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The projectable hull of an archimedean ℓ-group with weak unit

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Abstract. The much-studied projectable hull of an ℓ-group $G \leq pG$ is an essential extension, so that, in the case that G is archimedean with weak unit, "$G \in \mathbf{W}$", we have for the Yosida representation spaces a "covering map" $YG \leftarrow YpG$. We have earlier [8] shown that (1) this cover has a characteristic minimality property, and that (2) knowing YpG, one can write down pG. We now show directly that for $\mathscr{A}$, the boolean algebra in the power set of the minimal prime spectrum $\mathrm{Min}(G)$, generated by the sets $U(g) = \{P \in \mathrm{Min}(G) : g \notin P\}$ $(g \in G)$, the Stone space $\mathcal{S}\mathscr{A}$ is a cover of YG with the minimal property of (1); this extends the result from [1] for the strong unit case. Then, applying (2) gives the pre-existing description of pG, which includes the strong unit description of [1]. The present methods are largely topological, involving details of covering maps and Stone duality.

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1 Introduction

For $G \in \mathbf{W}$, the topological space $\mathrm{Min}(G)$ has the open base consisting of all $U(g) = \{P \in \mathrm{Min}(G) : g \notin P\}$ $(g \in G)$, and these sets are clopen. We denote $\mathrm{Min}(G) \setminus U(g)$ by $V(g)$. Let $\mathscr{A}$ be the boolean algebra defined in the abstract, $\mathcal{S}\mathscr{A}$ its Stone space. We then have

$$\begin{tikzcd} \mathrm{Min}(G) \arrow[r, hook, "\text{dense}"] \arrow[d, two heads, "\lambda"'] & \mathcal{S}\mathscr{A} \\ YG & \end{tikzcd}$$

where $\lambda(P)$ is the unique $M \in YG$ with $P \subseteq M$. The map λ is a continuous surjection and $\mathscr{A} \ni A \mapsto clA \subseteq \mathcal{S}\mathscr{A}$ is the isomorphism $\mathscr{A} \cong \mathrm{Clop}\,\mathcal{S}\mathscr{A}$. The following will be shown in Sections 5 and 6.

Theorem 1.1. *The map $\lambda : \mathrm{Min}(G) \mapsto YG$ extends continuously to a map $\overline{\lambda} : \mathcal{S}\mathscr{A} \mapsto YG$.*

$$\begin{tikzcd} \mathrm{Min}(G) \arrow[r, hook, "\text{dense}"] \arrow[d, two heads, "\lambda"'] & \mathcal{S}\mathscr{A} \arrow[dl, dashed, two heads, "\overline{\lambda}"] \\ YG & \end{tikzcd}$$

The map $\overline{\lambda}$ is irreducible (a covering map), and $(\mathcal{S}\mathscr{A}, \overline{\lambda})$ is the minimum among those zero-dimensional covers (W, h) of YG which have $cl_W h^{-1}\,\mathrm{coz}\,g$ open for all $g \in G$.

That is the property characterizing YpG ([8, Theorem 3.6 and Corollary 2.5]), whence we obtain immediately the following.

Theorem 1.2. *The projectable hull pG is the $\mathbf{W}$-object of extended real-valued functions on $\mathcal{S}\mathscr{A} = YpG$ of the form*

$$f = \sum (g_i \circ \overline{\lambda})\chi_{U_i}$$

for a finite sum, $g_i \in G$, $\{U_i\}$ a clopen partition of $\mathcal{S}\mathscr{A}$.

This extends [1] by a simple appeal to [8]. We also have shown this in [9] via a different approach.

2 Background and Preliminaries

In this section we set the notation and concepts needed from the theory of ℓ-groups. Our aim is to give a quick overview of the projectable hull of an archimedean lattice-ordered group with weak unit. Our standard references for the theory of ℓ-groups are [4] and [2].

Let G be an abelian ℓ-group. A convex ℓ-subgroup P of G is called prime if $a \wedge b \in P$ implies either $a \in P$ or $b \in P$. The set of all prime subgroups of G is called the prime spectrum of G and is denoted by $\mathrm{Spec}(G)$. Assuming Zorn's Lemma, primes exist in all ℓ-groups. In particular, given $0 < g \in G$, there are convex ℓ-subgroups which are maximal with respect to not containing g. These subgroups are known as values of g and we denote the set of them by $\mathrm{Val}(g)$. Observe that $\mathrm{Val}(g) = \mathrm{Val}(|g|)$.

