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Tangled closure algebras

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Dedicated to Bernhard Banaschewski on the occasion of his 90th birthday

Abstract. The tangled closure of a collection of subsets of a topological space is the largest subset in which each member of the collection is dense. This operation models a logical 'tangle modality' connective, of significance in finite model theory. Here we study an abstract equational algebraic formulation of the operation which generalises the McKinsey-Tarski theory of closure algebras. We show that any dissectable tangled closure algebra, such as the algebra of subsets of any metric space without isolated points, contains copies of every finite tangled closure algebra. We then exhibit an example of a tangled closure algebra that cannot be embedded into any complete tangled closure algebra, so it has no MacNeille completion and no spatial representation.

1 Introduction

McKinsey and Tarski [17, 18] defined a *closure algebra* as a Boolean algebra equipped with a unary function $\mathbf{C}$ that satisfies axioms of Kuratowski [15] for the operation of forming the topological closure of a set. They graph-

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ically revealed the intricacy of the structure of many familiar topological spaces by defining a notion of 'dissectable' closure algebra, showing that any such algebra contains copies of every finite closure algebra, and proving that any metric space without isolated points has a dissectable algebra of subsets. This work has been described [13] as the first attempt to do pointless topology, a subject that has been a significant theme in the work of Bernhard Banaschewski.

Our aim here is to generalise this theory to a study of *tangled closure*. In a topological space this operation assigns to each finite collection Γ of subsets a set $\mathbf{C}^t\Gamma$ which is the largest subset in which each member of Γ is dense. When Γ has one member, $\mathbf{C}^t\{\gamma\}$ is just the usual topological closure $\mathbf{C}\gamma$ of γ. In an order topology, determined by some quasi-ordering relation R, a point x belongs to the tangled closure $\mathbf{C}^t\Gamma$ if and only if there exists an 'endless R-path' $xRx_1 \cdots x_nRx_{n+1} \cdots$ starting from x such that the path enters each set belonging to Γ infinitely often.

This order-theoretic interpretation has been used to model a propositional connective known as the *tangle modality*, which was introduced by Dawar and Otto [4] in an analysis of logical formulas whose satisfaction is invariant under certain 'bisimulation' relations between models. A well-known result of van Benthem [26, 27] states that a first-order formula is invariant under bisimulations between arbitrary models if and only if that formula is equivalent to a formula of the basic language of propositional modal logic. This result continues to hold for bisimulation-invariance over any elementary class of models, such as the quasi-orderings, as well as over the class of all finite models. But on restriction to the class of all finite quasi-orderings (and some of its subclasses), the picture changes. Propositional formulas involving the tangle modality, which are bisimulation-invariant, become first-order definable in this setting, and van Benthem's result no longer holds. Instead, a first-order formula is bisimulation-invariant over the finite quasi-orderings if and only if it is equivalent to a formula of the language that enriches basic modal logic by the addition of the tangle modality. Moreover, [4] showed that the bisimulation-invariant fragment of monadic second order logic, which is equivalent over arbitrary models to the much more powerful modal mu-calculus, collapses over finite quasi-orderings to the first-order fragment, so is also equivalent to the language with the tangle modality. The name 'tangle' was introduced by Fernández-Duque [6, 7] who axiomatised the tangle modal logic of finite quasi-orderings. Subsequently we have made an extensive study [9–12] of a range of logics with this connective.

That accounts for the motivating origin of $\mathbf{C}^t\Gamma$, but here we subject it to an abstract algebraic analysis, defining a tangled closure algebra as a pair $(A, \mathbf{C}^t)$ with $\mathbf{C}^t$ an operation on finite subsets of a Boolean algebra A, with the restriction of $\mathbf{C}^t$ to one-element sets being a closure operator $\mathbf{C}$. We require $\mathbf{C}^t$ to satisfy equational conditions ensuring that $\mathbf{C}^t\Gamma$ is the greatest fixed point of the function $a \mapsto \bigwedge_{\gamma\in\Gamma}\mathbf{C}(\gamma \wedge a)$. We study homomorphisms and subalgebras of tangled closure algebras, and use the logical Lindenbaum-Tarski algebra construction to produce freely generated tangled closure algebras. Our main results extend those of McKinsey and Tarski by showing that if a tangled closure algebra $(A, \mathbf{C}^t)$ is dissectable, then any finite tangled closure algebra can be isomorphically embedded into the *relativised* algebra of all elements below some open element of $(A, \mathbf{C}^t)$. Furthermore, if $(A, \mathbf{C}^t)$ is totally disconnected (for example, the algebra of subsets of any zero-dimensional metric space without isolated points), then the embedding can be mapped into the relativisation to *any* non-zero open element.

As is well known, every Boolean algebra A has order-complete extensions, including the extension given by the Stone representation theory, and the MacNeille completion, which is a complete Boolean algebra B extending A with each element of B being the join of a subset of A. A closure algebra also has complete extensions of both these kinds. But in our final section we construct a tangled closure algebra that has no embedding into any complete tangled closure algebra at all. In particular, it cannot be represented as an algebra of subsets of a topological space.

2 Tangled Closure

Let A be a Boolean algebra with signature $\wedge$, $\vee$, $-, 0, 1$. Define the Boolean implication operation in A by $a \Rightarrow b = -a \vee b$, and put $a \Leftrightarrow b = (a \Rightarrow b) \wedge (b \Rightarrow a)$. Let $\bigvee E$ and $\bigwedge E$ denote the join and meet of a subset E of A when these exist. We sometimes write them as $\bigvee_A E$ and $\bigwedge_A E$ to clarify which algebra they are being defined in.

A *closure operator* on A is a function $\mathbf{C} : A \to A$ that is additive, normal, inflationary and idempotent, that is, satisfies the equational conditions

$$\mathbf{C}(a \vee b) = \mathbf{C}a \vee \mathbf{C}b, \quad \mathbf{C}0 = 0, \quad a \leq \mathbf{C}a = \mathbf{C}\mathbf{C}a.$$

$\mathbf{C}$ is then *monotonic*, that is, $a \leq b$ implies $\mathbf{C}a \leq \mathbf{C}b$, and *finitely additive* in the sense that $\mathbf{C}\bigvee\Gamma = \bigvee\{\mathbf{C}\gamma : \gamma \in \Gamma\}$ for all finite $\Gamma \subseteq A$. The pair $(A, \mathbf{C})$ is called a *closure algebra*. An element $a \in A$ is called *closed* if $a = \mathbf{C}a$, which is equivalent to having $a = \mathbf{C}b$ for some b.

In a closure algebra, $\mathbf{C}$ has a dual *interior* operation $\mathbf{I} : A \to A$ defined by $\mathbf{I}a = -\mathbf{C} - a$. This is also mononotonic; *multiplicative* in the sense that $\mathbf{I}\bigwedge\Gamma = \bigwedge\{\mathbf{I}\gamma : \gamma \in \Gamma\}$ for all finite Γ; and has $\mathbf{I}1 = 1$ and $\mathbf{I}\mathbf{I}a = \mathbf{I}a \leq a$. An element a is called *open* if $a = \mathbf{I}a$, which is equivalent to having $a = \mathbf{I}b$ for some b.

A basic property of all closure algebras that we make use of is that

$$\mathbf{I}a \wedge \mathbf{C}b \leq \mathbf{C}(a \wedge b). \tag{2.1}$$

In addition to the original paper [17], there is extensive information about closure algebras in Chapter III of [21], where they are called *topological Boolean algebras*.

Let $\mathcal{P}_{fin}A$ be the set of all finite subsets of A. A function $\mathbf{C}^t : \mathcal{P}_{fin}A \to A$ induces a unary function $\mathbf{C} : A \to A$ by putting $\mathbf{C}a = \mathbf{C}^t\{a\}$, and hence a dual operation $\mathbf{I}$ that has $\mathbf{I}a = -\mathbf{C}^t\{-a\}$. We will write these operations as $\mathbf{C}^t_A$, $\mathbf{C}_A$, $\mathbf{I}_A$, when needing to distinguish which algebra we are in.

We say that $\mathbf{C}^t$ is a *tangled closure operator*, and $(A, \mathbf{C}^t)$ is a *tangled closure algebra*, if its induced $\mathbf{C}$ is a closure operator on A, and the following hold for all $\Gamma \in \mathcal{P}_{fin}A$ and $a \in A$:

Fix $\mathbf{C}^t\Gamma \leq \bigwedge_{\gamma\in\Gamma}\mathbf{C}(\gamma \wedge \mathbf{C}^t\Gamma)$,

Ind $\mathbf{I}(a \Rightarrow \bigwedge_{\gamma\in\Gamma}\mathbf{C}(\gamma \wedge a)) \wedge a \leq \mathbf{C}^t\Gamma$.

