

# Choice principles and lift lemmas

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Dedicated to Bernhard Banaschewski, master of choice,  
on the occasion of his 90th birthday

**Abstract.** We show that in **ZF** set theory without choice, the Ultrafilter Principle (**UP**) is equivalent to several compactness theorems for Alexandroff discrete spaces and to Rudin's Lemma, a basic tool in topology and the theory of quasicontinuous domains. Important consequences of Rudin's Lemma are various lift lemmas, saying that certain properties of posets are inherited by the free unital semilattices over them. Some of these principles follow not only from **UP** but also from **DC**, the Principle of Dependent Choices. On the other hand, they imply the Axiom of Choice for countable families of finite sets, which is not provable in **ZF** set theory.

## 1 Introduction

If not otherwise stated, we are working rigorously in Zermelo–Fraenkel set theory (**ZF**) *without assuming any choice principles a priori*. In that environment, many order-theoretical, algebraic and topological theorems are equivalent to the Ultrafilter Theorem or Ultrafilter Principle (**UP**), postu-

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lating that every proper set-theoretical filter be contained in an ultrafilter; see Herrlich [26], and Howard and Rubin [28] for a long list of equivalent forms of **UP**. Three of the earliest references are Rubin, Scott and Tarski [38, 40, 41]. For special results relevant to the present study refer to Banaschewski and Erné [2–6, 12, 15, 16, 18, 20].

The equivalence of certain statements to **UP** appears sometimes rather unexpectedly. A useful tool for the investigation of such equivalences is the set  $\delta P$  of all Scott-open filters of a poset  $P$ ; recall that the Scott-open sets  $U \subseteq P$ , characterized by the equivalence  $x \in U \Leftrightarrow D \cap U \neq \emptyset$  for all directed sets  $D \subseteq P$  having a join  $x = \bigvee D$ , form the *Scott topology*  $\sigma P$ . A powerful lattice-theoretical equivalent of **UP**, established in [12] for locales and in [5] for quantales, is

**SL** *the Separation Lemma: any element outside a Scott-open filter in a locale lies below a prime element outside that filter.*

A few notational comments might be welcome. *Locales* or *frames* are complete lattices in which arbitrary joins distribute over finite meets (Johnstone [32], Picado and Pultr [37]). Given a quasiordered set (*qoset*), that is, a set with a reflexive and transitive order relation  $\leq$ , we adopt the usual notations  $\downarrow Y$  and  $\uparrow Y$  for the lower set (down-set) and the upper set (up-set), respectively, generated by a subset  $Y$ ; thus,  $x \in \downarrow Y$  means that there is a  $y \in Y$  with  $x \leq y$ , and dually for  $\uparrow Y$ . An *ideal* is an (up-)directed lower set, and a *filter* a filtered (that is, down-directed) upper set. In particular,  $\downarrow y = \downarrow \{y\}$  is the *principal ideal*, and  $\uparrow y = \uparrow \{y\}$  is the *principal filter* generated by  $y$ . If the order relation is antisymmetric, we have a partially ordered set (*poset*). A poset is *up-complete*, *directed complete* or a *dcpo* if each directed subset has a join. An element  $p$  of a poset  $P$  is *prime* if  $P \setminus \downarrow p$  is a filter; notice that for us (like in number theory and algebra, but deviating from [23]), a unit (top element) is never prime. A *completely prime* or  $\wedge$ -*prime* element generates a principal ideal that is complementary to a principal filter. The dual notion is *supercompact* or  $\bigvee$ -*prime*.

By a space, we always mean a topological space  $X$ , and we denote by  $\mathcal{T}$  or  $\mathcal{O}X$  its topology (the locale / frame of open sets). When we speak about compactness properties, no separation axiom is assumed. The *saturation* of a subset is the intersection of its neighborhoods. The set of all compact saturated subsets of  $X$  is denoted by  $\mathcal{Q}$  or  $\mathcal{Q}X$ . All order-theoretical statements

about spaces refer to the *specialization order*, given by  $x \leq y \Leftrightarrow x \in cl\{y\}$ . Thus, for subsets  $Y$  of  $X$ , the upper set  $\uparrow Y$  is the saturation of  $Y$ ; in particular, the principal filter  $\uparrow y$  generated by a point  $y$  is the (*neighborhood core*) of  $y$ , while the principal ideal  $\downarrow y$  is the closure of  $\{y\}$ . For collections  $\mathcal{Y}$  of subsets of  $X$ , put

$$\mathcal{Y}\mathcal{T} = \{U \in \mathcal{T} : \exists Y \in \mathcal{Y} (Y \subseteq U)\}.$$

Specifically, for  $Y \subseteq X$  and  $x \in X$ , the open neighborhood systems are

$$Y\mathcal{T} = \{Y\}\mathcal{T} = \{U \in \mathcal{T} : Y \subseteq U\}, \quad x\mathcal{T} = \{x\}\mathcal{T} = \{U \in \mathcal{T} : x \in U\}.$$

In [20], we demonstrate that **UP** is responsible for the coincidence of various degrees of (topological) sobriety. A  $T_0$ -space  $X$  or its topology  $\mathcal{T}$  is called  $\delta$ -*sober* or *strictly sober* [18, 20] if any  $\mathcal{V} \in \delta\mathcal{T}$  contains each open  $U$  with  $\bigcap \mathcal{V} \subseteq U$ . Strictly sober spaces are in fact *sober* (the only irreducible closed sets are the point closures) and *strongly well-filtered*, that is, for every set  $\mathcal{I} \subseteq \mathcal{Q}$  such that  $\mathcal{I}\mathcal{T}$  is filtered (that is, directed by dual inclusion), any  $U \in \mathcal{T}$  with  $\bigcap \mathcal{I} \subseteq U$  contains some member of  $\mathcal{I}$  (cf. [25]); and clearly, such spaces are *well-filtered*, that is, they satisfy the previous condition whenever  $\mathcal{I}$  itself is filtered (cf. [23]).

The following generalizations will be of interest: let  $\mathcal{K}$  be a collection of compact saturated subsets of a space  $X$  with topology  $\mathcal{T}$ . Then we say  $X$  or  $\mathcal{T}$  is *locally  $\mathcal{K}$ -compact* if each  $U \in \mathcal{T}$  is the directed union of all  $V \in \mathcal{T}$  for which there is a  $C \in \mathcal{K}$  with  $V \subseteq C \subseteq U$ ; and  $X$  or  $\mathcal{T}$  is  *$\mathcal{K}$ -well-filtered* (respectively, *strongly  $\mathcal{K}$ -well-filtered*) if it is  $T_0$  and for any  $\mathcal{I} \subseteq \mathcal{K}$  such that  $\mathcal{I}$  (respectively,  $\mathcal{I}\mathcal{T}$ ) is a filter base, there is a saturated  $C$  with  $\mathcal{I}\mathcal{T} = C\mathcal{T}$  (which implies  $C = \bigcap \mathcal{I} \in \mathcal{Q}$ ).

