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Filters of Coz(X)

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Dedicated to Bernhard Banaschewski on the occasion of his 90th birthday

Abstract. In this article we investigate filters of cozero sets for real-valued continuous functions, called *coz*-filters. Much is known for z-ultrafilters and their correspondence with maximal ideals of $C(X)$. Similarly, a correspondence will be established between *coz*-ultrafilters and minimal prime ideals of $C(X)$. We will further notice various properties of *coz*-ultrafilters in relation to P-spaces and F-spaces. In the last two sections, the collection of *coz*-ultrafilters will be topologized, and then compared to the hull-kernel and the inverse topologies placed on the collection of minimal prime ideals of $C(X)$ and general lattice-ordered groups.

1 Introduction and notation

In parallel to the notion of z-filters, we first consider the relationship between the *coz*-filters and the prime ideals of $C(X)$. In particular, we will show that maximal *coz*-filters (*coz*-ultrafilters) have a direct correspondence with the minimal prime ideals of $C(X)$. The inspiration for this comes from the

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known relationship between maximal ideals of $C(X)$ and z-ultrafilters. This is the content of Section 2.

The focus of Section 3 is topologizing the collection of all *coz*-ultrafilters. In particular, we endow topologies on these collections which are similar to the inverse and the hull-kernel topologies. It is shown that these topological spaces are homeomorphic to spaces of minimal prime ideals of $C(X)$.

In the last section we prove some general results for lattice-ordered groups. In particular, we show that the space of ultrafilters of G^+, with respect to the Stone topology, is homeomorphic to $\mathrm{Min}(G)^{-1}$, for an ℓ-group G. Moreover, for a **W**-object G, there is a one-to-one correspondence between the ultrafilters of $\mathrm{Coz}(G)$ and the ultrafilters of G^+.

Throughout this paper X is a Tychonoff space, that is, X is a completely regular Hausdorff space. Also, all rings R are commutative with identity. The ring of real-valued continuous functions on X is denoted by $C(X)$, or just simply C. We mention a few definitions that are essential in this article. For the rest of the concepts see [1], [11], [13], and [14].

If $f \in C(X)$, then the *zero set* of f is $z(f) = \{x \in X : f(x) = 0\}$ and the *cozero set* of f is $coz(f) = \{x \in X : f(x) \neq 0\}$; $z(f)$ and $coz(f)$ are set-theoretic complements of each other in X. Denote the collection of all zero sets of X as $\mathrm{Z}(X)$ and the collection of all cozero sets of X as $\mathrm{Coz}(X)$. A nonempty subfamily $\mathcal{F} \subseteq \mathrm{Coz}(X)$ is a *coz-filter* on X if the following holds:

1. $\emptyset \notin \mathcal{F}$;
2. if $A, B \in \mathcal{F}$, than $A \cap B \in \mathcal{F}$; and
3. if $A \in \mathcal{F}$, $B \in \mathrm{Coz}(X)$, and $A \subseteq B$, then $B \in \mathcal{F}$.

Observe that the above definition is the counterpart of the concept of *z-filter* from [11]. A *prime coz-filter* is a *coz*-filter $\mathcal{F}$ such that if $coz(f) \cup coz(g) \in \mathcal{F}$, then $coz(f) \in \mathcal{F}$ or $coz(g) \in \mathcal{F}$. A *coz*-filter $\mathcal{U} \subseteq \mathrm{Coz}(X)$ is a *coz-ultrafilter* if whenever $\mathcal{U} \subset \mathcal{F}$, where $\mathcal{F}$ is a *coz*-filter, then $\mathcal{F} = \mathrm{Coz}(X)$.

A subset M of a ring R is called a *multiplicative set* if $1 \in M$, and for all $f, g \in M$, the product $fg \in M$. A ring is called *reduced* if it has no non-zero nilpotent elements; hence, $C(X)$ is a reduced ring. A proper ideal P of R is called a *prime ideal* if $ab \in P$ implies $a \in P$ or $b \in P$. A *minimal prime* *ideal* of a ring is a prime ideal which contains no other smaller prime ideal. The collection of all minimal prime ideals of R is denoted by $\mathrm{Min}(R)$.

The following results are well-known and will be used often in this article.

Lemma 1.1. *If $f \in C(X)$, then the following are equivalent:*

1. *f is a non-zero-divisor of $C(X)$, that is, $Ann(f) = 0$.*

2. *$coz(f)$ is dense in X, that is, $cl(coz(f)) = X$.*

3. *$z(f)$ has empty interior, that is, $int(z(f)) = \emptyset$.*

Similarly, notice that given $f \in C(X)$, $f = 0$ precisely when $coz(f)$ is an empty set; this is equivalent to $z(f) = X$.

For more on Lemma 1.1 refer to Section 5 in [7] and Theorem 2.2 in [9].

Proposition 1.2. *Let $\mathcal{U}$ be a coz-filter. Then $\mathcal{U}$ is a coz-ultrafilter if and only if given any $coz(f) \notin \mathcal{U}$, there exists some $coz(g) \in \mathcal{U}$ such that $fg = 0$ (equivalently, $coz(f) \cap coz(g) = \emptyset$). As a consequence, $coz(f)$ is dense in X if and only if $coz(f) \in \mathcal{U}$ for every coz-ultrafilter $\mathcal{U}$.*

It is known that an ultrafilter is a prime filter. This is true for *coz*-ultrafilters as well. The next result follows from the preceding proposition.