These conditions are evidently equational, for example, Fix is equivalent to $\mathbf{C}^t\Gamma \wedge \bigwedge_{\gamma\in\Gamma}\mathbf{C}(\gamma \wedge \mathbf{C}^t\Gamma) = \mathbf{C}^t\Gamma$. The pair $(A, \mathbf{C})$ will be called the *closure algebra reduct* of $(A, \mathbf{C}^t)$.

In the papers [9]-[12] we assumed for simplicity that $\mathbf{C}^t\Gamma$ was only defined for non-empty Γ, but we do not make that restriction here. Note that by putting $a = 1$ and $\Gamma = \emptyset$ in axiom Ind we infer $\mathbf{C}^t\emptyset = 1$.

Lemma 2.1. *In any tangled closure algebra, it holds in general that*

$$\mathbf{C}^t\Gamma = \bigvee\{a \in A : a \le \textstyle\bigwedge_{\gamma\in\Gamma}\mathbf{C}(\gamma \wedge a)\}. \tag{2.2}$$

Proof. Let $f_\Gamma(a) = \bigwedge_{\gamma\in\Gamma}\mathbf{C}(\gamma \wedge a)$. This defines a function $f_\Gamma : A \to A$ which is monotonic. Say that a is a *post-fixed point for* Γ if $a \le f_\Gamma(a)$. Let $S_\Gamma = \{a \in A : a \le f_\Gamma(a)\}$ be the set of all post-fixed points for Γ. (2.2) asserts that $\mathbf{C}^t\Gamma$ is the join of S_Γ.

Now Fix states that $\mathbf{C}^t\Gamma \le f_\Gamma(\mathbf{C}^t\Gamma)$, hence $\mathbf{C}^t\Gamma \in S_\Gamma$. But Ind implies that $\mathbf{C}^t\Gamma$ is an upper bound of S_Γ, for if $a \in S_\Gamma$, then $\mathbf{I}(a \Rightarrow f_\Gamma(a)) = \mathbf{I}1 = 1$, so Ind reduces in this case to the assertion that $a \le \mathbf{C}^t\Gamma$.

Thus $\mathbf{C}^t\Gamma$ is both a member of S_Γ and an upper bound of it, hence is its least upper bound. □

Corollary 2.2. $\mathbf{C}^t\Gamma = \bigwedge_{\gamma\in\Gamma}\mathbf{C}(\gamma \wedge \mathbf{C}^t\Gamma)$. *Moreover* $\mathbf{C}^t\Gamma$ *is the greatest (post-)fixed point of* f_Γ.

Proof. As above $\mathbf{C}^t\Gamma \le f_\Gamma(\mathbf{C}^t\Gamma)$, so monotonicity of f_Γ yields $f_\Gamma\mathbf{C}^t\Gamma \le f_\Gamma(f_\Gamma\mathbf{C}^t\Gamma)$, showing $f_\Gamma\mathbf{C}^t\Gamma$ is also a post-fixed point of f_Γ. Thus $f_\Gamma\mathbf{C}^t\Gamma \le \mathbf{C}^t\Gamma$, altogether then $\mathbf{C}^t\Gamma = f_\Gamma\mathbf{C}^t\Gamma$, and so $\mathbf{C}^t\Gamma$ is a fixed point of f_Γ. Since all such fixed points belong to S_Γ, $\mathbf{C}^t\Gamma$ is the greatest of them, as well as of the post-fixed points. □

Lemma 2.1 implies that $\mathbf{C}^t$ is uniquely determined by the unary $\mathbf{C}$ it induces. Furthermore:

Theorem 2.3. *Any* ***complete*** *closure algebra* $(A, \mathbf{C})$ *expands uniquely to a tangled closure algebra* $(A, \mathbf{C}^t)$ *inducing* $\mathbf{C}$*, by taking* (2.2) *as the* ***definition*** *of* $\mathbf{C}^t\Gamma$.

Proof. For each $\Gamma \in \mathcal{P}_{fin}A$, define $\mathbf{C}^t\Gamma = \bigvee S_\Gamma$, where $S_\Gamma = \{a : a \le f_\Gamma(a)\}$ as above. Then we need to derive Fix and Ind for $\mathbf{C}^t$ thus defined. First, if $a \in S_\Gamma$, then $a \le \mathbf{C}^t\Gamma$, so $f_\Gamma(a) \le f_\Gamma(\mathbf{C}^t\Gamma)$. But $a \le f_\Gamma(a)$, so this shows that $a \le f_\Gamma(\mathbf{C}^t\Gamma)$, for all $a \in S_\Gamma$. Hence $\bigvee S_\Gamma \le f_\Gamma(\mathbf{C}^t\Gamma)$, that is, $\mathbf{C}^t\Gamma \le f_\Gamma(\mathbf{C}^t\Gamma)$, which is Fix.

The derivation of Ind is more lengthy, and uses some basic properties of closure algebras. Given Γ and a, let $b = (a \Rightarrow \bigwedge_{\gamma\in\Gamma}\mathbf{C}(\gamma \wedge a))$. Ind asserts $\mathbf{I}b \wedge a \le \mathbf{C}^t\Gamma$, so to prove this it is enough to show that $\mathbf{I}b \wedge a$ belongs to

S_Γ, that is, $\mathbf{I}b \wedge a$ is a post-fixed point of f_Γ. Taking an arbitrary $\gamma' \in \Gamma$ we have

$$\begin{array}{ll} \mathbf{I}b \wedge a & \\ = \mathbf{I}b \wedge (a \Rightarrow \bigwedge_{\gamma\in\Gamma}\mathbf{C}(\gamma \wedge a)) \wedge a & \text{as } \mathbf{I}b = \mathbf{I}b \wedge b \\ \leq \mathbf{I}b \wedge \bigwedge_{\gamma\in\Gamma}\mathbf{C}(\gamma \wedge a) & \text{by Boolean algebra} \\ \leq \mathbf{II}b \wedge \mathbf{C}(\gamma' \wedge a) & \mathbf{I}b = \mathbf{II}b \text{ and Boolean algebra} \\ \leq \mathbf{C}(\mathbf{I}b \wedge \gamma' \wedge a) & \text{by (2.1).} \end{array}$$

This shows that $\mathbf{I}b \wedge a \leq \mathbf{C}(\gamma' \wedge \mathbf{I}b \wedge a)$ for all $\gamma' \in \Gamma$, hence

$$\mathbf{I}b \wedge a \leq \bigwedge\nolimits_{\gamma\in\Gamma}\mathbf{C}(\gamma \wedge \mathbf{I}b \wedge a).$$

But that says $\mathbf{I}b\wedge a \leq f_\Gamma(\mathbf{I}b\wedge a)$, that is, $\mathbf{I}b\wedge a \in S_\Gamma$, hence $\mathbf{I}b\wedge a \leq \bigvee S_\Gamma = \mathbf{C}^t\Gamma$, which is Ind.

It remains to show that $\mathbf{C}$ is the closure operator induced by $\mathbf{C}^t$. Let Γ be any singleton $\{\gamma\}$. Then if $a \in S_{\{\gamma\}}$, $a \leq \mathbf{C}(\gamma \wedge a) \leq \mathbf{C}\gamma$. So $\mathbf{C}\gamma$ is an upper bound of $S_{\{\gamma\}}$. But $\mathbf{C}\gamma \leq \mathbf{C}(\gamma \wedge \mathbf{C}\gamma)$, since $\gamma \leq \mathbf{C}\gamma$, showing that $\mathbf{C}\gamma$ also belongs to $S_{\{\gamma\}}$. Hence $\mathbf{C}\gamma = \bigvee S_{\{\gamma\}} = \mathbf{C}^t\{\gamma\}$ as required. □

Example 2.4 (Spatial Tangled Closure)**.** The paradigm of a closure algebra is $(A_S, \mathbf{C}_S)$ where S is any topological space. Here A_S is the Boolean powerset algebra of all subsets of S, and $\mathbf{C}_S(a)$ is the topological closure of the set $a \subseteq S$, the intersection of all closed supersets of a. This is a complete closure algebra in which $\bigvee E = \bigcup E$ and $\bigwedge E = \bigcap E$ for all $E \subseteq A_S$. By Theorem 2.3, $\mathbf{C}_S$ has a unique expansion to a tangled closure operator $\mathbf{C}^t_S$. A point belongs to $\mathbf{C}^t_S\Gamma$ if and only if it belongs to some set a such that for all $\gamma \in \Gamma$, $a \subseteq \mathbf{C}_S(\gamma \cap a)$, so γ is dense in a in the sense that any open neighbourhood of any point of a contains a point in γ and a. Since $\mathbf{C}^t\Gamma$ is the greatest post-fixed point for Γ, it is the largest set in which every member of Γ is dense.