We put $S(a) = \{P \in \mathrm{Spec}(G) : a \notin P\}$. Observe that $S(a) = S(|a|)$ and that for any $0 < a, b \in G$, $S(a) \cap S(b) = S(a \wedge b)$ and $S(a) \cup S(b) = S(a \vee b)$. Thus, we can topologize $\mathrm{Spec}(G)$ by taking as a base of open sets the collection $\{S(a) : a \in G\}$. Further, $\mathrm{Spec}(G)$ forms a root system, that is, given a prime $P \in \mathrm{Spec}(G)$ the set of prime subgroups containing P forms a chain under inclusion. Thus, there is a map $\mu : S(a) \to \mathrm{Val}(a)$ that takes a $P \in S(a)$ to the unique value of a containing P, denoted by $\mu(P)$. For each $0 \neq a \in G$, the space $S(a)$ is quasi-compact. Since $\mathrm{Val}(a) \subseteq S(a)$, $\mathrm{Val}(a)$ inherits the subspace topology from $S(a)$, and this is identical to the hull-kernel topology on $\mathrm{Val}(a)$. Moreover, $\mathrm{Val}(a)$ is Hausdorff; we shall have more to say in Section 4.

$\mathrm{Min}(G)$ is the collection of minimal prime subgroups topologized with the topology inherited from $\mathrm{Spec}(G)$. Minimal prime subgroups are characterized amongst the primes as those P that have the property that for each $0 < g \in P$, there is some $h \in G \setminus P$ such that $g \wedge h = 0$. It follows that if $0 < u \in G$ is a weak order unit then it does not belong to any minimal prime subgroup.

Another way of constructing convex ℓ-subgroups is as follows. Given $S \subseteq G$, we define the polar of S as

$$S^{\perp} = \{g \in G : |g| \wedge |s| = 0 \text{ for all } s \in S\}.$$

This is clearly nonempty as $0 \in S^{\perp}$ for any subset $S \subseteq G$, and $S^{\perp}$ is a convex ℓ-subgroup, called a polar. When $S = \{g\}$ we instead write $g^{\perp}$;

notice that $g^\perp = |g|^\perp$. If $g^\perp = \{0\}$, then g is called a weak order unit of G. A strong order unit is a weak order unit.

Let $\mathbf{W}$ be the category whose objects are pairs (G, u), where G is an archimedean ℓ-group and $u \in G^+$ is a weak order unit, and a morphism between objects (G, u) and (H, v) be an ℓ-group homomorphism $\rho : G \to H$ for which $\rho(u) = v$. For $(G, u) \in \mathbf{W}$ put $YG = \mathrm{Val}(u)$. We have the Yosida functor from $\mathbf{W}$ to the category of compact Hausdorff spaces, which we now explain.

Put $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\}$ with the obvious topology and order. For a space X,

$$D(X) = \{f : X \to \overline{\mathbb{R}} : f \text{ is continous and } f^{-1}(\mathbb{R}) \text{ is dense in } X\}.$$

This is a lattice when ordered pointwise. In general, $D(X)$ need not be a group as addition is only partially defined. A subset $A \subseteq D(X)$ which is a sublattice, is closed under pointwise addition and subtraction, and contains $\mathbf{1}$ is a $\mathbf{W}$-object in $D(X)$, and then we write $G \leq D(X)$.

See [10] for details of the following.

Theorem 2.1 (The Yosida functor)**.** (a) *Suppose* $(G, u) \in \mathbf{W}$. *Then, there is an isomorphism* $G \cong \hat{G} \leq D(YG)$ *with* $\hat{u} = \mathbf{1}$, *and* $\hat{G}$ *separates the points of* YG.

(b) *Suppose* $(G, u) \xrightarrow{\rho} (H, v) \in \mathbf{W}$. *Then, there is a unique continuous* $YG \xleftarrow{Y\rho} YH$ *for which* $\widehat{\rho(g)} = (Y\rho) \circ \hat{g}$ *for each* $g \in G$. *If* ρ *is an injection, then* $Y\rho$ *is a surjection.*

We frequently write simply $G \in \mathbf{W}$ and "$G \leq D(YG)$" (that is, drop the "u" and identify G with its $\hat{G}$.)

The ℓ-group is called projectable if for all $g \in G$, $G = g^\perp + g^{\perp\perp}$. Every representable ℓ-group has a projectable hull $G \leq pG$, the unique minimum essential extension to a projectable ℓ-group. When G is archimedean, so is pG, and when $G \in \mathbf{W}$, the unit of G is a unit of pG because the embedding is essential, and we construe $G \leq pG$ in $\mathbf{W}$.

Now, ([11]) $G \xrightarrow{\rho} H$ in $\mathbf{W}$ is essential if and only if $YG \xleftarrow{Y\rho} YH$ is irreducible (the image of a proper closed set is proper). Thus, $G \leq pG$ (in $\mathbf{W}$) produces an irreducible surjection $YG \overset{\sigma}{\twoheadleftarrow} YpG$; we reserve "$\sigma$" for this map. This places our situation in a topological context, as follows.

In compact Hausdorff spaces, for irreducible $X \overset{f}{\twoheadleftarrow} Y$, (Y, f) is called a cover of X. For two covers (Y_i, f_i) of X, if there is a $Y_1 \overset{h}{\leftarrow} Y_2$ with $f_2 = f_1 \circ h$, then h is also irreducible and we write $(Y_1, f_1) \leq (Y_2, f_2)$, and say the two are equivalent if h is a homeomorphism. The collection of equivalence classes of covers is a set, denoted by $\operatorname{cov} X$, and it is a complete lattice. For details, see [7] and [15].

