l \in S \setminus I\} \,\dot\cup\, I, \quad C = \{(t, y) |\; t \in S \setminus I\} \,\dot\cup\, I.$$

It is easy to show that B and C are cyclic subacts of A_S such that

$$B = (1, x)S \cong S_S \cong (1, y)S = C,$$

$$A_S = \langle (1, x), (1, y) \rangle = (1, x)S \cup (1, y)S = B \cup C.$$

Now, since S_S satisfies Condition (G-PWP), the subacts B and C satisfy Condition (G-PWP), too, and so $A_S = B \cup C$ satisfies Property U-(G-PWP).

Also since

$$A_S = (1,x)S \cup (1,y)S,\ (1,x)S \cap (1,y)S = I,$$

it is easy to show that A_S is indecomposable, but it is not locally cyclic. $\square$

3 Characterization of monoids by U-(G-PWP) of right acts

We know that Condition (G-PWP) implies torsion freeness, but from the following example we can see that Property U-(G-PWP) of acts does not imply torsion freeness in general. So it is natural to ask for monoids over which U-(G-PWP) of acts implies torsion freeness and other properties. In this section we answer these questions.

We recall from [6] that a right S-act A_S is *torsion free* if for $a, b \in A_S$ and a right cancellable element c of S, the equality $ac = bc$ implies that $a = b$.

Example 3.1. Let $(\mathbb{N}, .)$ be the monoid of natural numbers under multiplication, and consider $A_S = \mathbb{N} \coprod^{2\mathbb{N}} \mathbb{N}$. Then A_S satisfies U-(G-PWP), by Theorem 2.2. But $(1,x) \neq (1,y)$ and $(1,x)2 = 2 = (1,y)2$, so A_S is not torsion free.

This example shows also that for a commutative monoid S, there may exists an indecomposable right S-act A_S generated by exactly two elements, such that A_S satisfies U-(G-PWP), but it is neither locally cyclic nor torsion free.

Theorem 3.2. *For any monoid S the following statements are equivalent*:

(1) *all right S-acts are torsion free*;

(2) *all right S-acts satisfying U-(G-PWP) are torsion free*;

(3) *all finitely generated right S-acts satisfying U-(G-PWP) are torsion free*;

(4) *all right S-acts generated by at most two elements satisfying U-(G-PWP) are torsion free*;

(5) *all right S-acts generated by exactly two elements satisfying U-(G-PWP) are torsion free*;

(6) *all indecomposable right S-acts satisfying U-(G-PWP) are torsion free*;

(7) *all finitely generated indecomposable right S-acts satisfying U-(G-PWP) are torsion free*;

(8) *all indecomposable right S-acts generated by at most two elements satisfying U-(G-PWP) are torsion free*;

(9) *all indecomposable right S-acts generated by exactly two elements satisfying U-(G-PWP) are torsion free*;

(10) *all right cancellable elements of S are right invertible.*

Proof. The implications (1)⇒(2)⇒ (3)⇒(4)⇒(5), (1)⇒ (6)⇒(7)⇒(8)⇒(9), and (5)⇒(9) are obvious.

(9)⇒(10) Let $c \in S$ be a right cancellable element such that $cS \neq S$ and consider $A_S = S \coprod^{cS} S$. Obviously, A_S is indecomposable which is generated by two elements $(1, x)$ and $(1, y)$. So A_S satisfies U-(G-PWP), by Theorem 2.2, and so, by assumption, it is torsion free. Hence the equality $(1, x)c = c = (1, y)c$ implies $(1, x) = (1, y)$, which is a contradiction. Thus $cS = S$ and so c is right invertible, as required.