Example 2.5 (Quasi-orders and Alexandroff Spaces)**.** A quasi-order is a reflexive transitive binary relation R on a set S. The pair (S, R) is a *quasi-ordered set.* Each $x \in S$ has the set $R(x) = \{y : xRy\}$ of *R-successors.* Then $y \in R(x)\cap R(z)$ implies $y \in R(y) \subseteq R(x)\cap R(z)$, so the collection $\{R(x) : x \in S\}$ of successor sets is a basis for a topology on S, the *Alexandroff topology.* Its open sets are the *up-sets*, those subsets a of S such that are closed upwards in the quasi-ordering in the sense that $x \in a$ implies $R(x) \subseteq a$. Its closed sets are the *down-sets*, the sets a for which $xRy \in a$ implies $x \in a$.

Its closure operator $\mathbf{C}_R$ has $\mathbf{C}_R(a) = R^{-1}(a) = \{x : \exists y(xRy \in a)\}$, giving the closure algebra $(A_S, \mathbf{C}_R)$. Hence by the preceding example, the tangled closure operator $\mathbf{C}^t_R$ of this space has

$$\mathbf{C}^t_R \Gamma = \bigcup\{a \subseteq S : a \subseteq \bigcap\nolimits_{\gamma\in\Gamma} R^{-1}(\gamma \cap a)\}.$$

To give an alternative characterisation of $\mathbf{C}^t_R$, define an *endless R-path* to be a sequence $\{x_n : n < \omega\}$ in S such that $x_n R x_{n+1}$ for all n. (The terms x_n of the sequence need not be distinct. Indeed S may be finite.) Then it can be shown that

> $x \in \mathbf{C}^t_R\Gamma$ if and only if there exists an endless R-path $\{x_n : n < \omega\}$ in S with $x = x_0$, such that for each $\gamma \in \Gamma$ there are infinitely many $n < \omega$ such that $x_n \in \gamma$ (see [7, §4.1]).

The next theorem records properties that will be used in Section 5 in constructing a tangled closure algebra with no complete extension.

Theorem 2.6. *In any tangled closure algebra $(A, \mathbf{C}^t)$, the following hold for all $\Gamma \in \mathcal{P}_{fin}A$.*

(1) *$\mathbf{C}^t\Gamma$ is closed, that is, $\mathbf{C}\mathbf{C}^t\Gamma = \mathbf{C}^t\Gamma$.*

(2) *If $\Gamma' = \{\gamma' : \gamma \in \Gamma\} \subseteq A$, then*

$$\bigwedge\nolimits_{\gamma\in\Gamma}\mathbf{I}(\gamma \Leftrightarrow \gamma') \leq \mathbf{I}\big(\mathbf{C}^t\Gamma \Leftrightarrow \mathbf{C}^t\Gamma'\big).$$

Proof. (1) We have $\mathbf{C}^t\Gamma \leq \mathbf{C}\mathbf{C}^t\Gamma$ as $\mathbf{C}$ is a closure operator, so we need to show the reverse inequality $\mathbf{C}\mathbf{C}^t\Gamma \leq \mathbf{C}^t\Gamma$. To see this, by Lemma 2.1, it suffices to show that $\mathbf{C}\mathbf{C}^t\Gamma$ is a post-fixed point for Γ. Now for any $\gamma \in \Gamma$, by Fix and $\mathbf{C}$-monotonicity $\mathbf{C}\mathbf{C}^t\Gamma \leq \mathbf{C}\mathbf{C}(\gamma \wedge \mathbf{C}^t\Gamma) = \mathbf{C}(\gamma \wedge \mathbf{C}^t\Gamma)$. But by closure algebra properties $\mathbf{C}(\gamma \wedge \mathbf{C}^t\Gamma) \leq \mathbf{C}(\gamma \wedge \mathbf{C}\mathbf{C}^t\Gamma)$. Altogether this implies that $\mathbf{C}\mathbf{C}^t\Gamma \leq \mathbf{C}(\gamma \wedge \mathbf{C}\mathbf{C}^t\Gamma)$ for all $\gamma \in \Gamma$. Hence $\mathbf{C}\mathbf{C}^t\Gamma \in S_\Gamma$ as required.

(2) Let $a = \bigwedge_{\gamma\in\Gamma}\mathbf{I}(\gamma \Leftrightarrow \gamma')$. As $\mathbf{I}$ is multiplicative, $a = \mathbf{I}\bigwedge_{\gamma\in\Gamma}(\gamma \Leftrightarrow \gamma')$, so a is open and therefore $a = \mathbf{I}a$. Now for each $\gamma \in \Gamma$, using Fix we have

$$a \wedge \mathbf{C}^t\Gamma \leq \mathbf{I}(\gamma \Leftrightarrow \gamma') \wedge \mathbf{C}(\gamma \wedge \mathbf{C}^t\Gamma) \leq \mathbf{C}((\gamma \Leftrightarrow \gamma') \wedge \gamma \wedge \mathbf{C}^t\Gamma)$$

by (2.1). Since $(\gamma \Leftrightarrow \gamma') \wedge \gamma \leq \gamma'$ and $\mathbf{C}$ is monotonic, this implies $a \wedge \mathbf{C}^t\Gamma \leq \mathbf{C}(\gamma' \wedge \mathbf{C}^t\Gamma)$. Thus $a \wedge \mathbf{C}^t\Gamma \leq \bigwedge_{\gamma\in\Gamma}\mathbf{C}(\gamma' \wedge \mathbf{C}^t\Gamma)$, so $a \leq \mathbf{C}^t\Gamma \Rightarrow \bigwedge_{\gamma\in\Gamma}\mathbf{C}(\gamma' \wedge \mathbf{C}^t\Gamma)$. Hence

$$a = \mathbf{I}a \leq \mathbf{I}(\mathbf{C}^t\Gamma \Rightarrow \textstyle\bigwedge_{\gamma\in\Gamma}\mathbf{C}(\gamma' \wedge \mathbf{C}^t\Gamma)).$$

Then $a \wedge \mathbf{C}^t\Gamma \leq \mathbf{I}(\mathbf{C}^t\Gamma \Rightarrow \bigwedge_{\gamma\in\Gamma}\mathbf{C}(\gamma' \wedge \mathbf{C}^t\Gamma)) \wedge \mathbf{C}^t\Gamma \leq \mathbf{C}^t\Gamma'$ by Ind for Γ'. It follows that $a \leq \mathbf{C}^t\Gamma \Rightarrow \mathbf{C}^t\Gamma'$. Interchanging Γ and Γ' here, and using $\gamma \Leftrightarrow \gamma' = \gamma' \Leftrightarrow \gamma$, we likewise show $a \leq \mathbf{C}^t\Gamma' \Rightarrow \mathbf{C}^t\Gamma$. Hence $a \leq \mathbf{C}^t\Gamma \Leftrightarrow \mathbf{C}^t\Gamma'$. Therefore $a = \mathbf{I}a \leq \mathbf{I}(\mathbf{C}^t\Gamma \Leftrightarrow \mathbf{C}^t\Gamma')$, which is the desired result. $\square$

3 Homomorphisms, Subalgebras, Free Algebras

A *homomorphism* $f : (A, \mathbf{C}^t_A) \to (B, \mathbf{C}^t_B)$ between algebras of the type of tangled closure algebras is a Boolean algebra homomorphism $f : A \to B$ that preserves the $\mathbf{C}^t$-operations in the sense that

$$f(\mathbf{C}^t_A\Gamma) = \mathbf{C}^t_B\{f\gamma : \gamma \in \Gamma\}.$$

If f is injective we call it an *embedding*. If it is surjective, then it preserves validity of equations, hence if $(A, \mathbf{C}^t_A)$ is a tangled closure algebra, then so is $(B, \mathbf{C}^t_B)$. If f is bijective then it is an *isomorphism*.

A homomorphism of tangled closure algebras preserves the associated closure operators, meaning that $f(\mathbf{C}_A(a)) = \mathbf{C}_B f(a)$. In general, a Boolean homomorphism $f : A \to B$ that is a closure algebra homomorphism in this sense need not preserve tangled closure, as we will see later in Section 5. However, if f is a closure algebra *isomorphism* from $(A, \mathbf{C}_A)$ onto $(B, \mathbf{C}_B)$, then it will preserve tangled closure and be a tangled closure algebra isomorphism from $(A, \mathbf{C}^t_A)$ onto $(B, \mathbf{C}^t_B)$. This follows by (2.2), since Boolean isomorphisms preserve all existing joins.