Thus, for $G \in \mathbf{W}$, $(YpG, \sigma) \in \operatorname{cov} YG$, and its position in $\operatorname{cov} YG$ is of central importance to this paper, as will be explained in Sections 5 and 6.

3 Lemmas on irreducible maps

We collect some rather dry topological items. A reader might skip this, and refer back when needed. In this section $X \overset{f}{\twoheadrightarrow} Y$ is a continuous surjection of Tychonoff spaces.

Definition 3.1. Here are some properties that f might possess.

1. f has (α) means: if W is a nonempty open subset of X, then there is a nonempty open subset of Y, say V, with $f^{-1}(V) \subseteq W$.
2. f is *irreducible* means: if F is closed and proper in X, then $f(F)$ is proper in Y.
3. f is *skeletal* means: if D is dense in Y, then $f^{-1}(D)$ is dense in X.

In the next section we show that λ has (α).

Definition 3.2. For $W \subseteq X$, set

$$OfW \equiv \{y \in Y | f^{-1}(\{y\}) \subseteq W\}$$

and notice that $OfW = Y \setminus f(X \setminus W)$. Furthermore, the surjectivity of f implies $OfW \subseteq f(W)$.

The proofs of the following are straightforward. For more information see [7, 2.6].

Lemma 3.3. (1) *f has (α) if and only if for each nonempty open subset $W \subseteq X$,* $\operatorname{int} OfW \neq \emptyset$.

(2) *If f is closed and W open, then OfW is open.*

(3) *If f is closed and irreducible and W is open, then $f^{-1}(OfW)$ is dense in W and $f(W) \subseteq cl_Y OfW$.*

Proposition 3.4. (a) *If f has (α), then f is irreducible and skeletal.*
(b) *If f is closed, then irreducibility implies that f has (α).*

Proof. (a) Suppose f has (α) and let $D \subseteq Y$ be a dense subset. For an open nonempty subset $W \subseteq X$, the condition (α), implies there is a nonempty open subset of Y, say V, such that $f^{-1}(V) \subseteq W$. By density there is some $y \in D \cap V$. Choose $x \in W$ such that $f(x) = y$. Then $x \in f^{-1}(D) \cap W$. Consequently, $f^{-1}(D)$ is dense in X.

Next, suppose F is a proper closed subset of X and set $W = X \setminus F$, nonempty and open. By (1) of Lemma 3.3, we gather that $\emptyset \neq \text{int}OfW \subseteq OfW = Y \setminus f(X \setminus W) = Y \setminus f(F)$, whence $f(F)$ is proper.

(b) Suppose f is closed and irreducible and let $W \subseteq X$ be nonempty and open. Setting $F = X \setminus W$, a proper closed subset, the hypothesis implies that $f(F)$ is both proper and closed. Therefore, $OfW = Y \setminus f(X \setminus W) = Y \setminus f(F)$ is nonempty and open. By (1) of Lemma 3.3, we conclude that f has (α). $\square$

The next two propositions show that (α) goes both up and down in certain cases.

Proposition 3.5. *Suppose X is dense in L, and there is a continuous extension of f to L, say $\tilde{f} : L \twoheadrightarrow Y$. If f has (α), then $\tilde{f}$ has (α).*

Proof. We shall use the following property of density twice in our proof. For any nonempty open subset O of L, $cl_L(O \cap X) = cl_L O$.

Assume that f has (α). To show that $\tilde{f}$ has (α) let T be nonempty and open subset of L set $W = T \cap X$, nonempty and open in X by density. Choose $\emptyset \neq W' \subseteq W$ such that $cl_L W' \subseteq W$. Since f has (α) there is a nonempty open subset of Y, say V, such that $f^{-1}(V) \subseteq W'$. Notice that density together with the fact that $\tilde{f}^{-1}(V) \subseteq L$ is open, yields that $cl_L \tilde{f}^{-1}(V) = cl_L(\tilde{f}^{-1}(V) \cap X)$. Thus,

$$\emptyset \neq \tilde{f}^{-1}(V) \subseteq cl_L(\tilde{f}^{-1}(V) \cap X) = cl_L f^{-1}(V) \subseteq cl_L(W' \cap X) = cl_L W' \subseteq T$$

where density is used again for the last equality. $\square$

Proposition 3.6. *Suppose E is a regular closed in X. If f has (α), then the restriction of f to E onto $f(E)$ also has (α).*

Proof. We shall denote the restriction of f to E by f' and set $Y' = f(E)$. Then we have a continuous surjection $f' : E \to Y'$.

Assume that f has (α) and let O be a nonempty subset of E. Let O' be an open subset of X for which $O' \cap E = O$. Set $W = O' \cap \operatorname{int} E$, a nonempty open subset of X. Since f has (α), there is a nonempty subset of Y, say V, such that $f^{-1}(V) \subseteq W$. Notice that $V \subseteq Y'$ and so

$$f'^{-1}(V) = f^{-1}(V) \cap E = f^{-1}(V) \subseteq O.$$

□

4 Properties of the map λ

For a **W**-object G, or (G, u), we have the map $\mu : S(u) \to \mathrm{Val}(u) = YG$, from Section 2. The restriction of μ to $\mathrm{Min}(G)$ is the map of Section 1, $\lambda : \mathrm{Min}(G) \to YG$.

