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One-point compactifications and continuity for partial frames

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Dedicated to Bernhard Banaschewski on the occasion of his 90th birthday

Abstract. Locally compact Hausdorff spaces and their one-point compactifications are much used in topology and analysis; in lattice and domain theory, the notion of continuity captures the idea of local compactness. Our work is located in the setting of pointfree topology, where lattice-theoretic methods can be used to obtain topological results. Specifically, we examine here the concept of continuity for partial frames, and compactifications of regular continuous such.

Partial frames are meet-semilattices in which not all subsets need have joins. A distinguishing feature of their study is that a small collection of axioms of an elementary nature allows one to do much that is traditional for frames or locales. The axioms are sufficiently general to include as examples σ-frames, κ-frames and frames.

In this paper, we present the notion of a continuous partial frame by means of a suitable "way-below" relation; in the regular case this relation can be characterized using separating elements, thus avoiding any use of pseudocomplements (which need not exist in a partial frame). Our first main

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result is an explicit construction of a one-point compactification for a regular continuous partial frame using generators and relations. We use strong inclusions to link continuity and one-point compactifications to least compactifications. As an application, we show that a one-point compactification of a zero-dimensional continuous partial frame is again zero-dimensional.

We next consider arbitrary compactifications of regular continuous partial frames. In full frames, the natural tools to use are right and left adjoints of frame maps; in partial frames these are, in general, not available. This necessitates significantly different techniques to obtain largest and smallest elements of fibres (which we call balloons); these elements are then used to investigate the structure of the compactifications. We note that strongly regular ideals play an important rôle here. The paper concludes with a proof of the uniqueness of the one-point compactification.

1 Introduction

The use of locally compact Hausdorff spaces in topology and analysis is ubiquitous, and the fact that all such have one-point compactifications is of particular interest. In lattice and domain theory, the notion of continuity captures, to some extent, the idea of local compactness. Our work is located in the setting of pointfree topology, where lattice-theoretic methods can be used to obtain topological results. Specifically, we examine here the concept of continuity for partial frames, and compactifications of regular continuous such.

Partial frames are meet-semilattices in which not all subsets need have joins; a selection function, $\mathcal{S}$, specifies, for all meet-semilattices, certain subsets under consideration; an $\mathcal{S}$-frame then must have joins of all such subsets and binary meet must distribute over these. This approach has been used by several authors with a variety of purposes in mind; it has led to different authors using different sets of axioms for their selection functions in order to produce tractable theories. See, for instance, [24, 28, 30, 31] and [11]. We have found this avenue of enquiry both illuminating and fruitful.

In this paper we present the notion of a continuous partial frame by means of a suitable "way-below" relation; in the regular case this relation can be characterized using separating elements, thus avoiding any use of pseudocomplements (which need not exist in a partial frame). Our first main result is an explicit construction of a one-point compactification for a regular continuous partial frame using generators and relations. (See Definition 4.1.) Strong inclusions have become a fundamental tool in understanding compactifications (see, for instance, [3, 5, 12] and [27]); for the partial frame case, see [18]. We use them here to link continuity and one-point compactifications to least compactifications. As an application, we show that a one-point compactification of a zero-dimensional continuous $\mathcal{S}$-frame is again zero-dimensional.

We next consider arbitrary compactifications of regular continuous $\mathcal{S}$-frames and use fibres (which we call balloons) to analyse these. In full (that is, not partial) frames, the natural tools to use here are right and left adjoints of frame maps; in $\mathcal{S}$-frames these are, in general, not available, since, for instance, $\mathcal{S}$-frame maps need not preserve arbitrary joins. This necessitates significantly different techniques to obtain largest and smallest elements of balloons (see Proposition 8.4, where strongly regular ideals play an important rôle); these elements are then used to investigate the structure of the compactifications. We conclude with the uniqueness of the one-point compactification, the proof of which requires the use of results concerning arbitrary compactifications just mentioned. We have not found a proof of this for full frames in the literature.

We finish this introduction with a few remarks concerning the literature on one-point compactifications in pointfree topology. In an early paper ([25]) Paseka and Šmarda define weak local compactness for locales and show that, in the presence of regularity, this coincides with continuity. Amongst other things they construct and provide properties of an object which they call the one-point compactification. In [19] the authors consider the same idea *en route* to describing the unit circle using generators and relations. A different path is pursued by Banaschewski (see [5]) who uses strong inclusions and strongly regular ideals to construct compactifications of completely regular frames. In the same paper he shows that a regular frame is continuous if and only if it has a least compactification, using a construction which, in a later paper, [9], is called a one-point extension or compactification. Baboolal (in [2, 3]) continues in this direction to provide n-point compactifications and further analysis of the one-point compactification. Our approach draws on all these threads in extending such notions and results to the setting of partial frames, while avoiding entirely the use of constructs such as right adjoints and pseudocomplements.

2 Background

See [26] and [20] as references for frame theory; see [8] and [6] for σ-frames; see [23] for κ-frames; see [22] and [1] for general category theory.

A detailed discussion of our approach to partial frames can be found in [13] and [14]; more information on the topic can be found in [15], [16], [17] and [18]. For earlier work in this field see [24], [30], [29], and [28]. We note that terminology is not standard: in [30] partial frames are called $\mathcal{Z}$-frames, the appropriate morphisms are termed $\mathcal{Z}$-frame maps and selection functions are referred to as "subset selections." We refer the reader to [14] for details on our and other authors' terminology.

We give now the details necessary for this paper. A *meet-semilattice* L is a partially ordered set in which all finite subsets have a meet. In particular, we regard the empty set as finite, so a meet-semilattice comes equipped with a top element, which we denote by 1. We also insist that a meet-semilattice should have a bottom element, which we denote by 0. Technically, one might wish to refer to these as *bounded* meet-semilattices, but for the purposes of this paper, all meet-semilattices are bounded and the term "bounded" will be omitted from now on. A function $f : L \to M$ between meet-semilattices is a *meet-semilattice map* if it preserves finite meets, as well as the top and the bottom elements.

Definition 2.1. A *selection function* is a rule, which we usually denote by $\mathcal{S}$, which assigns to each meet-semilattice A a collection $\mathcal{S}A$ of subsets of A, such that the following conditions hold (for all meet-semilattices A and B):
(S1) For all $x \in A$, $\{x\} \in \mathcal{S}A$.
(S2) If $G, H \in \mathcal{S}A$ then $\{x \wedge y : x \in G, y \in H\} \in \mathcal{S}A$.
(S2′) If $G, H \in \mathcal{S}A$ then $\{x \vee y : x \in G, y \in H\} \in \mathcal{S}A$.
(S3) If $G \in \mathcal{S}A$ and, for all $x \in G$, $x = \bigvee H_x$ for some $H_x \in \mathcal{S}A$, then

$$\bigcup_{x \in G} H_x \in \mathcal{S}A.$$

(S4) For any meet-semilattice map $f : A \to B$,

$$\mathcal{S}(f[A]) = \{f[G] : G \in \mathcal{S}A\} \subseteq \mathcal{S}B.$$

Remark 2.2. (a) Once a selection function, $\mathcal{S}$, has been fixed, we speak informally of the members of $\mathcal{S}A$ as the *designated* subsets of A.

(b) In a meet-semilattice binary joins may not exist, in which case, in the light of (S2′) it is possible that a designated set may be empty; this is not a problem.

We now impose the following axioms on any selection function:

(SCount): Every (at most) countable subset is designated.

(SCov): Every subset of an $\mathcal{S}$-cover is again designated, where an $\mathcal{S}$-*cover* is a designated subset whose join is 1.

(SRef) Let $X, Y \subseteq A$. If $X \leq Y$ with X designated in A there is a designated subset C of A such that $X \leq C \subseteq Y$. (By $X \leq Y$ we mean, as usual, that for each $x \in X$ there exists $y \in Y$ such that $x \leq y$.)

Definition 2.3. Let $\mathcal{S}$ be a selection function. Then

(1) An $\mathcal{S}$-*frame*, L, is a meet-semilattice that satisfies the following two conditions:

(a) For all $G \in \mathcal{S}L$, G has a join in L (that is, $\bigvee G$ exists).

(b) For all $x \in L$, for all $G \in \mathcal{S}L$, $x \wedge \bigvee G = \bigvee\limits_{y \in G} (x \wedge y)$.

(2) Let L and M be $\mathcal{S}$-frames. An $\mathcal{S}$-*frame map* $f : L \to M$ is a meet-semilattice map such that, for all $G \in \mathcal{S}L$, $f(\bigvee G) = \bigvee\limits_{y \in G} f(y)$.

(3) $\mathcal{S}$**Frm** is the category of $\mathcal{S}$-frames as objects and $\mathcal{S}$-frame maps as morphisms.

Note 2.4. Here are some examples of different selection functions and their corresponding $\mathcal{S}$-frames.

1. In the case that all joins are specified, we are of course considering the notion of a frame.

2. In the case that (at most) countable joins are specified, we have the notion of a σ-frame.

3. In the case that joins of subsets with cardinality less than some (regular) cardinal κ are specified, we have the notion of a κ-frame.

Definition 2.5. Let $\mathcal{S}$ be a selection function and L an $\mathcal{S}$-frame. We shall call a subset M of L a *sub $\mathcal{S}$-frame of L* if M is an $\mathcal{S}$-frame and the identical embedding $i : M \to L$ is an $\mathcal{S}$-frame map.

We impose the following further axiom on all selection functions:

(S5): For any $\mathcal{S}$-frame L, if M is a sub $\mathcal{S}$-frame of L, $G \subseteq M$ and G is a designated subset of L, then G is a designated subset of M.

Definition 2.6. Let A be a meet-semilattice. A non-empty subset $D \subseteq A$ is a *downset* if for any $x, y \in A$, $x \leq y \in D$ implies $x \in D$. A downset D of A is called *$\mathcal{S}$-generated* if there exists a designated subset S of A such that $D = \{x \in A : \ x \leq s \text{ for some } s \in S\}$.