Theorem 3.1. *Any finite tangled closure algebra* $(A, \mathbf{C}^t_A)$ *is isomorphic to the powerset algebra* $(A_S, \mathbf{C}^t_R)$ *of some finite quasi-ordered set* (S, R) *(see Example* 2.5*).*

Proof. Being finite, A is isomorphic to the powerset algebra A_S where S is the set of atoms of A. The closure operator $\mathbf{C}_A$ induced by $\mathbf{C}^t_A$ is transferred by the isomorphism to a closure operator $\mathbf{C}'$ on A_S. Here $\mathbf{C}'$ is equal to the operator $\mathbf{C}_R = R^{-1}$ of a quasi-order on S defined by xRy if and only if $x \in \mathbf{C}'\{y\}$. This follows from work in [14, Section 3] on complete and atomic algebras, and is set out explicitly in [5, Lemma 1].

Since the closure algebras $(A, \mathbf{C}_A)$ and $(A_S, \mathbf{C}_R)$ are isomorphic, it then follows that $(A, \mathbf{C}^t_A)$ and $(A_S, \mathbf{C}^t_R)$ are isomorphic, as noted above. □

Another case in which a closure algebra homomorphism between tangled closure algebras must preserve tangled closure occurs when the domain of the homomorphism is *finite*, as we now show.

Theorem 3.2. *Let $(A, \mathbf{C}^t_A)$ and $(B, \mathbf{C}^t_B)$ be tangled closure algebras and $f : A \to B$ be a closure algebra homomorphism between the associated closure algebra reducts $(A, \mathbf{C}_A)$ and $(B, \mathbf{C}_B)$. Suppose A is finite. Then f preserves the tangled closure operations $\mathbf{C}^t_A$ and $\mathbf{C}^t_B$.*

Proof. We need to show that if $\Gamma \in \mathcal{P}_{fin}A$, then $f\mathbf{C}^t_A\Gamma = \mathbf{C}^t_B f\Gamma$, where $f\Gamma = \{f\gamma : \gamma \in \Gamma\}$. We use the fact that f is monotonic and preserves finite meets and closure operators. Applying this to Fix for $\mathbf{C}^t_A\Gamma$ gives that in B,

$$f\mathbf{C}^t_A\Gamma \leq \textstyle\bigwedge_{\gamma\in\Gamma}\mathbf{C}_B(f\gamma \wedge f\mathbf{C}^t_A\Gamma).$$

This means that $f\mathbf{C}^t_A\Gamma$ is a post-fixed point for $f\Gamma$ in B, so by Lemma 2.1, $f\mathbf{C}^t_A\Gamma \leq \mathbf{C}^t_B f\Gamma$.

For the reverse inequality $\mathbf{C}^t_B f\Gamma \leq f\mathbf{C}^t_A\Gamma$, let $D = \{a \in A : \mathbf{C}^t_B f\Gamma \leq fa\}$. Put $d = \bigwedge D$, which exists in A as A is finite. Then in B we have

$$\mathbf{C}^t_B f\Gamma \leq \textstyle\bigwedge\{fa : a \in D\} = fd \tag{3.1}$$

as f preserves finite meets. Now by Fix for $\mathbf{C}^t_B f\Gamma$, (3.1) and preservation properties of f we get

$$\begin{aligned}\mathbf{C}^t_B f\Gamma &\leq \textstyle\bigwedge_{\gamma\in\Gamma}\mathbf{C}_B(f\gamma \wedge \mathbf{C}^t_B f\Gamma)\\ &\leq \textstyle\bigwedge_{\gamma\in\Gamma}\mathbf{C}_B(f\gamma \wedge fd) = f\big(\bigwedge_{\gamma\in\Gamma}\mathbf{C}_A(\gamma \wedge d)\big).\end{aligned}$$

This shows that $\bigwedge_{\gamma\in\Gamma}\mathbf{C}_A(\gamma \wedge d) \in D$. Hence $d = \bigwedge D \leq \bigwedge_{\gamma\in\Gamma}\mathbf{C}_A(\gamma \wedge d)$, proving that d is a post-fixed point for Γ. Thus $d \leq \mathbf{C}^t_A\Gamma$ by Lemma 2.1.

Then by (3.1) and monotonicity of f,

$$\mathbf{C}^t_B f\Gamma \leq fd \leq f\mathbf{C}^t_A\Gamma,$$

completing the proof that $\mathbf{C}^t_B f\Gamma \leq f\mathbf{C}^t_A\Gamma$ and hence $\mathbf{C}^t_B f\Gamma = f\mathbf{C}^t_A\Gamma$.[1] □

We will say that $(A, \mathbf{C}^t_A)$ is a *subalgebra* of $(B, \mathbf{C}^t_B)$ if A is a Boolean subalgebra of B that is closed under $\mathbf{C}^t_B$, that is, $\mathbf{C}^t_B\Gamma \in A$ for all $\Gamma \in \mathcal{P}_{fin}A$, and $\mathbf{C}^t_A$ is the restriction of $\mathbf{C}^t_B$ to A. Equivalently this means that $A \subseteq B$ and the inclusion $A \hookrightarrow B$ is a homomorphism $(A, \mathbf{C}^t_A) \to (B, \mathbf{C}^t_B)$ as above. This implies that the reduct $(A, \mathbf{C}_A)$ is a subalgebra of $(B, \mathbf{C}_B)$. But we will see in Section 5 that it is possible to have $(A, \mathbf{C}_A)$ a subalgebra of $(B, \mathbf{C}_B)$ while $(A, \mathbf{C}^t_A)$ is not a subalgebra of $(B, \mathbf{C}^t_B)$.

We also need the notion of the *relativisation* of an algebra to one of its elements. This abstracts from the notion of a topological subspace, that is, the relativisation of a topology to a subset. To describe it, let $(A, \mathbf{C}^t_A)$ be an abstract tangled closure algebra with closure algebra reduct $(A, \mathbf{C}_A)$. If $\alpha \in A$, let $A_\alpha = \{b \in A : b \leq \alpha\}$ be the Boolean algebra of elements below α, in which joins and meets are the same as in A, and the complement of b in A_α is $\alpha - b = \alpha \wedge -b$. The implication operation $\Rightarrow_\alpha$ of A_α has

$$b \Rightarrow_\alpha c = (\alpha - b) \vee c \leq -b \vee c = b \Rightarrow c.$$

A closure operator $\mathbf{C}_\alpha$ is defined on A_α by putting $\mathbf{C}_\alpha b := \alpha \wedge \mathbf{C}_A b$. The dual operator $\mathbf{I}_\alpha$ to $\mathbf{C}_\alpha$ has the property that if α is an open element of A, that is, $\mathbf{I}_A\alpha = \alpha$, then $\mathbf{I}_\alpha b = \mathbf{I}_A b$ for all $b \in A_\alpha$ [21, p. 96], that is, $\mathbf{I}_\alpha$ is the restriction of $\mathbf{I}_A$ to A_α. Define an operation $\mathbf{C}^t_\alpha$ on $\mathcal{P}_{fin}A_\alpha$ by putting $\mathbf{C}^t_\alpha\Gamma := \alpha \wedge \mathbf{C}^t_A\Gamma$. Then $(A_\alpha, \mathbf{C}^t_\alpha)$ is the *relativisation of* $(A, \mathbf{C}^t_A)$ *to* α.

Theorem 3.3. *If α is open, then $(A_\alpha, \mathbf{C}^t_\alpha)$ is a tangled closure algebra with closure algebra reduct $(A_\alpha, \mathbf{C}_\alpha)$.*

Proof. $\mathbf{C}^t_\alpha$ induces the unary operation $b \mapsto \alpha \wedge \mathbf{C}^t_A\{b\} = \alpha \wedge \mathbf{C}_A b$, which is the closure operator $\mathbf{C}_\alpha$ above. Then $\mathbf{C}^t_\alpha$ satisfies Fix, since for all finite

[1] This proof can be adapted to yield the following result. If $\alpha : A \to A$ and $\beta : B \to B$ are monotonic functions on complete lattices A and B, and $f : A \to B$ is a complete lattice homomorphism such that $f \circ \alpha = \beta \circ f$, then f preserves the greatest and least fixed points of α and β.

