Categories and
General Algebraic Structures
with Applications
Volume 7, Special issue on the Occasion of Banaschewski's 90th Birthday, July 2017, 125-140.

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Perfect secure domination in graphs

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Dedicated to Bernhard Banaschewski on the occasion of his 90th birthday

Abstract. Let $G = (V, E)$ be a graph. A subset S of V is a dominating set of G if every vertex in $V \setminus S$ is adjacent to a vertex in S. A dominating set S is called a secure dominating set if for each $v \in V \setminus S$ there exists $u \in S$ such that v is adjacent to u and $S_1 = (S \setminus \{u\}) \cup \{v\}$ is a dominating set. If further the vertex $u \in S$ is unique, then S is called a perfect secure dominating set. The minimum cardinality of a perfect secure dominating set of G is called the perfect secure domination number of G and is denoted by $\gamma_{ps}(G)$. In this paper we initiate a study of this parameter and present several basic results.

1 Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to Chartrand and Lesniak [4]

The open neighborhood of a vertex $v \in V$ is given by $N(v) = \{u \in V : uv \in E\}$ and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. Given $S \subseteq V$ and $v \in S$, a vertex $u \in V$ is an S-private neighbor of v if $N[u] \cap S = \{v\}$.

Keywords: Secure domination, perfect secure domination, secure domination number, perfect secure domination number.
Mathematics Subject Classification [2010]: 05C69.
Received: 16 October 2016, Accepted: 1 March 2017
ISSN Print: 2345-5853 Online: 2345-5861

The set of all S-private neighbors of v is denoted by $PN(v, S)$. If further $u \in V \setminus S$, then u is called an S-external private neighbor (abbreviated S-epn) of v. The set of all S-epns of v is denoted by $EPN(v, S)$. A set $S \subseteq V$ is called a dominating set of G if every vertex in $V \setminus S$ is adjacent to a vertex in S. A dominating set S is called a minimal dominating set of G if $S \setminus \{v\}$ is not a dominating set for all $v \in S$. The minimum cardinality of a minimal dominating set of G is called the domination number of G and is denoted by $\gamma(G)$.

Strategies for protection of a graph $G = (V, E)$ by placing one or more guards at every vertex of a subset S of V, where a guard at v can protect any vertex in its closed neighborhood have resulted in the study of several concepts such as Roman domination, weak Roman domination and secure domination. The concept of secure domination is motivated by the following situation. Given a graph $G = (V, E)$ we wish to place one guard at each vertex of a subset S of V in such a way that S is a dominating set of G and if a guard at v moves along an edge to protect an unguarded vertex u, then the new configuration of guards also forms a dominating set. In other words, for each $u \in V \setminus S$ there exists $v \in S$ such that v is adjacent to u and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. In this case we say that u is S-defended by v or v S-defends u. A dominating set S in which every vertex in $V \setminus S$ is S-defended by a vertex in S is called a secure dominating set of G. The secure domination number $\gamma_s(G)$ is the minimum cardinality of a secure dominating set of G. This concept was introduced by Cockayne et al. [7]. It has been further investigated by several authors [1–3, 5, 6, 9, 10].

Weichsel [11] introduced the concept of perfect domination in graphs. A dominating set S is called a perfect dominating set of a graph G if every vertex in $V \setminus S$ is adjacent to exactly one vertex in S. The minimum cardinality of a perfect dominating set of G is called the perfect domination number of G and is denoted by $\gamma_p(G)$.

In this paper we introduce the concept of perfect secure domination number and initiate a study of this parameter.

We need the following definitions and results.

Definition 1.1. Let G_1 and G_2 be two graphs with disjoint vertex sets. Then the graph G obtained by joining every vertex of G_1 with every vertex of G_2 is called the join of G_1 and G_2 and is denoted by $G_1 + G_2$.

Definition 1.2. Let G_1 and G_2 be two graphs with disjoint vertex sets V_1

and V_2 respectively. Then the Cartesian product $G_1 \Box G_2$ is defined to be the graph with vertex set $V_1 \times V_2$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if $u_1 = u_2$ and v_1, v_2 are adjacent in G_2 or $v_1 = v_2$ and u_1, u_2 are adjacent in G_1.

Theorem 1.3. [7] *For the path P_n we have $\gamma_s(P_n) = \left\lceil \frac{3n}{7} \right\rceil$.*

Theorem 1.4. [7] *For the cycle C_n we have $\gamma_s(C_n) = \left\lceil \frac{3n}{7} \right\rceil$.*

2 Perfect secure domination number of standard graphs

In this section, we present some basic results on perfect secure domination and determine the perfect secure domination number of some standard graphs including paths, cycles, complete bipartite graphs, caterpillars and wheels. We end this section by showing that for two given positive integers a and b with $a \leq b$, there exists a graph G with $\gamma_p(G) = a$ and $\gamma_{ps}(G) = b$.

Definition 2.1. Let $G = (V, E)$ be a graph. A subset S of V is called a perfect secure dominating set (psd-set) of G if for every vertex $v \in V \setminus S$, there exists a unique vertex $u \in S$ such that u and v are adjacent and $(S \setminus \{u\}) \cup \{v\}$ is a dominating set of G. The minimum cardinality of a psd-set of G is called the pefect secure domination number of G and is denoted by $\gamma_{ps}(G)$.