Let $g \in G$. We have these subsets of YG.

$$\mathrm{coz}(g) = \{M \in YG : g \notin M\} \text{ and } Z(g) = YG \setminus \mathrm{coz}(g);$$

and the subsets of $\mathrm{Min}(G)$,

$$U(g) = \{P \in \mathrm{Min}(G) : g \notin P\} \text{ and } V(g) = \mathrm{Min}(G) \setminus U(g).$$

Summing up:

Proposition 4.1. (a) *The space YG is compact Hausdorff, with $\{\mathrm{coz}(g) : g \in G\}$ an open basis.*

(b) *The space* $\mathrm{Min}(G)$ *is zero-dimensional Hausdorff, with $\{U(g) : g \in G\}$ an open basis.*

(c) *The map $\lambda : \mathrm{Min}(G) \to YG$ is a continuous surjection.*

We establish some other properties of λ.

Theorem 4.2. *Let $(G, u) \in$ **W**. For each $g \in G$, we have*

(1) $\lambda^{-1}(\mathrm{coz}(g)) \subseteq U(g)$*;* $\mathrm{coz}(g) \subseteq \lambda(U(g))$*,*

(2) $\lambda(U(g))$ *is compact, hence a closed subset of* YG*.*

(3) $\lambda^{-1}(\mathrm{int}Z(g)) \subseteq V(g)$*,*

(4) $\lambda(U(g)) = cl_{YG}\,\mathrm{coz}\, g$*.*

Proof. (1) Let $Q \in \lambda^{-1}(\mathrm{coz}(g))$. This means that $\lambda(Q) \in \mathrm{coz}(g)$ and so $g \notin \lambda(Q)$. Since $Q \leq \lambda(Q)$, it follows that $g \notin Q$, that is, $Q \in U(g)$. Next, if $P \in \mathrm{coz}(g)$, then for any minimal prime $Q \leq P$ (which indeed exists), it follows that $\lambda(Q) = P$. Since $g \notin P$, we gather that $Q \in U(g)$.

(2) Fix the map $\mu : S(u) \to YG$. We claim that $\lambda(U(g)) = \mu(S(|g| \wedge u))$. Since $S(|g| \wedge u)$ is quasi-compact, so is $\mu(S(|g| \wedge u)$ by continuity. Therefore, $\lambda(U(g))$ is a compact subset, whence a closed subset of YG. As for the claim for any prime subgroup P, if $|g| \wedge u \notin P$, then $u \notin P$ and so $\mu(P) \in YG$. Furthermore, for any $Q \in \mathrm{Min}(G)$ with $Q \leq P$, we know that $g \notin P$, thus $Q \in U(g)$. For any $Q \in U(g)$, it is also the case that $Q \in S(|g| \wedge u)$.

(3) Let $Q \in \lambda^{-1}(\mathrm{int}Z(g))$, that is, $\lambda(Q) \in \mathrm{int}Z(g)$. Since sets of the form $\mathrm{coz}(h)$ form a base for the open sets of YG, we can find an $0 < h$ such that $\lambda(Q) \in \mathrm{coz}(h) \subseteq Z(g)$. In the Yosida representation, it follows that $h \wedge g = 0$. Now $Q \in \lambda^{-1}(\mathrm{coz}(h)) \subseteq U(h)$ by (1) and therefore, $g \in Q$ by primality, that is, $Q \in V(g)$.

(4) By (1), $\mathrm{coz}(g) \subseteq \lambda(U(g)$. By (2), $\lambda(U(g))$ is closed and therefore, $cl_Y G\,\mathrm{coz}(g) \subseteq \lambda(U(g))$. For the reverse direction, let $P \in \lambda(U(g))$ and choose $Q \in U(g)$ such that $\lambda(Q) = P$. If $P \in YG \setminus cl_{YG}\,\mathrm{coz}(g)$, then $P \in \mathrm{int}Z(g)$, and thus by (3), $Q \in V(g)$, a contradiction. □

Corollary 4.3. *Let* $(G, u) \in \mathbf{W}$*. Then, the map* $\lambda : \mathrm{Min}(G) \to YG$ *has* (α)*.*

Proof. Let W be a nonempty open subset of $\mathrm{Min}(G)$. Choose $g \in G$ such that $U(g)$ is nonempty and $U(g) \subseteq W$. Observe that $g \neq 0$ and so $\mathrm{coz}(g) \neq \emptyset$. By (1) of Theorem 4.2, $\lambda^{-1}(\mathrm{coz}(g)) \subseteq U(g)$. Therefore, λ has (α). □

5 $\mathcal{S}\mathscr{A}$ is a cover of YG

We restate and prove half of Theorem 1.1. The following is pivotal ([5, 3.2] and [15, 4.1 (m)])