$\Gamma \subseteq A_\alpha$ and all $\gamma \in \Gamma$, using Fix for $\mathbf{C}^t_A$ shows that

$$\begin{aligned}\mathbf{C}^t_\alpha \Gamma &= \alpha \wedge \mathbf{C}^t_A \Gamma \leq \alpha \wedge \mathbf{C}_A(\gamma \wedge \mathbf{C}^t_A \Gamma) = \alpha \wedge \mathbf{C}_A(\gamma \wedge \alpha \wedge \mathbf{C}^t_A \Gamma) \\ &= \alpha \wedge \mathbf{C}_A(\gamma \wedge \mathbf{C}^t_\alpha \Gamma) = \mathbf{C}_\alpha(\gamma \wedge \mathbf{C}^t_\alpha \Gamma).\end{aligned}$$

To show $\mathbf{C}^t_\alpha$ satisfies Ind, we need the assumption that α is open, implying that $\mathbf{I}_\alpha$ is the restriction of $\mathbf{I}_A$ to A_α. Let $x = \mathbf{I}_\alpha(b \Rightarrow_\alpha \bigwedge_{\gamma\in\Gamma} \mathbf{C}_\alpha(\gamma \wedge b)) \wedge b$ where $b \leq \alpha$. Then

$$\begin{aligned}x &= \mathbf{I}_A(b \Rightarrow_\alpha \textstyle\bigwedge_{\gamma\in\Gamma} \mathbf{C}_\alpha(\gamma \wedge b)) \wedge b \\ &\leq \mathbf{I}_A(b \Rightarrow \textstyle\bigwedge_{\gamma\in\Gamma} \mathbf{C}_A(\gamma \wedge b)) \wedge b \\ &\leq \mathbf{C}^t_A \Gamma\end{aligned}$$

by Ind for $\mathbf{C}^t_A$. Hence $x \leq \alpha \wedge \mathbf{C}^t_A \Gamma = \mathbf{C}^t_\alpha \Gamma$, which gives Ind for $\mathbf{C}^t_\alpha$. $\square$

A *free* tangled closure algebra over any set V can be constructed using a propositional modal logic and the standard Lindenbaum-Tarski algebra construction. To outline this, take an arbitrary V and regard its members as (propositional) variables that can range over the elements of an algebra. From these variables we construct *formulas* $\varphi, \psi, \dots$ using

- the Boolean connectives $\wedge$, $\vee$, $\neg$, $\to$, $\leftrightarrow$, and a constant $\bot$, interpreted as the corresponding operations in a Boolean algebra;
- unary modalities $\Diamond$ and $\Box$ interpreted as $\mathbf{C}$ and $\mathbf{I}$;
- a new connective $\langle t \rangle$, interpreted as $\mathbf{C}^t$, which provides formation of a formula $\langle t \rangle \Gamma$ for each finite set Γ of formulas.

We denote by S4t be the propositional logic obtained by adding to a suitable axiomatisation of the (non-modal) two-valued propositional calculus the axiom schemes

K $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,

T $\varphi \to \Diamond\varphi$,

4 $\Diamond\Diamond\varphi \to \Diamond\varphi$,

Fix $\langle t \rangle \Gamma \to \Diamond(\gamma \wedge \langle t \rangle \Gamma)$, all $\gamma \in \Gamma$,

Ind $\Box(\varphi \to \bigwedge_{\gamma\in\Gamma} \Diamond(\gamma \wedge \varphi)) \to (\varphi \to \langle t\rangle\Gamma)$,

and the inference rule of $\Box$-generalisation (from φ infer $\Box\varphi$). We write S4t $\vdash \varphi$ to mean that formula φ is derivable as a theorem of this logic, which is studied in detail in [9–11].

An equivalence relation $\equiv$ on formulas is defined by putting $\varphi \equiv \psi$ if and only if S4$t \vdash \varphi \leftrightarrow \psi$. If $|\varphi| = \{\psi : \varphi \equiv \psi\}$ is the equivalence class of φ, then the Lindenbaum-Tarski algebra of S4t is the set

$$A_t = \{|\varphi| : \varphi \text{ is a formula}\}$$

of all equivalence classes, with the operations

$$\begin{aligned}
|\varphi| \wedge |\psi| &= |\varphi \wedge \psi| \\
|\varphi| \vee |\psi| &= |\varphi \vee \psi| \\
-|\varphi| &= |\neg\varphi| \\
0 &= |\bot| \\
1 &= |\neg\bot| \\
\mathbf{C}^t_{A_t}\{|\varphi| : \varphi \in \Gamma\} &= |\langle t\rangle\Gamma|.
\end{aligned}$$

$(A_t, \mathbf{C}^t_{A_t})$ is a well-defined tangled closure algebra having an injective function $\eta : V \to A_t$ given by $\eta(v) = |v|$. This is for the most part standard theory [21, §§VI.10, XI.7]. That $\mathbf{C}^t_{A_t}$ is well-defined follows because if $\Gamma' = \{\varphi' : \varphi \in \Gamma\}$ and S4$t \vdash \varphi \leftrightarrow \varphi'$ for all $\varphi \in \Gamma$, then S4$t \vdash \langle t\rangle\Gamma \leftrightarrow \langle t\rangle\Gamma'$. The axioms **Fix** and **Ind** for S4t ensure that $\mathbf{C}^t_{A_t}$ is a tangled closure operator.

The image $\{|v| : v \in V\}$ of η generates the algebra $(A_t, \mathbf{C}^t_{A_t})$, which is free over V in the sense that for any tangled closure algebra $(A, \mathbf{C}^t_A)$ and any function $f : V \to A$, there is a unique tangled closure algebra homomorphism $f' : (A_t, \mathbf{C}^t_{A_t}) \to (A, \mathbf{C}^t_A)$ such that $f' \circ \eta = f$. The function f itself is extended to map all formulas into A by interpreting the connectives by the corresponding operations of $(A, \mathbf{C}^t_A)$, and then f' is defined by putting $f'|\varphi| = f(\varphi)$. Identifying v with $\eta(v)$ allows us to view V as a subset of A_t that freely generates $(A_t, \mathbf{C}^t_{A_t})$.

Remark 3.4. A tangled closure algebra differs from the type of algebra conventionally studied in universal algebra, since the operation $\mathbf{C}^t$ is not finitary, that is, not n-ary for any $n < \omega$. But it gives rise to the sequence of finitary operations $\{\mathbf{C}^t_n : n < \omega\}$, where $\mathbf{C}^t_n$ is the n-ary operation defined by $\mathbf{C}^t_n(a_1, \dots, a_n) = \mathbf{C}^t\{a_1, \dots, a_n\}$ for $n > 0$, and $\mathbf{C}^t_0$ is the nullary operation with $\mathbf{C}^t_0(\emptyset) = 1$. We could define a tangled closure algebra as a conventional algebra with infinite signature, having the form $(A, \{\mathbf{C}^t_n : n < \omega\})$, satisfying axioms Fix_n and Ind_n stated in terms of $\mathbf{C}^t_n$ for each n, and satisfying axioms $\mathbf{C}^t_n(a_1, \dots, a_n) = \mathbf{C}^t_n(a_{\sigma 1}, \dots, a_{\sigma n})$ expressing the invariance of $\mathbf{C}^t_n$ under any permutation of its arguments. It is evident that this alternative approach is equivalent to the presentation we have given here. But it helps clarify that the class of tangled closure algebras is an equational class, or variety, in the traditional sense. □

The logic S4t has the finite model property: any non-theorem of the logic is falsifiable in the powerset algebra $(A_S, \mathbf{C}^t_R)$ of some finite quasi-ordered set (S, R) (see [6, 9, 10] for a proof). From this, it can be concluded that the variety of tangled closure algebras is generated by its finite members.

4 Dissectable Algebras

A closure algebra $(B, \mathbf{C}_B)$ is *dissectable* if for any non-zero open element α of B, and any natural numbers r and s, there exist non-zero elements $\alpha_1, \dots, \alpha_r, \beta_0, \dots, \beta_s$ of B such that

- these elements form a partition of α, that is, they are pairwise disjoint (any two have meet 0) and the join of all of them is α;[2]
- $\alpha_1, \dots, \alpha_r$ are all open; and
- $\mathbf{C}_B\alpha_i - \alpha_i = \mathbf{C}_B\beta_j = \mathbf{C}_B\alpha - (\alpha_1 \vee \cdots \vee \alpha_r)$ for all $i \leq r$ and $j \leq s$.

Originally Tarski formulated the dissectability property with $s = 0$, and proved that this holds for the powerset algebra of the real line and of its dense-in-themselves subspaces. Density-in-itself means that there are no isolated points, that is, no open singletons. Samuel Eilenberg then proved that the property holds for any separable dense-in-itself metric space, and this was presented in [23]. The more general formulation with arbitrary

[2] When $r = 0$, the sequence $\alpha_1, \dots, \alpha_r$ is empty.

finite s was given in [17], where it was shown to hold for separable dense-in-themselves metric spaces. Another proof was given in [21] that eliminated the separability restriction. New kinds of dissectability theorems along these lines are presented in [10–12].