**Theorem 1.1.** *Strictly sober spaces are strongly  $\mathcal{K}$ -well-filtered.*

*Conversely, locally  $\mathcal{K}$ -compact  $\mathcal{K}$ -well-filtered spaces are strictly sober.*

The first statement is obvious:  $\mathcal{I}\mathcal{T}$  is a Scott-open filter in  $\mathcal{T}$  if it is a filter base and  $\mathcal{I} \subseteq \mathcal{Q}$ ; the second one is established in [20] (see [18] for a point-free version).

**Corollary 1.2.** *A locally compact space is strictly sober if and only if it is (strongly) well-filtered.*

A subset  $S$  of a poset  $P$  is  $\wedge$ -dense (respectively  $\wedge$ -dense) if each element of  $P$  is a meet of a subset (respectively finite subset) of  $S$ . The *spectrum* of a complete lattice  $L$  is the set  $P$  of its primes, equipped with the *hull-kernel topology*  $\{P \setminus \uparrow x : x \in L\}$ ; it is always sober [23, V-4.4].  $L$  is called *spatial* if it is an isomorphic copy of a topology or, equivalently, if the spectrum is  $\wedge$ -dense in  $L$ . If, moreover,  $L$  satisfies the condition in the Separation Lemma **SL**, we speak of a *strictly spatial lattice*. The following lemma, established choice-freely in [20], provides a basic connection.

**Lemma 1.3.** *A space is strictly sober if and only if its topology is strictly spatial. Hence, the category of strictly sober spaces is dual to the category of strictly spatial frames.*

Let us record some fundamental results from [20] we shall need in due course. For convenience, call a class  $\mathbf{X}$  of spaces *cs-stable* if it contains all closed subspaces of any  $\mathbf{X}$ -space, that is, of any member of  $\mathbf{X}$ . A *transversal* of a collection of sets is a subset of its union meeting all its members.

**Proposition 1.4.** *Consider the following statements on a class  $\mathbf{X}$  of spaces:*

**SLX** *the Separation Lemma for topologies of sober  $\mathbf{X}$ -spaces,*

**SSX** *the Strict Sobriety Theorem: sober  $\mathbf{X}$ -spaces are strictly sober,*

**WX** *the Well-filter Theorem: sober  $\mathbf{X}$ -spaces are well-filtered,*

**CIX** *the Compact Intersection Theorem: any filter base of compact saturated subsets of a sober  $\mathbf{X}$ -space has a compact intersection,*

**NIX** *the Non-void Intersection Theorem: any filter base of compact saturated subsets of a sober  $\mathbf{X}$ -space has a non-void intersection,*

**ITX** *the Irreducible Transversal Theorem: any set of compact subsets of an  $\mathbf{X}$ -space whose saturations form a filter base has an irreducible transversal.*

(1) *The first five statements are equivalent if  $\mathbf{X}$  is cs-stable, consists of locally compact spaces and contains for each  $X \in \mathbf{X}$  the ordinal sum with a non-compact space; and all six are equivalent if additionally  $\mathbf{X}$  is closed under sobrification.*

(2) *If  $\mathbf{X}$  is cs-stable and contains all powers of a space  $\mathbb{V}$  with specialization poset  $\mathfrak{V}$ , all six statements are equivalent to **UP** and to the Tychonoff*

*Product Theorem for any class of sober spaces that contains a two-element discrete space.*

Rudin's Lemma [39] states that any collection of finite subsets of a qoset whose generated upper sets form a filter base has a directed transversal [23, 39]. This lemma is a useful tool in topology and in domain theory [23, 24]. We shall see that Rudin's Lemma and a topological variant of it, due to Heckmann and Keimel [25] and resembling **ITX**, are equivalent to the above theorems, hence to **UP**; in fact, Rudin's Lemma is just a reformulation of **ITA**, the Irreducible Transversal Theorem for the class **A** of Alexandroff discrete spaces (*A-spaces*). The principles **SLA**, **SSA**, **WA**, **CIA**, and **NIA** for *A-spaces* are mutually equivalent, but, surprisingly, they are strictly weaker than **ITA** in **ZF**; indeed, we shall see that they follow not only from **UP** but also from **DC**, the Principle of Dependent Choices, and they are equivalent to the Noetherian Lift Lemma, saying that the free unital semilattice over a noetherian poset is noetherian, too. These principles and even weaker lift lemmas imply König's Infinity Lemma and so the Axiom of Choice for countable families of finite sets. As there are models in which the latter fails, none of these principles is provable in **ZF**.

Valuable references concerning choice axioms in set theory, order theory and topology are Fraïssé [22], Herrlich [26], Howard and Rubin [28], Jech [30], and Moore [36].

## 2 Rudin's Lemma

In this section, we shall apply the Irreducible Transversal Theorem (see the introduction) to the special class **A** of *Alexandroff discrete spaces* (*A-spaces*), in which arbitrary intersections of open sets are open. From Alexandroff's seminal work [1], we learn that these are just the quasiordered sets (*qosets*), topologized by the collection of all upper sets as open sets. In that situation, the compact saturated sets are precisely the *feet*, where a *foot* (*Fuß* in German) in a qoset is a *finitely generated upper set*. On the other hand, the irreducible subsets are just the directed ones, whence the closed irreducible sets are the ideals.

Given a collection  $\mathcal{Y}$  of sets, we mean by a *cutset* [15] of  $\mathcal{Y}$  a set that meets each member of  $\mathcal{Y}$ ; a *transversal* of  $\mathcal{Y}$  is a cutset contained in  $\bigcup \mathcal{Y}$ . Any cutset contains a transversal. A collection  $\mathcal{E}$  of subsets of a qoset is

said to be *quasifiltered* if the collection  $\{\uparrow E : E \in \mathcal{E}\}$  is filtered by inclusion (in other words,  $\mathcal{E}$  is directed by the quasiorder  $\sqsubseteq$  with  $E \sqsubseteq F \Leftrightarrow F \subseteq \uparrow E$ ; cf. [23, 25]). By the previous remarks, **ITA**, the Irreducible Transversal Theorem for A-spaces, implies

**RL** *the Rudin Lemma: any quasifiltered collection  $\mathcal{E}$  of non-void finite sets in a qoset has a directed transversal.*

Originally, Mary Rudin [39] proved that lemma by transfinite tools, based on **AC**. The proof was modified in [23, III–3] but still involved the full **AC**. Our approach relies on the weaker **UP** or, more directly, on the Irreducible Transversal Theorem, which opens applications to a wider range of problems. We shall make use of an obvious self-refinement of **RL**, namely (cf. [24, 25, 33])

**SRL** *the Strong Rudin Lemma: if  $\mathcal{E}$  is a quasifiltered set of finite subsets of a qoset and  $U$  is an upper set not containing any  $E \in \mathcal{E}$  then  $\mathcal{E}$  has a directed transversal disjoint from  $U$ .*