We impose one further axiom on our selection functions:

(SDown) The union of a designated collection of $\mathcal{S}$-generated downsets is $\mathcal{S}$-generated.

The reader wanting more detail on the rôle played by the various axioms is referred to our earlier work, for instance: [13] for (S5), [14] for (SCov), [15] for (SRef) and (SCount), [17] for (SDown).

Definition 2.7. (1) For a, b elements of a lattice A, we write $a \prec b$ if there exists $s \in A$ such that $a \wedge s = 0$ and $s \vee b = 1$. We call such an s a "separating element."

(2) An $\mathcal{S}$-frame L is *regular* if for each $a \in L$ there is a designated subset T of L such that $a = \bigvee T$ and $t \prec a$ for each $t \in T$.

(3) An $\mathcal{S}$-frame L is *compact* if every $\mathcal{S}$-cover has a finite subcover.

(4) An $\mathcal{S}$-frame map $g : M \to L$ is *dense* if $g(a) = 0 \Rightarrow a = 0$, *codense* if $g(a) = 1 \Rightarrow a = 1$.

(5) An $\mathcal{S}$-frame map $g : M \to L$ is a *compactification* of L if M is a compact, regular $\mathcal{S}$-frame and g is a dense, onto $\mathcal{S}$-frame map.

(6) An $\mathcal{S}$-frame map $g : M \to L$ is a *one-point compactification* of L if it is a compactification of L and there exists a maximal element m in M such that $g : {\downarrow}m \to L$ is an isomorphism, where ${\downarrow}m = \{a \in M : \ a \leq m\}$.

Remark 2.8. The definition given above of a one-point compactification appears for full frames in [9]. The term is used in a different, though ultimately equivalent, sense in [19] and [25].

3 Introducing continuity for $\mathcal{S}$-frames

Here we provide the basics that establish a natural extension of the usual notion of continuity to partial frames. Of particular importance in the next section is Theorem 3.8 whose frame precursor can be found in [3] and [19].

Definition 3.1. (1) For a, b elements of an $\mathcal{S}$-frame L, we write $a \ll b$ and say that a is *way below* b if this condition holds: whenever $b \leq \bigvee S$ for some designated subset S of L, then $a \leq \bigvee F$ for some finite subset F of S.

(2) item An $\mathcal{S}$-frame L is *continuous* if for each $a \in L$ there is a designated subset T of L such that $a = \bigvee T$ and $t \ll a$ for each $t \in T$.

Remark 3.2. (a) It is possible to rewrite the definition of the way below relation slightly as follows, using Lemma 3.2 of [18]: $a \ll b$ if and only if whenever $b \leq \bigvee T$ for some designated, updirected subset T of L, then $a \leq t$ for some $t \in T$.

(b) It is immediate to see that, in any $\mathcal{S}$-frame, $\ll$ is closed under finite join; that is, $0 \ll 0$ and if $a \ll c$, $b \ll c$ then $a \vee b \ll c$.

(c) In the case where $\mathcal{S}$ selects countable subsets, the corresponding way below relation has been denoted in various ways: [21] uses the notation $\ll_c$, [10] uses $\ll_\sigma$ whereas [4] uses the plain $\ll$; we follow the latter usage for all selection functions.

Lemma 3.3. (a) *The way below relation $\ll$ interpolates in a continuous $\mathcal{S}$-frame.*

(b) *Suppose M is a continuous $\mathcal{S}$-frame and $a \in M$. Let $h : M \to \downarrow a$ be given by $h(x) = x \wedge a$. Then $\downarrow a$ is a continuous $\mathcal{S}$-frame and h is an $\mathcal{S}$-frame map.*

Proof. (a) Suppose $a \ll b$ in a continuous $\mathcal{S}$-frame L. Write $b = \bigvee S$ for some designated subset S of L such that $s \ll b$ for each $s \in S$. For each $s \in S$, write $s = \bigvee H_s$ for some designated subset H_s of L such that $h \ll s$ for all $h \in H_s$. By Axiom (S3), $K = \bigcup_{s \in S} H_s$ is a designated subset of L; and $\bigvee K = b$. Then there exists a finite subset $\{s_1, \ldots, s_n\}$ of S such that $a \leq h_{s_1} \vee \cdots \vee h_{s_n} \ll s_1 \vee \cdots \vee s_n$ where $h_{s_i} \in H_{s_i}$. Also $s_1 \vee \cdots \vee s_n \ll b$, so $s_1 \vee \cdots \vee s_n$ is an appropriate interpolating element.

(b) It is clear that $\downarrow a$ is a meet-semilattice and h a meet-semilattice map. If S is a designated subset of $\downarrow a$, Axiom (S4) ensures the existence of a designated subset T of L with $\{t \wedge a : t \in T\} = S$. By Axioms (S1) and (S2), $\{t \wedge a : t \in T\}$ (and hence S) is a designated subset of L; so we see that designated subsets of $\downarrow a$ are also designated subsets of L. Then $\bigvee S = \bigvee\{t \wedge a : t \in T\} = a \wedge \bigvee T$. So the designated subsets of $\downarrow a$ have joins, with join in $\downarrow a$ the same as join in L, and it follows then that binary meet distributes over designated join in $\downarrow a$. So $\downarrow a$ is an $\mathcal{S}$-frame. That h is an $\mathcal{S}$-frame map then follows, from binary meet distributing over designated join in L.

Next we show that the way below relation in $\downarrow a$ coincides with that in L. Suppose $x \ll b$ in $\downarrow a$, and $b \leq \bigvee T$ for some designated subset T of L. Then $b = b \wedge a = \bigvee\{t \wedge a : t \in T\}$ and $\{t \wedge a : t \in T\}$ is a designated subset of $\downarrow a$, by Axiom (S4). So $x \leq (t_1 \wedge a) \vee \cdots \vee (t_n \wedge a) \leq t_1 \vee \cdots \vee t_n$ for some $t_1, \ldots, t_n \in T$. Conversely, suppose $x \ll b$ in L, $b \leq a$, and $b \leq \bigvee S$ for some designated subset S of $\downarrow a$. By Axiom (S4), there exists a designated subset T of L with $\{t \wedge a : t \in T\} = S$. Then $b \leq \bigvee\{t \wedge a : t \in T\} \leq \bigvee T$, so $x \leq t_1 \vee \cdots \vee t_n$ for some $t_1, \ldots, t_n \in T$. Thus $x = x \wedge a \leq (a \wedge t_1) \vee \cdots \vee (a \wedge t_n) = s_1 \vee \cdots \vee s_n$ for some $s_1, \ldots, s_n \in S$.

Finally, to show $\downarrow a$ is continuous, we begin with $b \in \downarrow a$. Then $b = \bigvee_L\{t \in L : t \ll b \text{ in } L\} = \bigvee_{\downarrow a}\{t \in \downarrow a : t \ll b \text{ in } \downarrow a\}$, by the above. □

Remark 3.4. A similar proof to that of Lemma 3.3 shows that, if M is an $\mathcal{S}$-frame and $a \in M$, the map $h : M \to \uparrow a$ given by $h(x) = x \vee a$, is an $\mathcal{S}$-frame map, and $\uparrow a$ is an $\mathcal{S}$-frame. If M is continuous, so is $\uparrow a$.

In the presence of regularity, the way below relation has a simple characterization (see Lemma 3.6) using elements s for which $\uparrow s$ is compact. (In [19] these elements are called "cocompact.") The next lemma collects some straightforward facts about such.

Lemma 3.5. *Let L be an $\mathcal{S}$-frame and $a, b \in L$. Then*

(a) *If $\uparrow a$ and $\uparrow b$ are compact, then $\uparrow(a \wedge b)$ is compact.*

(b) *If L is not compact and $a \ll 1$, then $\uparrow a$ is not compact.*

(c) *If $\uparrow a$ is compact and $a \leq b$, then $\uparrow b$ is compact.*

Proof. (a) Suppose $\bigvee W = 1$ for some designated subset of $\uparrow(a \wedge b)$. Then $\bigvee\{a \vee w : w \in W\} = 1$, so $(a \vee w_1) \vee \cdots \vee (a \vee w_n) = 1$ for some $w_1, \ldots, w_n \in W$. Also $\bigvee\{b \vee w : w \in W\} = 1$, so $(b \vee w_{n+1}) \vee \cdots \vee (b \vee w_m) = 1$ for some

$w_{n+1}, \ldots, w_m \in W$. Then $(a \wedge b) \vee w_1 \vee \cdots \vee w_m = 1$, so $w_1 \vee \cdots \vee w_m = 1$, as required.

(b) Suppose $a \ll 1$ and $\uparrow a$ is compact. Suppose S is a designated subset of L with $\bigvee S = 1$. Then $\bigvee\{s \vee a : s \in S\} = 1$ and $\{s \vee a : s \in S\}$ is a designated subset of $\uparrow a$, so $(s_1 \vee a) \vee \cdots \vee (s_n \vee a) = 1$ for some $s_1, \ldots, s_n \in S$. Since $a \ll 1$, we have $a \leq s_{n+1} \vee \cdots \vee s_m$ for some $s_{n+1}, \ldots, s_m \in S$. Then $s_1 \vee \cdots \vee s_n \vee s_{n+1} \vee \cdots \vee s_m \geq s_1 \vee \cdots \vee s_n \vee a = 1$. So L is compact.

(c) Suppose S is a designated subset of $\uparrow b$ and $\bigvee S = 1$. The map $h : \uparrow a \to \uparrow b$ given by $h(x) = x \vee b$ is an onto $\mathcal{S}$-frame map. (See Remark 3.4.) By Axiom (S4), there exists a designated subset T of $\uparrow a$ such that $\{t \vee b : t \in T\} = S$. Now $\{t \vee b : t \in T\}$ is a designated subset of $\uparrow a$ and $\bigvee\{t \vee b : t \in T\} = 1$. By compactness of $\uparrow a$, we obtain $(t_1 \vee b) \vee \cdots \vee (t_n \vee b) = 1$ for some $t_1, \ldots, t_n \in T$, and thus $s_1 \vee \cdots \vee s_n = 1$ for some $s_1, \ldots, s_n \in S$. □

We turn our attention to regular continuous $\mathcal{S}$-frames. The characterization of the way below relation in regular continuous frames (see [5]) involves the use of pseudocomplements, which are unavailable in this context; instead we use separating elements.