It was shown in [17] that if $(B, \mathbf{C}_B)$ is dissectable then every finite closure algebra is isomorphic to a subalgebra of the relativised algebra $(B_\alpha, \mathbf{C}_\alpha)$ for some non-zero open element α of B, and that any such $(B_\alpha, \mathbf{C}_\alpha)$ is itself dissectable. Moreover, a *well-connected* finite closure algebra is embeddable into $(B_\alpha, \mathbf{C}_\alpha)$ for *every* non-zero open α. Well-connectedness means that $\mathbf{C}a \wedge \mathbf{C}b = 0$ implies $a = 0$ or $b = 0$. Equivalently, it means that the meet of any two non-zero closed elements is non-zero. In a finite closure algebra, this means that there is a *least* non-zero closed element, a property called *strong compactness* in [21, p. 110]. For the powerset closure algebra $(A_S, \mathbf{C}_R)$ of a quasi-order set (S, R), as in Example 2.5, this means that the quasi-order is *point-generated* in the sense that there is a point $x \in S$ such that $R(x) = S$, so that every $y \in S$ has xRy. To see why, let a be a least non-empty closed subset of S in the Alexandroff topology. Take any $x \in a$. Then for any $y \in S$, the set $\{z : zRy\}$ is closed and non-empty, so includes a, showing that xRy. Hence $R(x) = S$. Conversely, if $R(x) = S$, then the set $\{z : zRx\}$ is a non-empty closed set included in all others. In summary: if S is finite, then $(A_S, \mathbf{C}_R)$ is well-connected if and only if (S, R) is point-generated.

Using our result from the previous section on homomorphisms with finite domains, we can readily lift the McKinsey-Tarski analysis to tangled closure algebras.

Theorem 4.1. *Let $(B, \mathbf{C}^t_B)$ be a tangled closure algebra whose closure algebra reduct $(B, \mathbf{C}_B)$ is dissectable. Then any finite tangled closure algebra with a well-connected closure algebra reduct is isomorphically embeddable into the relativised algebra $(B_\alpha, \mathbf{C}^t_\alpha)$ of any non-zero open element α of $(B, \mathbf{C}_B)$.*

Proof. Let α be any non-zero open element of $(B, \mathbf{C}_B)$. By Theorem 3.3 $(B_\alpha, \mathbf{C}^t_\alpha)$, is a tangled closure algebra, with closure algebra reduct $(B_\alpha, \mathbf{C}_\alpha)$.

Now let $(A, \mathbf{C}^t_A)$ be a finite tangled closure algebra whose closure algebra reduct $(A, \mathbf{C}_A)$ is well-connected. Then by [17, Theorem 3.7] there is a closure algebra embedding $f : (A, \mathbf{C}_A) \to (B_\alpha, \mathbf{C}_\alpha)$. By our Theorem 3.2

this f preserves the tangled closure operations $\mathbf{C}^t_A$ and $\mathbf{C}^t_\alpha$, so provides the result. $\square$

Note that by putting $\alpha = 1$ in this theorem, so that $B_\alpha = B$, we conclude that any well-connected finite tangled closure algebra is isomorphic to a subalgebra of $(B, \mathbf{C}^t_B)$ itself. We can now apply this result to show that any finite tangled closure algebra has *some* embedding into a relativised algebra of any dissectable tangled closure algebra.

Theorem 4.2. *Let $(B, \mathbf{C}^t_B)$ be a dissectable tangled closure algebra. Then any finite tangled closure algebra is isomorphically embeddable into the relativised algebra $(B_\alpha, \mathbf{C}^t_\alpha)$ of some open element α of $(B, \mathbf{C}_B)$.*

Proof. By Theorem 3.1, it suffices to give the proof for finite algebras of the form $(A_S, \mathbf{C}^t_R)$. If (S, R) is point-generated, the result follows from Theorem 4.1. Otherwise, we add a generating point. Let x be any object not in S, put $S^* = S \cup \{x\}$, and let $R^* = R \cup (\{x\} \times S^*)$. Then (S^*, R^*) is a quasi-ordered set point-generated by x, with no member of S being R^*-related to x. The finite tangled closure algebra $(A_{S^*}, \mathbf{C}^t_{R^*})$ is well-connected, so by Theorem 4.1 with $\alpha = 1$, there is a tangled closure embedding $h : (A_{S^*}, \mathbf{C}^t_{R^*}) \to (B, \mathbf{C}^t_B)$. The image $(B', \mathbf{C}^t_{B'})$ of h is a tangled closure subalgebra of $(B, \mathbf{C}^t_B)$ isomorphic to $(A_{S^*}, \mathbf{C}^t_{R^*})$, where $\mathbf{C}^t_{B'}$ is the restriction of $\mathbf{C}^t_B$ to $B' = h(B)$.

Now S is a subset of S^* which is closed upwards under R^*, so S is an open element of $(A_{S^*}, \mathbf{C}_{R^*})$, that is, $\mathbf{I}_{R^*}(S) = S$. But h preserves the interior operations $\mathbf{I}_{R^*}$ and $\mathbf{I}_B$, so then $h(S)$ is an open element of $(B, \mathbf{C}_B)$, that is, $\mathbf{I}_B h(S) = h(S)$. Let $\alpha = h(S) \in B'$. Then as $(A_{S^*}, \mathbf{C}^t_{R^*})$ is isomorphic to $(B', \mathbf{C}^t_{B'})$ under h, the relativisation of $(A_{S^*}, \mathbf{C}^t_{R^*})$ to S is isomorphic to the relativisation of $(B', \mathbf{C}^t_{B'})$ to α, which is a subalgebra of the relativisation $(B_\alpha, \mathbf{C}^t_\alpha)$ of $(B, \mathbf{C}^t_B)$ to the open element α.

But the relativisation of $(A_{S^*}, \mathbf{C}^t_{R^*})$ to S is exactly $(A_S, \mathbf{C}^t_R)$. For, the relativisation $(A_{S^*})_S$ of the powerset algebra A_{S^*} of S^* to S is just the powerset algebra A_S of S. Also, the relativisation of $\mathbf{C}^t_{R^*}$ to S is the map $\Gamma \mapsto S \cap \mathbf{C}^t_{R^*}\Gamma$ for $\Gamma \subseteq A_S$. But $S \cap \mathbf{C}^t_{R^*}\Gamma = \mathbf{C}^t_R\Gamma$ because S is closed upwards under R^* and an endless R^*-path that starts in S must remain in S and be an endless R-path.

Altogether then, this shows that $(A_S, \mathbf{C}^t_R)$ is isomorphic to a subalgebra of $(B_\alpha, \mathbf{C}^t_\alpha)$. $\square$

The proof of Theorem 4.1 can be extended to all finite tangled closure algebras if the dissectable algebra $(B, \mathbf{C}_B)$ is assumed to be *totally disconnected*, which means that every non-zero open element is the join of two disjoint non-zero open elements. The totally disconnected dissectable algebras include the closure algebras of all dense-in-themselves metric spaces that are totally disconnected in the spatial sense that distinct points can be separated by a clopen set. Examples of such spaces include the rational line, the Cantor space and the Baire space ω^ω.

Theorem 4.3. *Let α be any non-zero open element of a tangled closure algebra $(B, \mathbf{C}^t_B)$ whose closure algebra reduct is totally disconnected and dissectable. Then any finite tangled closure algebra is isomorphically embeddable into the relativised algebra $(B_\alpha, \mathbf{C}^t_\alpha)$.*

Proof. The reduct $(B_\alpha, \mathbf{C}_\alpha)$ is totally disconnected and dissectable, so by [17, Theorem 3.8], if $(A, \mathbf{C}^t_A)$ is any finite tangled closure algebra, there is a closure algebra embedding $f : (A, \mathbf{C}_A) \to (B_\alpha, \mathbf{C}_\alpha)$. By Theorem 3.2, this f preserves the tangled closure operations $\mathbf{C}^t_A$ and $\mathbf{C}^t_\alpha$, so provides the result. $\square$

5 No Completion

A completion of a Boolean algebra A is any *complete* Boolean algebra B extending A, that is, having A as a subalgebra, such that each member of B is the join of a set of members of A. This last condition is equivalent to A being *dense* in B in the sense that each non-zero member of B is above some non-zero member of A. It implies that B is a *regular* extension of A, that is, the inclusion $A \hookrightarrow B$ preserves any joins (hence meets) that exist in A, so that if $a = \bigvee_A E$ in A, then $a = \bigvee_B E$ in B. Any Boolean algebra A has a completion, and any two completions of A are isomorphic by a function that is the identity on A (see for example, [3, 8, 22]). This unique-up-to-isomorphism algebra is often called the *MacNeille completion* of A, after its construction in [16]. It has various abstract characterisations, some due to Banaschewski [1, 2].