Heckmann and Keimel study in [25], for any space  $X$  and diverse subsets  $\mathcal{K}$  of  $\mathcal{P}X$ , the *upper Vietoris space*  $V_{\mathcal{K}}X$  with ground set  $\mathcal{K}$  and basic open sets

$$\mathcal{K}U = \{K \in \mathcal{K} : K \subseteq U\} \quad (U \in \mathcal{T} = \mathcal{O}X),$$

and a topological variant of **RL**, called

**TRL** *the Topological Rudin Lemma: if  $\mathcal{I}$  is an irreducible set in the upper Vietoris space  $V_{\mathcal{Q}}X$  then every cutset of  $\mathcal{I}$  that is closed in  $X$  contains an irreducible closed cutset of  $\mathcal{I}$ .*

We shall compare **RL** and **SRL** with **TRL**, but also with

**SLL** *the Sober Lift Lemma: the upper Vietoris space  $V_{\mathcal{Q}}X$  of any sober space  $X$  is sober,*

and with

**SLI** *the Separation Lemma for Irreducible sets: if  $\mathcal{V}$  is a Scott-open filter of open sets in a space then for each open set  $U$  outside  $\mathcal{V}$  there is an irreducible closed cutset of  $\mathcal{V}$  disjoint from  $U$ .*

**Proposition 2.1.** (1)  $\mathcal{I}$  is irreducible in  $V_{\mathcal{K}}X$  if and only if  $\mathcal{IT}$  is filtered ( $\mathcal{I} \subseteq \mathcal{K}$ ).

(2) A  $T_0$ -space  $X$  is strongly well-filtered if and only if the space  $V_{\mathcal{Q}}X$  is sober.

*Proof.* For (1) use the equivalence

$$\bigcap \mathcal{Y} \in \mathcal{IT} \Leftrightarrow \mathcal{I} \cap \bigcap \{\mathcal{K}U : U \in \mathcal{Y}\} \neq \emptyset \quad (\mathcal{Y} \subseteq \mathcal{T}),$$

and then for (2) the equivalence

$$\mathcal{IT} = \mathcal{CT} \Leftrightarrow \forall U \in \mathcal{T} (\mathcal{I} \cap \mathcal{Q}U \neq \emptyset \Leftrightarrow C \in \mathcal{Q}U) \Leftrightarrow cl\mathcal{I} = cl\{C\}.$$

In [25], (2) was deduced from Zorn's Lemma. □

Now, invoking Theorem 1.1 and Proposition 1.4, and making  $V_{\mathcal{Q}}$  functorial by setting  $V_{\mathcal{Q}}f : V_{\mathcal{Q}}X \rightarrow V_{\mathcal{Q}}Y, C \mapsto \uparrow f[C]$  for continuous  $f: X \rightarrow Y$ , we obtain

**Corollary 2.2.** UP is equivalent to SLL, hence to the statement that  $V_{\mathcal{Q}}$  gives rise to an endofunctor of the category of sober spaces.

Recall that a *monotone convergence space* is a  $T_0$ -space in which every monotone net has a supremum (relative to the specialization order) to which it converges [23]. It is easy to see that these spaces are just the *d-spaces* or *temperate spaces* in the sense of Wyler [43], that is,  $T_0$ -spaces in which the closure of any directed subset is the closure of a point, which is then the supremum of the directed set; equivalently, the specialization poset of a d-space is up-complete and its Scott topology contains the original topology (see [17, 23]).

**Lemma 2.3.** The d-spaces are exactly the  $\mathcal{C}$ -well-filtered spaces for the collection  $\mathcal{C}$  of cores. Hence, the following implications hold for arbitrary spaces:

$$\begin{array}{ccccccc} \text{strictly sober} & \Rightarrow & \text{strongly well-filtered} & \Rightarrow & \text{well-filtered} & \Rightarrow & \mathcal{H}\text{-well-filtered} \\ & & \downarrow & & & & \downarrow \\ \text{sober} & \Rightarrow & \text{d-space} & \Leftrightarrow & \text{monotone convergence space} & \Leftrightarrow & \mathcal{C}\text{-well-filtered.} \end{array}$$

*Proof.* If  $X$  is a d-space and  $\mathcal{F}$  is a filter base of cores then there is a directed set  $D$  with  $\mathcal{F} = \{\uparrow d : d \in D\}$ . Hence,  $x = \bigvee D$  exists, and for each open  $U$ , we have  $\uparrow x = \bigcap \mathcal{F} \in U$  if and only if  $\uparrow d \subseteq U$  for some  $d \in D$ . Thus,  $X$  is  $\mathcal{C}$ -well-filtered.

Conversely, if  $X$  is  $\mathcal{C}$ -well-filtered and  $D$  is a directed subset of  $X$ , then the set  $D^{\uparrow\downarrow} = \bigcap \{\downarrow y : D \subseteq \downarrow y\}$  is closed, and  $V = X \setminus D^{\uparrow\downarrow}$  is open. Assuming the inclusion  $D^\uparrow = \bigcap \{\uparrow d : d \in D\} \subseteq V$  leads to  $\uparrow d \subseteq V$  for a  $d \in D$  (as  $X$  is  $\mathcal{C}$ -well-filtered), hence to  $D \cap V \neq \emptyset$ , in contrast to  $V = X \setminus D^{\uparrow\downarrow}$ . Thus,  $D^\uparrow \cap D^{\uparrow\downarrow}$  is non-empty; the only element in that intersection is the supremum  $x$  of  $D$ , and any neighborhood of  $x$  contains a set  $\uparrow d \cap D$  with  $d \in D$ ; thus,  $D$  converges to  $x$ .  $\square$

Johnstone's famous example of an up-complete poset with non-sober Scott space [31] is not well-filtered either (there is a filter base of "rectangular" compact saturated sets with empty intersection). However, this Scott space is  $\mathcal{H}$ -well-filtered, being a d-space, as will follow from Theorem 2.4 (**UP** may be circumvented by the specific structure of that countable example).

Though in this paper we are mainly interested in A-spaces, let us have a look at more general topological situations. A subset of a space is called *supercompact* if its saturation is a core (a principal filter), and *hypercompact* if its saturation is finitely generated, in other words, a foot relative to the specialization order.

Clearly, every supercompact set is hypercompact, and every hypercompact set is compact, but not conversely. In a *locally supercompact* space, every point has a neighborhood base of supercompact sets, hence also a neighborhood base of cores; therefore, such spaces are also referred to as *core spaces* or *C-spaces*. Similarly, in a *locally hypercompact space* or *H-space* [19], *alias locally finite-bottomed space* (Isbell [29]), every point has a neighborhood base of hypercompact sets, hence also a neighborhood base of feet.

Denoting by  $\mathcal{H}$  the collection of all hypercompact saturated subsets, we see that the H-spaces are just the locally  $\mathcal{H}$ -compact spaces. All A-spaces are B-spaces (spaces with a minimal base), all B-spaces are C-spaces, and all C-spaces are H-spaces; for more details, see [14, 17, 19]. Each of these four classes satisfies the hypotheses in Proposition 1.4 (1), but only the last three contain all powers of  $\mathbb{V}$ , which is required for (2) in the same proposition.



Heckmann and Keimel [25] have derived a few of the statements in the next theorem (namely **TRL**, **SWT**, **SLL**) from Zorn's Lemma, which is equivalent to **AC**.