Theorem 5.1 (Taimonov's Theorem)**.** *For Tychonoff spaces, suppose $f : X \to Y$ is continuous with Y compact, and X dense in L. Then, f extends continuously over L if and only if E, F closed and disjoint in Y implies $cl_L f^{-1}(E) \cap cl_L f^{-1}(F) = \emptyset$.*

Theorem 5.2. *Let $(G, u) \in \mathbf{W}$. Then,*

(a) *there is a continuous $\tilde{\lambda} : \mathcal{S}\mathscr{A} \twoheadrightarrow YG$ extending λ.*

(b) *$\tilde{\lambda}$ has (α), thence is skeletal and irreducible, whence $(\mathcal{S}\mathscr{A}, \tilde{\lambda}) \in \operatorname{cov} YG$.*

Proof. (a) Suppose E and F are disjoint closed sets in YG. That YG is a compact Hausdorff space provides us with a $g \in G^+$ with $E \subseteq \operatorname{coz} g$, $F \subseteq intZ(g)$. Then, by (1) and (2) of Theorem 4.2, $\lambda^{-1}(E) \subseteq \lambda^{-1}(\operatorname{coz}(g)) \subseteq U(g)$, and $\lambda^{-1}(F) \subseteq \lambda^{-1}(\operatorname{int}Z(g)) \subseteq V(g)$. Since $U(g)$ are complementary members of $\mathscr{A}$, we have $cl_{\mathcal{S}\mathscr{A}}U(g) \cap cl_{\mathcal{S}\mathscr{A}}V(g) = \emptyset$, by Stone Representation. By Taimonov's Theorem, we have the extension $\tilde{\lambda}$.

(b) By Corollary 4.3, λ has (α), and since (α) goes up (Proposition 3.5), $\tilde{\lambda}$ also has (α) and thus is irreducible and skeletal (Proposition 3.4). □

6 $(\mathcal{S}\mathscr{A}, \tilde{\lambda})$ is (YpG, σ); a Theorem about minimal covers

We are going to apply the following.

Theorem 6.1. [8, 3.6] *(YpG, σ) is the minimum in $\operatorname{cov} YG$ among covers (W, h) with W zero-dimensional and satisfying for all $g \in G$, $cl_{YG}h^{-1}(\operatorname{coz}(g))$ is open.*

(This result is also visible (with some thought) in [14, 4.6].) For brevity's sake we shall denote the condition: for all $g \in G$, $cl_{YG}h^{-1}(\operatorname{coz}(g))$ is open, by (†).

Proposition 6.2. *For all $g \in G$, $cl_{\mathcal{S}\mathscr{A}}\tilde{\lambda}^{-1}(\operatorname{coz}(g)) = cl_{\mathcal{S}\mathscr{A}}U(g)$, so this is open.*

Proof. Note that this result is in fact a corollary to Theorem 4.2.

Let $M = \mathrm{Min}(G)$ and $S = \mathcal{S}\mathscr{A}$. Then, by Theorem 4.2, we have $cl_M\lambda^{-1}(\mathrm{coz}(g)) = U(g)$. Now, M is dense in S, thus, for all W open in S, $cl_S W = cl_S W \cap M = cl_S(cl_M W \cap M)$. Apply this to $W = \tilde{\lambda}^{-1}(\mathrm{coz}(g))$, for which $W \cap M = cl_M\lambda^{-1}(\mathrm{coz}(g))$. $\square$

Towards the minimality condition in Theorem 6.1, we have the following topological/boolean algebraic theorem. (We need only (a) implies (b) but we prove the equivalence.)

Theorem 6.3. *Suppose* $(Z, f) \in \mathrm{cov}\, Y$, *$Z$ is zero-dimensional, and $\mathcal{U}$ is an open base for Y. The following two statements are equivalent.*

(a) *For all* $U \in \mathcal{U}$, $cl_Z f^{-1}(U)$ *is open, and* $\{cl_Z f^{-1}(U) : U \in \mathcal{U}\}$ *generates* $\mathrm{Clop}\, Z$ (*qua boolean algebra*).

(b) (Z, f) *is the minimum in* $\mathrm{cov}\, Y$ *among covers* (W, h), *with W zero-dimensional and satisfying for all* $U \in \mathcal{U}$, $cl_W h^{-1}(U)$ *is open.*

Proof. (a) $\Rightarrow$ (b) Suppose $(Z, f) \in \mathrm{cov}\, Y$ satisfies (a), and let (W, h) be as in (b). Let $\mathscr{B}$ be the sub-boolean algebra of $\mathrm{Clop}\, W$ generated by the collection $\{cl_W h^{-1}(U)\}$. Note that $\mathscr{B}$ is dense in $\mathrm{Clop}\, W$ because $\mathcal{U}$ is a basis for Y. This means that the embedding $\mathscr{B} \leq \mathrm{Clop}\, W$ has its Stone dual surjection $\mathcal{S}\mathscr{B} \overset{s}{\twoheadleftarrow} W$ irreducible (see [16]). We shall show that $\mathrm{Clop}\, Z \cong \mathscr{B}$, which means that s is, up to homeomorphism, a map $Z \twoheadleftarrow W$, showing that $(Z, f) \leq (W, h)$ in $\mathrm{cov}\, Y$.