If $(A, \mathbf{C}_A)$ is a closure algebra and B is any complete extension of A, then $\mathbf{C}_A$ can be extended to a closure operator on B by putting

$$\mathbf{C}_B b = \bigwedge\nolimits_B \{\mathbf{C}_A a : b \leq a \in A\} \tag{5.1}$$

for all $b \in B$. This definition was given in [17] where it was applied to the Stone representation of A to lift $\mathbf{C}_A$ to the powerset algebra of the representing set, ultimately showing that any closure algebra is embeddable into the complete algebra of subsets of some topological space. It was later used in [20] to extend $\mathbf{C}_A$ to the MacNeille completion of A, applying this to construct a regular complete extension of any Heyting algebra, and then using the regularity to obtain completeness theorems in algebraic semantics for versions of intuitionistic logic and the modal logic S4 with first-order quantifiers. In more recent literature on MacNeille completions [25], $\mathbf{C}_B$ as given by (5.1) is called the *upper MacNeille extension* of $\mathbf{C}_A$.

There is no unique definition of MacNeille extension for operations on Boolean algebras. Monk [19] showed that for algebras, such as cylindric algebras, in which the operations are completely additive (preserve all joins), it is fruitful to use the *lower MacNeille extension* which lifts an operation $\mathbf{O}_A$ to the operation $\mathbf{O}_B b = \bigvee_B \{\mathbf{O}a : b \geq a \in A\}$.

We now define a *completion* of a closure algebra $(A, \mathbf{C}_A)$ to be a closure algebra $(B, \mathbf{C}_B)$ such that B is a Boolean completion of A, $(A, \mathbf{C}_A)$ is a subalgebra of $(B, \mathbf{C}_B)$, and (5.1) holds for each $b \in B$.

Theorem 5.1. *Any closure algebra $(A, \mathbf{C}_A)$ has a completion, and any two such completions are isomorphic by a function that is the identity on A.*

Proof. Let B be a Boolean completion of A, and *define* a closure operator $\mathbf{C}_B$ on B by (5.1). Then $\mathbf{C}_B a = \mathbf{C}_A a$ for $a \in A$, so $(A, \mathbf{C}_A)$ is a subalgebra of $(B, \mathbf{C}_B)$ and $(B, \mathbf{C}_B)$ is a completion of $(A, \mathbf{C}_A)$. If $(B', \mathbf{C}_{B'})$ is another one, then there is a Boolean isomorphism $f : B \to B'$ that is the identity on A. Hence f preserves joins and meets, and for any $b \in B$ and $a \in A$, we have $b \leq a$ if and only if $f(b) \leq a$. Then we can shown that f preserves closure operators as follows.

$$
\begin{array}{ll}
f(\mathbf{C}_B b) & \\
= f \bigwedge_B \{\mathbf{C}_A a : b \leq a \in A\} & \text{by (5.1),} \\
= \bigwedge_{B'} \{f(\mathbf{C}_A a) : b \leq a \in A\} & \text{as } f \text{ preserves meets,} \\
= \bigwedge_{B'} \{\mathbf{C}_A a : f(b) \leq a \in A\} & \text{as } f \text{ fixes } A \\
= \mathbf{C}_{B'} f(b) & \text{by (5.1) for } B'.
\end{array}
$$

Thus f is a closure algebra isomorphism. □

It would thus seem natural to define a completion of a tangled closure algebra $(A, \mathbf{C}^t_A)$ to be a tangled closure algebra $(B, \mathbf{C}^t_B)$ such that

(i) $(A, \mathbf{C}^t_A)$ is a subalgebra of $(B, \mathbf{C}^t_B)$,

(ii) the closure algebra $(B, \mathbf{C}_B)$ induced by $\mathbf{C}^t_B$ is a completion of the closure algebra $(A, \mathbf{C}_A)$ induced by $\mathbf{C}^t_A$,

and perhaps some other conditions as well. However, we will now construct a tangled closure algebra $(A, \mathbf{C}^t_A)$ for which there is no *complete* tangled closure algebra $(B, \mathbf{C}^t_B)$ satisfying (i), let alone (i) and (ii).

Lemma 5.2. *There exists a tangled closure algebra* $(A_0, \mathbf{C}^t)$ *having a subset* $\{p_n : n < \omega\} \cup \{q\}$ *and an ultrafilter* x_0 *such that* $\mathbf{C}^t\{q, -q\} \notin x_0$ *while* $\Sigma \subseteq x_0$*, where*

$$\Sigma = \{p_0\} \cup \{\mathbf{I}(p_{2n} \Rightarrow \mathbf{C}(p_{2n+1} \wedge q)), \mathbf{I}(p_{2n+1} \Rightarrow \mathbf{C}(p_{2n+2} \wedge -q)) : n < \omega\}.$$

Proof. Let $(A_0, \mathbf{C}^t)$ be the free tangled closure algebra generated by a set $\{p_n : n < \omega\} \cup \{q\}$ of distinct elements. This exists as explained in Section 3. Let a_n be $\mathbf{I}(p_n \Rightarrow \mathbf{C}(p_{n+1} \wedge q))$ if n is even, and $\mathbf{I}(p_n \Rightarrow \mathbf{C}(p_{n+1} \wedge -q))$ if n is odd, where $\mathbf{C}$ is the closure operator induced by $\mathbf{C}^t$ and $\mathbf{I}$ is the interior operation dual to $\mathbf{C}$. Then $\Sigma = \{p_0\} \cup \{a_n : n < \omega\} \subseteq A_0$.

It suffices to show that the set $\Sigma \cup \{-\mathbf{C}^t\{q, -q\}\}$ has the finite meet property in A_0: every finite subset has non-zero meet. For then $\Sigma \cup \{-\mathbf{C}^t\{q, -q\}\}$ is included in an ultrafilter x_0 of A_0 which includes Σ but does not contain $\mathbf{C}^t\{q, -q\}$ as it contains $-\mathbf{C}^t\{q, -q\}$.

For each positive integer m, let $\Sigma_m = \{p_0\} \cup \{a_n : n < m\}$. Any finite subset of $\Sigma \cup \{-\mathbf{C}^t\{q, -q\}\}$ is a subset of $\Sigma_m \cup \{-\mathbf{C}^t\{q, -q\}\}$ for some m, so it suffices now to show that $\bigwedge(\Sigma_m \cup \{-\mathbf{C}^t\{q, -q\}\}) \neq 0$ for any m.

Define a quasi-ordered set (S_m, R_m) by $S_m = \{0, \ldots, m\}$ and $xR_m y$ if and only if $x \leq y$. Put $p'_n = \{n\}$ for all $n \leq m$ and let $q' = \{n \leq m : n \text{ is odd}\}$. Then by the freeness property there exists a tangled closure algebra homomorphism f from $(A_0, \mathbf{C}^t)$ to the powerset algebra $(A_{S_m}, \mathbf{C}^t_{R_m})$ of (S_m, R_m) such that $f(p_n) = p'_n$ for all $n \leq m$ and $f(q) = q'$.

For $n < m$, let $a'_n = f(a_n)$. Since f preserves the closure algebra operations, a'_n is the subset of S_m specified by replacing p_k by p'_k and q by q' in a_n. Let $\Sigma'_m = \{p'_0\} \cup \{a'_n : n < m\}$. It is evident that $0 \in \bigcap(\Sigma'_m \cup \{-\mathbf{C}^t_{R_m}\{q', -q'\}\})$, since 0 belongs to p'_0 and to each a'_n, and any endless R_m-path is ultimately constant, so cannot move in and out of q' endlessly, hence $0 \notin \mathbf{C}^t_{R_m}\{q', -q'\}$.

But if we had $\bigwedge(\Sigma_m \cup \{-\mathbf{C}^t\{q, -q\}\}) = 0$ in A_0, then as f preserves all the operations involved, we would have $\bigcap(\Sigma'_m \cup \{-\mathbf{C}^t_{R_m}\{q', -q'\}\}) = \emptyset$, a contradiction. $\square$

Now taking the algebra $(A_0, \mathbf{C}^t)$ given by this lemma, let U_0 be the set of ultrafilters of A_0. Define a relation R on U_0 by putting xRy if and only if $\{a : \mathbf{I}a \in x\} \subseteq y$, or equivalently if and only if $\{\mathbf{C}a : a \in y\} \subseteq x$. Then it is standard theory that R is a quasi-order on U_0, and has, for all $a \in A_0$ and $x \in U_0$,

$$\mathbf{I}a \in x \quad \text{if and only if} \quad \text{for all } y \in U_0,\ xRy \text{ implies } a \in y. \tag{5.2}$$

$$\mathbf{C}a \in x \quad \text{if and only if} \quad \text{for some } y \in U_0,\ xRy \text{ and } a \in y. \tag{5.3}$$

Let $U = \{y \in U_0 : x_0Ry\}$, where x_0 is the ultrafilter given by Lemma 5.2. For $a \in A_0$, put

$$|a| = \{x \in U : a \in x\}.$$

Then $A = \{|a| : a \in A_0\}$ is a Boolean subalgebra of the powerset algebra of U, since $U \setminus |a| = |-a|$ and $|a| \cap |b| = |a \wedge b|$. The map $a \mapsto |a|$ is a Boolean algebra homomorphism from A_0 onto A.