**Theorem 2.4.** *Each of the following principles is equivalent to **UP**:*

**SLI** *the Separation Lemma for Irreducible sets,*

**ITT** *the Irreducible Transversal Theorem for arbitrary spaces,*

**ITA** *the Irreducible Transversal Theorem for A-spaces,*

**RL** *the Rudin Lemma,*

**SRL** *the Strong Rudin Lemma,*

**TRL** *the Topological Rudin Lemma,*

**HW** *the  $\mathcal{H}$ -Well-filter Theorem:  $d$ -spaces are  $\mathcal{H}$ -well-filtered,*

**FT** *the Foot Theorem: filtered meets of feet in  $d$ -spaces are compact,*

**SH** *the Strict Sobriety Theorem for  $H$ -spaces: locally hypercompact  $d$ -spaces are strictly sober,*

**SWT** *the Strong Well-filter Theorem: sober spaces are strongly well-filtered,*

**SLL** *the Sober Lift Lemma.*

*Proof.* First, we shall establish the following implication circuit:

**UP**  $\Rightarrow$  **SLI**  $\Rightarrow$  **TRL**  $\Rightarrow$  **ITT**  $\Rightarrow$  **ITA**  $\Rightarrow$  **RL**  $\Rightarrow$  **SRL**  $\Rightarrow$  **HW**  $\Rightarrow$  **SH**  $\Rightarrow$  **UP**.

By Proposition 1.4, **UP** implies **SLS**, the Separation Lemma for sober topologies, and **SLI** is just a reformulation of **SLS** (passing to complements).

**SLI**  $\Rightarrow$  **TRL**: If  $\mathcal{I}$  is an irreducible set in  $V_{\mathcal{Q}}X$  then  $\mathcal{V} = \mathcal{IT}$  is a Scott-open filter in  $\mathcal{T} = \mathcal{O}X$  (Proposition 2.1). For a closed cutset  $C$  of  $\mathcal{I}$ , the set  $U = X \setminus C$  lies in  $\mathcal{T} \setminus \mathcal{V}$ , so that, by **SLI**, there is an irreducible closed cutset  $A$  of  $\mathcal{V}$  with  $A \cap U = \emptyset$ . Assuming  $A \cap B = \emptyset$  for a  $B \in \mathcal{I}$ , we obtain  $B \subseteq V = X \setminus A \in \mathcal{V}$  but  $A \cap V = \emptyset$ , impossible for a cutset  $A$  of  $\mathcal{V}$ . Thus,  $A$  is a cutset of  $\mathcal{I}$  containing  $C$ .

**TRL**  $\Rightarrow$  **ITT**  $\Rightarrow$  **ITA**: Let  $\mathcal{E}$  be a quasifiltered collection of non-void compact sets in a space  $X$ . We may assume  $X = \bigcup \mathcal{E}$ . Then  $\mathcal{I} = \{\uparrow F : F \in \mathcal{E}\}$  is a filter base of compact saturated subsets of  $X$ , hence  $\mathcal{I}$  is irreducible

in  $V_{\mathcal{Q}}X$ . By **TRL**,  $\mathcal{I}$  has an irreducible closed cutset, which is then an irreducible transversal of  $\mathcal{E}$ .

**ITA**  $\Rightarrow$  **RL**  $\Rightarrow$  **SRL**: See the introductory comments in this section.

**SRL**  $\Rightarrow$  **HW**: If  $\mathcal{F}$  is a filter base of feet in a d-space  $X$ , and  $U$  is an open (hence upper) set with  $F \not\subseteq U$  for all  $F \in \mathcal{F}$ , then, by **SRL**, there is a directed transversal  $D$  of  $\mathcal{F}$  with  $D \cap U = \emptyset$ . As  $X$  is a d-space,  $x = \bigvee D$  exists and lies in  $\bigcap \mathcal{F}$  (because each  $F \in \mathcal{F}$  meets  $D$ , whence  $x \in \uparrow F = F$ ). Since  $U$  is disjoint from  $D$ , the point  $x$  cannot belong to the open (and so, as we are in a d-space, Scott-open) set  $U$ . Thus,  $\bigcap \mathcal{F} \not\subseteq U$ . By contraposition,  $X$  is  $\mathcal{H}$ -well-filtered.

**HW**  $\Rightarrow$  **SH**: By Theorem 1.1,  $\mathcal{H}$ -well-filtered H-spaces are strictly sober.

**SH**  $\Rightarrow$  **UP**: The class of all locally hypercompact d-spaces contains all powers of the space  $\mathbb{V}$ , since these are locally supercompact sober spaces. By Proposition 1.4 (2), it follows that **SH** is equivalent to **UP**.

**HW**  $\Rightarrow$  **FT** is straightforward.

**FT**  $\Rightarrow$  **UP**: For the two-element space  $\mathbb{D}$  of maximal elements in  $\mathbb{V}$ , the sets  $\mathbb{D}^E \times \mathbb{V}^{I \setminus E}$  ( $E$  finite) form a filter base of feet with meet  $\mathbb{D}^I$ . Now, apply Proposition 1.4.

**SWT**  $\Leftrightarrow$  **SLL**  $\Leftrightarrow$  **UP**: Theorem 1.1, Proposition 1.4, Corollary 2.2.  $\square$

H-spaces share many useful properties with the more special C-spaces, but most of these properties rely on **RL**, which, for example, implies that the d-H-spaces are exactly the Scott spaces of quasicontinuous domains, as defined in [23].

### 3 Sober A-spaces and noetherian posets

Henceforth, we shall focus on

**DC** *the Principle of (Countable) Dependent Choices: if  $R$  is a relation on  $X$  such that  $xR = \{y : xRy\} \neq \emptyset$  for all  $x \in X$  then each  $x \in X$  is the first member of a sequence  $(x_n)$  with  $x_n R x_{n+1}$ .*

As shown in [14] (see also [17, 19]), the sober A-spaces are exactly the Scott spaces of noetherian posets, which we now are going to study more thoroughly. We shall apply the previously introduced principles to A-spaces and investigate their relationships to maximality conditions on a poset  $P$ .

An  $\omega$ -chain in  $P$  is the image of the chain  $\omega$  of natural numbers under an isotone (alias monotone) map into  $P$ , and a *properly ascending chain* in  $P$  is an isomorphic copy of  $\omega$  in  $P$ . We abbreviate “Ascending Chain Condition” by *ACC*. Consider the following properties:

- (1)  $P$  is *co-well-founded*; any non-empty subset has a maximal element.
- (2)  $P$  is *noetherian*: every directed subset has a maximal element.
- (3)  $P$  satisfies the *ACC*: every  $\omega$ -chain in  $P$  has a maximal element.

A maximal element of a chain or directed set is already the greatest element. Clearly, (1) implies (2), which in turn implies (3), and **DC** makes all three properties equivalent. As the nomenclature indicates, the *ACC* excludes properly ascending chains. Notice, however, that Birkhoff [7] and others defined the *ACC* by (1), although the equivalence of (1) and (3) fails in **ZF** without choice.