Corollary 1.3. *If $\mathcal{U}$ is a coz-ultrafilter, then it is a prime coz-filter.*

2 Prime ideals and *coz*-filters

Gillman and Jerison devoted the second chapter of their text *Rings of Continuous Functions* to the relationship between the ideals of C and the z-filters. Here we provide a few of their results as a reference for the inspiration of the results which follow:

Theorem 2.1 (Theorem 2.3, [11])**.** (1) *If I is an ideal in C, then $Z(I) = \{z(f) : f \in I\}$ is a z-filter on X.*

(2) *If $\mathcal{F}$ is a z-filter on X, then $Z^{\leftarrow}(\mathcal{F}) = \{f : z(f) \in \mathcal{F}\}$ is an ideal in C.*

If I is an ideal of C, then I is a *z-ideal* if $z(f) \in \mathrm{Z}(I)$ implies $f \in I$, that is to say if $I = \mathrm{Z}^{\leftarrow}(\mathrm{Z}(I))$. [Note, $\mathcal{F} = \mathrm{Z}(\mathrm{Z}^{\leftarrow}(\mathcal{F}))$ is always true for a z-filter $\mathcal{F}$.]

With z-ideals more can be said about the relationship between the prime ideals of C and the prime z-filters.

Theorem 2.2 (Theorem 2.12, [11]). (1) *If P is a prime ideal of C, then $Z(P)$ is a prime z-filter.*

(2) *If $\mathcal{F}$ is a prime z-filter, then $Z^{\leftarrow}(\mathcal{F})$ is a prime z-ideal.*

Furthermore, it has been shown that the maximal ideals of C are in one-to-one correspondence to the z-ultrafilters on X.

The results from Gillman and Jerison are well-known, but the relationship between the ideals of C and the *coz*-filters on X are yet to be considered; this section is devoted to such an investigation. Through the results in this section we notice the difference between the z-filters and the *coz*-filters; to start, observe that we require the ideal of C to be prime to obtain a *coz*-filter of $\mathrm{Coz}(X)$.

If P is an ideal of C, then P^c is the set-theoretic complement of P in C.

Theorem 2.3. *If P is a prime ideal of $C(X)$, then $Coz(P^c) = \{coz(f) : f \in P^c\}$ is a coz-filter.*

Proof. The first two axioms of being a *coz*-filter can be easily verified. To show the third axiom we will require the primeness of P. Suppose $coz(f) \subseteq coz(g)$ for some $f \notin P$ and $g \in C(X)$. Since P is prime, using 5D in [11], it follows that $f^2 + g^2 \in P^c$. Therefore $coz(g) \in Coz(P^c)$, since $coz(g) = coz(f^2 + g^2)$. □

This result is different than the $\mathrm{Z}(I)$ case, since I need only be an ideal for $\mathrm{Z}(I)$ to be a z-filter. Primeness is needed since this is related to a multiplicative set which will become evident. The following example will demonstrate this fact.

Example 2.4. Let $C = C(\mathbb{N})$, where $\mathbb{N}$ is the natural numbers with the discrete topology. Recall that the characteristic function on $S \subseteq \mathbb{N}$ is χ_S. In this case, every characteristic function is continuous, since $\mathbb{N}$ is discrete. Now, consider $I =< \chi_{\{1\}} >$, the ideal generated by $\chi_{\{1\}}$. Then $z(\chi_{\{1\}}) = \mathbb{N}\backslash\{1\}$ and for any $f \in I$, $z(\chi_{\{1\}}) \subseteq z(f)$. It follows that $\chi_{\{2\}}$ and $\chi_{\{3\}}$

are not in I. So, $coz(\chi_{\{2\}}), coz(\chi_{\{3\}}) \in \mathrm{Coz}(I^c)$. However, $coz(\chi_{\{2\}}) \cap coz(\chi_{\{3\}}) = \{2\} \cap \{3\} = \emptyset$. Therefore, $\mathrm{Coz}(I^c)$ is not a *coz*-filter. Do note that $\chi_{\{1,2\}}\chi_{\{1,3\}} = \chi_{\{1\}} \in I$, but $\chi_{\{1,2\}}, \chi_{\{1,3\}} \notin I$, so I is not prime.

We know that there is a one-to-one correspondence between the z-ideals of C and the z-filters of X. Such a correspondence does not exist for *coz*-filters. In order to establish a relation between z-ideals and *coz*-filters, we need the prime condition on both. The next result is a counterpart of Theorem 2.12 in [11]; there is a one-to-one correspondence between the prime z-ideals of C and the prime *coz*-filters of X.

Theorem 2.5. *If P is a prime z-ideal of C, then Coz(P^c) is a prime coz-filter. If $\mathcal{F}$ is a prime coz-filter, and $M =$Coz$^{\leftarrow}(\mathcal{F}) = \{f : coz(f) \in \mathcal{F}\}$, then $P = C\backslash M$ is a prime z-ideal.*

Moreover, Coz$^{\leftarrow}$(Coz$(P^c)) = P^c$ and Coz(Coz$^{\leftarrow}(\mathcal{F})) = \mathcal{F}$, for all prime z-ideals P and all prime coz-filters $\mathcal{F}$.

Proof. The proof of the first part is left to the interested reader.

We show that P is a prime z-ideal. To show that P is an ideal, let $f, g \in P$. Then $coz(f), coz(g) \notin \mathcal{F}$. This implies that $coz(f) \cup coz(g) \notin \mathcal{F}$, since $\mathcal{F}$ is prime. Note that $coz(f - g) \subseteq coz(f) \cup coz(g)$. Since $\mathcal{F}$ is *coz*-filter, $coz(f - g) \notin \mathcal{F}$. Hence $f - g \notin M$; in other words, $f - g \in P$. Now let $f \in P$ and $h \in C$. Since $coz(f) \notin \mathcal{F}$ and $coz(fh) \subseteq coz(f)$, therefore $coz(fh) \notin \mathcal{F}$. Consequently, $fh \in P$. Furthermore, P is a proper ideal since given any unit $u \in C$, $coz(u) = X \in \mathcal{F}$, which means P does not contain any unit. Observe that for all $f, g \in C$, $fg \notin P$ whenever $f, g \notin P$, proving that P is prime. Thus P is a z-ideal by the definition of M.