Since V is trivially a psd-set of G, $\gamma_{ps}(G)$ is defined for all graphs G. It follows from the definition that $\gamma_s(G) \leq \gamma_{ps}(G)$.

Since $\gamma_s(G) = 1$ if and only if $G \cong K_n$, it follows that $\gamma_{ps}(G) = 1$ if and only if G is complete.

Note 2.2. A perfect secure dominating set need not be a perfect dominating set and vice versa. For example, for the path $P_6 = (v_1, v_2, \ldots, v_6)$, $S = \{v_1, v_4, v_6\}$ is a perfect secure dominating set but not a perfect dominating set. Also for the path $P_5 = (v_1, v_2, v_3, v_4, v_5)$, $S = \{v_1, v_4\}$ is a perfect dominating set, but not a perfect secure dominating set.

Note 2.3. If G is a graph of order n which is not complete and $\Delta = n - 1$, then every vertex v of degree $n-1$ belongs to every perfect secure dominating set S of G. Since G is not complete, $|S| \geq 2$. Now, if $v \notin S$, then v is defended by every vertex in S which is a contradiction.

Note 2.4. Let G be a graph of order n with k support vertices u_1, $u_2, \ldots, u_k$ and a unique leaf w_i adjacent to $u_i, 1 \leq i \leq k$. Then $V(G) \setminus \{w_1, w_2, \ldots, w_k\}$ is a perfect secure dominating set of G and hence $\gamma_{ps}(G) \leq n - k$.

For any graph G of order n, we have $1 \leq \gamma_{ps}(G) \leq n$. Also $\gamma_{ps}(G) = 1$ if and only if $G = K_n$. The following lemma gives a family of graphs with $\gamma_{ps}(G) = n$.

Lemma 2.5. *Let G be a graph of order n which is not complete. If G has at least two vertices u and v of degree $n-1$, then $\gamma_{ps}(G) = n$.*

Proof. Let S be a perfect secure dominating set of G. It follows from Note 2.3 that both u and v are in S. Now, if $|S| \neq n$, then any vertex in $V \setminus S$ is defended by both u and v which is a contradiction. Thus $|S| = n$ and $\gamma_{ps}(G) = n$. $\square$

The following examples show that removal of an edge or a vertex may increase γ_{ps} arbitrarily.

Example 2.6. For any $n \geq 4$, $\gamma_{ps}(K_n) = 1$ and by Lemma 2.5, $\gamma_{ps}(K_n \setminus \{e\}) = n$, where e is any edge of K_n.

Example 2.7. Consider the graph G given in Figure 1. Then $\gamma_{ps}(G) = 2$ and $S = \{x, y\}$ is a perfect secure dominating set of G. Now $\gamma_{ps}(G \setminus \{v\}) = n + 2$, since $V(G) \setminus \{v\}$ is the only perfect secure dominating set of $G \setminus \{v\}$.

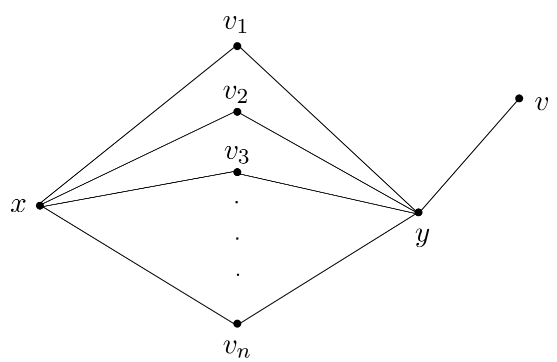

Figure 1.

This also shows that the converse of Lemma 2.5 is not true.

We now proceed to determine $\gamma_{ps}(G)$ for some standard graphs. If a vertex v is S-defended by u we say that v is defended by u.

Theorem 2.8. *For any path P_n with $n \geq 2$, we have $\gamma_{ps}(P_n) = \left\lceil \frac{3n}{7} \right\rceil$.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$. Clearly $\gamma_{ps}(P_n) \geq \gamma_s(P_n) = \left\lceil \frac{3n}{7} \right\rceil$ by Theorem 1.3. If $n \equiv x(mod\ 7)$, let $S = \{v_m : m - x \equiv 2, 4$ or $6(mod\ 7)$ and $m \geq x\}$. We consider the following cases.

Case 1. $n \equiv 0(mod\ 7)$.

Let $v_x \notin S$.

If $x \equiv 0(mod\ 7)$, then v_x can only be defended by v_{x-1} since $v_{x+1} \notin S$ and v_{x-2} is still dominated by v_{x-3}.

If $x \equiv 1(mod\ 7)$, then v_1 can only be defended by v_2 and for $x > 7$, v_x can only be defended by v_{x+1}, since $v_{x-1} \notin S$ and v_{x+2} is still dominated by v_{x+3}.

If $x \equiv 3(mod\ 7)$, then v_x cannot be defended by v_{x-1}, since then v_{x-2} is not dominated. So v_x can only be defended by v_{x+1} as v_{x+2} is still dominated by v_{x+3}.