Let $R(\cdot)$ denote the boolean algebra of regular closed sets of the space $(\cdot)$. By [15, §6], whenever $F \overset{t}{\twoheadleftarrow} K$ is irreducible between compact spaces, then $R(K) \ni E \mapsto t(E) \in R(T)$, and this defines a boolean algebra isomorphism, again denoted by $t : R(K) \to R(T)$, thence carries a generating set in $R(K)$ to a generating set in $R(T)$. Note, "generating" refers to the boolean operations in the $R(\cdot)$s.

Applying this to our construction, we have boolean algebra isomorphisms

$$\begin{tikzcd} R(Y) & R(Z) \arrow[l, "f"'] \\ R(W) \arrow[u, "h"] & \end{tikzcd}$$

with $\mathscr{B} \cong h[\mathscr{B}]$, the latter generated by $\{h(cl_W h^{-1}(U)) : U \in \mathcal{U}\}$, and $\mathrm{Clop}\, Z \cong f[\mathrm{Clop}\, Z]$, the latter generated by $\{f(cl_Z f^{-1}(U)) : U \in \mathcal{U}\}$. Note that, for all $U \in \mathcal{U}$, $h(cl_W h^{-1}(U)) = cl_Y U = f(cl_Z f^{-1}(U))$. Therefore, $\mathrm{Clop}\, Z \cong f[\mathrm{Clop}\, Z] = h[\mathscr{B}] \cong \mathscr{B}$, as desired.

(b) $\Rightarrow$ (a) (This mimics the proof of 3.6 (c) in [8].)

We show this: suppose $(Z, f) \in \mathrm{cov}\, Y$ with Z zero-dimensional and each $cl_Z f^{-1}(U)$ open ($U \in \mathcal{U}$). Let $\mathscr{A}$ be the sub-boolean algebra of $\mathrm{Clop}\, Z$ generated by the set $\{cl_Z f^{-1}(U) : U \in \mathcal{U}\}$, and let $\mathcal{S}\mathscr{A} \overset{h}{\leftarrow} Z$ be the Stone dual of $\mathscr{A} \leq \mathrm{Clop}\, Z$. (This is irreducible because $\mathcal{U}$ is a basis.) Then, if there is an s with $s \circ h = f$ as

$$\begin{tikzcd} Y & Z \arrow[l, "f"'] \arrow[dl, "h"] \\ \mathcal{S}\mathscr{A} \arrow[u, "s"] & \end{tikzcd}$$

it then follows that s is irreducible, since f and h are ([7, 2.6]). Thus, if (Z, f) satisfies the minimality condition in (b), then h must be a homeomorphism, which means that $\mathscr{A} \cong \mathrm{Clop}\, Z$, as desired.

Now, h is a quotient map (being a surjection of compact spaces). Thus, the existence of the s is equivalent to: $f(p_1) \neq f(p_2)$ implies $h(p_1) \neq h(p_2)$. So suppose $f(p_1) \neq f(p_2)$. Since $\mathcal{U}$ is a basis, there are disjoint $U_1, U_2 \in \mathcal{U}$ with $f(p_i) \in U_i$ ($i = 1, 2$). Then $f^{-1}(U_1) \cap cl_Z f^{-1}(U_2) = \emptyset$. Since $cl_Z f^{-1}(U_2)$ is open, $cl_Z f^{-1}(U_1) \cap cl_Z f^{-1}(U_2) = \emptyset$. Thus, p_1, p_2 lie in disjoint elements of $\mathscr{A}$, whence $h(p_1) \neq h(p_2)$. $\square$

Theorem 6.4. $(\mathcal{S}\mathscr{A}, \tilde{\lambda})$ *is* (YpG, σ).

Proof. $(\mathcal{S}\mathscr{A}, \tilde{\lambda})$ is certainly a zero-dimensional cover of YG satisfying (†) (Proposition 6.2). By Proposition 4.1, $\mathcal{U} = \{\mathrm{coz}(g) : g \in G\}$ is an open base for YG. By design and Stone duality $\{cl_{\mathcal{S}\mathscr{A}} U(g)\}_{g \in G}$ generates the boolean algebra of clopen sets of $\mathcal{S}\mathscr{A}$. Thus, by Theorem 6.3, applied to $Z = \mathcal{S}\mathscr{A}$, we conclude that $(\mathcal{S}\mathscr{A}, \tilde{\lambda})$ is (YpG, σ). $\square$

7 Representations of pG

We give three representations, each derived from Theorem 7.1. The following notation from [8, §2] is convenient. Given skeletal $YG \overset{\tau}{\twoheadleftarrow} X$ we have

$G \cong G \circ \tau \leq D(X)$ ($G \circ \tau$ consists of all $g \circ \tau$, and $g \mapsto g \circ \tau$ preserves the **W**-operations.) Suppose X is zero-dimensional, and put

$$\begin{aligned}(G \circ \tau)_X \equiv \{\textstyle\sum (g_i \circ \tau) \cdot \chi(W_i) : \sum &\text{ is finite}, \{W_i\} \text{ is a clopen partition} \\ &\text{of X}, g_i \in G\} \\ \leq D(X).\end{aligned}$$

Theorem 7.1. [8, 3.5] *Granted* $YG \overset{\sigma}{\twoheadleftarrow} YpG$*,*

$$pG = (G \circ \sigma)_{YpG}.$$

(This result is also visible in [14, §2 and 5.11]. Also, a version for rings is [9, 5.3].)