Finally, it is easy to check that $\mathrm{Coz}(\mathrm{Coz}^{\leftarrow}(\mathcal{F}))= \mathcal{F}$, for all prime *coz*-filters $\mathcal{F}$. On the other hand, if $f \in \mathrm{Coz}^{\leftarrow}(\mathrm{Coz}(P^c))$, then $coz(f) = coz(g)$ for some $g \notin P$. Since P is a z-ideal, $f \in P^c$. Thus, $\mathrm{Coz}^{\leftarrow}(\mathrm{Coz}(P^c))\subseteq P^c$. The other inclusion can be verified easily. Hence the result follows. □

The following example shows that the prime condition of a *coz*-filter is required in the preceding theorem.

Example 2.6. Consider $C(\mathbb{N})$ and let $\mathcal{F} = \{A \subseteq \mathbb{N} : \{2n : n \in \mathbb{N}\} \subseteq A\}$, this is the collection of all the subsets of $\mathbb{N}$ which contain all of the even natural numbers. It is clear that $\mathcal{F}$ is a *coz*-filter since all subsets of $\mathbb{N}$ are clopen. Next, take $\{2, 4, 6, 8, \ldots\} \subseteq \mathcal{F}$, this is equal to $\{2\} \cup \{4, 6, 8, \ldots\}$. However, both $\{2\}$ and $\{4, 6, 8, \ldots\}$ are not in $\mathcal{F}$, so $\mathcal{F}$ is not a prime *coz*-filter.

Next, take $M = \{f : coz(f) \in \mathcal{F}\}$. Since $\{2\}$ and $\{4, 6, 8 \ldots\}$ are not in $\mathcal{F}$, $\chi_{\{2\}}$ and $\chi_{\{4,6,8,\ldots\}}$ are not in M, hence they are in $C \backslash M$. Further, we see that $coz\left(\chi_{\{2\}} - \chi_{\{4,6,8,\ldots\}}\right) = \{2n : n \in \mathbb{N}\}$, this implies that $\chi_{\{2\}} - \chi_{\{4,6,8,\ldots\}} \in M$ and not in $C \backslash M$. So, $C \backslash M$ is not an ideal.

Note that the set M, in Theorem 2.5, is a multiplicative set. Recall some general results on multiplicative sets which will be used later, but will not be referenced to directly.

Theorem 2.7 (Chapter 3, Example 1, [1]; Page 2, Theorem 2, [14])**.**
(1) *If P is a prime ideal of C, then $C \backslash P$ is a multiplicative set.*
(2) *$M \subseteq C$ is a maximal multiplicative set if and only if $C \backslash M$ is a minimal prime ideal of C.*

For more on general multiplicative sets refer to [1] or [16].

Next, we consider *coz*-ultrafilters and how they relate to prime ideals. Gillman and Jerison provide us with the following:

Theorem 2.8 (Theorem 2.5, [11])**.** *If M is a maximal ideal in C, then $Z(M)$ is a z-ultrafilter on X.*

We will establish a correspondence between the minimal prime ideals of C and the *coz*-ultrafilters; or, another way to view the situation is that there is a relationship between the maximal multiplicative sets and the *coz*-ultrafilters. Recall the following known results:

Theorem 2.9 (Theorem 14.7, [11])**.** *Every minimal prime ideal of C is a z-ideal.*

Lemma 2.10 (Corollary 2.2, [13])**.** *Let R be a reduced ring and let P be a prime ideal of R. Then P is minimal if and only if for each $x \in P$ there exists an $r \in R \backslash P$ such that $xr = 0$.*

Now we state the main theorem for this section.

Theorem 2.11. (1) *If $\mathcal{U}$ is a coz-ultrafilter of X and $M = \{f : coz(f) \in \mathcal{U}\}$ with $P = C \backslash M$, then P is a minimal prime ideal.*
(2) *If P is a minimal prime ideal of C, then $Coz(P^c)$ is a coz-ultrafilter.*

Proof. (1) Suppose $\mathcal{U}$ is a *coz*-ultrafilter on X. By Theorems 1.3 and 2.5, it follows that P is a prime z-ideal. To show that P is a minimal prime ideal, let $f \in P$. Since $f \notin M$, it follows that $coz(f) \notin \mathcal{U}$. Using Proposition 1.2, there exists some $g \in C(X)$ such that $coz(g) \in \mathcal{U}$ and $fg = 0$. Consequently, by Lemma 2.10, P is minimal.

(2) Suppose P is a minimal prime ideal of C and let $coz(g) \notin \mathrm{Coz}(P^c)$. It follows that $g \in P$. So, by Lemma 2.10, there exists $t \in \mathrm{Ann}(g)$ with $t \notin P$. Consequently, $coz(t) \cap coz(g) = coz(tg) = coz(0) = \emptyset$. Since $t \notin P$, we have $coz(t) \in \mathrm{Coz}(P^c)$. By Proposition 1.2, $\mathrm{Coz}(P^c)$ is a *coz*-ultrafilter. □

The remainder of the section will focus on *coz*-filters related to P-spaces and F-spaces. The inspiration for the following investigation is given by the sets $A_p = \{z(f) : p \in z(f)\}$, defined in [11]. Note that A_p is a z-filter for every $p \in X$.