If $x \equiv 5(mod\ 7)$, then v_x cannot be defended by v_{x+1}, since then v_{x+2} is not dominated. So v_x can only be defended by v_{x-1} as v_{x-2} is still dominated by v_{x-3}. Thus S is a perfect secure dominating set of P_n of order $\left\lceil \frac{3n}{7} \right\rceil$.

Case 2. $n \equiv 1(mod\ 7)$.

Add to S the vertices v_2 and v_4 and delete v_3. Then v_1 is defended only by v_2. Also v_3 cannot be defended by v_2, since otherwise v_1 will not be dominated. Thus v_3 is defended only by v_4. Since $v_8 \notin S$, v_6 is defended only by v_5. Similarly, v_8 is defended only by v_7 and v_9 is defended only by v_{10}. Now by Case 1, each of the vertices in $V(P_n) \setminus S$ is defended by a unique vertex in S. Thus S is a perfect secure dominating set and $|S| = \left\lceil \frac{3n}{7} \right\rceil$.

Case 3. $n \equiv 2(mod\ 7)$.

Add to S the vertex v_1. Since $v_3 \notin S$, v_2 is defended only by v_1. Since $v_2 \notin S$, v_3 is defended only by v_4 and v_5 is dominated by v_6. Further by Case 1, each of the vertices in $V(P_n) \setminus S$ is defended by a unique vertex in S. Thus S is a perfect secure dominating set and $|S| = \left\lceil \frac{3n}{7} \right\rceil$.

Case 4. $n \equiv 3(mod\ 7)$.

Add to S the vertices v_1 and v_2. Since $v_4 \notin S$, v_3 is defended only by v_2. Since $v_3 \notin S$, v_4 is defended only by v_5 and v_6 is dominated by v_7. Further by Case 1, each of the vertices in $V(P_n) \setminus S$ is defended by a unique vertex in S. Thus S is a perfect secure dominating set and $|S| = \left\lceil \frac{3n}{7} \right\rceil$.

Case 5. $n \equiv 4(mod\ 7)$.

Add to S the vertices v_1 and v_3. Since both v_4 and v_5 are not in S, v_2 is defended only by v_1. Since $v_5 \notin S$, v_4 is defended only by v_3 and since $v_4 \notin S$, v_5 is defended only by v_6. Further by Case 1, each of the vertices in $V(P_n) \setminus S$ is defended by a unique vertex in S. Thus S is a perfect secure dominating set and $|S| = \left\lceil \frac{3n}{7} \right\rceil$.

Case 6. $n \equiv 5(mod\ 7)$.

Add to S the vertices v_2, v_3 and v_4. Then v_1 is defended only by v_2, v_5 is defended only by v_4, and v_6 is defended only by v_7. Further by Case 1, each of the vertices in $V(P_n) \setminus S$ is defended by a unique vertex in S. Thus S is a perfect secure dominating set and $|S| = \left\lceil \frac{3n}{7} \right\rceil$.

Case 7. $n \equiv 6(mod\ 7)$.

Add to S the vertices v_1, v_4 and v_5. Then v_2 is defended only by v_1, v_3 is defended only by v_4, v_6 is defended only by v_5 and v_7 is defended only by v_8. Further by Case 1, each of the vertices in $V(P_n) \setminus S$ is defended by a unique vertex in S. Thus S is a perfect secure dominating set and $|S| = \left\lceil \frac{3n}{7} \right\rceil$. □

Theorem 2.9. *For any cycle C_n with $n \geq 4$,*

$$\gamma_{ps}(C_n) = \begin{cases} \left\lceil \frac{3n}{7} \right\rceil + 1 & \text{if } n \equiv 2(mod\ 7) \\ \left\lceil \frac{3n}{7} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Since $\gamma_{ps}(C_n) \geq \gamma_s(C_n)$, it follows from Theorem 1.4 that $\gamma_{ps}(C_n) \geq \left\lceil \frac{3n}{7} \right\rceil$. Now let S be the same set defined in Theorem 2.8 for each of the respective cases. If $n \equiv 0, 1$ or $5(mod\ 7)$, then neither v_1 nor v_n are in S. Then v_1 is defended only by v_2 and v_n is defended only by v_{n-1}. Since S is a perfect secure dominating set of P_n, all the remaining vertices are uniquely defended. If $n \equiv 4$ or $6(mod\ 7)$, then $v_1 \in S$ and $S_1 = (S \setminus \{v_1\}) \cup \{v_2\}$ is a perfect secure dominating set of C_n. If $n \equiv 3(mod\ 7)$, then $v_{n-1} \in S$ and $S_1 = (S \setminus \{v_{n-1}\}) \cup \{v_{n-2}\}$ is a perfect secure dominating set of C_n. In all these cases $|S| = |S_1| = \left\lceil \frac{3n}{7} \right\rceil$ and hence $\gamma_{ps}(C_n) \leq \left\lceil \frac{3n}{7} \right\rceil$. Thus $\gamma_{ps}(C_n) = \left\lceil \frac{3n}{7} \right\rceil$.