Lemma 3.6. *Let L be a regular, continuous $\mathcal{S}$-frame. Then*

(a) *For $a, b \in L$, $a \ll b$ if and only if there exists $s \in L$ with $s \wedge a = 0$, $s \vee b = 1$ and $\uparrow s$ is compact.*

(b) *The way below relation is closed under binary meet; that is, if $a \ll b$, $a \ll c$ then $a \ll b \wedge c$.*

Proof. (a) ($\Longrightarrow$) Suppose $a \ll b$. Since L is a regular $\mathcal{S}$-frame, there is a designated subset T of L such that $b = \bigvee T$ and $t \prec b$ for each $t \in T$. Then $a \leq t_1 \vee \cdots \vee t_n \prec b$ for some $t_1, \ldots, t_n \in T$. So $a \prec b$.
Since L is a continuous $\mathcal{S}$-frame, the way below relation interpolates, so there exists $c \in L$ with $a \ll c \ll b$. So $a \prec c$ and there exists $s \in L$ with $a \wedge s = 0$ and $s \vee c = 1$. We show $\uparrow s$ is compact. Suppose U is a designated subset of $\uparrow s$ with $\bigvee U = 1$. Since the map $h : L \to \uparrow s$ given by $h(x) = x \vee s$ is an onto $\mathcal{S}$-frame map, there exists a designated subset W of L such that $\{w \vee s : w \in W\} = U$. By Axiom (S2′), $\{w \vee s : w \in W\}$ is a designated subset of L, and since $c \ll b$, also $c \ll 1$. So $c \leq (w_1 \vee s) \vee \cdots \vee (w_n \vee s)$ for some $w_1, \ldots, w_n \in W$. Then $s \vee c = 1$ gives $(w_1 \vee s) \vee \cdots \vee (w_n \vee s) = 1$, so $u_1 \vee \cdots \vee u_n = 1$ for some $u_1, \ldots, u_n \in U$.

($\Longleftarrow$) Suppose there exists $s \in L$ with $s \wedge a = 0$, $s \vee b = 1$ and $\uparrow s$ is compact. Let $b \leq \bigvee T$ for some designated subset T of L. Then $s \vee \bigvee T = 1$, so $\bigvee\{s \vee t : t \in T\} = 1$. Since $\{s \vee t : t \in T\}$ is a designated subset of $\uparrow s$ by Axiom (S4), $(s \vee t_1) \vee \cdots \vee (s \vee t_n) = 1$ for some $t_1, \ldots, t_n \in T$. Then $a \wedge (s \vee t_1 \vee \cdots \vee t_n) = a$, so $a \leq t_1 \vee \cdots \vee t_n$.

(b) Suppose $a \ll b$ and $a \ll c$, with elements $s, t \in L$ such that $a \wedge s = 0$, $b \vee s = 1$, $a \wedge t = 0$, $c \vee t = 1$ and $\uparrow s$, $\uparrow t$ are compact. Then $a \wedge (s \vee t) = 0$, $(b \wedge c) \vee (s \vee t) = 1$ and $\uparrow(s \vee t)$ is compact, so $a \ll b \wedge c$. □

Example 3.1. This example shows that the condition characterizing $a \ll b$ in Lemma 3.6 cannot be replaced by a condition requiring that *all* separating elements s (that is, s such that $a \wedge s = 0$ and $s \vee b = 1$) satisfy the condition that $\uparrow s$ be compact. (Of course this cannot be done for full frames either.)

Let L be the power set of the natural numbers. Then L is a regular, continuous σ-frame. Let $A = \{1\}$ and $B = \mathbb{N} \setminus \{2\}$. Then $A \ll B$. Further, $\mathbb{N} \setminus \{1\}$ and $\{2\}$ are both separating elements showing $A \prec B$; but $\uparrow(\mathbb{N} \setminus \{1\})$ is compact, whereas $\uparrow(\{2\})$ is not. □

Lemma 3.7. *Any compact, regular $\mathcal{S}$-frame is continuous.*

Proof. Suppose that L is a compact, regular $\mathcal{S}$-frame. Continuity will follow from the fact that, for $a, b \in L$, $a \ll b$ if and only if $a \prec b$:

($\Longrightarrow$) This only requires regularity of L, and follows as in the proof of Lemma 3.6(a).

($\Longleftarrow$) Suppose $b \leq \bigvee S$ for some designated subset S of L. There exists $t \in L$ with $a \wedge t = 0$ and $t \vee b = 1$. Then $t \vee \bigvee S = 1$, so $\bigvee\{t \vee s : s \in S\} = 1$ and $\{t \vee s : s \in S\}$ is a designated subset of L. By compactness of L, $(t \vee s_1) \vee \cdots \vee (t \vee s_n) = 1$ for some $s_1, \ldots, s_n \in S$. Meeting with a gives $a \leq s_1 \vee \cdots \vee s_n$. □

We provide an explicit and concrete example of a regular, continuous σ-frame that is not a frame, and that is not a compact σ-frame.

Example 3.2. Let $\mathcal{L}$ consist of all countable and cocountable subsets of the real line. Then $\mathcal{L}$ is a Boolean algebra, and a completely regular σ-frame. The rather below relation here amounts to set inclusion. So the Stone-Čech compactification of $\mathcal{L}$ is $\mathcal{J}_\sigma\mathcal{L}$, the collection of countably generated ideals of $\mathcal{L}$, with the join map $j : \mathcal{J}_\sigma\mathcal{L} \to \mathcal{L}$ as the compactification map. (See [7] and [18].)

The collection $\{\downarrow\{i\} : \ i \text{ an irrational}, i \geq 0\}$ has no join in $\mathcal{J}_\sigma\mathcal{L}$, so $\mathcal{J}_\sigma\mathcal{L}$ is a σ-frame that is not a frame.

Let $I = \downarrow(\mathbb{R}\setminus\mathbb{Q})$. Clearly I is a member of $\mathcal{J}_\sigma\mathcal{L}$. Then the open quotient $\downarrow I$ is a regular, continuous σ-frame, by Lemma 3.3(b); further, $\downarrow I$ is not a compact σ-frame, and is not a frame. □

We conclude this section with a result that proves surprisingly useful.

Theorem 3.8. *Let L be a regular, continuous $\mathcal{S}$-frame. Suppose $b \in L$ and $\uparrow b$ is compact. If $a \in L$ is such that $a \prec b$, then there exists $c \in L$ such that $a \leq c \prec b$ and $\uparrow c$ is compact.*

Proof. Since $a \prec b$, there exists $s \in L$ with $a \wedge s = 0$ and $s \vee b = 1$. Since L is continuous, $s = \bigvee T$ for some designated subset T of L with $t \ll s$ for all $t \in T$. Then $b \vee \bigvee T = 1$, so $\bigvee\{b \vee t : \ t \in T\} = 1$. Since $\{b \vee t : \ t \in T\}$ is designated and $\uparrow b$ is compact, $(b \vee t_1) \vee \cdots \vee (b \vee t_n) = 1$ for some $t_1, \ldots, t_n \in T$. Then $b \vee (t_1 \vee \cdots \vee t_n) = 1$ and $t_1 \vee \cdots \vee t_n \ll s$. By Lemma 3.6, there exists $u \in L$ such that $u \wedge (t_1 \vee \cdots \vee t_n) = 0$, $u \vee s = 1$, and $\uparrow u$ is compact.

Let $c = u \wedge b$. Then $\uparrow c$ is compact, by Lemma 3.5(a). Also, $c \prec b$ since $c \wedge (t_1 \vee \cdots \vee t_n) = u \wedge b \wedge (t_1 \vee \cdots \vee t_n) = 0$ and $b \vee (t_1 \vee \cdots \vee t_n) = 1$. Finally, $a = a \wedge (u \vee s) = a \wedge u$, so $a \leq u$, and since $a \leq b$ by assumption, $a \leq b \wedge u = c$. Thus $a \leq c \prec b$, and $\uparrow c$ is compact, as required. □

4 Constructing the one-point compactification using generators and relations

Here we present a construction of the one-point compactification of a non-compact, regular continuous $\mathcal{S}$-frame, in which all the elements are given absolutely explicitly. The section concludes with the result that an $\mathcal{S}$-frame is non-compact, regular, and continuous if and only if it has a one-point compactification.

Definition 4.1. (Construction of the one-point compactification of an $\mathcal{S}$-frame)

Let L be a regular, continuous $\mathcal{S}$-frame which is not compact.

1. Then $L^\bullet = \{o_x : \ x \in L\} \cup \{p_x : \ x \in L \text{ with } \uparrow x \text{ compact}\}$, with

- Bottom o_0
- Top p_1
- Binary meet given, for $x, y \in L$, by

$$\begin{array}{rcll} o_x \wedge o_y & = & o_{x\wedge y} & \\ p_x \wedge p_y & = & p_{x\wedge y} & \text{for } \uparrow x, \uparrow y \text{ compact} \\ o_x \wedge p_y & = & o_{x\wedge y} & \text{for } \uparrow y \text{ compact} \end{array}$$

- Joins calculated as follows:

  If $S = \{o_x : x \in X\}$ is a designated subset of $L^\bullet$, then $\bigvee S = o_a$ for $a = \bigvee X$.

  If $S = \{o_x : x \in X\} \cup \{p_y : y \in Y\}$ is a designated subset of $L^\bullet$ with $Y \neq \emptyset$, then $\bigvee S = p_b$ for $b = \bigvee (X \cup Y)$.

2. The map $h : L^\bullet \to L$ given by $h(o_x) = x$ and $h(p_x) = x$, is a compactification of L.

Remark 4.2. (a) As usual in a presentation by generators and relations, the symbols o_x and p_x are intended to satisfy the conditions: $x \neq y \Longrightarrow o_x \neq o_y$ and $x \neq y \Longrightarrow p_x \neq p_y$; and $o_x \neq p_y$ regardless of x and y.