(10)⇒(1) It is clear by [6, IV, 6.1]. □

Recall from [6] that a right S-act A_S satisfies *Condition* (P) if $as = a't$, for $a, a' \in A_S$, and $s, t \in S$, there exist $a'' \in A_S$, and $u, v \in S$ such that $a = a''u$, $a' = a''v$, and $us = vt$. Also we recall from [2] that a right S-act A_S satisfies *Condition* (P') if $as = a't$ and $sz = tz$, for $a, a' \in A_S$, and $s, t, z \in S$, imply that there exist $a'' \in A_S$ and $u, v \in S$, such that $a = a''u$, $a' = a''v$, and $us = vt$. Also recall from [9] that a right S-act A_S satisfies *Condition* (WP) if $af(s) = a'f(t)$, for $a, a' \in A_S$, $s, t \in S$, and homomorphism $f : {}_S(Ss \cup St) \to {}_SS$, implies that there exist $a'' \in A_S$, $u, v \in S$, and $s', t' \in \{s, t\}$ such that $f(us') = f(vt')$, $a \otimes s = a'' \otimes us'$, and $a' \otimes t = a'' \otimes vt'$ in $A_S \otimes {}_S(Ss \cup St)$.

Theorem 3.3. *For any monoid S the following statements are equivalent*:

(1) *all right S-acts satisfying U-(G-PWP) are WPF*;

(2) *all right S-acts satisfying U-(G-PWP) are WKF*;

(3) *all right S-acts satisfying U-(G-PWP) are $PWKF$*;

(4) *all right S-acts satisfying U-(G-PWP) are TKF*;

(5) *all right S-acts satisfying U-(G-PWP) satisfy Condition (P)*;

(6) *all right S-acts satisfying U-(G-PWP) satisfy Condition (WP)*;

(7) *all right S-acts satisfying U-(G-PWP) satisfy Condition (PWP)*;

(8) *all right S-acts satisfying U-(G-PWP) satisfy Condition (P')*;

(9) *all right S-acts satisfying U-(G-PWP) satisfy Condition* (G-PWP);
(10) *S is a group.*

Proof. The implications (1)⇒(2)⇒(3)⇒(4)⇒(7)⇒(9), (1)⇒(5)⇒(6)⇒(7), and (5)⇒(8)⇒(7) are obvious.

(9)⇒(10) Suppose for $s \in S$, $sS \neq S$. Consider $A_S = S \coprod^{sS} S$. By Theorem 2.2, A_S satisfies U-(G-PWP), and so by the assumption, A_S satisfies Condition (G-PWP). Thus the equality $(1,x)s = (1,y)s$ implies that there exist $a \in A_S$, $u, v \in S$, and $n \in \mathbb{N}$ such that $(1,x) = au$, $(1,y) = av$, and $us^n = vs^n$. Then the equalities $(1,x) = au$ and $(1,y) = av$ imply, respectively, that there exist $l, l' \in S \setminus I$ such that $a = (l,x)$ and $a = (l',y)$, which is a contradiction. Thus $sS = S$ and so S is a group, as required.

(10)⇒(1) It is true by [8, Proposition 9]. □

Notice that by the proof of Theorem 3.3 and also (6) of Theorem 2.2, Theorem 3.3 is also true for finitely generated acts, acts generated by at most two elements, and acts generated exactly by two elements.

Theorem 3.4. *For any monoid S the following statements are equivalent*:
(1) *all right S-acts satisfying U-(G-PWP) are free*;
(2) *all right S-acts satisfying U-(G-PWP) are projective generator*;
(3) *all right S-acts satisfying U-(G-PWP) are projective*;
(4) *all right S-acts satisfying U-(G-PWP) are strongly flat*;
(5) $S = \{1\}$.

Proof. The implications (1)⇒(2)⇒(3)⇒(4) and (5)⇒(1) are obvious.

(4)⇒(5) By the assumption, all right S-acts satisfying U-(G-PWP) are WPF. So S is a group, by Theorem 3.3. Thus all right S-acts satisfy Condition (PWP), by [8, Proposition 9], and so all right S-acts satisfy Condition (G-PWP) and so they satisfy U-(G-PWP). Hence, by the assumption, all right S-acts are strongly flat and so $S = \{1\}$ by [6], IV, 10.5]. □

Notice that, by the proof of Theorem 3.3 and also (6) of Theorem 2.2, Theorem 3.4 is also true for all finitely generated acts, acts generated by at most two elements, and acts generated exactly by two elements.

We recall from [6] that a monoid S is called *regular* if for every $s \in S$, there exists $x \in S$ such that $s = sxs$.