Now, let $n \equiv 2(mod\ 7)$. Let $n = 7k + 2$, so that $\left\lceil \frac{3n}{7} \right\rceil + 1 = 3k + 2$. Then $\left(\bigcup_{i=0}^{k-2} \{v_{7i+2}, v_{7i+4}, v_{7i+6}\} \right) \cup \{v_{7k-5}, v_{7k-3}, v_{7k-2}, v_{7k-1}, v_{7k+1}\}$ is a perfect secure dominating set of C_n of cardinality $3k+2$. Hence $\gamma_{ps}(C_n) \le 3k+2 = \lceil \frac{3n}{7} \rceil + 1$.

Now let S be a γ_{ps}-set of C_n, S be independent, and $v_1 \in S$. Then exactly one of v_3, v_4 is in S. Suppose $v_4 \in S$. Since v_3 is defended by v_4, it follows that $v_6 \in S$ and v_5 is defended by v_6. Hence $v_8 \in S$. Now if $v_{10} \in S$, then v_7 is defended by v_6 and v_8, which is a contradiction. Thus $v_{10} \notin S$. Thus by the previous argument v_{11}, v_{13}, and v_{15} are in S. Continuing this process v_{7i+4}, v_{7i+6}, and v_{7i+8} are in S where $1 \le i \le k-1$. Thus $\{v_{7k-3}, v_{7k-1}, v_{7k+1}, v_1\} \subseteq S$ and v_{7k} is defended by both v_{7k-1} and v_{7k+1} which is a contradiction. Hence S is not independent and we may assume that $v_1, v_2 \in S$. Then $T = S \backslash \{v_1, v_2\}$ is a perfect secure dominating set of the path $P_{n-4} = (v_4, v_5, v_6, \ldots, v_{n-1})$, and so $|T| \ge \left\lceil \frac{3(n-4)}{7} \right\rceil = \lceil \frac{21k-6}{7} \rceil = 3k$. Thus $\gamma_{ps}(C_n) = |S| \ge 3k+2$. Hence $\gamma_{ps}(C_n) = \lceil \frac{3n}{7} \rceil + 1$. □

Theorem 2.10. *For the complete bipartite graph $G = K_{r,s}$ with $r \le s$ we have*

$$\gamma_{ps}(G) = \begin{cases} s & \textit{if } r = 1 \\ 2 & \textit{if } r = s = 2 \\ r+s & \textit{otherwise.} \end{cases}$$

Proof. Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be the bipartition of G.

Let $S = \begin{cases} (Y \setminus \{y_s\}) \cup \{x_1\} & \text{if } r = 1 \\ \{x_1, y_1\} & \text{if } r = s = 2 \\ V(G) & \text{otherwise.} \end{cases}$

Clearly S is a perfect secure dominating set of G and hence $\gamma_{ps}(G) \le |S|$. Now, let S be any γ_{ps}-set of G.

If $r = 1$, then $|S| \ge s$. Now, suppose $r, s \ge 2$. Since S is a perfect dominating set of G, $|S \cap X| = 1$ or $|X|$ and $|S \cap Y| = 1$ or $|Y|$. Also if $s \ge 3$, then $|S \cap X| = |X|$ and $|S \cap Y| = |Y|$. Hence it follows that

$$\gamma_{ps}(G) \ge \begin{cases} s & \text{if } r = 1 \\ 2 & \text{if } r = s = 2 \\ r+s & \text{otherwise.} \end{cases}$$

This completes the proof. □

Theorem 2.11. *Let $G = K_{n_1,n_2,\ldots,n_r}$, where $1 \le n_1 \le n_2 \le \cdots \le n_r$ and $r \ge 3$. Let $n = \sum_{i=1}^{r} n_i$. Then*

$$\gamma_{ps}(G) = \begin{cases} 1 & \text{if } n_i = 1 \text{ for all } i \\ 2 & \text{if } n_1 = 1 \text{ and } n_2 = 2 \\ n & \text{otherwise.} \end{cases}$$

Proof. Let $V_1, V_2, \ldots, V_r$ be the partite sets of G with $|V_i| = n_i$. If $n_i = 1$ for all i, then $G = K_r$ and $\gamma_{ps}(G) = 1$. If $n_1 = 1$ and $n_2 = 2$, then G is not complete and hence $\gamma_{ps}(G) > 1$. Further $S = \{v_1, v_2\}$, where $V_1 = \{v_1\}$ and $v_2 \in V_2$ is a perfect secure dominating set of G and hence $\gamma_{ps}(G) = 2$.

Now suppose $(n_1, n_2) \ne (1, 2)$ and $n_i \ge 2$ for at least one i. Let S be a γ_{ps}-set of G and suppose $S \ne V$. If $S \subseteq V_i$ for some i, then $S = V_i$ and every vertex of $V \backslash S$ is defended by every vertex of S. Hence $S \cap V_i \ne \emptyset$ for at least two values of i, say i_1 and i_2. Let $x \in S \cap V_{i_1}$ and $y \in S \cap V_{i_2}$. Then any vertex of $V \setminus (S \cup V_{i_1} \cup V_{i_2})$ is defended by both x and y. Hence it follows that $S = V$ and $\gamma_{ps}(G) = n$. □

We now proceed to determine the value of γ_{ps} for caterpillars. Two support vertices s_1, s_2 of a caterpillar T are said to be consecutive if all the internal vertices of the unique s_1-s_2 path are of degree 2.