Lemma 7.2. (a) $G \cong G \circ \lambda \leq D(\mathrm{Min}(G))$.
(b) $G \cong G \circ \tilde{\lambda} \leq D(\mathcal{S}\mathscr{A}(G))$.

Proof. (a) The map λ has the property (α) (Corollary 4.3), thus is skeletal (Proposition 3.4), and so $g \circ \lambda \in D(\mathrm{Min}(G))$. The resulting map $g \mapsto g \circ \lambda$ preserves the **W**-operations and is clearly a bijection.
(b) As (a), since $\tilde{\lambda}$ is skeletal (Theorem 5.2). □

(We note that Lemma 7.2(a) is the Johnson-Kist representation of G on $\mathrm{Min}(G)$; see [12] and [13]).

Since $(\mathcal{S}\mathscr{A}(G), \tilde{\lambda})$ is (YpG, σ) (Section 6), we have immediately the following corollary.

Corollary 7.3. (a) $pG = (G \circ \tilde{\lambda})_{\mathcal{S}\mathscr{A}(G)} \leq D(\mathcal{S}\mathscr{A}(G))$.
(b) $pG = (G \circ \lambda)_{\mathrm{Min}(G)} \leq D(\mathrm{Min}(G)$.

(In [1], it is proved that for $G \in \mathbf{W}^*$ (note, $\mathbf{W}^*$, not $\mathbf{W}$), a simpler version of Corollary 7.3(a) holds; this is without *a priori* knowledge of YpG. It then follows that for $G \in \mathbf{W}^*$, $YpG = \mathcal{S}\mathscr{A}$. See the discussion in [9].)

Now we shall represent pG as continuous functions on "certain dense subsets" of YG. It is clear that this can be done: $G \leq pG$ is an essential extension, and the maximum essential extension of G consists of all continuous real-valued functions on dense G_δs in YG modulo $f_1 \approx f_2$ if $f_1 = f_2$ on $\mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$, the intersection of the respective domains. Also, pG

embeds in the strongly projectable hull of G which consists of all "finitely G-local" functions on dense open sets in YG, modulo $\approx$. (See discussions in [6], [9], and [17].)

The first issue for pG is to specify the "certain dense subsets."

Let $\mathcal{B}$ be the family of open sets in YG generated by finite intersections and unions from $\{\text{coz}(g) : g \in G\} \cup \{\text{int}Z(g) : g \in G\}$. Let L be the family of continuous functions on certain subsets of YG as follows. The notation $f \in L$ means: the domain of f has the form $\bigcup B_i$ for pairwise disjoint $B_1, \ldots, B_n \in \mathcal{B}$ with $\bigcup B_i$ dense in YG; and there are $g_1, \ldots, g_n \in G$ for which $f|_{B_i} = g|_{B_i}$ for all $i = 1, \ldots, n$. Next, we define an equivalence relation on the $f \in L$ as above: $f_1 \approx f_2$ if they agree on the intersection of their domains.

Theorem 7.4. *The set of equivalence classes $L/\approx$ is a $\mathbf{W}$-object isomorphic to pG.*

Proof. We outline the one-to-one correspondence, omitting many details. This correspondence comes from that between $\text{Clop}\,\mathcal{S}\mathscr{A}(G)$ and $\mathcal{B}$, and the description in Theorem 6.3(a).

Notation for the nonce: In $\mathcal{B}$, $U_1 = \{\text{coz}(g) : g \in G\}$ and $U_2 = \{\text{int}Z(g) : g \in G\}$; in $\widehat{\mathscr{A}} \equiv \text{Clop}\,\mathcal{S}\mathscr{A}(G)$, $\widehat{U_1} = \{cl_{\mathcal{S}\mathscr{A}}U(g) : g \in G\}$ and $\widehat{U_2} = \{cl_{\mathcal{S}\mathscr{A}}V(g) : g \in G\}$. From §3, $\tilde{\lambda}(cl_{\widehat{\mathscr{A}}}U(g)) = \text{cl}_{YG}\,\text{coz}(g)$ and $\tilde{cl}_{\widehat{\mathscr{A}}}V(g) = \text{cl}_{YG}\text{int}Z(g)$. By definition above, $\mathcal{B}$ is generated by $U_1 \cup U_2$, with finite intersections and unions; from [16, p.14], $\widehat{\mathscr{A}}$ is likewise generated (*qua* boolean algebra) by $\widehat{U_1} \cup \widehat{U_2}$.