(b) It is clear that, for any $x \in L$ with $\uparrow x$ compact, $o_x \leq p_x$ since $o_x \wedge p_x = o_x$.

(c) An example at the end of this section (see Example 4.1) will illustrate the intuition behind this construction in the spatial case.

(d) The proof that the construction above does indeed give a one-point compactification is provided below. The structure is defined by the three rules for binary meets given above; the description of the partial order, top, bottom, and joins follows as a consequence.

Theorem 4.3. *Let L be a regular, continuous $\mathcal{S}$-frame which is not compact, and $L^\bullet$, $h : L^\bullet \to L$ be defined as in Definition* 4.1*. Then*

(a) *$L^\bullet$ is a compact, regular $\mathcal{S}$-frame.*

(b) *$h : L^\bullet \to L$ is a dense, onto $\mathcal{S}$-frame map, and so a compactification of L.*

(c) *$h : L^\bullet \to L$ is a one-point compactification of L.*

Proof. (a) We note that in the rule $p_x \wedge p_y = p_{x\wedge y}$, the right-hand expression is defined, since compactness of $\uparrow x, \uparrow y$ implies $\uparrow(x\wedge y)$ is compact, by Lemma 3.5(a). From the definition of binary meet, these follow, for $x, y \in L$:

$$\begin{array}{rclr} x \le y & \iff & o_x \le o_y & \\ x \le y & \iff & p_x \le p_y & \text{for } \uparrow x, \uparrow y \text{ compact} \\ & p_x \nleq o_y & & \text{for } \uparrow x \text{ compact} \\ & o_x \le p_x & & \text{for } \uparrow x \text{ compact} \end{array}$$

A straightforward use of the definition of join then shows, for $x, y \in L$:

$$\begin{array}{rclr} o_x \vee o_y & = & o_{x\vee y} & \\ p_x \vee p_y & = & p_{x\vee y} & \text{for } \uparrow x, \uparrow y \text{ compact} \\ o_x \vee p_y & = & p_{x\vee y} & \text{for } \uparrow y \text{ compact} \end{array}$$

Next we establish the existence of designated joins in $L^\bullet$.

Suppose $S = \{o_x : x \in X\} \cup \{p_y : y \in Y\}$ is a designated subset of $L^\bullet$ with $Y \neq \emptyset$. Since $L^\bullet$ is a meet-semilattice and h a meet-semilattice map, $h[S] = X \cup Y$ is a designated subset of L, by Axiom (S4). So $b = \bigvee(X \cup Y)$ exists. Certainly p_b is an upper bound of S. Since $Y \neq \emptyset$, no element of the form o_t can be an upper bound for S. Suppose then that $p_t \ge o_x$ and $p_t \ge p_y$ for all $x \in X, y \in Y$. Then $t \ge x, t \ge y$ for $x \in X, y \in Y$, and hence $t \ge \bigvee(X \cup Y) = b$, so $p_t \ge p_b$. Thus $p_b = \bigvee S$. A similar argument shows that if $S = \{o_x : x \in X\}$ is a designated subset of L, then $\bigvee S = o_a$ for $a = \bigvee X$.

Next we show that binary meet distributes over designated join in $L^\bullet$.

Suppose $S = \{o_x : x \in X\} \cup \{p_y : y \in Y\}$ is a designated subset of $L^\bullet$ with $Y \neq \emptyset$. Direct calculation shows that:

$$\begin{array}{rclr} o_t \wedge \bigvee S & = & o_a & \text{for } a = t \wedge \bigvee(X \cup Y) \\ & = & \bigvee\{o_t \wedge s : s \in S\} & \\ p_t \wedge \bigvee S & = & p_a & \text{for } a = t \wedge \bigvee(X \cup Y) \\ & = & \bigvee\{p_t \wedge s : s \in S\} & \end{array}$$

This uses the fact that, if $X \cup Y$ is a designated subset of L, then $\{t \wedge c : c \in X \cup Y\}$ is also a designated subset of L. A similar argument shows that binary meet distributes over designated joins of sets of the form $\{o_x : x \in X\}$.

So far, we have established that $L^\bullet$ is an $\mathcal{S}$-frame.

Next we show that $L^\bullet$ is compact.

Suppose $S = \{o_x : x \in X\} \cup \{p_y : y \in Y\}$ is a designated subset of $L^\bullet$ with $\bigvee S = p_1$. Then $Y \neq \emptyset$; for $t \in Y$, $\uparrow t$ is compact. Now the map $g : L \to \uparrow t$ given by $g(a) = a \vee t$, preserves designated sets (see Remark 3.4), so $\{t \vee c : c \in X \cup Y\}$ is a designated subset of $\uparrow t$. Now $\bigvee S = p_1 = p_b$ for $b = \bigvee (X \cup Y)$; so $\bigvee (X \cup Y) = 1$. Then $\bigvee \{t \vee c : c \in X \cup Y\} = 1$. Applying compactness of $\uparrow t$ gives $(t \vee x_1) \vee \cdots \vee (t \vee x_n) \vee (t \vee y_{n+1}) \vee \cdots \vee (t \vee y_m) = 1$ for some $x_1, \ldots, x_n \in X, y_{n+1}, \ldots, y_m \in Y$. Then $o_{x_1} \vee \cdots \vee o_{x_n} \vee p_{y_{n+1}} \vee \cdots \vee p_{y_m} \vee p_t = p_1$, as required.

Next we show that $L^\bullet$ is regular.

Claim:

$$\begin{aligned}
o_a \prec o_b \text{ in } L^\bullet &\iff a \ll b \text{ in } L \\
p_a \prec p_b \text{ in } L^\bullet &\iff a \prec b \text{ in } L, \text{ for } \uparrow a, \uparrow b \text{ compact} \\
o_a \prec p_b \text{ in } L^\bullet &\iff a \prec b \text{ in } L, \text{ for } \uparrow b \text{ compact}
\end{aligned}$$

Proof: We have

$$\begin{aligned}
o_a \prec o_b &\iff \text{there exists } t \in L \text{ with } o_a \wedge p_t = o_0 \text{ and } o_b \vee p_t = p_1 \\
&\iff \text{there exists } t \in L \text{ with } a \wedge t = 0,\ b \vee t = 1, \\
&\qquad \text{and } \uparrow t \text{ compact} \\
&\iff a \ll b
\end{aligned}$$

The last step uses Lemma 3.6(a).

$$\begin{aligned}
p_a \prec p_b &\iff \text{there exists } t \in L \text{ with } p_a \wedge o_t = o_0 \text{ and } p_b \vee o_t = p_1 \\
&\iff \text{there exists } t \in L \text{ with } a \wedge t = 0 \text{ and } b \vee t = 1 \\
&\iff a \prec b
\end{aligned}$$

The argument for $o_a \prec p_b$ is similar to that for $p_a \prec p_b$.

To establish regularity of $L^\bullet$, we begin with $a \in L$; first show that o_a can be written in the appropriate form; then do the same for p_a.

Write $a = \bigvee U$ for some designated subset U of L with $u \ll a$ for all $u \in U$. By Axiom (S4), there exists a designated subset V of $L^\bullet$ such that $h[V] = U$. Since, for any $u \in U$, $u \ll 1$, Lemma 3.5(b) shows that $\uparrow u$ is not compact, so $h^{-1}(\{u\}) = \{o_u\}$, and hence $V = \{o_u : u \in U\}$. Then $o_a = o_{\bigvee U} = \bigvee\{o_u : u \in U\}$ and $o_u \prec o_a$ for all $u \in U$.

Now consider $a \in L$ with $\uparrow a$ compact. Write $a = \bigvee T$ for some designated subset T of L with $t \prec a$ for all $t \in T$. Apply Theorem 3.8: for each such t, obtain $r_t \in L$ with $t \leq r_t \prec a$ and $\uparrow r_t$ compact. Since $T \leq \{r_t : t \in T\}$, by Axiom (SRef), there exists a designated subset R of $\{r_t : t \in T\}$ such that $T \leq R$. Note that $\bigvee R = a$.

By Axiom (S4), there exists a designated subset S of $L^\bullet$ such that $h[S] = R$. Suppose $S = \{o_x : x \in X\} \cup \{p_y : y \in Y\}$ for some $X, Y \subseteq L$ with $X \cup Y = R$. Let $Z = \{p_x : x \in X\} \cup \{p_y : y \in Y\}$. We note that, for $x \in R$, $\uparrow x$ is compact, so the elements p_x in this set do exist. Then $S \leq Z$, so by Axiom (SRef) there exists a designated subset W of Z such that $S \leq W$. Now $p_a = \bigvee W$ and $w \prec p_a$ for all $w \in W$.

So we have established that $L^\bullet$ is a compact, regular $\mathcal{S}$-frame.

(b) The description of joins and meets given in Definition 4.1 now makes it clear that the map $h : L^\bullet \to L$ given by $h(o_x) = x$ and $h(p_x) = x$, is an onto $\mathcal{S}$-frame map. It is dense because $h(p_x) = 0$ cannot occur, since this would imply that $x = 0$ and $\uparrow 0$ is compact, contrary to assumption.

(c) We note that o_1 is a maximal element of $L^\bullet$: $o_1 \geq o_x$ for all $x \in L$ and if $p_x \geq o_1$ for some $x \in L$, then $p_x \wedge o_1 = o_1$, so $o_{x \wedge 1} = o_1$, and hence $x = 1$.

Further, $h : \downarrow o_1 \to L$ is an $\mathcal{S}$-frame isomorphism. So $h : L^\bullet \to L$ is a one-point compactification of L. □

Remark 4.4. Arguments similar to those used in the proof of Theorem 4.3 show that $h : L^\bullet \to L$ preserves all existing joins (whether designated or not) and all existing meets.

Corollary 4.5. *A regular, non-compact $\mathcal{S}$-frame is continuous if and only if it has a one-point compactification.*

Proof. ($\Longrightarrow$) Follows by Theorem 4.3.