Theorem 2.12. *Let T be a caterpillar with k support vertices $s_1, s_2, \ldots, s_k$ such that s_i and s_{i+1} are consecutive and $d(s_i, s_{i+1}) = a_i$. Then $\gamma_{ps}(T) = l + \sum_i \gamma_{ps}(P_{a_i - 1})$, where the summation is taken over all i with $a_i > 1$ and l is the number of leaves of T.*

Proof. Let $S = \{s_1, s_2, \ldots, s_k\}$ and let L denote the set of all leaves of T. Choose one leaf v_i adjacent to $s_i, 1 \le i \le k$. Let P_i be the s_i-s_{i+1} path in T, where $a_i > 1$. Then $P_i' = P_i \setminus \{s_i, s_{i+1}\}$ is a subpath of P_i with a_{i-1} vertices. Let X_i be a γ_{ps}-set of P_i'. Then

$$D = S \cup (L \setminus \{v_1, v_2, \ldots, v_k\})(\bigcup_i X_i)$$

where the union is taken over all i with $a_i > 2$ is a perfect secure dominating set of T. Hence $\gamma_{ps}(T) \le |D| = l + \sum_i \gamma_{ps}(P_{a_i - 1})$. Now, let D_1 be any γ_{ps}-set of T. Obviously $D_1 \supseteq D$ and hence the reverse inequality follows. □

We now proceed to determine γ_{ps} for wheels. We observe that $\gamma_{ps}(W_4) = 1$ and $\gamma_{ps}(W_5) = 5$.

Theorem 2.13. *Let $n \geq 6$. Then*

$$\gamma_{ps}(W_n) = \begin{cases} k & \text{if } n = 3k \\ k+1 & \text{if } n = 3k+1 \\ k+2 & \text{if } n = 3k+2. \end{cases}$$

Proof. Let $W_n = C_{n-1} + \{v_n\}$ where $C_{n-1} = (v_1, v_2, \ldots, v_{n-1}, v_1)$. Let

$$S = \begin{cases} \{v_1\} \cup \{v_{3i} : 2 \leq i \leq k\} & \text{if } n = 3k \\ \{v_1, v_6\} \cup \{v_{3i+1} : 2 \leq i \leq k\} & \text{if } n = 3k+1 \\ \{v_1, v_6, v_n\} \cup \{v_{3i+1} : 2 \leq i \leq k\} & \text{if } n = 3k+2. \end{cases}$$

It can be easily verified that S is a minimal perfect secure dominating set of W_n and hence

$$\gamma_{ps}(W_n) \leq |S| = \begin{cases} k & \text{if } n = 3k \\ k+1 & \text{if } n = 3k+1 \\ k+2 & \text{if } n = 3k+2. \end{cases}$$

Now to prove the reverse inequality we first assume that $n = 3k$. Let S be a minimal perfect secure dominating set of W_n. If $v_n \notin S$, then any vertex of $V(C_{n-1}) \cap S$ defends v_n which is a contradiction. Hence $v_n \in S$. Suppose $|S| < k$. Then $|S \cap V(C_{n-1})| \leq k - 2$. Let $l_1, l_2, \ldots, l_s$ be the lengths of the segments of C_{n-1} determined by the vertices of $S \setminus \{v_n\}$ where $s = |S| - 1$. Since $l_1 + l_2 + \cdots + l_s = 3k - 1$ and $s < k - 1$, it follows that $l_i \geq 4$ for at least one i. Now if each $l_i = 4$ then the middle vertex of each of the segment is not defended by any vertex of S. Hence it follows that $l_i \geq 5$ for at least one i, say $i = 1$. Also if $l_1 \geq 6$ then S is not a secure dominating set. Hence $l_1 = 5$. Now consider the path P of C_{n-1} obtained by removing all the six vertices of the segment of length l_1. Then $S \cap V(P)$ is a perfect secure dominating set of P and hence $|S \cap V(P)| \geq \left\lceil \frac{3(3k-7)}{7} \right\rceil$. Thus $k - 4 \geq \left\lceil \frac{9k}{7} - 3 \right\rceil = \left\lceil \frac{9k}{7} \right\rceil - 3$, and so $k \geq \left\lceil \frac{9k}{7} \right\rceil + 1$, a contradiction. Hence $|S| \geq k$. The proof is similar if $n = 3k+1$ or $n = 3k+2$. $\square$

Theorem 2.14. *Given two positive integers a and b with $a \leq b$, there exists a graph G with $\gamma_s(G) = a$ and $\gamma_{ps}(G) = b$.*

Proof. **Case 1.** $a = b$.

Choose n such that $\lceil \frac{3n}{7} \rceil = a$. Then it follows from Theorems 1.3 and 2.8 that $\gamma_{ps}(P_n) = \gamma_s(P_n) = a$.

Case 2. $a + 1 = b$.

Let G be the graph obtained from the cycle $C_7 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_1)$ by adding a new vertex v_8 and joining it to v_i, where $2 \leq i \leq 7$, joining v_1 to v_i where $i \neq 4$ and attaching $a-2$ pendant vertices $u_1, u_2, \ldots, u_{a-2}$ adjacent to v_8. Then $S_1 = \{v_1, v_8, u_1, u_2, \ldots, u_{a-2}\}$ is a γ_s-set of G and $S_2 = \{v_1, v_3, v_4, u_1, u_2, \ldots, u_{a-2}\}$ is γ_{ps}-set of G.