Theorem 3.5. *For any monoid S the following statements are equivalent*:

(1) *all right S-acts are PWF;*

(2) *all right S-acts satisfying U-(G-PWP) are PWF*;

(3) *all finitely generated right S-acts satisfying U-(G-PWP) are PWF*;

(4) *all right S-acts generated by at most two elements satisfying U-(G-PWP) are PWF*;

(5) *all right S-acts generated by exactly two elements satisfying U-(G-PWP) are PWF*;

(6) *S is regular.*

Proof. The implications (1)⇒(2)⇒(3)⇒(4)⇒(5) are obvious.

(5)⇒(6) Let $s \in S$. If $sS = S$, then it is obvious that s is regular. Thus we suppose that $sS \neq S$ and let $A_S = S \coprod^{sS} S$. By Theorem 2.2, A_S satisfies U-(G-PWP), and so, by the assumption, A_S is principally weakly flat. Thus by ([6], III, 12.19), sS is left stabilizing, and so there exists $l \in sS$ such that $s = ls$. Hence there exists $x \in S$ such that $l = sx$, and so $s = ls = sxs$, that is, S is regular.

(6)⇒(1) It is true by [6, IV, 6.6] □

We recall from [11] that a right S-act A_S is called *GP-flat* if $a \otimes s = a' \otimes s$ in $A_S \otimes {}_SS$, for $a, a' \in A_S$ and $s \in S$, implies that there exists $n \in \mathbb{N}$ such that $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_SSs^n$. Also, a monoid S is called *generally regular* if for every $s \in S$ and $n \in \mathbb{N}$, there exists $x \in S$ such that $s^n = sxs^n$.

Theorem 3.6. *For any monoid S the following statements are equivalent*:

(1) *all right S-acts are GP-flat*;

(2) *all right S-acts satisfying U-(G-PWP) are GP-flat*;

(3) *all finitely generated right S-acts satisfying U-(G-PWP) are GP-flat*;

(4) *all right S-acts generated by at most two elements satisfying U-(G-PWP) are GP-flat*;

(5) *all right S-acts generated by exactly two elements satisfying U-(G-PWP) are GP-flat*;

(6) *S is generally regular.*

Proof. The implications (1)⇒(2)⇒(3)⇒(4)⇒(5) are obvious.

(5)⇒(6) Let $s \in S$. If $sS = S$, then it is obvious that s is generally regular. Thus we suppose that $sS \neq S$ and let $A_S = S \coprod^{sS} S$. By Theorem 2.2, A_S satisfies U-(G-PWP), and so, by the assumption, A_S is GP-flat.

Thus, by ([11], Lemma 2.4), for $s \in sS$ there exist $n \in \mathbb{N}$ and $j \in sS$ such that $s^n = js^n$. Hence there exists $x \in S$ such that $j = sx$, that is, $s^n = sxs^n$.

(6)⇒(1) Since S is generally regular, by [11, Theorem 3.4], all right S-acts are GP-flat, and so all right S-acts satisfying U-(G-PWP) are GP-flat, as required. □

We recall from [6] that a right S-act A_S is *divisible* if for every element $a \in A_S$ and any left cancellable element $c \in S$ there exists $b \in A_S$ such that $a = bc$.

Theorem 3.7. *For any monoid S the following statements are equivalent*:

(1) *all right S-acts are divisible;*

(2) *all right S-acts satisfying U-(G-PWP) are divisible*;

(3) *all finitely generated right S-acts satisfying U-(G-PWP) are divisible*;

(4) *all right S-acts generated by at most two elements satisfying U-(G-PWP) are divisible*;

(5) *all right S-acts generated by exactly two elements satisfying U-(G-PWP) are divisible*;

(6) *all left cancellable elements of S are left invertible.*

Proof. The implications (1)⇒(2)⇒(3)⇒(4)⇒(5) are obvious.