($\Longleftarrow$) Follows by Lemma 3.7 and Lemma 3.3(b). □

Example 4.1. Here we provide a very simple example illustrating our construction above.
Let L be the open subsets of $[0, 1)$ with the usual topology, and with one-point compactification $[0, 1]$. For $x = (\frac{1}{2}, 1)$ think of o_x as being $(\frac{1}{2}, 1)$ and p_x as $(\frac{1}{2}, 1]$. Note that $\uparrow x$ is isomorphic to the open sets of $[0, \frac{1}{2}]$, which is compact. By contrast, for $y = [0, \frac{1}{2})$, o_y can be thought of as $[0, \frac{1}{2})$ and since $[\frac{1}{2}, 1)$ is not compact, p_y does not exist.
As we hope this example makes clear, the intention is that o_x represents the original open set, now as a subset of its compactification, and p_x represents the open set with the "point at infinity" added when this is indeed an open set of the compactification. We chose the letter "o" to stand for "original" and "p" for "partner." □

Remark 4.6. In Definition 2.7 the notion of a one-point compactification was given; in this section we have constructed such a compactification. One-point compactifications are in fact unique, as will be shown in Theorem 8.9.

5 Compactifications and strong inclusions

In this section, we briefly summarize some definitions and results from [18] (Sections 5, 6, and 7) concerning strong inclusions and compactifications. A compactification of an $\mathcal{S}$-frame L can be associated with a strong inclusion on L, and conversely, in such a way that these associations are mutual inverses, making the collection of strong inclusions on L isomorphic, as partially ordered set, to the collection of compactifications of L.

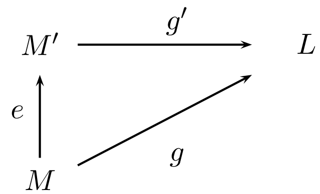
Definition 5.1. A *strong inclusion* on an $\mathcal{S}$-frame L is a binary relation $\lhd$ on L such that, for all $a, b, c, d \in L$:
(SI1) $a \le b \lhd c \le d \Rightarrow a \lhd d$.
(SI2) $\lhd$ is a sublattice of $L \times L$.
(SI3) $a \lhd b \Rightarrow a \prec b$
(SI4) $\lhd$ interpolates on L; that is, if $a \lhd b$ there exists $c \in L$ with $a \lhd c \lhd b$.
(SI5) If $a \lhd b$, there exist $c, d \in L$ with $c \lhd d, a \wedge d = 0$ and $b \vee c = 1$.
(SI6) For $a \in L, a = \bigvee S$ for some designated set S in L with $s \in S \Rightarrow s \lhd a$.

Definition 5.2. Let L be an $\mathcal{S}$-frame, $\lhd$ a strong inclusion on L. An ideal I of L is called $\mathcal{S}$*-generated* if there exists a designated subset S of L with

$I = \{x \in L : x \leq s \text{ for some } s \in S\}$. An $\mathcal{S}$-generated ideal I of L is called $\lhd$-*strongly regular* if, for each $x \in I$ there exists $y \in I$ with $x \lhd y$. The collection of all $\lhd$-strongly regular $\mathcal{S}$-generated ideals of L will be denoted by $\mathcal{R}_{\lhd}\mathcal{J}_{\mathcal{S}}L$. The join map $\bigvee : \mathcal{R}_{\lhd}\mathcal{J}_{\mathcal{S}}L \to L$ is a compactification of L. We denote it by $g_{\lhd}$.

Definition 5.3. If $g : M \to L$ is a compactification of L, then $\lhd_g = (g \times g)[\prec]$ is a strong inclusion on L.

Definition 5.4. Let L be an $\mathcal{S}$-frame with $g : M \to L$ and $g' : M' \to L$ compactifications of L. A preorder $\leq$ on the compactifications of L is defined as follows: $g \leq g'$ if there exists an $\mathcal{S}$-frame map $e : M \to M'$ such that the following diagram commutes:



In the above diagram, such an e, if it exists, is unique and one-one. For $g : M \to L$ and $g' : M' \to L$ compactifications of L, if $g \leq g'$ and $g' \leq g$ then the $\mathcal{S}$-frame map e satisfying $g'e = g$ is an isomorphism; we then say that $g : M \to L$ and $g' : M' \to L$ are isomorphic.

Definition 5.5. Let L be an $\mathcal{S}$-frame. Then

1. Cpns(L) denotes the partially ordered set obtained from the preorder on the compactifications of L, given above.

2. StrIncl(L) denotes the partially ordered set of all strong inclusions on L ordered by inclusion.

Proposition 5.6. *Let L be an $\mathcal{S}$-frame. Then*

(a) *If $\lhd$ is a strong inclusion on L, then $\lhd_{g_{\lhd}} = \lhd$.*

(b) *If $g : M \to L$ is a compactification of L, then $g_{\lhd_g} = g$.*

(c) *StrIncl(L) and Cpns(L) are isomorphic partially ordered sets.*

6 Least strong inclusions, least compactifications and one-point compactifications

In Section 4, it was established in Corollary 4.5 that a regular $\mathcal{S}$-frame is continuous if and only if it is compact or has a one-point compactification. It is not a priori clear that a one-point compactification is necessarily also the smallest such. In this section we provide a proof of this, using strong inclusions as a basic tool.

Definition 6.1. Let L be a regular, continuous $\mathcal{S}$-frame which is not compact and $h : L^\bullet \to L$ the one-point compactification of Definition 4.1. Define the strong inclusion $\blacktriangleleft$ on L by $\blacktriangleleft = (h \times h)[\prec]$, where $\prec$ is the rather below relation on $L^\bullet$.

Lemma 6.2. *Let L be a regular, continuous $\mathcal{S}$-frame which is not compact. For $a, b \in L$, $a \blacktriangleleft b$ if and only if*
(i) *$a \prec b$, and*
(ii) *$\uparrow b$ is compact, or there exists $s \in L$ with $a \wedge s = 0$, $s \vee b = 1$ and $\uparrow s$ is compact.*
This is equivalent to saying that $a \blacktriangleleft b$ if and only if
(iii) *$a \ll b$ or*
(iv) *$a \prec b$ and $\uparrow b$ is compact.*

Proof. Apply Lemma 3.6(a) and the explicit description of the rather below relation on $L^\bullet$ given in the proof of Theorem 4.3. □

Remark 6.3. We note that, in the case that the $\mathcal{S}$-frame is regular and compact, the relation $\blacktriangleleft$ described in Lemma 6.2 is still a strong inclusion on L; it obviously reduces to $\prec$. It would also have been possible to *define* $\blacktriangleleft$ on any regular, continuous $\mathcal{S}$-frame, by the description of Lemma 6.2, and then prove explicitly that the six conditions required for a strong inclusion hold. This is the approach taken in [5], where the case for full frames is discussed.

We now examine the link between one-point and least compactifications.

Proposition 6.4. *Let L be a regular, continuous $\mathcal{S}$-frame. The strong inclusion $\blacktriangleleft$ is the least strong inclusion on L.*

Proof. Suppose $\lhd$ is another strong inclusion on L. Begin with $a \blacktriangleleft b$ in L.

Case 1: $a \ll b$. Write $b = \bigvee T$ for some designated subset T of L with $t \lhd b$ for all $t \in T$, using condition (SI6) for strong inclusions. Then $a \leq t_1 \vee \cdots \vee t_n \lhd b$ for some $t_1, \ldots, t_n \in T$; so $a \lhd b$.

Case 2: $a \prec b$ and $\uparrow\! b$ is compact. There exists $s \in L$ with $a \wedge s = 0$ and $s \vee b = 1$. Write $s = \bigvee U$ for some designated subset U of L with $u \lhd s$ for all $u \in U$. Then $b \vee \bigvee U = 1$, so $\bigvee \{b \vee u : u \in U\} = 1$, and since $\{b \vee u : u \in U\}$ is a designated subset of $\uparrow\! b$, we obtain $(b \vee u_1) \vee \cdots \vee (b \vee u_m) = 1$ for some $u_1, \ldots, u_m \in U$. Also $w = u_1 \vee \cdots \vee u_m \lhd s$. Using condition (SI5) for strong inclusions gives elements $c, d \in L$ with $c \lhd d$, $w \wedge d = 0$ and $c \vee s = 1$. Then $a \prec c \lhd d \prec b$, so $a \lhd b$. □

The following lemma is needed in the proof of Proposition 6.7.

Lemma 6.5. *Let M be a compact, regular $\mathcal{S}$-frame and $a \in M$. Define $M_a = \{x \in M : x \leq a$ or $x \vee a = 1\}$. Then M_a is a compact, regular sub $\mathcal{S}$-frame of M.*

Proof. That M_a is a sub meet-semilattice of M is straightforward to check. Let S be a designated subset of M_a. Since the inclusion map $M_a \to M$ is a meet-semilattice map, Axiom (S4) makes S a designated subset of M; so $\bigvee_M S$ exists. We check that $\bigvee S \in M_a$: If $s \leq a$ for all $s \in S$, then $\bigvee S \leq a$. If there exists $s \in S$ with $s \vee a = 1$, then $a \vee \bigvee S = 1$.
So M_a is a sub $\mathcal{S}$-frame of M, and M_a inherits compactness from M automatically.

We note that a subset of M_a is a designated subset of M_a if and only if it is a designated subset of M, by the Axioms (S4) and (S5).

To show regularity of M_a, we begin with $y \in M_a$ and consider two cases.

Case 1: $y \leq a$. Since M is regular, $y = \bigvee T$ for some designated subset T of M with $t \prec y$ in M, for all $t \in T$. For such t, $t \in M_a$ and there exists $c \in M$ with $t \wedge c = 0$ and $c \vee y = 1$. Then $c \vee a = 1$, so $c \in M_a$. So $t \prec y$ in M_a.