As explained in [19], the sober  $\mathbf{A}$ -spaces correspond not only (via the specialization functor) to the noetherian posets, but also (via the open set locale functor) to the *superspatial locales*, in which the primes are  $\bigwedge$ -dense and completely prime ( $\bigwedge$ -prime). We now supplement these facts by analogous correspondences for co-well-founded posets and for posets satisfying the *ACC*.

Recall that a space is *scattered* if each non-empty subspace has an isolated point. A poset is *strongly atomic* if each non-singleton interval  $[x, y] = \uparrow x \cap \downarrow y$  has an atom (an element covering  $x$ ). A *superalgebraic lattice* is a complete lattice whose elements are joins of supercompact ( $\bigvee$ -prime) elements. As in [14], a *superatomic lattice* is a superalgebraic and strongly atomic lattice (the notion of *superatomic boolean algebra* is related but different). By an  $\omega$ -sober space we mean a  $T_0$ -space in which the closure of any  $\omega$ -chain is a point closure; equivalently, every monotone sequence has a supremum to which it converges. A complete lattice is  $\omega$ -spatial if it has a  $\bigwedge$ -dense subset of primes satisfying the *DCC* (the dual of the *ACC*). Some of the implications in the next proposition are known, but we are interested in a systematic comparison within a choice-free setting.

**Proposition 3.1.** *For a  $T_0$ - $\mathbf{A}$ -space  $X$  with specialization poset  $P$  and open set locale  $\mathcal{T}$ , the following equivalences and implications hold:*

$$\begin{array}{lll}
(1) P \text{ is co-well-founded} & \Leftrightarrow (1a) X \text{ is scattered} & \Leftrightarrow (1b) \mathcal{T} \text{ is superatomic} \\
\Downarrow & \Downarrow & \Downarrow \\
(2) P \text{ is noetherian} & \Leftrightarrow (2a) X \text{ is sober} & \Leftrightarrow (2b) \mathcal{T} \text{ is superspatial} \\
\Downarrow & \Downarrow & \Downarrow \\
(3) P \text{ satisfies the ACC} & \Leftrightarrow (3a) X \text{ is } \omega\text{-sober} & \Leftrightarrow (3b) \mathcal{T} \text{ is } \omega\text{-spatial}.
\end{array}$$

**DC** is equivalent to (3)  $\Rightarrow$  (1) and makes therefore all nine conditions equivalent.

*Proof.* (1)  $\Rightarrow$  (1a): Let  $Y$  be a non-empty subset of  $X$  and  $x$  a maximal element of  $Y$ . Then  $\{x\} = Y \cap \uparrow x$  is open in  $Y$  (since  $\uparrow x$  is open in  $X$ ).

(1a)  $\Rightarrow$  (1b): Observe that  $\subset$  denotes *proper* inclusion. For open sets  $U \subset V$  pick an isolated  $x$  in  $V \setminus U$  and an open set  $W \supseteq U$  with  $\{x\} = W \cap V \setminus U$ . Then  $W \cap V = U \cup \{x\}$  is an atom of the interval  $[U, V]$ .

(1b)  $\Rightarrow$  (1): For  $\emptyset \neq Y \subseteq X$ , the lower set  $A = \downarrow Y$  is a non-empty closed set. Dualizing the strong atomicity of  $\mathcal{T}$ , one obtains a maximal  $C$  among the closed proper subsets of  $A$ . Any element  $x \in A \setminus C$  must then be maximal in  $A$  and so in  $Y$  (assuming  $x < y \in A$  would lead to  $C \subset C \cup \downarrow x \subset C \cup \downarrow y \subseteq A$ ).

(2)  $\Leftrightarrow$  (2a): In A-spaces the directed lower sets are the irreducible closed sets, and the lower sets with greatest elements are the point closures.

(2a)  $\Rightarrow$  (2b): In a sober A-space  $X$ , each prime open set is complementary to a point closure, hence  $\wedge$ -prime (as arbitrary intersections of open sets are open).

(2b)  $\Rightarrow$  (2): If  $D$  is a directed subset of  $P$  then  $P \setminus \downarrow D$  is prime in the open set locale, hence  $\wedge$ -prime by (2b) and so complementary to a principal ideal.

(3)  $\Leftrightarrow$  (3a): Let  $x_0 \leq x_1 \leq x_2 \dots$  be an ascending sequence in  $P$ . Then the closure of  $\{x_n : n \in \omega\}$  is a point closure  $\downarrow y$  if and only if  $x_n$  coincides with  $y$  eventually, because  $\uparrow y$  is open in the A-space  $X$ .

(3)  $\Rightarrow$  (3b): The set  $\{X \setminus \downarrow x : x \in X\}$  is  $\wedge$ -dense in  $\mathcal{T}$ , consists of primes and satisfies the *DCC* (being dually isomorphic to  $P$ ).

(3b)  $\Rightarrow$  (3): If  $\mathcal{P}$  is an arbitrary  $\wedge$ -dense set of primes in  $\mathcal{T}$  then the set  $\{X \setminus \downarrow x : x \in X\}$  must be contained in  $\mathcal{P}$ , because it consists of  $\wedge$ -primes in the lattice  $\mathcal{T}$ . And if  $\mathcal{P}$  satisfies the *DCC* then so does the poset  $\{X \setminus \downarrow x : x \in X\}$ , whence the dually isomorphic poset  $P$  satisfies the *ACC*. The equivalence of (3)  $\Rightarrow$  (1) to **DC** is well-known (see below and [28]).  $\square$

**Theorem 3.2.** *DC is equivalent to either of the global implications*

**NW** *every noetherian poset is co-well-founded,*

**AW** *every poset satisfying the ACC is co-well-founded.*

*A consequence of these statements is*

**AN** *every poset satisfying the ACC is noetherian,*

*which is strictly stronger than*

**DS** *the Denumerable Subset Axiom: infinite sets have denumerable subsets  
(or, Dedekind finite sets are finite)*

*which in turn is strictly stronger than*

**C**<sub><ω</sub><sup>ω</sup> *the Axiom of Choice for countable families of finite sets.*

*Proof.* We know that **DC** implies **AW**, which clearly implies **NW**.

**NW** ⇒ **DC**: Let  $R$  be a relation on a set  $X$  such that for each  $x \in X$  there is a  $y \in X$  with  $xRy$ . Then the poset  $T$  of all finite sequences  $(x_0, \dots, x_n)$  in  $X$  with  $x_k R x_{k+1}$  for all  $k < n$ , ordered by  $(x_0, \dots, x_n) \leq (y_0, \dots, y_m)$  if  $n \leq m$  and  $x_k = y_k$  for all  $k \leq n$ , has no maximal element. By **NW**,  $T$  contains a directed set with no maximal element. But the only directed sets in  $T$  are the  $\omega$ -chains, and a properly ascending chain  $(x_0, \dots, x_{n_0}) < (x_0, \dots, x_{n_1}) < \dots$  in  $T$  gives a sequence  $(x_n : n \in \omega)$  with  $x_n R x_{n+1}$ .