In a similar flavor we define B_p to be the collection of all cozero neighborhoods of p, that is, $B_p = \{coz(f) : p \in coz(f)\}$. Observe that B_p is a prime *coz*-filter with $\bigcap B_p = \{p\}$. On the other hand, B_p is not necessarily a *coz*-ultrafilter. For an example, consider $\mathbb{R}$ with the usual topology and $1 \in \mathbb{R}$. Then B_1 is not a *coz*-ultrafilter, since the cozero set $(-\infty, 1)$ intersects every member of B_1.

Recall that for $p \in X$, $M_p = \{f \in C : f(p) = 0\}$ is a maximal ideal and $O_p = \{f \in C : p \in \mathrm{int}(z(f))\}$ is a z-ideal, with $O_p \subseteq M_p$. Refer to 4.6-4.7 and 4I in [11] for more on M_p and O_p. Further, if $M_p = O_p$, then p is called a *P-point*. We say that X is a *P-space* if every element of X is a P-point (refer to 4L.2 in [11]).

Observe that for $p \in X$,

$$\mathrm{Coz}^{\leftarrow}(B_p) = C \setminus M_p \text{ and } \mathrm{Coz}(C \setminus M_p) = B_p.$$

Hence, using Theorem 2.11 and 4I in [11], we have the following characterization of when B_p is a *coz*-ultrafilter.

Theorem 2.12. *Let $p \in X$. Then B_p is a coz-ultrafilter if and only if p is a P-point. Further, B_p is a coz-ultrafilter for all $p \in X$ if and only if X is a P-space.*

Following the pattern of the nomenclature, we define $C_p = \{coz(f) : p \in \mathrm{cl}(coz(f))\}$. Note that $B_p \subseteq C_p$ for all $p \in X$. In general, C_p is not a *coz*-filter. For example, consider $C(\mathbb{R})$. We have $0 \in [-1,0] = \mathrm{cl}((-1,0))$ and $0 \in [0,1] = \mathrm{cl}((0,1))$, and so $(-1,0),(0,1) \in C_0$. However, $\mathrm{cl}((-1,0) \cap (0,1)) = \mathrm{cl}(\emptyset) = \emptyset$, which is not in C_0.

Observe that C_p satisfies the prime condition: $p \in \mathrm{cl}(coz(f) \cup coz(g)) = \mathrm{cl}(coz(f)) \cup \mathrm{cl}(coz(g))$. Consequently, $p \in \mathrm{cl}(coz(f)$ or $p \in \mathrm{cl}(coz(g)$.

Our next result gives a characterization of when C_p is a *coz*-filter.

Proposition 2.13. *Let $p \in X$. Then C_p is a prime coz-filter if and only if O_p is a prime ideal.*

Proof. ($\Rightarrow$) Suppose $fg \in O_p$. Then $p \in \mathrm{int}(z(fg))$, and so $p \notin \mathrm{cl}(coz(fg))$. Consequently $coz(fg) \notin C_p$, which gives $coz(f) \cap coz(g) \notin C_p$. So, $coz(f) \notin C_p$ or $coz(g) \notin C_p$, since C_p is a *coz*-filter. Hence $p \notin \mathrm{cl}(coz(f))$ or $p \notin \mathrm{cl}(coz(g))$. So, $p \in \mathrm{int}(z(f))$ or $p \in \mathrm{int}(z(g))$, and therefore $f \in O_p$ or $g \in O_p$, *that is*, O_p is prime.

($\Leftarrow$) It is evident that $\emptyset \notin C_p$. If $coz(f), coz(g) \in C_p$, then $p \notin \mathrm{int}(z(f))$ and $p \notin \mathrm{int}(z(g))$. Consequently, $f, g \notin O_p$. Since O_p is prime, $fg \notin O_p$. Thus, $p \notin \mathrm{int}(z(fg))$. It follows that $p \in \mathrm{cl}(coz(fg)) = \mathrm{cl}(coz(f) \cap coz(g))$, and so C_p is closed under intersection. Lastly, let $coz(f) \in C_p$ and $coz(g) \in \mathrm{Coz}(X)$, with $coz(f) \subseteq coz(g)$. Since $\mathrm{cl}(coz(f)) \subseteq \mathrm{cl}(coz(g))$, we have that $coz(g) \in C_p$. $\square$

Recall that a space X is called an F'*-space* if O_p is a prime ideal, for all $p \in X$. A space X is called an F*-space* if every finitely generated ideal of C is a principal ideal. An F-space is also an F'-space, and the two notions are equivalent if X is compact. Equivalent conditions for X to be an F-space or an F'-space are well-known (see [10] and [11]). In the following theorem we provide some more equivalent conditions for F'-spaces in terms of C_p.

Recall that when O_p is a prime ideal, it is a minimal prime ideal of C. Moreover, $\mathrm{Coz}(C \setminus O_p) = C_p$ and $\mathrm{Coz}^{\leftarrow}(C_p) = C \setminus O_p$. Using these facts and Theorem 2.11, the following theorem holds.

Theorem 2.14. *For a* (*compact*) *space X, the following are equivalent*:

(1) *X is an F'-space* (*F-space*).
(2) *C_p is a prime coz-filter, for all $p \in X$.*
(3) *C_p is a coz-ultrafilter, for all $p \in X$.*

3 The spaces of *coz*-ultrafilters

In this section, we investigate the relationship between the spaces of $\mathrm{Min}(C(X))$, with respect to the hull-kernel and the inverse topologies, and the spaces of *coz*-ultrafilters.