We now transfer the tangled closure operation $\mathbf{C}^t$ on A_0 to one on A, by defining

$$\mathbf{C}^t_A\{|\gamma| : \gamma \in \Gamma\} = |\mathbf{C}^t\Gamma| \tag{5.4}$$

for all $\Gamma \in \mathcal{P}_{fin}(A_0)$. We need to check that this is well-defined, that is, that if $|\gamma| = |\gamma'|$ for all $\gamma \in \Gamma$, and $\Gamma' = \{\gamma' : \gamma \in \Gamma\}$, then $|\mathbf{C}^t_A\Gamma| = |\mathbf{C}^t_A\Gamma'|$. But we have $|\gamma| = |\gamma'|$ if and only if γ and γ' belong to the same members of U, which is equivalent to requiring that $\gamma \Leftrightarrow \gamma'$ belongs to every member of U. By (5.2) with $x = x_0$, this is equivalent to having $\mathbf{I}(\gamma \Leftrightarrow \gamma') \in x_0$. Hence the well-definedness follows because $(A_0, \mathbf{C}^t)$ satisfies

$$\textstyle\bigwedge_{\gamma\in\Gamma}\mathbf{I}(\gamma \Leftrightarrow \gamma') \leq \mathbf{I}\big(\mathbf{C}^t\Gamma \Leftrightarrow \mathbf{C}^t\Gamma'\big)$$

by Theorem 2.6 (2), so if $\mathbf{I}(\gamma \Leftrightarrow \gamma') \in x_0$ for all $\gamma \in \Gamma$, then $\mathbf{I}(\mathbf{C}^t\Gamma \Leftrightarrow \mathbf{C}^t\Gamma') \in x_0$.

The unary operation $\mathbf{C}_A$ induced by $\mathbf{C}^t_A$ is given by

$$\mathbf{C}_A|a| = \mathbf{C}^t_A\{|a|\} = |\mathbf{C}^t\{a\}| = |\mathbf{C}a|,$$

and its dual has $\mathbf{I}_A|a| = |\mathbf{I}a|$. Equation (5.4) ensures that $a \mapsto |a|$ is a homomorphism from $(A_0, \mathbf{C}^t, \mathbf{C})$ onto $(A, \mathbf{C}^t_A, \mathbf{C}_A)$. Hence $(A, \mathbf{C}^t_A, \mathbf{C}_A)$ is a closure algebra satisfying Fix and Ind, so is a tangled closure algebra.

Theorem 5.3. *If $(B, \mathbf{C}^t_B)$ is any tangled closure algebra for which B is complete, then there is no tangled closure embedding of $(A, \mathbf{C}^t_A)$ into $(B, \mathbf{C}^t_B)$.*

Proof. Assume for the sake of contradiction that there exists a $f : (A, \mathbf{C}^t_A) \to (B, \mathbf{C}^t_B)$ that is a tangled closure embedding. Then we show that

$$f\mathbf{C}^t_A\{|q|, |-q|\} \neq \mathbf{C}^t_B\{f|q|, f|-q|\},$$

which contradicts the assumption that f preserves tangled closure.

By Theorem 2.6(1), $\mathbf{C}\mathbf{C}^t\{q, -q\} = \mathbf{C}^t\{q, -q\} \notin x_0$, so by (5.3), for all $y \in U$, we have $\mathbf{C}^t\{q, -q\} \notin y$. Thus $|\mathbf{C}^t\{q, -q\}| = \emptyset$. Therefore

$$f\mathbf{C}^t_A\{|q|, |-q|\} = f|\mathbf{C}^t\{q, -q\}| = f\emptyset = 0$$

in B. Hence to prove that f is not a tangled closure homomorphism it suffices to show that $\mathbf{C}^t_B\{f|q|, f|-q|\} \neq 0$.

To show this, put $b_n = f|p_n|$ for each $n < \omega$, and let $b = \bigvee\{b_n : n < \omega\}$. Then b exists in B as B is complete. We prove that b is a post-fixed point for $\{f|q|, f|-q|\}$, that is,

$$b \leq \mathbf{C}_B(b \wedge f|q|) \wedge \mathbf{C}_B(b \wedge f|-q|). \tag{5.5}$$

Now if n is even, then since $a_n \in x_0$ it follows by (5.2) that $p_n \Rightarrow \mathbf{C}(p_{n+1} \wedge q) \in y$ for all $y \in U$, hence $|p_n| \subseteq |\mathbf{C}(p_{n+1} \wedge q)| = \mathbf{C}_A(|p_{n+1}| \cap |q|)$. Similarly, if n is odd, then $|p_n| \subseteq \mathbf{C}_A(|p_{n+1}| \cap |-q|)$. Since f is a closure algebra homomorphism, this implies that for all $n < \omega$,

$$b_n \leq \mathbf{C}_B(b_{n+1} \wedge f|q|), \quad \text{if } n \text{ is even;} \tag{5.6}$$

$$b_n \leq \mathbf{C}_B(b_{n+1} \wedge f|-q|), \quad \text{if } n \text{ is odd.} \tag{5.7}$$

Thus if n is even, then by (5.6) $b_n \leq \mathbf{C}_B(b_{n+1} \wedge f|q|) \leq \mathbf{C}_B(b \wedge f|q|)$. Also then as $n+1$ is odd we use (5.7) with $n+1$ in place of n to infer that

$$\mathbf{C}_B(b_{n+1}) \leq \mathbf{C}_B\mathbf{C}_B(b_{n+2} \wedge f|-q|) \leq \mathbf{C}_B(b \wedge f|-q|).$$

Since $b_n \leq \mathbf{C}_B(b_{n+1})$ follows from (5.6), altogether these facts imply that

$$b_n \leq \mathbf{C}_B(b \wedge f|q|) \wedge \mathbf{C}_B(b \wedge f|-q|) \tag{5.8}$$

when n is even. But a similar proof shows that (5.8) also holds when n is odd. Hence it holds for all $n < \omega$, from which (5.5) follows.

Thus b is indeed a post-fixed point for $\{f|q|, f|-q|\}$, and therefore $b \leq \mathbf{C}^t_B\{f|q|, f|-q|\}$ by Lemma 2.1. But as $p_0 \in x_0$ we have $x_0 \in |p_0|$, hence $|p_0| \neq \emptyset$. So as f is injective,

$$0 = f\emptyset \neq f|p_0| = b_0 \leq b \leq \mathbf{C}^t_B\{f|q|, f|-q|\}.$$

This proves $\mathbf{C}^t_B\{f|q|, f|-q|\} \neq 0$, which completes the proof as explained. $\square$

Thus $(A, \mathbf{C}^t_A)$ has no homomorphic embedding into any complete tangled closure algebra. In particular it is not embeddable into the algebra $(A_S, \mathbf{C}^t_S)$ of subsets of any topological space S, including not being embeddable into the algebra $(A_S, \mathbf{C}^t_R)$ of subsets of any quasi-ordered set (S, R).

Let B be any complete extension of the Boolean algebra A; take $\mathbf{C}_B$ to be the closure operator on B extending $\mathbf{C}_A$ defined by (5.1); and let $\mathbf{C}^t_B$ be the expansion of $\mathbf{C}_B$ given by (2.2). Then $(B, \mathbf{C}^t_B)$ is a tangled closure algebra by Theorem 2.3. The inclusion $A \hookrightarrow B$ provides the promised example of a map $(A, \mathbf{C}^t_A) \to (B, \mathbf{C}^t_B)$ that is a homomorphism of the associated closure algebra reducts but is not a tangled closure homomorphism (by Theorem 3.2, such an example must have infinite A). It also provides the promised example in which $(A, \mathbf{C}_A)$ is a subalgebra of $(B, \mathbf{C}_B)$ while $(A, \mathbf{C}^t_A)$ is not a subalgebra of $(B, \mathbf{C}^t_B)$.

Finally we note that this absence of complete extensions is not attributable to the fact that tangled closure algebras can be seen as having infinite signature (Remark 3.4). The construction needs just the element $\mathbf{C}^t_A\{|q|, |-q|\}$ which requires only the binary operation $(a, b) \mapsto \mathbf{C}^t_A\{a, b\}$ of this signature for its formation.

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