**AN** ⇒ **DS**: First, we show that **AN** implies **C**<sub><ω</sub><sup>ω</sup>.

Let  $(F_n : n \in \omega)$  be a sequence of non-empty finite sets, which may be assumed to be pairwise disjoint. On their union  $U = \bigcup \{F_n : n \in \omega\}$  define an order relation  $R$  by  $xRy$  if and only if  $x=y$  or  $x \in F_m, y \in F_n$  for some  $m < n$ . Then  $U$  is directed by  $R$  (indeed, for  $x \in F_k$  and  $y \in F_m$ , any  $z \in F_n$  with  $n > k$  and  $n > m$  satisfies  $xRz$  and  $yRz$ ) but has no maximal element. Hence, by **AN**,  $U$  contains a properly ascending chain, which provides a choice function for a subsequence of  $(F_n)$ . As remarked by Brunner [8], this suffices to deduce **C**<sub><ω</sub><sup>ω</sup>. Now, if  $X$  is any infinite set then the collection of all finite subsets is directed by inclusion but has no greatest member, hence cannot be noetherian. **AN** gives an ascending sequence  $E_0 \subset E_1 \subset \dots$  of finite subsets of  $X$ , and we may select via **C**<sub><ω</sub><sup>ω</sup> a sequence of elements  $x_n \in E_{n+1} \setminus E_n$ , hence a denumerable subset of  $X$ .

Suitable models of set theory demonstrate that **DS** does not even imply the well-orderedness of chains with *DCC* (Höft and Howard [27] prove that and several related implications and independencies but denote by **DS** the statement that chains with *DCC* are well-ordered). All the less can **DS** imply the stronger (dual of) **AN**. On the other hand, it is known that  $\mathbf{C}_{<\omega}^{\omega}$  is strictly weaker than **DS** but not provable in **ZF** (see [28] for these facts).  $\square$

A similar clue as for  $\mathbf{DC} \Leftrightarrow \mathbf{AW} \Leftrightarrow \mathbf{NW}$  leads to Tarski's observation [42] that **DC** is equivalent to the existence of maximal elements in all non-empty posets not containing any infinite chains. However, the precise position of the principle **AN** seems to be unknown: it might be equivalent to or else strictly weaker than **DC**.

Let us conclude this section with an application of Proposition 3.1. The lower sets of a poset  $P$  form the *Alexandroff completion* of  $P$ . A  $\wedge$ -decomposition of an element  $x$  in a poset or lattice is a subset with meet  $x$ ; and  $x$  is  $\wedge$ -irreducible if it is contained in any  $\wedge$ -decomposition of  $x$ .

**Lemma 3.3.** (1) *A lattice  $L$  (equivalently, its dual) is superalgebraic if and only if it is isomorphic to the Alexandroff completion of a poset, which is then isomorphic to the poset of all  $\vee$ -primes and also to that of all  $\wedge$ -primes of  $L$ .*

(2) *A frame is strongly atomic if and only if its elements have least (hence unique irredundant)  $\wedge$ -decompositions into  $\wedge$ -irreducible elements.*

Part (1) is easy and well-known; (2) was mentioned in [11] and proven in [13, 7.1.29].

**Corollary 3.4.** *For a complete lattice  $L$ , the following are equivalent:*

- (a)  *$L$  is isomorphic to the Alexandroff completion of a well-founded poset.*
- (b)  *$L$  is a superatomic lattice.*
- (c)  *$L$  is a superspatial lattice with scattered spectrum.*
- (d)  *$L$  is a superspatial lattice with co-well-founded spectrum.*
- (e) *Each element has a unique irredundant  $\wedge$ -decomposition into  $\wedge$ -primes.*

**DC** is equivalent to the duality between the category of posets with *DCC* and that of superatomic frames via the Alexandroff completion functor.

*Proof.* By Lemma 3.3 (1), there is no loss of generality in assuming that  $L$  is the Alexandroff completion of a poset  $P$ . Via dualization, the equivalence of (a)–(d) and the last claim is now immediate from Proposition 3.1, which also assures that a lattice is superspatial if and only if it is superalgebraic and its spectrum consists of  $\wedge$ -primes.

(b)  $\Leftrightarrow$  (e) follows from Lemma 3.3 (2).  $\square$

For the categorical duality statement in **ZFC** see also [14].

#### 4 Footstep induction and the Noetherian lift lemma

Throughout this section,  $P$  denotes a poset and  $\mathcal{S} = \mathcal{F}^\uparrow P$  the meet-semilattice of all finitely generated upper sets (feet) of  $P$ , ordered by *dual* inclusion. Thus,  $\emptyset$  is the *greatest* element of  $\mathcal{S}$ , and the map  $\eta_P : P \rightarrow \mathcal{S}$ ,  $x \mapsto \uparrow x$ , embeds  $P$  in  $\mathcal{S}$ . In fact,  $\eta_P$  is a *universal embedding of  $P$  in a unital semilattice*, whence  $\mathcal{S}$  may be referred to as “*the*” *free unital semilattice over  $P$* : for each isotone map  $\varphi$  from  $P$  into a unital semilattice  $T$ , there is a unique homomorphism  $\hat{\varphi} : \mathcal{S} \rightarrow T$  such that  $\varphi = \hat{\varphi} \circ \eta_P$  (given by  $\hat{\varphi}(F) = \bigwedge \varphi[F]$ ). Moreover,  $\mathcal{F}^\uparrow$  gives rise to a functorial equivalence between the category of posets and the category of unital meet-semilattices whose primes form a  $\wedge$ -dense subset.

Non-trivial is the fact that, under weak choice principles, the maximality properties (1), (2), (3) discussed in the previous section may be “lifted” from a poset  $P$  to  $\mathcal{S}$ , the free unital semilattice over it. Birkhoff established the dual result in the 1948 edition of his seminal monograph *Lattice Theory* [7], by rather technical arguments involving ordinals and making heavy use of choice. Simpler arguments were given later by Ern e [9] and Fraiss e [22]. The proof in [9] is based on

*FIP the Footstep Induction Property:*

*if a subset  $\mathcal{S}'$  of  $\mathcal{S} = \mathcal{F}^\uparrow P$  satisfies the conditions*

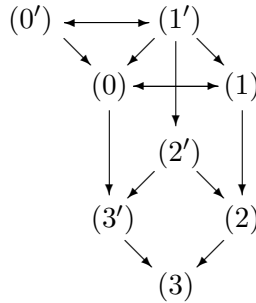
- (i)  $x \in P$  and  $\{Y \in \mathcal{S} : Y \subset \uparrow x\} \subseteq \mathcal{S}'$  imply  $\uparrow x \in \mathcal{S}'$ ,
- (ii)  $Y \in \mathcal{S}$  and  $\{\uparrow x : x \in Y\} \subseteq \mathcal{S}'$  imply  $Y \in \mathcal{S}'$ ,

*then  $\mathcal{S}'$  already coincides with  $\mathcal{S}$ .*

**Proposition 4.1.** *Consider the following conditions on a poset  $P$  and the free unital semilattice  $\mathcal{S} = \mathcal{F}^\uparrow P$  over  $P$ :*

- |                                    |   |
|------------------------------------|---|
| (0) $P$ has the <i>FIP</i> .       | (0') $\mathcal{S}$ has the <i>FIP</i> .       |
| (1) $P$ is co-well-founded.        | (1') $\mathcal{S}$ is co-well-founded.        |
| (2) $P$ is noetherian.             | (2') $\mathcal{S}$ is noetherian.             |
| (3) $P$ satisfies the <i>ACC</i> . | (3') $\mathcal{S}$ satisfies the <i>ACC</i> . |

In **ZF**, all implications displayed in the diagram below are true:



In **ZF+DC** all eight conditions are equivalent.