Let $\mathfrak{U}(X)$ denote the collection of all *coz*-ultrafilters of X. Let $\Phi : \mathrm{Min}(C(X)) \to \mathfrak{U}(X)$ be defined by $\Phi(P) = \mathrm{Coz}(P^c) = \{coz(f) : f \in P^c\}$. Clearly Φ is a bijection, implied by Theorem 2.11.

Given any $f \in C(X)$, let us use the conventional notation $V_{\mathfrak{U}}(f) = \{F \in \mathfrak{U}(X) : coz(f) \in F\}$. The set-theoretic complement of $V_{\mathfrak{U}}(f)$ is denoted by $U_{\mathfrak{U}}(f)$. Notice the following properties of $V_{\mathfrak{U}}$ and $U_{\mathfrak{U}}$:

Proposition 3.1. *For $f, g \in C$ we have*:

(1) $V_{\mathfrak{U}}(f) \cap V_{\mathfrak{U}}(g) = V_{\mathfrak{U}}(fg)$ *and* $U_{\mathfrak{U}}(f) \cup U_{\mathfrak{U}}(g) = U_{\mathfrak{U}}(fg)$.

(2) $V_{\mathfrak{U}}(f) \cup V_{\mathfrak{U}}(g) = V_{\mathfrak{U}}\left(f^2 + g^2\right)$ *and* $U_{\mathfrak{U}}(f) \cap U_{\mathfrak{U}}(g) = U_{\mathfrak{U}}(f^2 + g^2)$.

(3) $V_{\mathfrak{U}}(f) = \emptyset$ *if and only if* $f = 0$ *if and only if* $U_{\mathfrak{U}}(f) = \mathfrak{U}(X)$.

(4) $V_{\mathfrak{U}}(f) = \mathfrak{U}(X)$ *if and only if f is a non-zero-divisor if and only if* $U_{\mathfrak{U}}(f) = \emptyset$.

Proof. (1) This is due to the fact that given any *coz*-filter F, $coz(f), coz(g) \in F$ if and only if $coz(fg) \in F$.

(2) This is due to the fact that given any prime *coz*-filter F, $coz(f) \cup coz(g) \in F$ if and only if $coz(f) \in F$ or $coz(g) \in F$.

If $V_{\mathfrak{U}}(f) = \emptyset$, then $coz(f) \notin F$ for every $F \in \mathfrak{U}(X)$. Consequently, $f \in P$ for every $P \in \mathrm{Min}(C(X))$ and hence $f \in \bigcap \mathrm{Min}(C(X)) = 0$. The other direction is clear.

(3) This follows directly from Lemma 1.1 and Proposition 1.2. □

Based on the properties described above, we can talk about two different topologies on $\mathfrak{U}(X)$:

(i) the collection $\{V_{\mathfrak{U}}(f) : f \in C\}$ forms a basis for open sets on $\mathfrak{U}(X)$, this topology is denoted by $\mathfrak{U}(X)^{-1}$.

(ii) the collection $\{U_{\mathfrak{U}}(f) : f \in C\}$ also forms a basis for open sets on $\mathfrak{U}(X)$. We use $\mathfrak{U}(X)$ to denote this topology.

$\mathfrak{U}(X)$ is the same topology considered by Wallman [19], Samuel [18], and Brooks [4], generated by the collection $\{V_{\mathfrak{U}}(f) : f \in C\}$ as a basis for closed sets. It is known that $\mathfrak{U}(X)$ is a compact T_1 space. We show a more interesting fact about $\mathfrak{U}(X)$ with a different viewpoint of this topology.

These two topological spaces on $\mathfrak{U}(X)$ have similar flavors as the spaces of $\mathrm{Min}(C(X))$. In [12] the concept of the hull-kernel topology, also referred to as the Stone topology, for $\mathrm{Min}(R)$ is studied for commutative rings. Recall that the *hull-kernel topology* on $\mathrm{Min}(C(X))$, denoted by $\mathrm{Min}(C(X))$, has subsets of the form $U(f) = \{P \in \mathrm{Min}(C(X)) : f \notin P\}$ as basic open sets, for $f \in C$. It is well-known that $\mathrm{Min}(C(X))$ is a zero dimensional, Hausdorff topological space (see Corollary 2.4 in [12]).

On the other hand, the *inverse topology* on $\mathrm{Min}(C(X))$, denoted by $\mathrm{Min}(C(X))^{-1}$, is generated by the subsets of the form

$$V(f) = \{P \in \mathrm{Min}(C(X)) : f \in P\}$$

as basic open sets, for $f \in C$. The inverse topology on $\mathrm{Min}(G)$ is considered in [17] for a lattice-ordered group G. Also, $\mathrm{Min}(C(X))^{-1}$ is studied in [3]; $\mathrm{Min}(C(X))^{-1}$ is a compact, T_1 space.

Theorem 3.2. *The following hold for any Tychonoff space X:*

(1) $\mathfrak{U}(X)^{-1}$ *is homeomorphic to Min$(C(X))$.*

(2) $\mathfrak{U}(X)$ *is homeomorphic to Min$(C(X))^{-1}$.*

Proof. Consider $\Phi : \mathrm{Min}(C(X)) \to \mathfrak{U}(X)$ as defined at the beginning of Section 4. We have noticed earlier that Φ is a well-defined bijection.