Case 2: $a \vee y = 1$. Again, $y = \bigvee T$ for some designated subset T of M with $t \prec y$ in M, for all $t \in T$. Since $a \vee \bigvee T = 1$ and $\{a \vee t : t \in T\}$ is a designated subset of M, $a \vee (t_1 \vee \cdots \vee t_n) = 1$ for some $t_1, \ldots, t_n \in T$; so $u = t_1 \vee \cdots \vee t_n \in M_a$ and $u \prec y$ in M. So there exists $d \in M$ with $u \wedge d = 0$ and $d \vee y = 1$. Then $d = d \wedge (a \vee u) = d \wedge a$, so $d \leq a$, making $d \in M_a$.

Thus $u \prec y$ in M_a. Now $\{t \vee u : t \in T\}$ is a designated subset of M, and hence M_a, $t \vee u \prec y$ in M_a, for all $t \in T$ and $y = \bigvee\{t \vee u : t \in T\}$. □

Remark 6.6. If M is a regular, normal $\mathcal{S}$-frame, then $M_a = \{x \in M : x \leq a \text{ or } x \vee a = 1\}$ is a regular sub $\mathcal{S}$-frame of M. See [9] for such a statement in the context of full frames. A similar proof to that of Lemma 6.5 shows this; we do not pursue it here because normality is not needed anywhere else in this paper. We refer the reader to Section 9 of [14] for a discussion of normality for $\mathcal{S}$-frames.

Proposition 6.7. *Any least compactification of a non-compact $\mathcal{S}$-frame is a one-point compactification.*

Proof. Let $h : M \to L$ be the least compactification of a non-compact $\mathcal{S}$-frame L. Then h is obviously not an isomorphism, and so it is not one-one, and hence not codense. (See Proposition 8.10 of [14].)

Take $a \in M$ with $h(a) = 1$ but $a < 1$, and form the compact, regular sub $\mathcal{S}$-frame M_a of M described in Lemma 6.5. Then the restriction $h : M_a \to L$ is a dense $\mathcal{S}$-frame map. We show that it is also onto. For $x < 1$ in L, there exists $b \in M$ with $h(b) = x$. Then $h(b \wedge a) = h(b) \wedge h(a) = x$ and $b \wedge a \in M_a$ (in fact, $b \wedge a \in \downarrow a$). So $h : M_a \to L$ is a compactification of L; since $h : M \to L$ was the least compactification of L, it follows that $M = M_a$. Then a is a maximal element of M, since it is clearly a maximal element of M_a.

The restriction $h : \downarrow a \to L$ is an $\mathcal{S}$-frame map which is onto, as noted above. To show that it is an isomorphism, it suffices to show it is codense, by Proposition 8.10 of [14]. This result applies because $\downarrow a$, being a quotient of a regular $\mathcal{S}$-frame, is again regular. So suppose $c \leq a$ and $h(c) = 1$. Then, by the same argument as above, $M = M_c$. So $a \in M_c$. If $a \vee c = 1$, then $a = 1$; which is a contradiction. So $a \leq c$, making $a = c$ as desired.

In total, we have shown that M has a maximal element a such that the map $h : \downarrow a \to L$ is an isomorphism, making $h : M \to L$ a one-point compactification of L. □

To summarize, here is the main result so far:

Theorem 6.8. *For a regular $\mathcal{S}$-frame L, the following are equivalent.*

(a) *L is continuous.*

(b) *L has a least strong inclusion.*
(c) *L has a least compactification.*
(d) *L is compact or has a one-point compactification.*

Proof. (a) $\Longrightarrow$ (b): By Proposition 6.4.
(b) $\Longleftrightarrow$ (c): From Corollary 7.8 in [18].
(c) $\Longrightarrow$ (d): By Proposition 6.7.
(d) $\Longrightarrow$ (a): By Lemma 3.7 and Lemma 3.3(b). □

Corollary 6.9. *If the Stone-Čech compactification and the one-point compactification of an $\mathcal{S}$-frame L coincide, then L has a unique compactification.*

7 A zero-dimensional interlude

A compactification of a zero-dimensional partial frame need not be zero-dimensional; here we show that the one-point compactification of a zero-dimensional continuous $\mathcal{S}$-frame is zero-dimensional. The corresponding result for full frames may be folklore; we have not been able to find a reference in the literature for this.

Definition 7.1. An $\mathcal{S}$-frame L is called *zero-dimensional* if for each $a \in L$ there is a designated subset T of L such that $a = \bigvee T$ and $t \prec t$ for each $t \in T$.

We note that the condition $t \prec t$ is equivalent to saying that t is complemented; that is, there exists an element c in the $\mathcal{S}$-frame with $t \wedge c = 0$ and $c \vee t = 1$.

Lemma 7.2. (a) *In any $\mathcal{S}$-frame L, $a \prec b \ll 1$ implies that $a \ll b$.*
(b) *In any $\mathcal{S}$-frame L, if c is a complemented element of L and $c \ll 1$, then $c \ll c$.*

Proof. (a) Suppose there exists $s \in L$ with $a \wedge s = 0$ and $s \vee b = 1$. Suppose $b \leq \bigvee T$ for some designated subset T of L. Then $s \vee \bigvee T = 1$, so $\bigvee\{s \vee t : \ t \in T\} = 1$ and $\{s \vee t : \ t \in T\}$ is a designated subset of L. Since $b \ll 1$, $b \leq (s \vee t_1) \vee \cdots \vee (s \vee t_n)$ for some $t_1, \ldots, t_n \in T$. Then $a = a \wedge b \leq a \wedge (s \vee t_1 \vee \cdots \vee t_n) = a \wedge (t_1 \vee \cdots \vee t_n)$, so $a \leq t_1 \vee \cdots \vee t_n$.
(b) Direct from (a). □

Lemma 7.3. *Let L be a zero-dimensional $\mathcal{S}$-frame and $a, b \in L$.*

(a) *If $a \prec b$ and $\uparrow b$ is compact, there exists a complemented element c of L with $a \leq c \leq b$.*

(b) *If $a \ll b$, there exists a complemented element c of L with $a \leq c \leq b$.*

Proof. (a) Suppose $s \in L$ satisfies $a \wedge s = 0$ and $s \vee b = 1$. Write $s = \bigvee T$ for some designated subset T of L with $t \prec t$ for all $t \in T$. Then $b \vee \bigvee T = 1$, so $\bigvee\{b \vee t : t \in T\} = 1$, and since $\uparrow b$ is compact, $(b \vee t_1) \vee \cdots \vee (b \vee t_n) = 1$ for some $t_1, \ldots, t_n \in T$. Let $c = t_1^* \wedge \cdots \wedge t_n^*$ where t_i^* is here the complement, not merely the pseudocomplement of t_i. Then c is complemented. Since $a \wedge \bigvee T = 0$, $a \leq t_i^*$ for $i = 1, \ldots, n$ and so $a \leq c$. Also $c = c \wedge (b \vee t_1 \vee \cdots \vee t_n) = c \wedge b$, so $c \leq b$.

(b) Write $b = \bigvee T$ for some designated subset T of L with $t \prec t$ for all $t \in T$. Then $a \leq t_1 \vee \cdots \vee t_n$ for some $t_1, \ldots, t_n \in T$; also $t_1 \vee \cdots \vee t_n \leq b$ and $t_1 \vee \cdots \vee t_n$ is complemented. □

Proposition 7.4. *The one-point compactification of a zero-dimensional, continuous $\mathcal{S}$-frame is zero-dimensional.*

Proof. Let L be a zero-dimensional, continuous $\mathcal{S}$-frame that is not compact. This argument proceeds along similar lines to that of the proof of the regularity of $L^\bullet$ in Theorem 4.3; we refer the reader to that proof for a description of the rather below relation on $L^\bullet$. From there, we have, for $a \in L$,

$$o_a \prec o_a \text{ in } L^\bullet \iff a \ll a \text{ in } L$$
$$p_a \prec p_a \text{ in } L^\bullet \iff a \prec a \text{ in } L, \text{ for } \uparrow a \text{ compact}$$

Let $a \in L$. We first show that we can express o_a in the appropriate form; then do the same for p_a.

Write $a = \bigvee U$ for some designated subset U of L with $u \ll a$ for all $u \in U$. Since L is continuous, $\ll$ interpolates (by Lemma 3.3(a)), so for each such $u \in U$, there exists $w_u \in L$ with $u \ll w_u \ll a$. By Lemma 7.3(b), there exists a complemented element $c_u \in L$ with $u \leq c_u \leq w_u \ll a$. Then $c_u \ll 1$, so by Lemma 7.2(b), $c_u \ll c_u$. Now $U \leq \{c_u : u \in U\}$, so by Axiom (SRef), there exists a designated subset B of $\{c_u : u \in U\}$ such that $U \leq B$. Then $\bigvee B = a$.

By Axiom (S4) there exists a designated subset V of $L^\bullet$ such that $h[V] = B$, where $h : L^\bullet \to L$ is the one-point compactification of L (described in Definition 4.1). By Lemma 3.5(b), $\uparrow b$ is non-compact, for all $b \in B$, so $h^{-1}(\{b\}) = \{o_b\}$. Thus $V = \{o_b : b \in B\}$. Then $o_a = o_{\bigvee B} = \bigvee\{o_b : b \in B\}$, and $o_b \prec o_b$ for all $b \in B$.

Now consider $a \in L$ with $\uparrow a$ compact. Write $a = \bigvee T$ for some designated subset T of L with $t \prec a$ for all $t \in T$. Apply Theorem 3.8: For each such t, obtain $r_t \in L$ with $t \leq r_t \prec a$ and $\uparrow r_t$ compact. By Lemma 7.3(a), there exists a complemented element c_t of L with $t \leq r_t \leq c_t \leq a$. Since $\uparrow r_t$ is compact, $\uparrow c_t$ is compact, also. Then $T \leq \{c_t : t \in T\}$, so by Axiom (SRef), there exists a designated subset B of $\{c_t : t \in T\}$ such that $T \leq B$. Then $\bigvee B = a$.