(5)⇒(6) Let $c \in S$ be any left cancellable element. If $Sc = S$, then it is obvious that c is left invertible. Thus we suppose that $Sc \neq S$ and let $A_S = S \coprod^{Sc} S$. By Theorem 2.2, A_S satisfies U-(G-PWP), and so, by the assumption, A_S is divisible. Since $(1, x) \in A_S$, there exists $\alpha \in A_S$ such that $(1, x) = \alpha c$. Hence for $l \in S \setminus Sc$ we have $\alpha = (l, x)$ and so $(1, x) = \alpha c = (l, x)c = (lc, x) \Rightarrow 1 = lc$. Thus c is left invertible.

(6)⇒(1) It is true by [6, III, 2.2]. □

We recall from [6] that a right S-act A_S is called *simple* if it contains no subacts other than A_S itself, and A_S is called *completely reducible* if it is a disjoint union of simple subacts.

Theorem 3.8. *For any monoid S the following statements are equivalent*:

(1) *all right S-acts are completely reducible*;

(2) *all right S-acts satisfying U-(G-PWP) are completely reducible*;

(3) *all finitely generated right S-acts satisfying U-(G-PWP) are completely reducible*;

(4) *all right S-acts generated by at most two elements satisfying U-(G-PWP) are completely reducible*;

(5) *all right S-acts generated by exactly two elements satisfying U-(G-PWP) are completely reducible*;

(6) *S is a group.*

Proof. The implications (1)⇒(2)⇒(3)⇒(4)⇒(5) are obvious.

(5)⇒(6) Let $s \in S$. If $sS \neq S$ then let $A_S = S \coprod^{sS} S$. By Theorem 2.2, A_S satisfies U-(G-PWP) and so, by the assumption, A_S is completely reducible. Hence A_S is a disjoint union of simple subacts. But we know that A_S is indecomposable. Hence A_S is simple such that sS is a subact other than A_S, which is contradiction. So $sS = S$ and S is a group, as required.

(6)⇒(1) It is true by [6, I, 5.34]. □

Notice that, in the previous theorem, $A_S = \theta_1 \dot{\cup} \theta_2$ satisfies U-(G-PWP) right S-act generated by at most two elements, which is completely reducible.

We recall from [6] that a right S-act A_S is *faithful* if for $s, t \in S$ the equality $as = at$, for all $a \in A$, implies that $s = t$, and A_S is *strongly faithful* if for $s, t \in S$ the equality $as = at$, for some $a \in A$, implies that $s = t$. It is obvious that every strongly faithful right $S-$act is faithful.

Notice that Property U-(G-PWP) of cyclic acts does not imply faithfulness in general. For, if S is a non-trivial monoid, then it is obvious that right cyclic S-act Θ_S satisfies U-(G-PWP), but it is not faithful, because $|S| > 1$. Thus Property U-(G-PWP) of cyclic acts does not imply strong faithfulness in general.

Theorem 3.9. *For any monoid S the following statements are equivalent*:

(1) *all right S-acts are (strongly) faithful;*

(2) *all right S-acts satisfying U-(G-PWP) are (strongly) faithful*;

(3) *all finitely generated right S-acts satisfying U-(G-PWP) are (strongly) faithful*;

(4) *all right S-acts generated by at most two elements satisfying U-(G-PWP) are (strongly) faithful*;

(5) *all right S-acts generated by exactly two elements satisfying U-(G-PWP) are (strongly) faithful*;

(6) $S = \{1\}$.

Proof. The implications (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$ (4)$\Rightarrow$(5) are obvious.

(5)$\Rightarrow$(6) For any monoid S, $A_S = \theta_1 \dot{\cup} \theta_2$ is a right S-act, generated by exactly two elements, satisfies U-(G-PWP). So, by the assumption, A_S is (strongly) faithful. If $S \neq \{1\}$ then there exist $s, t \in S$ such that $s \neq t$. But it is obvious that for every $a \in A_S$, $as = at$, which is a contradiction. So $S = \{1\}$, as required.

(6)$\Rightarrow$(1) It is obvious. $\square$

We recall from [13] that a right S-act A_S is *strongly torsion free* if the equality $as = bs$, for all $a, b \in A_S$ and all $s \in S$, implies $a = b$.