(1) For $f \in C$, we claim that $\Phi^{-1}(V_{\mathfrak{U}}(f)) = U(f)$. Consider $P \in \Phi^{-1}(V_{\mathfrak{U}}(f))$, then $\mathrm{Coz}(P^c) \in V_{\mathfrak{U}}(f)$. It follows that $coz(f) \in \mathrm{Coz}(P^c)$, and so $coz(f) = coz(g)$ for some $g \in P^c$. If $f \in P$, then $g \in P$ since P is a z-ideal. Therefore, $f \notin P$ which means $P \in U(f)$. Conversely suppose $P \in U(f)$, that is $f \notin P$. Since $coz(f) \in \mathrm{Coz}(P^c)$, we have $\Phi(P) = \mathrm{Coz}(P^c) \in V_{\mathfrak{U}}(f)$. Accordingly, $P \in \Phi^{-1}(V_{\mathfrak{U}}(f))$. Hence, $\Phi : \mathrm{Min}(C(X)) \to \mathfrak{U}(X)^{-1}$ is a continuous function.

Finally we show that Φ is an open map. Let $\Phi(Q) \in \Phi(U(f)) \subseteq \mathfrak{U}(X)^{-1}$ for some $Q \in U(f)$. Since $f \notin Q$, we have $coz(f) \in \mathrm{Coz}(Q^c)$, which means that $\Phi(Q) \in V_{\mathfrak{U}}(f)$. Now, let $F \in V_{\mathfrak{U}}(f)$ be arbitrary. Then there exists some $P \in \mathrm{Min}(C(X))$ such that $F = \Phi(P) = \mathrm{Coz}(P^c)$. Note that $coz(f) \in F$ implies that $f \notin P$, since P is a z-ideal. Consequently $P \in U(f)$, and so $F = \Phi(P) \in \Phi(U(f))$.

(2) The homeomorphic aspect of $\Phi : \mathrm{Min}(C(X))^{-1} \to \mathfrak{U}(X)$ can be shown similarly and is left to the interested reader. $\square$

Corollary 3.3. $\mathfrak{U}(X)^{-1}$ *is a zero-dimensional, Hausdorff topological space.*

We direct our attention to [17] where it is shown that if X is an F-space, then $\text{Min}(C(X))^{-1}$ is homeomorphic to $\beta(X)$, the Stone-Čech compactification of X. Hence, we have the following result:

Corollary 3.4. *If X is a compact F-space, then $\mathfrak{U}(X)$ is homeomorphic to X.*

Definition 3.5 (Definition 2.3, [4])**.** Let L be a lattice on X. L is a *normal* lattice if for each pair of disjoint elements A and B of L, there exists $C, D \in L$ such that $A \subseteq C$, $B \subseteq D$, $A \cap D = \emptyset = B \cap C$, and $C \cup D \in U$ for every ultrafilter U of L.

In [4], it is shown that the collection of ultrafilters, with the Samuel topology, on a lattice L is Hausdorff precisely when L is a normal lattice. Here we are considering $\text{Coz}(X)$ as a lattice on the set $\mathcal{P}(X)$. In [3] the authors have shown various conditions for the space $\text{Min}(C(X))^{-1}$ to be Hausdorff. A further study on the inverse topology on the minimal prime subgroups of lattice-ordered groups is being conducted currently in [2]. Since we have established a homeomorphism between $\mathfrak{U}(X)$ and $\text{Min}(C(X))^{-1}$, summing up, we can say the following:

Proposition 3.6. *The following are equivalent*:

(1) $\mathfrak{U}(X)$ *is a Hausdorff space.*

(2) *Min*$(C(X))^{-1}$ *is a Hausdorff space.*

(3) *For each pair of disjoint cozero sets coz*(f) *and coz*(g)*, there exist* $f_1, g_1 \in C$ *such that* $coz(f) \subseteq int(z(f_1))$, $coz(g) \subseteq int(z(g_1))$, *and* $int(z(f_1)) \cap int(z(g_1)) = \emptyset$.

(4) *For each pair of disjoint cozero sets* $C, D \subseteq X$*, there exist cozero sets* $C_1, D_1 \subseteq X$ *such that* $C \subseteq C_1$, $D \subseteq D_1$, $C \cap D_1 = \emptyset$, $D \cap C_1 = \emptyset$, *and* $C_1 \cup D_1$ *is a dense subset of* X.

(5) *Coz*(X) *is a normal lattice.*

Proof. The equivalence of (1) and (2) follows from Theorem 3.2. That (2), (3), and (4) are equivalent is shown in [3]. In [4] it is shown that (1) and (5) are equivalent. We want to point out further that a direct equivalence of (4) and (5) can be shown using Proposition 1.2. □

4 Lattice-ordered groups

We finish our article with some general results on ultrafilters associated with lattice-ordered groups. These are generalizations of results from Sections 2 and 3. As a reference for the theory of lattice-ordered groups, see [6].

A *lattice-ordered group* or *ℓ-group* is a group $(G, +, 0)$ which is also a lattice $(G, \leq)$ such that for every $a, b, x, y \in G$, $a \leq b$ implies that $x+a+y \leq x+b+y$. In general, an ℓ-group is not necessarily abelian, but do note that C is an abelian ℓ-gorup. The positive cone of G is denoted by $G^+ = \{g \in G : g \geq 0\}$. An *$\ell$-subgroup* of G is a subgroup which is also a sublattice. An ℓ-subgroup H of G is *convex* if whenever $a \in G$ and $0 \leq a \leq h$ for some $h \in H^+$, then $a \in H$. Given $g \in G$, the set $g^\perp = \{h \in G : |g| \wedge |h| = 0\}$ is called the *polar* of g. It turns out that polars of G are convex ℓ-subgroups. We say an element $g \in G$ is a *weak order unit* if $g^\perp = 0$.