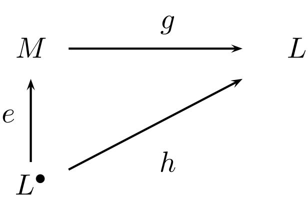
By Axiom (S4), there exists a designated subset S of $L^\bullet$ such that $h[S] = B$. Suppose $S = \{o_x : x \in X\} \cup \{p_y : y \in Y\}$ for some $X, Y \subseteq L$ with $X \cup Y = B$. Let $Z = \{p_x : x \in X\} \cup \{p_y : y \in Y\}$. We note that, since $\uparrow b$ is compact for all $b \in B$, the elements p_x in this set do exist. Then $S \leq Z$, so by Axiom (SRef) there exists a designated subset W of Z such that $S \leq W$. Now $p_a = \bigvee W$ and $w \prec w$ for all $w \in W$. □

8 Arbitrary compactifications

We return to the setting of regular continuous $\mathcal{S}$-frames to investigate compactifications other than the smallest such.

We begin by defining two functions associated with any compactification which ultimately will allow us to obtain results for partial frames which for full frames are obtained using right adjoints.

Definition 8.1. Let L be a regular, continuous $\mathcal{S}$-frame which is not compact. Let $h : L^\bullet \to L$ be its one-point compactification, as given in Definition 4.1. Let $g : M \to L$ be an arbitrary compactification of L. By Theorem 6.8, $h : L^\bullet \to L$ is the least compactification of L, so there exists a (unique) one-one $\mathcal{S}$-frame map $e : L^\bullet \to M$ such that $g \circ e = h$; that is, such that the following diagram commutes:



Define $\tilde{g} : L \to M$ and $\hat{g} : L \to M$ as follows:

For $x \in L$, $\tilde{g}(x) = e(o_x)$.

For $x \in L$, $\hat{g}(x) = \begin{cases} e(p_x) & \text{if } \uparrow x \text{ is compact} \\ e(o_x) & \text{if } \uparrow x \text{ is not compact} \end{cases}$

We note that the functions $\tilde{g}$ and $\hat{g}$ are not claimed to be $\mathcal{S}$-frame maps; their properties are given in Lemma 8.2 below.

Lemma 8.2. (a) *The function $\tilde{g} : L \to M$ preserves binary meets, designated joins, including the bottom element, but not the top element.*

(b) *The function $\hat{g} : L \to M$ is a meet-semilattice map; that is, it preserves binary meets, the bottom element and the top element.*

(c) *The functions $\tilde{g}$ and $\hat{g}$ are one-one.*

Proof. We refer the reader to Definition 4.1 for the lattice-theoretic properties of $L^\bullet$.

(a) For $x, y \in L$,

$$\tilde{g}(x \wedge y) = e(o_{x\wedge y}) = e(o_x \wedge o_y) = e(o_x) \wedge e(o_y) = \tilde{g}(x) \wedge \tilde{g}(y),$$

for S a designated subset of L,

$$\tilde{g}(\bigvee S) = e(o_{\bigvee S}) = e(\bigvee_{s\in S} o_s) = \bigvee_{s\in S} e(o_s) = \bigvee_{s\in S} \tilde{g}(s),$$

and $\tilde{g}(0) = e(o_0) = 0$. In $L^\bullet$, $o_1 \neq p_1$ and $e(p_1) = 1$, since e is an $\mathcal{S}$-frame map. Since e is one-one, this shows that $e(o_1) \neq 1$.

(b) $\hat{g}(0) = e(o_0) = 0$, since $\uparrow 0$ is not compact. Also, $\hat{g}(1) = e(p_1) = 1$, since $\uparrow 1$ is compact. Let $x, y \in L$.

If $\uparrow x$ and $\uparrow y$ are compact, so is $\uparrow(x \wedge y)$, by Lemma 3.5(a). Then

$$\hat{g}(x \wedge y) = e(p_{x\wedge y}) = e(p_x \wedge p_y) = e(p_x) \wedge e(p_y) = \hat{g}(x) \wedge \hat{g}(y).$$

If $\uparrow x$ is compact, but $\uparrow y$ is not compact, then $\uparrow(x \wedge y)$ is not compact, so

$$\hat{g}(x \wedge y) = e(o_{x\wedge y}) = e(p_x \wedge o_y) = e(p_x) \wedge e(o_y) = \hat{g}(x) \wedge \hat{g}(y).$$

If neither $\uparrow x$ nor $\uparrow y$ are compact, then neither is $\uparrow(x \wedge y)$, and so

$$\hat{g}(x \wedge y) = e(o_{x\wedge y}) = e(o_x \wedge o_y) = e(o_x) \wedge e(o_y) = \hat{g}(x) \wedge \hat{g}(y).$$

(c) This follows from the construction of $L^\bullet$ and the fact that e is one-one. □

Definition 8.3. For a compactification $g : M \to L$ of a regular and continuous $\mathcal{S}$-frame L, we define for $x \in L$,

$$\text{Balloon}_x = \{a \in M : \ g(a) = x\}.$$

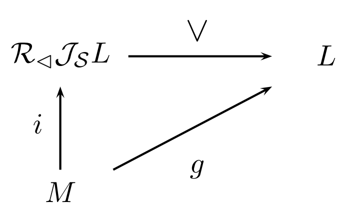
The next result shows that the elements $\tilde{g}(x)$ and $\hat{g}(x)$ play a special rôle in our understanding of the balloons in a compactification: $\tilde{g}(x)$ is the smallest element in Balloon_x, and, if $\uparrow x$ is compact, then $\hat{g}(x)$ is the largest element of Balloon_x.

Proposition 8.4. *Let $g : M \to L$ be a compactification of a regular, continuous $\mathcal{S}$-frame L, and $x \in L$.*

(a) *If $a \in \text{Balloon}_x$, then $\tilde{g}(x) \leq a$.*

(b) *If $\uparrow x$ is compact and $a \in \text{Balloon}_x$, then $a \leq \hat{g}(x)$.*

Proof. We use the description of compactifications using ideals (see Section 5). To be specific: Let $\lhd$ be the strong inclusion (previously denoted $\lhd_g$) of the compactification $g : M \to L$. The map i in the commuting diagram below is an isomorphism:



So we replace M by $\mathcal{R}_{\lhd}\mathcal{J}_{\mathcal{S}}L$, the $\lhd$-strongly regular $\mathcal{S}$-generated ideals of L, and $g : M \to L$ by the join map $\mathcal{R}_{\lhd}\mathcal{J}_{\mathcal{S}}L \to L$.

Further, let $j : \mathcal{R}_{\prec}\mathcal{J}_{\mathcal{S}}L^{\bullet} \to L^{\bullet}$ be the join map from the $\prec$-strongly regular $\mathcal{S}$-generated ideals of $L^{\bullet}$; this is an isomorphism, since $L^{\bullet}$ is compact, regular.

Let $\mathcal{J}_{\mathcal{S}}h : \mathcal{R}_{\prec}\mathcal{J}_{\mathcal{S}}L^{\bullet} \to \mathcal{R}_{\lhd}\mathcal{J}_{\mathcal{S}}L$ be the $\mathcal{S}$-frame map given by $\mathcal{J}_{\mathcal{S}}h(J) = {\downarrow}h[J]$. (See Lemma 4.3 of [18].)

Consider the diagram:

$$\begin{array}{ccc} \mathcal{R}_{\lhd}\mathcal{J}_{\mathcal{S}}L & \xrightarrow{\ \bigvee\ } & L \\ {\scriptstyle \mathcal{J}_{\mathcal{S}}h}\uparrow & & \uparrow{\scriptstyle h} \\ \mathcal{R}_{\prec}\mathcal{J}_{\mathcal{S}}L^{\bullet} & \underset{\simeq}{\xrightarrow{\ j\ }} & L^{\bullet} \end{array}$$

Here the $\mathcal{S}$-frame map $e : L^{\bullet} \to M$ of Definition 8.1 is explicitly given by $\mathcal{J}_{\mathcal{S}}h \circ j^{-1}$.

(a) Suppose $I \in \mathcal{R}_{\lhd}\mathcal{J}_{\mathcal{S}}L$ and $\bigvee I = x$. (This is the "ideal version" of saying $g(a) = x$.) We show that $(\mathcal{J}_{\mathcal{S}}h \circ j^{-1})(o_x) \subseteq I$. (This is the "ideal version" of saying $\tilde{g}(x) \leq a$.)

Suppose $y \in \mathcal{J}_{\mathcal{S}}h(j^{-1}(o_x))$. Then $y \leq h(t)$ for some $t \in j^{-1}(o_x)$. Since $j^{-1}(o_x)$ is $\prec$-strongly regular, there exists $u \in j^{-1}(o_x)$ with $t \prec u$. Since j is the map taking joins, $u \leq o_x$. From the structure of $L^{\bullet}$, $t = o_z$ for some $z \in L$; so $y \leq h(o_z)$ for some $o_z \prec o_x$. Then $y \leq z$ and $z \ll x$ and $x = \bigvee I$. Now $z \leq i_1 \vee \ldots \vee i_n$ for some $\{i_1, \ldots, i_n\} \subseteq I$. Then $y \in I$, as required.

(b) Suppose $\uparrow x$ is compact, $I \in \mathcal{R}_{\lhd}\mathcal{J}_{\mathcal{S}}L$, and $\bigvee I = x$. We show that $I \subseteq (\mathcal{J}_{\mathcal{S}}h \circ j^{-1})(p_x)$. (This is the "ideal version" of saying $a \leq \hat{g}(x)$.) Suppose $y \in I$. Since I is $\lhd$-strongly regular, there exists $z \in I$ with $y \lhd z \leq x$. Then $y \prec x$ and ${\uparrow}x$ is compact, so by Theorem 3.8, there exists $w \in I$ with $y \leq w \prec x$ and $\uparrow w$ compact. Then p_w exists, $y \leq h(p_w)$, and $p_w \prec p_x$. From this, we can conclude that $p_w \in j^{-1}(p_x)$. Since $L^{\bullet}$ is compact, regular, the rather below relation and the way below relation on it coincide, so if $J \in \mathcal{R}_{\prec}\mathcal{J}_{\mathcal{S}}L^{\bullet}$ and $\bigvee J = b$ and $a \prec b$, then $a \in J$. This shows that $y \in (\mathcal{J}_{\mathcal{S}}h \circ j^{-1})(p_x)$, as required. □

Remark 8.5. (a) In the full frame case, tops and bottoms of balloons can be easily identified by using the right adjoint g_* and the left adjoint g^* of the compactification map g. The fact that $g^*(x) \in \mathrm{Balloon}_x$ depends on the compactification map being open. (See [2] and [26].)