Theorem 3.10. *For any monoid S the following statements are equivalent*:

(1) *all right S-acts are strongly torsion free*;

(2) *all right S-acts satisfying U-(G-PWP) are strongly torsion free*;

(3) *all finitely generated right S-acts satisfying U-(G-PWP) are strongly torsion free*;

(4) *all right S-acts generated by at most two elements satisfying U-(G-PWP) are strongly torsion free*;

(5) *all right S-acts generated by exactly two elements satisfying U-(G-PWP) are strongly torsion free*;

(6) *S is a group.*

Proof. The implications (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(5) are obvious.

(5)$\Rightarrow$(6) Let $s \in S$ be such that $sS \neq S$ and suppose $A_S = S \coprod^{sS} S$. By Theorem 2.2, A_S satisfies U-(G-PWP) and so, by the assumption, A_S is strongly torsion free. Now let

$$B = \{(l, x) |\ l \in S \setminus sS\} \dot{\cup} sS \cong S_S \cong \{(t, y) |\ t \in S \setminus sS\} \dot{\cup} sS = C.$$

Then, by [13, Proposition 2.1], B as a subact of A_S is strongly torsion free, and so S_S is strongly torsion free. Hence S is right cancellative by [13, Proposition 2.1]. But in the case of cancellablity of S, strong torsion freeness and torsion freeness coincide. So, by Theorem 3.2, every right cancellable element of S is right invertible, hence $sS = S$, which is a contradiction. Thus for every $s \in S$, $sS = S$ and so S is a group, as required.

(6)$\Rightarrow$(1) It is true by [13, Theorem 6.1]. $\square$

Theorem 3.11. *Let S be a right cancellative monoid. Then the following statements are equivalent*:

(1) *all right S-acts are flat*;

(2) *all right S-acts satisfying U-(G-PWP) are flat*;

(3) *all finitely generated right S-acts satisfying U-(G-PWP) are flat*;

(4) *all right S-acts generated by at most two elements satisfying U-(G-PWP) are flat*;

(5) *all right S-acts generated by exactly two elements satisfying U-(G-PWP) are flat*;

(6) *all right S-acts are WF*;

(7) *all right S-acts satisfying U-(G-PWP) are WF*;

(8) *all finitely generated right S-acts satisfying U-(G-PWP) are WF*;

(9) *all right S-acts generated by at most two elements satisfying U-(G-PWP) are WF*;

(10) *all right S-acts generated by exactly two elements satisfying U-(G-PWP) are WF*;

(11) *S is a group.*

Proof. The implications (1)⇒(2)⇒(3)⇒(4)⇒(5)⇒(10), also the implications (1)⇒(6)⇒(7)⇒(8)⇒(9)⇒(10) are obvious.

(10)⇒(11) Since, for right cancellative monoids, torsion freeness and strong torsion freeness of acts coincide, and also weak flatness implies torsion freeness, thus all right S-acts generated by exactly two elements satisfying U-(G-PWP) are strongly torsion free, and so S is a group, by Theorem 3.10.

(11)⇒(1) It is true by [8, Proposition 9]. □

We recall from [6] that an element s of a monoid S is called *left e-cancellable* for an idempotent $e \in S$ if $s = se$ and $ker\lambda_s \leq ker\lambda_e$. A monoid S is called *right PP* if every element $s \in S$ is left e-cancellable for some idempotent $e \in S$. Also we recall from [6] that a right S-act A_S satisfies *Condition* (E) if $as = at$, for $a \in A_S$ and $s, t \in S$, implies that there exist $a' \in A_S$ and $u \in S$ such that $a = a'u$ and $us = ut$.

Notice that U-(G-PWP) of cyclic acts does not imply regularity. Because, if $S = \{0, 1, x\}$ with $x^2 = 0$, then S is not right PP, and so, by [6, III, 19.3] and [6, IV, 11.15], S_S is not a regular act, but it satisfies U-(G-PWP). Now it is natural to ask for monoids S for which U-(G-PWP) of their acts implies regularity.