*Proof.* Note first that for any foot  $Y \in \mathcal{S}$ , the finite set  $\text{Min } Y$  of all minimal elements is the least set  $F$  satisfying  $Y = \uparrow F$ .

(0)  $\Rightarrow$  (1): Apply the *FIP* to the set  $\mathcal{S}'$  of all co-well-founded members of  $\mathcal{S}$ :

- (i) if all feet properly contained in  $\uparrow x$  are co-well-founded then so is  $\uparrow x$ ;
- (ii) if all  $\uparrow x$  with  $x \in Y$  are co-well-founded then so is  $Y$ .

By the *FIP*, all principal filters and the whole poset  $P$  are co-well-founded, too.

(1)  $\Rightarrow$  (0): Assume a proper subset  $\mathcal{S}'$  of  $\mathcal{S}$  satisfies (i) and (ii); then, by (1) and (ii), there is a maximal  $x_0$  with  $\uparrow x_0 \notin \mathcal{S}'$ . But then, by (i) and (ii) again, one would obtain a  $Y \in \mathcal{S} \setminus \mathcal{S}'$  with  $Y \subset \uparrow x_0$  and an  $x \in Y$  with  $\uparrow x \notin \mathcal{S}'$ , whence  $x_0 < x$ , contradicting the maximality of  $x_0$ .

We have proved the equivalence (0)  $\Leftrightarrow$  (1) and, in particular, (0')  $\Leftrightarrow$  (1').

(0)  $\Rightarrow$  (3'): We want to apply the *FIP* to the set  $\mathcal{S}'$  of all elements of  $\mathcal{S}$  that are not the first member of a properly ascending (that is,  $\subseteq$ -descending) sequence in  $\mathcal{S}$ . Condition (i) is obviously fulfilled. For (ii), assume  $Y_0 \in$



$\mathcal{S} \setminus \mathcal{S}'$ . There is a sequence  $Y_0 \supset Y_1 \supset Y_2 \dots$  in  $\mathcal{S}$  with  $\uparrow F_n = Y_n$  for finite  $F_n = \text{Min } Y_n$ . For each  $x \in F_0$ , define recursively a sequence  $Y_0(x) \supseteq Y_1(x) \supseteq Y_2(x) \dots$  in  $\mathcal{S}$  by

$$Y_0(x) = \uparrow x, \quad Y_{n+1}(x) = \uparrow(F_{n+1} \cap Y_n(x)).$$

Then, by induction,

$$(*) \quad Y_n = \bigcup \{Y_n(x) : x \in F_0\} \text{ for all } n.$$

In the step from  $n$  to  $n+1$ , note that for  $y \in Y_{n+1}$  there is a  $z \in F_{n+1}$  with  $z \leq y$ , and  $F_{n+1} \subseteq Y_n = \bigcup \{Y_n(x) : x \in F_0\}$  yields an  $x \in F_0$  with  $z \in F_{n+1} \cap Y_n(x)$ , whence  $y \in \uparrow z \subseteq \uparrow(F_{n+1} \cap Y_n(x)) = Y_{n+1}(x)$ .

Now, if  $\uparrow x \in \mathcal{S}'$  for all  $x \in F_0$  then each of the sets  $\{Y_n(x) : n \in \omega\}$  and, by (\*), the set  $\{Y_n : n \in \omega\}$  would be finite, a contradiction. By contraposition, (ii) is fulfilled, and then (1) yields  $\mathcal{S}' = \mathcal{S}$ ; in other words,  $\mathcal{S}$  satisfies the ACC.

The other implications in the diagram are clear since  $P$  is embedded in  $\mathcal{S}$ . **DC** ensures the implications (3)  $\Rightarrow$  (1) and (3')  $\Rightarrow$  (1'), which make all eight conditions equivalent.  $\square$

While we saw that the implication (1)  $\Rightarrow$  (3') holds in **ZF**, an additional reasoning in Section 5 will show that the implications (2)  $\Rightarrow$  (3') and (3)  $\Rightarrow$  (3') are equivalent to the unprovable Axiom of Choice for countable families of finite sets. Whether (1) implies (2') or even (1') (in **ZF**) remains open. But we are able to prove the equivalence of the implication (2)  $\Rightarrow$  (2') to several of the previously considered topological principles, restricted to A-spaces, using

**Lemma 4.2.** *The class **A** of A-spaces contains all discrete spaces, consists of locally supercompact spaces, and is stable under the formation of subspaces and of ordinal sums. In particular, the ordinal sum of an A-space and an infinite discrete, hence non-compact space is an A-space.*

*Proof.* Since in an A-space all cores are open, each point has a least neighborhood (namely, its core), which is supercompact. That subspaces of A-spaces are again A-spaces is clear; but notice that subspaces of locally supercompact spaces (C-spaces) need not be locally supercompact; in fact, every space is a subspace of a B-space, hence of a C-space [17]. The ordinal sum of two disjoint spaces  $X$  and  $Y$  has as open sets the open sets in  $Y$  and the sets  $U \cup Y$  with  $U$  open in  $X$ . From this it is evident that the ordinal sum of two A-spaces is an A-space.  $\square$

**Theorem 4.3.** *The following principles are equivalent:*

**SLA** *the Separation Lemma for superspatial locales,*

**SSA** *the Strict Sobriety Theorem for A-spaces,*

**WA** *the Well-filter Theorem for A-spaces,*

**CIA** *the Compact Intersection Theorem for A-spaces,*

**NIA** *the Non-void Intersection Theorem for A-spaces,*

**NL** *the Noetherian Lift Lemma: if  $P$  is noetherian then so is  $\mathcal{F}^\uparrow P$ ,*

**NF** *the Noetherian Foot Intersection Lemma: in every noetherian poset, the intersection of any filter base of feet is a foot,*

**NC** *the Noetherian Compactness Lemma: the free unital semilattice over a noetherian poset has a compact top (unit element).*

*These principles are not only strictly weaker than **UP** but also strictly weaker than **DC**, which is independent of **UP**.*

*Proof.* By Lemma 4.2, all required hypotheses are fulfilled in order to apply Proposition 1.4 (1) to the class  $\mathbf{X} = \mathbf{A}$  of A-spaces, which gives the equivalence of the first five principles listed in Theorem 4.3.

Passing from sober A-spaces to noetherian posets, **WA** asserts that for every  $\supseteq$ -directed, that is,  $\subseteq$ -filtered subset  $\mathcal{F}$  of  $\mathcal{S} = \mathcal{F}^\uparrow P$ , any upper set  $U$  with  $\bigcap \mathcal{F} \subseteq U$  contains some member of  $\mathcal{F}$ . But the latter simply means that  $\mathcal{F}$  has a greatest (!) element in the order of  $\mathcal{S}$  (as  $\bigcap \mathcal{F}$  itself is an upper set). Thus, **WA**  $\Leftrightarrow$  **NL**.