(b) Right and left adjoints are emphatically unavailable in the setting of partial frames; their use here is bypassed by the introduction of $\hat{g}$ and $\tilde{g}$. The fact that $\hat{g}$ and $\tilde{g}$ suffice for this purpose is by no means obvious and is the content of Proposition 8.4. We note that an understanding of the internal structure of the one-point compactification is essential here.

We are now in the position to prove that regular continuous $\mathcal{S}$-frames are open quotients of any of their compactifications.

Proposition 8.6. *Let $g : M \to L$ be a compactification of a regular, continuous $\mathcal{S}$-frame L. The restriction of the compactification map $g : M \to L$ to ${\downarrow}\tilde{g}(1)$ is an isomorphism; so ${\downarrow}\tilde{g}(1) \cong L$.*

Proof. We first show that if $t \in M$ and $t \leq \tilde{g}(1)$, then $t = \tilde{g}(x)$ for some $x \in L$; in fact $t = \tilde{g}(g(t))$. Since $\tilde{g}(g(t))$ is the smallest element of $\mathrm{Balloon}_{g(t)}$ and $t \in \mathrm{Balloon}_{g(t)}$, we have $\tilde{g}(g(t)) \leq t$, by Proposition 8.4(a). For the reverse inequality, write $t = \bigvee S$ for some designated subset S of M with $s \prec t$ for all $s \in S$. For such s, there exists $c \in M$ with $s \wedge c = 0$ and $c \vee t = 1$. Then $g(c) \vee g(t) = 1$ and so $\tilde{g}(g(c)) \vee \tilde{g}(g(t)) = \tilde{g}(1)$, by Lemma 8.2(a). Then $s = s \wedge (\tilde{g}(g(c)) \vee \tilde{g}(g(t))) = s \wedge \tilde{g}(g(t))$, using $\tilde{g}(g(c)) \leq c$. So $s \leq \tilde{g}(g(t))$, which gives $t = \bigvee S \leq \tilde{g}(g(t))$.

The fact that $\tilde{g} : L \to {\downarrow}\tilde{g}(1)$ is an isomorphism then follows from Lemma 8.2. □

We now investigate an arbitrary compactification of a regular continuous $\mathcal{S}$-frame by comparing its balloons.

Lemma 8.7. *Let $g : M \to L$ be a compactification of a regular, continuous $\mathcal{S}$-frame L. If $x \leq y$ in L, then there is a one-one function $k : Balloon_x \to Balloon_y$.*

Proof. Define $k : \mathrm{Balloon}_x \to \mathrm{Balloon}_y$ by $k(a) = a \vee \tilde{g}(y)$. If $a \in \mathrm{Balloon}_x$, then $g(a) = x$, so $g(k(a)) = g(a) \vee g(\tilde{g}(y)) = x \vee y = y$. Thus $k(a) \in \mathrm{Balloon}_y$. Suppose $k(a) = k(b)$ for some $a, b \in \mathrm{Balloon}_x$. Then $a \vee \tilde{g}(y) = b \vee \tilde{g}(y)$. However, $a \wedge \tilde{g}(y) = \tilde{g}(t)$, for some $t \in L$, by Proposition 8.6. So $g(a \wedge \tilde{g}(y)) = g(\tilde{g}(t))$, giving $x \wedge y = t$, and hence $t = x$. Thus $a \wedge \tilde{g}(y) = b \wedge \tilde{g}(y)$. Distributivity then guarantees that $a = b$. □

Lemma 8.8. *Let $g : M \to L$ be a compactification of a regular, continuous $\mathcal{S}$-frame L. If $\uparrow x$ is compact, then Balloon_x and Balloon_1 are isomorphic $\mathcal{S}$-frames.*

Proof. The function $k : \text{Balloon}_x \to \text{Balloon}_1$ given by $k(a) = a \vee \tilde{g}(1)$ (see Lemma 8.7) clearly preserves binary meets and designated joins, and sends $\tilde{g}(x)$, the bottom of Balloon_x, to $\tilde{g}(1)$, the bottom of Balloon_1. Also $k(\hat{g}(x)) = \hat{g}(x) \vee \tilde{g}(1) = e(p_x) \vee e(o_1) = e(p_x \vee o_1) = e(p_1) = \hat{g}(1)$; so k sends the top of Balloon_x to the top of Balloon_1. The function k is one-one (see Lemma 8.7). To show that here k is onto, suppose $c \in \text{Balloon}_1$. Then $g(c \wedge \hat{g}(x)) = g(c) \wedge g(\hat{g}(x)) = 1 \wedge x = x$, so $c \wedge \hat{g}(x) \in \text{Balloon}_x$. Further, $k(c \wedge \hat{g}(x)) = (c \wedge \hat{g}(x)) \vee \tilde{g}(1) = (c \vee \tilde{g}(1)) \wedge (\hat{g}(x) \vee \tilde{g}(1)) = c \wedge 1 = c$, using $c \in \text{Balloon}_1$, so $c \geq \tilde{g}(1)$. □

It is clear that Balloon_0 is always a singleton because compactification maps are dense. Lemma 8.8 provides balloons of largest size. The example below shows that balloons of intermediate size may exist as well.

Example 8.1. Let L consist of the open sets of $(0, 1)$ with the usual topology of the real line, and M the open sets of $[0, 1]$. Let $x = (0, \frac{1}{2})$.
Then $\text{Balloon}_x = \{(0, \frac{1}{2}), [0, \frac{1}{2})\}$, but $\text{Balloon}_1 = \{(0, 1), (0, 1], [0, 1), [0, 1]\}$. □

In our final result, we use the techniques of this section to prove that one-point compactifications of regular continuous $\mathcal{S}$-frames are indeed unique.

Theorem 8.9. *The one-point compactification of a regular, continuous $\mathcal{S}$-frame is unique.*

Proof. Suppose that L is a regular, continuous $\mathcal{S}$-frame and $g : M \to L$ a one-point compactification of L; meaning that there exists a maximal element $m \in M$ such that $g : \downarrow m \to L$ is an isomorphism. Let $\tilde{g}$ be defined as in Definition 8.1. We first establish some properties of the compactification $g : M \to L$ before showing that it is isomorphic to the compactification $h : L^\bullet \to L$ of Definition 4.1.

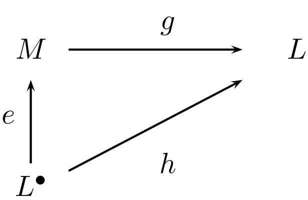
First, we check that $m = \tilde{g}(1)$: Since $g : \downarrow m \to L$ sends top elements to top elements, $g(m) = 1$. If $a \in M$ and $g(a) = 1$, then $g(a \wedge m) = g(a) \wedge g(m) = 1$. Since $g : \downarrow m \to L$ is an isomorphism, $a \wedge m = m$, so $m \leq a$.

This shows that m is the bottom element of Balloon_1, namely $\tilde{g}(1)$. (See Proposition 8.4.)

Next, we show that, for any $x \in L$, there is at most one element $a \in M$ such that $g(a) = x$ and $a \neq \tilde{g}(x)$. So suppose $g(a) = g(b) = x$, $a \neq \tilde{g}(x)$, $b \neq \tilde{g}(x)$. By Proposition 8.6, $a \wedge \tilde{g}(1) = \tilde{g}(t)$ for some $t \in L$; applying g gives $t = x$, so $a \wedge \tilde{g}(1) = \tilde{g}(x)$. Then $a \wedge \tilde{g}(1) = b \wedge \tilde{g}(1)$. But also $a \vee \tilde{g}(1) = b \vee \tilde{g}(1)$, because $\tilde{g}(1)$ is a maximal element of M. Distributivity then shows that $a = b$.

Finally, we show that if $x \in L$ is such that there exists $a \in M$ with $g(a) = x$ and $a \neq \tilde{g}(x)$, then $\uparrow x$ is compact. Suppose S is a designated subset of $\uparrow x$ and $\bigvee S = 1$. Then S is a designated subset of L (as in the proof of Lemma 3.3(b)). Since $g : \downarrow m \to L$ is an isomorphism, there exists a designated subset T of $\downarrow m$ with $g[T] = S$. Then $g(\bigvee T) = \bigvee g[T] = 1$, so $\bigvee T = m$. Since m is a maximal element of M, $a \vee \bigvee T = 1$. Since $\{a \vee t : t \in T\}$ is a designated subset of M, compactness of M gives $(a \vee t_1) \vee \cdots \vee (a \vee t_n) = 1$ for some $t_1, \ldots, t_n \in T$. Applying g gives $g(a) \vee g(t_1) \vee \cdots \vee g(t_n) = 1$, so $x \vee s_1 \vee \cdots \vee s_n = 1$, and hence $s_1 \vee \cdots \vee s_n = 1$, for some $s_1, \ldots, s_n \in S$.

Consider the commuting diagram (as in Definition 8.1):



The map e is a one-one $\mathcal{S}$-frame map; to show it is an isomorphism, it suffices to show it is onto. For any $x \in L$, $\tilde{g}(x) = e(o_x)$, so obviously $\tilde{g}(x)$ is in the range of e. If there exists $a \in M$ with $g(a) = x$ and $a \neq \tilde{g}(x)$, then the arguments above showed this a to be unique. Since $a \in \text{Balloon}_x$ and the arguments above showed $\uparrow x$ to be compact, p_x exists and $a = e(p_x)$, so a is also in the range of e. □

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