Lemma 3.12. *Let S be a monoid and A a right S-act. Then*

(1) *If S is right PP and A satisfies Condition (E), then aS satisfies Condition (E) for every $a \in A$.*

(2) *If aS satisfies Condition (E) for every $a \in A$, then A satisfies Condition (E).*

Proof. (1) Suppose that A satisfies Condition (E) and let $as = at$, for $a \in A$ and $s, t \in S$. Then there exist $a' \in A$ and $u \in S$ such that $a = a'u$ and $us = ut$. Since S is right PP, there exists $e \in E(S)$ such that $ker\lambda_e = ker\lambda_u$. Thus $es = et$, $u = ue$, and so $a = ae$. Hence aS satisfies Condition (E), as required.

(2) Let $as = at$, for $a \in A_S$ and $s, t \in S$. Since aS satisfies Condition (E), there exist $w_1, w_2 \in S$ such that $a = (aw_1)w_2$ and $w_2s = w_2t$. If $w_1w_2 = u$, then $a = au$ and $us = ut$, and so A_S satisfies Condition (E). □

Recall from [6] that a monoid S is called *left collapsible* if for any $p, q \in S$ there exists $r \in S$ such that $rp = rq$, and S satisfies Condition (K) if every left collapsible submonoid of S contains a left zero. Also an element $a \in A_S$ is called *act-regular*, if there exists a homomorphism $f : aS \to S$ such that $af(a) = a$, and A_S is called a *regular act* if every $a \in A_S$ is an act-regular element.

Theorem 3.13. *For any monoid S the following statements are equivalent*:

(1) *All right S-acts satisfying U-(G-PWP) are regular*;

(2) *S is right PP, satisfies Condition (K) and every right S-act satisfying Condition (G-PWP) satisfies Condition (E).*

Proof. (1)⇒(2) Since S_S satisfies U-(G-PWP), by the assumption, S_S is regular, and so, by [6, III, 19.3], all principal right ideals of S are projective. Thus, by [6, IV, 11.15], S is right PP. Also, by the assumption, all strongly flat cyclic right S-acts are regular. Thus, by [6, III, 19.3], all strongly flat cyclic right S-acts are projective, and so, by [6, IV, 11.2], S satisfies Condition K. Since, by [6, III, 19.3], all cyclic subacts of a regular right S-act is projective, then all cyclic subacts of a regular right S-act satisfy Condition (E), and so, by Lemma 3.12, every regular right S-act satisfies Condition (E). Thus, by assumption and Theorem 2.2, every right S-act satisfying Condition (G-PWP) satisfies Condition (E).

(2)⇒(1) Suppose that right S-act A_S satisfies U-(G-PWP) and let $a \in A_S$. Then there exists a family $\{B_i \mid i \in I\}$ of subacts of A_S such that

$A = \bigcup_{i\in I} B_i$ and B_i, $i \in I$, satisfies Condition (G-PWP). Also there exists $i_0 \in I$ such that $a \in B_{i_0}$. Since B_{i_0} satisfies Condition (G-PWP), then, by the assumption, B_{i_0} satisfies Condition (E). But S is right PP, and so, by Lemma 3.12, every cyclic subact of B_{i_0} satisfies Condition (E), and hence every cyclic subact of B_{i_0} is strongly flat, thus aS is strongly flat. Since S satisfies Condition (K), by [6, IV, 11.2], aS is projective, and so by [6, III, 19.3], A_S is regular. □

4 When other properties imply U-(G-PWP)

In this section we consider monoids over which other properties of acts imply U-(G-PWP). Meanwhile we give some equivalent descriptions of all right acts satisfying U-(G-PWP).

Recall from [12] that an element $a \in A_S$ is called *act-regular*, if there exists a homomorphism $f : aS \to S$ such that $af(a) = a$, and A_S is called a *regular act* if every $a \in A_S$ is an act-regular element. Also we recall from [3] that A_S is called *P-regular*, if all cyclic subacts of A_S satisfy Condition (P). In [3] we gave a characterization of monoids by P-regularity of their acts. A_S is called *strongly (P)-cyclic* if for any $a \in A_S$ there exists $z \in S$ such that $ker\lambda_a = ker\lambda_z$ and zS satisfies Condition (P). Since from $ker\lambda_a = ker\lambda_z$, it can be seen that $aS \cong zS$, thus aS satisfies Condition (P), and so strong P-cyclicity implies P-regularity. In [4] we gave a characterization of monoids by strongly P-cyclic right S-acts.