For $B = \text{coz}(g)$ (respectively, $\text{int}Z(g)$), put $\overleftarrow{B} = cl_{\mathcal{S}\mathscr{A}}U(g)$ (respectively, $\overleftarrow{B} = cl_{\mathcal{S}\mathscr{A}}V(g)$), and for $B = \bigcup\bigcap B_{ij}$, with each $B_{ij} \in U_1 \cup U_2$, put $\overleftarrow{B} = \bigcap\bigcup \overleftarrow{B}_{ij}$. For the other direction, for $W = cl_{\mathcal{S}\mathscr{A}(G)}U(g)$ (respectively, $cl_{\mathcal{S}\mathscr{A}(G)}V(g)$) set $\overrightarrow{W} = \text{coz}(g)$ (respectively, $\text{int}Z(g)$), and for $W = \bigcap\bigcup W_{ij}$ with each $W_{ij} \in \widehat{U_1} \cup \widehat{U_2}$ put $\overrightarrow{W} = \bigcap\bigcup \overrightarrow{W}_{ij}$.

Now consider $f = \sum(g_i \circ \tilde{\lambda} \cdot \chi(W_i)) \in pG$, per Theorem 6.3(a). Here $\{W_i\}$ is disjoint in $\widehat{\mathscr{A}}$ with $\bigcup W_i = \mathcal{S}\mathscr{A}(G)$, so $\{\overrightarrow{W}_i\}$ is disjoint in $\mathcal{B}$ with $\bigcup \overrightarrow{W}_i$ dense in YG. Therefore, we can define the element $\overrightarrow{f} \in L/\approx$ to be the equivalence class of the function which agrees with g_i on $\overrightarrow{W}_i$.

The reverse correspondence $L \to pG$ is clear. Vagaries in the above evaporate upon factoring L by $\approx$. $\square$

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References

[1] Ball, R.N., Marra, V., McNeill, D., and Pedrini, A., *From Freudenthal's Spectral Theorem to projectable hulls of unital archimedean lattice-groups, through compactification of minimal spectra*, arXiv: 1406-1352 V2.

[2] Bigard, A., Keimel, K., and Wolfenstein, S., "Groupes et Anneaux Réticulés", Lecture Notes in Math. 608, Springer-Verlag, Berlin-New York, 1977.

[3] Carral, M. and Coste, M., *Normal spectral spaces and their dimensions*, J. Pure Appl. Algebra 30(3) (1983), 227-235.

[4] Darnel, M., "Theory of Lattice-Ordered Groups", Monographs and Textbooks in Pure and Applied Mathematics 187, Marcel Dekker, Inc., New York, 1995.

[5] Engelking, R., "General Topology". Second edition. Sigma Series in Pure Mathematics 6, Heldermann Verlag, Berlin, 1989.

[6] Fine, N.J., Gillman, L., and Lambek, J., "Rings of Quotients of Rings of Functions", McGill Univ. Press, Montreal, 1966.

[7] Hager, A.W., *Minimal covers of topological spaces*, Papers on general topology and related category theory and topological algebra (New York, 1985/1987), 44-59, Ann. New York Acad. Sci. 552, New York Acad. Sci., New York, 1989.

[8] Hager, A.W., Kimber, C.M., and McGovern, W.Wm., *Weakly least integer closed groups*, Rend. Circ. Mat. Palermo (2), 52(3) (2003), 453-480.

[9] Hager, A.W. and McGovern, W.Wm., *The Yosida space and representation of the projectable hull of an archimedean ℓ-group with weak unit*, Quaest. Math., 40(1) (2017), 57-63.

[10] Hager, A.W. and Robertson, L., *Representing and ringifying a Riesz space*, Symp. Math 21 (1977), 411-431.

[11] Hager, A.W. and L. Robertson, *On the embedding into a ring of an archimedean ℓ-group*, Canad. J. Math. 31 (1979), 1-8.

[12] Johnson, D.G. and Kist, J.E., *Prime ideals in vector lattices*, Canad. J. Math. 14 (1962), 517-528.

[13] Luxemburg, W.A.J. and Zaanen, A.C., "Riesz Spaces". Vol. I., North-Holland Publishing Co., Amsterdam-London, 1971.

[14] Martínez, J., *Hull classes of Archimedean lattice-ordered groups with unit: a survey*, Ordered algebraic structures, 89-121, Dev. Math. 7, Kluwer Acad. Publ., Dordrecht, 2002.

[15] Porter, J. and Woods, R.G., "Extensions and Absolutes of Hausdorff Spaces", Springer-Verlag, New York, 1988.

[16] Sikorski, R. "Boolean Algebras", Third edition. 25 Springer-Verlag New York Inc., New York, 1969.

[17] Veksler, A.I. and Geĭler, V.A., *Order and disjoint completeness of linear partially ordered spaces*, Sib. Math. J. 13 (1972), 30-35. (Plenum translation).

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