Thus $\gamma_s(G) = a$ and $\gamma_{ps}(G) = a + 1 = b$.

Case 3. $a + 2 = b$.

If $a = 2$, then $\gamma_s(K_4 - \{e\}) = a$ and $\gamma_{ps}(K_4 - \{e\}) = 4 = b$. If $a \geq 3$, let G be the graph obtained from $K_{2,a}$ with bipartition $V_1 = \{v_1, v_2\}$ and $V_2 = \{u_1, u_2, \ldots, u_a\}$ by attaching a pendant vertex w_i adjacent to u_i, where $1 \leq i \leq a-2$.

Then $S_1 = \{v_1, v_2, u_1, u_2, \ldots, u_{a-2}\}$ is a γ_s-set of G and $S_2 = \{v_1, v_2, u_1, u_2, \ldots, u_a\}$ is a γ_{ps}-set of G.

Thus $\gamma_s(G) = a$ and $\gamma_{ps}(G) = a + 2 = b$.

Case 4. $b \geq a + 3$.

Let G be the graph obtained from $K_{2,b-a}$ with bipartition $V_1 = \{v_1, v_2\}$ and $V_2 = \{u_1, u_2, \ldots, u_{b-a}\}$ and $a-2$ copies of K_3 and joining one vertex w_i of each copy of K_3 to v_i. Then $S_1 = \{v_1, v_2, w_1, w_2, \ldots, w_{a-2}\}$ is a γ_s-set of G and $S_2 = S_1 \cup V_2$ is a γ_{ps}-set of G.

Thus $\gamma_s(G) = a$ and $\gamma_{ps}(G) = b$. $\square$

3 Perfect secure domination and graph operations

In this section we determine $\gamma_{ps}(G + H)$ and $\gamma_{ps}(G \square H)$ for any two graphs G and H.

Theorem 3.1. *Let G and H be two connected graphs. Then $\gamma_{ps}(G \square H) \leq \min\{\gamma_{ps}(G)|V(H)|, \gamma_{ps}(H)|V(G)|\}$.*

Proof. Let S be a perfect secure dominating set of G. We claim that $S_1 = S \times V(H)$ is a perfect secure dominating set of $G \square H$. Let $(u_i, v_j) \in (V(G) \times V(H)) \setminus S_1$. Then $u_i \notin S$. Let u_i be S-defended by the vertex u_r in S. Then the vertex (u_i, v_j) is S_1-defended by (u_r, v_j). Further any vertex in

S_1 which S_1-defends (u_i, v_j) is of the form (u_k, v_j) where $u_k \in S$ and u_k S-defends u_i. Hence (u_i, v_j) is S_1-defended by exactly one vertex in S_1, so that S_1 is a perfect secure dominating set of $G \Box H$. Thus $\gamma_{ps}(G \Box H) \leq |S_1| \leq \gamma_{ps}(G)|V(H)|$. Similarly $\gamma_{ps}(G \Box H) \leq \gamma_{ps}(H)|V(G)|$ and hence the result follows. $\Box$

The following theorem shows that the bound in Theorem 3.1 is sharp.

Theorem 3.2. *Let H be any graph of order at most n and $G = K_n \Box H$. Then $\gamma_{ps}(G) = |V(H)|$.*

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{w_1, w_2, \ldots, w_r\}$. Since $\gamma_{ps}(K_n) = 1$ and $|V(H)| \leq n$, it follows from Theorem 3.1 that $\gamma_{ps}(G) \leq \min\{|V(H)|, \gamma_{ps}(H)n\} = |V(H)|$. Now, let S be a γ_{ps}-set of G. Suppose $|S| < |V(H)| \leq n$. Then there exists $v_i \in V(K_n)$ such that $S \cap (\{v_i\} \times H) = \emptyset$. Now the only possible neighbors of any vertex (v_i, w_j) outside $\{v_i\} \times H$ is of the form (v_i, w_k) and hence no two vertices in $\{v_i\} \times H$ are adjacent to the same vertex in $V(G) \setminus (\{v_i\} \times H)$. Since $|S| < n$, it follows that at least one vertex in $\{v_i\} \times H$ is not dominated by a vertex in S, which is a contradiction. Thus $|S| \geq |V(H)|$ and $\gamma_{ps}(G) = |V(H)|$. $\Box$

We now proceed to determine $\gamma_{ps}(G_1 + G_2)$. For this purpose we introduce a property, which we denote by J. Any maximal complete subgraph of G is called a clique in G.

Property J**.** Let G be a graph of order n. A vertex $u \in V(G)$ is said to have property J if *deg* $u < n - 1$ and $V \setminus N[u]$ is a clique in G.

Theorem 3.3. *Let G and H be two graphs of order n_1 and n_2 respectively with $n_1, n_2 \geq 2$. Then $\gamma_{ps}(G + H) = 1$ or 2 or $n_1 + n_2$. Further*

(i) $\gamma_{ps}(G + H) = 1$ *if and only if G and H are complete.*

(ii) $\gamma_{ps}(G + H) = 2$ *if and only if both G and H have a vertex that satisfy property J.*

(iii) *If $\gamma_{ps}(G + H) \neq 1$ or 2, then $\gamma_{ps}(G + H) = n_1 + n_2$.*

Proof. Since $\gamma_{ps}(G) = 1$ if and only if G is complete, (i) follows.