Theorem 4.1. *Let S be a monoid. Then:*

(1) *all strongly faithful right S-acts satisfy U-(G-PWP).*

(2) *all P-regular right S-acts satisfy U-(G-PWP).*

(3) *all strongly P-cyclic right S-acts satisfy U-(G-PWP).*

(4) *all regular right S-acts satisfy U-(G-PWP).*

Proof. (1) Let A_S be a strongly faithful right S-act. For every $\alpha \in A_S$ define the mapping $\psi_\alpha : \alpha S \to S_S$ as $\psi_\alpha(\alpha s) = s$. It is obvious that ψ_α is an isomorphism and so for every $\alpha \in A_S$, $\alpha S \cong S_S$. Thus all cyclic subacts of A_S satisfy Condition (G-PWP). But $A_S = \bigcup_{\alpha\in A_S} \alpha S$, and so A_S satisfies U-(G-PWP), as required.

(2) Let A_S be a P-regular right S-act. By the definition, every cyclic subact of A_S satisfies Condition (P). Thus for every $\alpha \in A_S$, αS satisfies

Condition (*G-PWP*) and so $A_S = \bigcup_{\alpha \in A_S} \alpha S$, A_S satisfies U-(*G-PWP*), as required.

The items (3) and (4) are implied from (2), because every strongly P-cyclic or regular right S-act is P-regular. □

Theorem 4.2. *For any monoid S the following statements are equivalent*:

(1) *all right S-acts satisfy U-(G-PWP);*

(2) *all finitely generated right S-acts satisfy U-(G-PWP);*

(3) *all cyclic right S-acts satisfy Condition (G-PWP);*

(4) $(\forall x, y, t \in S)(\exists u, v \in S)(\exists n \in \mathbb{N})(x\rho(xt, yt)u \wedge y\rho(xt, yt)v \wedge ut^n = vt^n)$.

Proof. (1)⇒(2) is obvious.

(2)⇒(3) It is true by (7) of Theorem 2.2.

(3)⇔(4) It follows from the proof of Proposition 3.1 of [10].

(3)⇒(1) It is clear. □

We recall from [9] that for any monoid S and $x, y \in S$, $\rho_{TF}(x, y)$ denotes the smallest right congruence on S containing (x, y), where the cyclic right S-act $S/\rho_{TF}(x, y)$ is torsion free.

Theorem 2.2 and [1, Example 2.6] show that torsion freeness does not imply U-(*G-PWP*) in general.

Theorem 4.3. *For any monoid S the following statements are equivalent*:

(1) *all torsion free right S-acts satisfy U-(G-PWP);*

(2) *all torsion free finitely generated right S-acts satisfy U-(G-PWP);*

(3) *all torsion free cyclic right S-acts satisfy Condition (G-PWP);*

(4) $(\forall x, y, t \in S)(\exists u, v \in S)(\exists n \in \mathbb{N})\big((x, u), (y, v) \in \rho_{TF}(xt, yt) \wedge ut^n = vt^n\big)$.

Proof. The implication (1)⇒(2) is obvious.

(2)⇒(3) It is true by (7) of Theorem 2.2.

(3)⇒(1) Let the right S-act A_S be torsion free. It is obvious that every subact of A_S is also torsion free. Thus, by the assumption, αS satisfies Condition (*G-PWP*) for every $\alpha \in A_S$. Hence $A_S = \cup_{\alpha \in A_S} \alpha S$ satisfies U-(*G-PWP*).

(3)⇒(4) Let $x, y, t \in S$. Then $S/\rho_{TF}(xt, yt)$ is torsion free and so, by assumption, it satisfies Condition (*G-PWP*). Thus, by Proposition 3.1 of

[10], $xt\ \rho_{TF}(xt, yt)\ yt$ implies that there exist $u, v \in S$ and $n \in \mathbb{N}$ such that $(x, u), (y, v) \in \rho_{TF}(xt, yt)$ and $ut^n = vt^n$.