Recall the well-known Lemma on Ultrafilters where a one-to-one correspondence has been shown between the minimal prime subgroups of G, $\mathrm{Min}(G)$, and the ultrafilters of G^+, denoted as $ult(G^+)$.

Lemma 4.1 (Lemma on Ultrafilters, [5])**.** *Let G be an ℓ-group. For each minimal prime subgroup P, the set $u(P) = \{q > 0 : q \notin P\}$ is an ultrafilter. Conversely, if U is an ultrafilter, then $Q(U) = \bigcup\{x^\perp : x \in U\}$ is a minimal prime subgroup. The correspondences $P \to u(P)$ and $U \to Q(U)$ are mutually inverse bijections.*

We use $\Psi : \mathrm{Min}(G) \to ult(G^+)$ to be the bijective map described above, that is, $\Psi(P) = u(P) = G^+ \setminus P$. Given $g \in G^+$, let $N(g) = \{U \in ult(G^+) : g \in U\}$ be in the power set of $ult(G^+)$. Note the following properties of $N(g)$:

1. If g is a weak order unit, then $g \notin P$ for all $P \in \mathrm{Min}(G)$. Consequently, $N(g) = ult(G^+)$.

2. $N(0) = \emptyset$, since $0 \in P$ for all $P \in \mathrm{Min}(G)$.

3. $N(g) \cap N(h) = N(g \wedge h)$, since for any $P \in \mathrm{Min}(G)$, $g \wedge h \in P$ if and only if $g \in P$ or $h \in P$.

4. $N(g) \cup N(h) = N(g \vee h)$: If $g \notin U$ and $h \notin U$, then there exists $a, b \in U$ such that $a \wedge g = 0$ and $b \wedge h = 0$. So, $(g \vee h) \wedge (a \wedge b) = (g \wedge a \wedge b) \vee (h \wedge a \wedge b) = 0$. Since $a \wedge b \in U$, it follows that $g \vee h \notin U$.

The collection $\{N(g) : g \in G^+\}$ forms a closed base for a topology on $ult(G^+)$. Further, the inverse topology on $\mathrm{Min}(G)$, denoted by $\mathrm{Min}(G)^{-1}$, is defined in a similar manner as to that of $\mathrm{Min}(C(X))^{-1}$: the collection $\{V(g) : g \in G^+\}$ forms an open base for the topology, where $V(g) = \{P \in \mathrm{Min}(G) : g \in P\}$. The topological space $\mathrm{Min}(G)^{-1}$ is compact T_1.

Theorem 4.2. *The map* $\Psi : Min(G)^{-1} \to ult(G^+)$ *is a homeomorphism. In particular,* $\Psi(V(g)) = ult(G^+) \setminus N(g)$ *for all* $g \in G^+$.

Proof. We only need to show that $\Psi(V(g)) = ult(G^+)\setminus N(g)$ for all $g \in G^+$. Then using the fact that Ψ is a bijective map, it follows immediately that Ψ is continuous and hence a homeomorphism.

Let $V(g)$ be a basic open set of $\mathrm{Min}(G)^{-1}$ and $G^+\setminus Q = \Psi(Q) \in \Psi(V(g))$ for some $Q \in V(g)$. Since $g \in Q$, $g \notin G^+ \setminus Q \in ult(G^+)$. Therefore $\Psi(Q) = G^+\setminus Q \in ult(G^+)\setminus N(g)$. On the other hand, if $U \in ult(G^+)\setminus N(g)$, then $g \notin U = G^+ \setminus P$ for some $P \in \mathrm{Min}(G)^{-1}$, and so $g \in P$. Hence $P \in V(g)$ and $U = \Psi(P) \in \Psi(V(g))$. □

Corollary 4.3. *The space* $ult(C(X)^+)$ *is homeomorphic to* $Min(C(X))^{-1}$.

We next consider the category $\mathbf{W}$ whose objects are archimedean ℓ-groups with a designated weak order unit and whose morphisms are the ℓ-homomorphisms which preserves the designated unit. An ℓ-group G is *archimedean* if whenever $g, h \in G^+$ and $ng \leq h$ for all $n \in \mathbf{N}$, then $g = 0$. An archimedean ℓ-group is abelian. Let $(G, u) \in \mathbf{W}$. Using Zorn's Lemma, there exist convex ℓ-subgroups which are maximal with respect to not containing u. Such a convex ℓ-subgroup is called a *value* of u, and the collection of all such values is denoted by YG. We endow YG with the hull-kernel topology, which is a compact Hausdorff space. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ be the two-point compactification of $\mathbb{R}$. For a topological space X, we define

$$D(X) = \{f : X \to \overline{\mathbb{R}} : f \text{ is continuous and } f^{-1}(\mathbb{R}) \text{ is dense}\}.$$

Then $D(X)$ is a lattice under pointwise operation, but not necessarily a group. We say $H \subseteq D(X)$ is an ℓ-subgroup of $D(X)$ if H is an ℓ-group under pointwise operation. As an example, $C(X) \subseteq D(X)$ is an ℓ-subgroup. Given an $f \in D(X)$, we write $coz(f) = \{x \in X : f(x) \neq 0\}$, and is called the *cozero-set of* f. The set $z(f) = X \setminus coz(f)$ is called the *zero-set of* f.