Now, let $\gamma_{ps}(G + H) = 2$ and let $S = \{u, v\}$ be a perfect secure dominating set of $G + H$. If both $u, v \in V(G)$, then every vertex of H is defended by both u and v which is a contradiction. Similarly both u and v cannot be in $V(H)$. Hence we assume that $u \in V(G)$ and $v \in V(H)$. We claim that u and v satisfy the property J. If $deg_G(u) = n_1 - 1$, then any vertex of $V(G) \setminus \{u\}$ is defended by both u and v, which is a contradiction. Hence $deg_G(u) < n_1 - 1$ and $V(G) \setminus N_G[u] \neq \emptyset$. Now, suppose there exist two non-adjacent vertices x and y in $V(G) \setminus N_G[u]$. Then x and y are defended by v. However $\{u, x\}$ does not dominate y and $\{u, y\}$ does not dominate x, which is a contradiction. Thus $\langle V(G) \setminus N_G[u] \rangle$ is complete. Now, if $V(G) \setminus N_G[u]$ is not a clique in G, then there exists a vertex w in $N(u)$ such that w is adjacent to every vertex in $V(G) \setminus N_G[u]$. Clearly w is defended by both u and v, a contradiction. Thus $V(G) \setminus N_G[u]$ is a clique in G and u satisfies property J in G. Similarly v satisfies property J in H.

Conversely, suppose there exist vertices u in G and v in H that satisfy the property J. Let $S = \{u, v\}$. Then any vertex in $V(G) \setminus N_G[u]$ is uniquely defended by v. Further, since $V(G) \setminus N_G[u]$ is a clique, any vertex $w \in N(u)$ is non-adjacent to at least one vertex in $V(G) \setminus N_G[u]$ and hence w is uniquely defended by v. Similarly any vertex in $V(H) \setminus \{v\}$ is uniquely defended by u and hence S is a perfect secure dominating set of $G+H$. Thus $\gamma_{ps}(G + H) = 2$. It now remains to show that if $\gamma_{ps}(G + H) \neq 1$ or 2, then $\gamma_{ps}(G + H) = n_1 + n_2$. Suppose there exists a perfect secure dominating set S of $G + H$ with $|S| < n_1 + n_2$. Since $|S| \geq 3$, we may assume without loss of generality that $|S \cap V(G)| \geq 2$. Let $v_1, v_2 \in V(G) \cap S$. If $V(H) \subseteq S$, then any vertex in $V(G) \setminus S$ is defended by every vertex of H, a contradiction. Hence there exists $u \in V(H) \setminus S$ and now u is defended by both v_1, v_2 which is again a contradiction. Hence $\gamma_{ps}(G_1 + G_2) = n_1 + n_2$. □

Corollary 3.4. *If $r \geq 2$ and $s \geq 3$, then $\gamma_{ps}(K_{r,s}) = r + s$.*

Proof. Since $K_{r,s} = \overline{K}_r + \overline{K}_s$, no vertex of $\overline{K}_r$ satisfies property J and $\overline{K}_r + \overline{K}_s$ is not complete, it follows that $\gamma_{ps}(K_{r,s}) = r + s$. □

This gives an alternate proof of Theorem 2.10.

Property P. Let $G = (V, E)$ be a graph. A subset S of V is said to satisfy property P if $V(G) \setminus N_G[S]$ is a clique in G and $(N_G(a) \cap N_G(b)) \setminus S = \emptyset$ for all $a, b \in S$.

Note that if $|S| = 1$, then the property P is the same as the property J.

Theorem 3.5. *Let G be a graph of order n and let $H = G + K_1$ where $V(K_1) = \{v_1\}$. If $\gamma_{ps}(H) = 1$ or $n + 1$, then for any subset $S \subseteq V(G)$, the* *set $S_1 = S \cup \{v_1\}$ is a perfect secure dominating set of H if and only if S satisfies property P and $(N_G(a) \cap N_G(b)) \setminus (S \cup \{v_1\}) = \emptyset$ for all $a, b \in S$.*

Proof. Let $S_1 = S \cup \{v_1\}$ be a perfect secure dominating set of H. Suppose $N_G[S] = V(G)$. Let $w \in V(G) \setminus S$ and let v be a vertex in S that is adjacent to w. Then w is S-defended by both v and v_1 which is a contradiction. Hence $V(G) \setminus N_G[S] \neq \emptyset$. If there exist two vertices a, b in $V(G) \setminus N_G[S]$ which are nonadjacent, then both a and b are not S_1-defended. Hence $\langle V(G) \setminus N_G[S] \rangle$ is complete. Now if $V(G) \setminus N_G[S]$ is not a clique in G, then there exists a vertex $a \in N(S)$ such that a is adjacent to every vertex in $V(G) \setminus N_G[S]$. If b is any vertex in S which is adjacent to a, then a is S_1-defended by both b and v_1, which is a contradiction. Hence $V(G) \setminus N_G[S]$ is a clique in G. Now if there exist two vertices a, b in S such that $X = (N_G(a) \cap N_G(b)) \setminus S \neq \emptyset$, then any vertex of X is S_1-defended by both a and b. Hence $X = \emptyset$ and S satisfies the property P. Clearly $(N_G(a) \cap N_G(b)) \setminus (S \cup \{v_1\}) = \emptyset$.