(4)⇒(3) Let S/ρ be torsion free for the right congruence ρ on S and suppose $(xt)\rho(yt)$, for $x, y, t \in S$. Then there exist $u, v \in S$ and $n \in \mathbb{N}$ such that $(x, u), (y, v) \in \rho_{TF}(xt, yt)$ and $ut^n = vt^n$. But $\rho_{TF}(xt, yt) \subseteq \rho$, and so $x\rho u$, $y\rho v$. Thus, by Proposition 3.1 of [10], S/ρ satisfies Condition (*G-PWP*), as required. □

We recall from [13] that $\rho_{STF}(x, y)$ denotes the smallest right congruence on S containing (x, y) such that the cyclic right S-act $S/\rho_{STF}(x, y)$ is strongly torsion free.

Similar to Theorem 4.3, we have the following.

Theorem 4.4. *For any monoid S the following statements are equivalent*:

(1) *all strongly torsion free right S-acts satisfy U-(G-PWP);*

(2) *all strongly torsion free finitely generated right S-acts satisfy U-(G-PWP)*;

(3) *all strongly torsion free cyclic right S-acts satisfy Condition (G-PWP);*

(4) $(\forall x, y, t \in S)(\exists u, v \in S)(\exists n \in \mathbb{N})\big((x, u), (y, v) \in \rho_{STF}(xt, yt) \ \wedge\ ut^n = vt^n\big)$.

An act A_S is called $\Re$-torsion free if for any $a, b \in A_S$ and right cancellabe element $c \in S$, $ac = bc$ and $a\Re b$ imply that $a = b$ ($\Re$ is Green relation). It is clear that torsion freeness implies $\Re$-torsion freeness.

Also $\rho_{\Re TF}(x, y)$ denotes the smallest right congruence on S containing (x, y) such that the cyclic right S-act $S/\rho_{\Re TF}(x, y)$ is $\Re$-torsion free.

Lemma 4.5. *For any monoid S the following statements are equivalent*:

(1) *all $\Re$-torsion free cyclic right S-acts satisfy U-(G-PWP);*

(2) $(\forall x, y, t \in S)(\exists u, v \in S)(\exists n \in \mathbb{N})\big((x, u), (y, v) \in \rho_{\Re TF}(xt, yt) \ \wedge\ ut^n = vt^n\big)$.

Proof. (1)⇒(2) Suppose that all R-torsion free cyclic right S-acts satisfy Condition (*G-PWP*) and let $x, y, t \in S$. Then $S/\rho_{\Re TF}(xt, yt)$ is $\Re$-torsion free and so, by the assumption, it satisfies Condition (*G-PWP*). Thus, by Proposition 3.1 of [10], $xt\ \rho_{\Re TF}(xt, yt)\ yt$ implies that there exist $u, v \in S$ and $n \in \mathbb{N}$ such that $(x, u), (y, v) \in \rho_{\Re TF}(xt, yt)$ and $ut^n = vt^n$.

(2)$\Rightarrow$(1) Let S/ρ be $\Re$-torsion free for the right congruence ρ on S and suppose $(xt)\rho(yt)$, for $x, y, t \in S$. Then there exist $u, v \in S$ and $n \in \mathbb{N}$ such that $(x, u), (y, v) \in \rho_{\Re TF}(xt, yt)$ and $ut^n = vt^n$. But $\rho_{\Re TF}(xt, yt) \subseteq \rho$, and so $x\rho u$, $y\rho v$. Thus, by Proposition 3.1 of [10], S/ρ satisfies Condition $(G\text{-}PWP)$, as required. $\square$

Similar to Theorem 4.3, we have the following.

Theorem 4.6. *For any monoid S the following statements are equivalent*:

(1) *all $\Re$-torsion free right S-acts satisfy U-$(G\text{-}PWP)$*;

(2) *all $\Re$-torsion free finitely generated right S-acts satisfy U-$(G\text{-}PWP)$*;

(3) *all $\Re$-torsion free cyclic right S-acts satisfy Condition $(G\text{-}PWP)$*;

(4) $(\forall x, y, t \in S)(\exists u, v \in S)(\exists n \in \mathbb{N})\big((x, u), (y, v) \in \rho_{\Re TF}(xt, yt) \wedge ut^n = vt^n\big)$.

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