Theorem 4.4 (The Yosida Embedding Theorem). *Let $(G,u) \in \mathbf{W}$. Then there is an ℓ-isomorphism $(g \mapsto \hat{g})$ of G onto an ℓ-subgroup $\hat{G}$ of $D(YG)$ such that $\hat{u} = \mathbf{1}$ and $\hat{G}$ has the following separation property: for each $p \in YG$ and closed set $V \subseteq YG$ not containing p, there is some $g \in G$ for which $\hat{g}(p) = 1$ and $\hat{g}(V) = 0$. Further, YG is the unique compact space, up to homeomorphism, that satisfies these two properties.*

We identify $g \in G$ with $\hat{g} \in D(YG)$. Note that $\{coz(g) : g \in G\}$ forms an open basis for the topology of YG; any subset of YG of this form is called a *G-cozero-set*, and the collection of all such is denoted by $\mathrm{Coz}(G)$. We denote the collection of all cozero sets of YG by $\mathrm{Coz}(YG)$. In general, $\mathrm{Coz}(G) \subseteq \mathrm{Coz}(YG)$. We use $\mathfrak{U}(YG)$ and $\mathfrak{U}(G)$ to denote the ultrafilters on $\mathrm{Coz}(YG)$ and $\mathrm{Coz}(G)$, respectively. Using Theorem 3.2, we can immediately conclude that $\mathfrak{U}(YG)$ is homeomorphic to $\mathrm{Min}(C(YG))^{-1}$, since YG is a Tychonoff space.

Next, we focus on $\mathrm{Coz}(G)$-ultrafilters. Let $\Omega : ult(G^+) \to \mathfrak{U}(G)$ be defined by $\Omega(F) = \mathrm{Coz}(F) = \{coz(f) : f \in F\}$.

Lemma 4.5. *The map Ω defined above is well-defined.*

Proof. We first show that $\mathrm{Coz}(F)$ is a filter.

(1) $\mathrm{coz}(f) = \emptyset$ if and only if $f = 0$. Hence $\emptyset = \mathrm{coz}(0) \notin \mathrm{Coz}(F)$, since $0 \notin F$.

(2) Suppose $coz(f), coz(g) \in \mathrm{Coz}(F)$ for some $f, g \in F$. Since F is a filter, $f \wedge g \in F$. Hence, $coz(f) \cap coz(g) = coz(f \wedge g) \in \mathrm{Coz}(F)$.

(3) Suppose $f \in F$ and $g \in G^+$ with $coz(f) \subseteq coz(g)$. Note that $coz(g) = coz(f) \cup coz(g) = coz(f \vee g)$. Since $f \in F$ and $f \leq f \vee g$, it follows that $f \vee g \in F$, because F is a filter. Consequently, $coz(g) \in \mathrm{Coz}(F)$.

Finally, let $coz(g) \notin \mathrm{Coz}(F)$. Then $g \notin F$. Since F is an ultrafilter, there exists some $f \in F$ such that $g \wedge f = 0$. Therefore, $coz(f) \in \mathrm{Coz}(F)$ such that $coz(f) \wedge coz(g) = \emptyset$. Hence, $\mathrm{Coz}(F)$ is an ultrafilter. □

Lemma 4.6. *Let $U \in \mathfrak{U}(G)$. Then the set $F(U) = \{f \in G^+ : coz(f) \in U\}$ is an ultrafilter of G^+.*

Proof. The proof is similar to the preceding lemma, using the reverse implications of the statements. □

We can view the Lemma on Ultrafilters in an alternative manner.

Theorem 4.7. *Let* $(G, u) \in \mathbf{W}$. *For each ultrafilter* F *of* G^+,

$$F = F(Coz(F)) = \{f \in G^+ : coz(f) \in Coz(F)\}.$$

Conversely, for each ultrafilter U *of* $Coz(G)$, $U = Coz(F(U))$. *In other words,* $\Omega : ult(G^+) \to \mathfrak{U}(G)$ *is a bijection.*

Proof. It is clear, by the definition of $\mathrm{Coz}(F)$, that $F \subseteq \{f \in G^+ : coz(f) \in \mathrm{Coz}(F)\}$. For the reverse inclusion, let $f \in G^+$ be such that $coz(f) \in \mathrm{Coz}(F)$. Then there exists $g \in F$ such that $coz(f) = coz(g)$. If $f \notin F$, then there exists $f' \in F$ such that $f \wedge f' = 0$, since F is an ultrafilter. Now

$$coz(g \wedge f') = coz(g) \cap coz(f') = coz(f) \cap coz(f') = coz(f \wedge f') = \emptyset.$$

Consequently, $g \wedge f' \in \bigcap YG = u^{\perp} = 0 \in F$, which is a contradiction.

The converse can be verified easily. □

To summarize, we have the following one-to-one correspondences between ultrafilters and minimal prime ideals of a $\mathbf{W}$-object (G, u):

$$\mathrm{Min}(G) \xhookrightarrow{\Psi} ult\left(G^+\right) \xhookrightarrow{\Omega} \mathfrak{U}(G)$$

Endow a topology on $\mathfrak{U}(G)$ in a similar manner, with the collection of sets $\{F \in \mathfrak{U}(G) : coz(g) \in F\}$ as a basis for closed sets. It follows that this space is also compact T_1. We can say more about the spaces $\mathfrak{U}(G)$, $ult(G^+)$, and $\mathrm{Min}(G)^{-1}$.

Theorem 4.8. *All three spaces,* $\mathfrak{U}(G)$, $ult(G^+)$, *and* $Min(G)^{-1}$ *are homeomorphic to each other.*

Proof. We have shown already that $ult(G^+)$ is homeomorphic to $\mathrm{Min}(G)^{-1}$. In a similar fashion, it can be shown that $\mathfrak{U}(G)$ and $\mathrm{Min}(G)^{-1}$ are homeomorphic to each other. The proof of this is left to the interested reader. □

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