Conversely, let S be a subset of $V(G)$ such that S satisfies the property P and $(N_G(a) \cap N_G(b)) \setminus (S \cup \{v_1\}) = \emptyset$. Let $S_1 = S \cup \{v_1\}$ and let $x \notin S_1$. If $x \in N(S)$, choose $w \in S$ such that w is adjacent to x. Since $v_1 \in S_1$, it follows that x is S_1-defended by w. Also since S satisfies the property P, $N_G(w) \cap N_G(u) \setminus S = \emptyset$ for all $u \in S \setminus \{w\}$. Hence $x \notin N_G(u)$ and x is S_1-defended only by w. Now, suppose $x \notin N(S)$. Then x is S_1-defended only by v_1. Thus S_1 is a perfect secure dominating set of H. □

4 Changing and unchanging of perfect secure domination

In this section we examine the effects on $\gamma_{ps}(G)$ when a vertex or an edge is deleted from G. A detailed study of such results for the domination number is given in Chapter 5 of Haynes et al. [8].

For the complete graph K_n with $n \geq 3$, we have $\gamma_{ps}(K_n) = \gamma_{ps}(K_n \setminus \{v\}) = 1$. Now let G be the graph obtained from the complete graph K_n with $n \geq 3$ by adding a vertex x and joining x to two vertices v_1 and v_2 in K_n. It follows from Lemma 2.5 that $\gamma_{ps}(G) = n + 1$. Further $\{v_2, x\}$ is a perfect secure dominating set of $G \setminus \{v_1\}$ and $G \setminus \{v_1 v_2\}$ and hence $\gamma_{ps}(G \setminus \{v_1\}) = \gamma_{ps}(G \setminus \{v_1 v_2\}) = 2$. This example also shows that addition of an edge may result in increase of γ_{ps} and the increase can be made arbitrarily large. Also if G is the graph obtained from the complete bipartite graph

$K_{2,n}$ by adding a vertex x and joining x to a vertex v in the partite set V_1 with $|V_1| = 2$, then $\gamma_{ps}(G) = 2$ and $\gamma_{ps}(G \setminus \{x\}) = n + 2$.

We observe that $\gamma_{ps}(K_n) = 1$ and $\gamma_{ps}(K_n \setminus \{e\}) = n$ for all $n \geq 4$. For the cycle C_{7k} we have $\gamma_{ps}(C_{7k}) = \gamma_{ps}(C_{7k} \setminus \{e\}) = 3k$, by Theorems 2.8 and 2.9.

Thus γ_{ps} may increase or decrease arbitrarily or remain the same when a vertex or an edge is deleted. Hence as in the case of domination each of the sets $V(G)$ and $E(G)$ can be partitioned into three subsets as follows.

$V^0 = \{v \in V(G) : \gamma_{ps}(G \setminus \{v\}) = \gamma_{ps}(G)\}$,
$V^+ = \{v \in V(G) : \gamma_{ps}(G \setminus \{v\}) > \gamma_{ps}(G)\}$,
$V^- = \{v \in V(G) : \gamma_{ps}(G \setminus \{v\}) < \gamma_{ps}(G)\}$,
$E^0 = \{e \in E(G) : \gamma_{ps}(G \setminus \{e\}) = \gamma_{ps}(G)\}$,
$E^+ = \{e \in E(G) : \gamma_{ps}(G \setminus \{e\}) > \gamma_{ps}(G)\}$ and
$E^- = \{e \in E(G) : \gamma_{ps}(G \setminus \{e\}) < \gamma_{ps}(G)\}$.

We observe that some of these sets may be empty. For example for any graph G with $\gamma_{ps}(G) = n$, we have $V(G) = V^-$, so that $V^+ = V^0 = \emptyset$. For K_n with $n \geq 4$, we have $V(K_n) = V^0$ and $E(K_n) = E^+$.

Several properties of vertices in V^0, V^+, V^- and edges in E^0, E^+, E^- with respect to the domination number γ are given in Chapter 5 of Haynes et al. [8]. Similar investigation can be taken up for γ_{ps} and results in this direction will be reported in a subsequent paper.

5 Conclusion and scope

We have given several families of graphs G for which $\gamma_{ps}(G) = |V(G)| = n$. Hence the following problems arise naturally.

Problem 1. *Characterize graphs G of order n for which $\gamma_{ps}(G) = n$.*

Since $\gamma_s(G) \leq \gamma_{ps}(G)$ for any graph G, we have the following problem.

Problem 2. *Characterize graphs G for which $\gamma_{ps}(G) = \gamma_s(G)$.*

Further the concepts related to changing and unchanging of $\gamma_{ps}(G)$ for vertex or edge removal can be further investigated.

Acknowledgements

Kiran R. Bhutani and Peter Gartland gratefully acknowledge support from The Catholic University of America through Grant-in-Aid funding. The authors express their special thanks to the editor for timely processing and to the referee for thorough review of this manuscript.

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