Similarly, one proves the equivalences **CIA**  $\Leftrightarrow$  **NF** and **NIA**  $\Leftrightarrow$  **NC**. For this, notice that if a foot  $Y$  is the join of some set  $\mathcal{F} \subseteq \mathcal{S}$  (with respect to  $\supseteq$ ) then  $Y$  must be the intersection  $\bigcap \mathcal{F}$ : by definition,  $Y \subseteq \bigcap \mathcal{F}$ , and conversely, for any  $x \in \bigcap \mathcal{F}$ , the foot  $\uparrow x$  is an upper bound of  $\mathcal{F}$ , whence  $x \in \uparrow x \subseteq Y$ . A top element is compact if and only if it is not a directed join of smaller elements. In  $\mathcal{S}$ , this condition means that  $\emptyset$  is not the intersection of a filter base of (non-empty!) feet.

**UP** and **ITA** are equivalent, by Theorem 2.4, and **ITA** implies **NIA**. On the other hand, **DC** implies **NL**, by Proposition 4.1.

For the independence of **UP** and **DC** in **ZF**, consult [28].  $\square$

In Section 5, we shall see that the principles in Theorem 4.3 imply at least  $\mathbf{C}_{<\omega}^\omega$ , the Axiom of Choice for countable families of finite sets, but it might be that they imply even stronger axioms like **DS**.

Let us mention that weak choice principles make it possible to lift some other important properties from a poset to the free unital semilattice over it; for example, that is the case for certain continuity properties. A poset is *continuous*, respectively *quasicontinuous*, if each principal filter  $\uparrow x$  is the filtered intersection of cores, respectively of feet, that meet every directed set  $D$  with  $D^\uparrow \subseteq \uparrow x$ . Using facts from Section 2, one proves a result that Heckmann and Keimel [25] deduced from Zorn's Lemma.

**Corollary 4.4.** **UP** *implies that a dcpo is quasicontinuous if and only if the free unital semilattice over it is continuous (but not necessarily a dcpo).*

*In  $\mathbf{ZF} + \mathbf{UP}$ , the functor  $\mathcal{F}^\uparrow$  induces an equivalence between the category of quasicontinuous dcpo's and that of continuous unital semilattices whose prime elements form an up-complete  $\wedge$ -dense subset.*

## 5 König's Lemma

The equivalence theorems derived in the previous sections would be of minor interest if the involved statements themselves (not only their equivalence) could be established in **ZF** without choice. But that is not the case, because they all imply the weak choice principle  $\mathbf{C}_{<\omega}^\omega$ , which is not provable in **ZF**.

Call a (binary) relation  $R$  *locally finite* if the sets

$$xR = \{y : x R y\}$$

are finite for all  $x$  in the domain of  $R$ . The powers  $R^n$  are defined as iterated relation products. The *covering relation*  $R$  of a poset with strict order  $<$  is given by  $x R y$  if and only if  $x < y$  holds but no  $z$  satisfies  $x < z < y$ ; in case  $<$  is the transitive closure of  $R$ , we speak of a *concatenated poset*. For example, all finite posets are concatenated. In [9], it is shown that passing from a partial order to its covering relation yields a bijection between concatenated posets and *diagrams*, that is, pairs  $(X, R)$  where  $R$  is a relation on  $X$  with  $R \cap R^n = \emptyset$  for  $n \geq 2$ .

By a *tree* we mean here a poset with a least element in which all (principal) ideals are chains, and by an  $\omega$ -*tree* a poset with least element such

that all ideals are  $\omega$ -chains and the tree, respectively, its covering relation is locally finite, that is, each element has only a finite number of covers. Dénes König was a pioneer in the investigation of such trees (see [34]); today, they play an important role in various fields of logic, set theory, graph theory and computer science. Of particular interest are  $\omega$ -trees of words (cf. Jech [30, p.115]).

**Lemma 5.1.** (1) *The  $\omega$ -trees are the concatenated locally finite trees.*  
 (2) *Let  $S$  be a set and  $S^*$  the set of all finite sequences (words) over  $S$ , ordered by the prefix relation. If  $T$  is a lower set in  $S^*$  such that for each  $t \in T$  the set  $\{s \in S : (t, s) \in T\}$  is finite, then  $T$  is an  $\omega$ -tree.*

*Proof.* (1) That an  $\omega$ -tree is concatenated follows from the condition that each principal ideal must be a finite chain. Conversely, assume  $T$  is a concatenated tree with locally finite covering relation. Let  $D$  be an ideal of  $T$ , hence a chain. If  $D$  has a greatest element, it is a finite chain (by the concatenation property), and we are done. Otherwise, define recursively elements  $d_n \in D$  by

$$d_n = \min\{d \in D : d_k < d \text{ for } k < n\}.$$

(The minimum exists since the principal ideals are finite, whereas  $D$  is infinite.) It remains to verify  $\{d_n : n < \omega\} = D$ . By the recursive construction,  $d_n$  covers  $d_{n-1}$  in  $D$  and, as  $D$  is a lower set, also in  $T$ . Now, for any  $d \in D$ , if  $d_m < d$  for all  $m$  then  $\downarrow d$  would be infinite; hence, we must have  $d \leq d_n$  for some  $n$  and then  $d \in \downarrow d_n = \{d_k : k \leq n\}$ .

(2) is an easy consequence of (1). □

We shall see that the Noetherian Lift Lemma is closely related to a famous graph-theoretical tool, namely

**KL** *König's (Generalized) Infinity Lemma: if  $R$  is a locally finite relation and  $x$  is an element with  $xR^n \neq \emptyset$  for all  $n$ , then there is a sequence  $(x_n)$  with  $x_0 = x$  and  $x_n R x_{n+1}$  for all  $n$ .*

**Theorem 5.2.** *The following principles are mutually equivalent consequences of the Noetherian Lift Lemma **NL**, hence of **UP** and of **DC**:*

**C** $_{<\omega}^\omega$  *the Axiom of Choice for countable families of finite sets,*

- $\mathbf{U}_{<\omega}^\omega$  the principle that countable unions of finite sets are countable.  
 $\mathbf{T}_{<\omega}^\omega$  Tychonoff's Theorem for countable families of finite spaces,  
 $\mathbf{D}_{<\omega}^\omega$  the Principle of Dependent Choices for locally finite relations,  
 $\mathbf{KL}$  König's Infinity Lemma for locally finite relations,  
 $\mathbf{KT}$  König's Infinity Lemma for covering relations of  $\omega$ -trees,  
 $\mathbf{KT}'$  König's Infinity Lemma for covering relations of word trees,  
 $\mathbf{CL}$  the Chain Lift Lemma: if a poset  $P$  fulfils the ACC then so does  $\mathcal{F}^\uparrow P$ ,  
 $\mathbf{WL}$  the Weak Lift Lemma: if  $P$  is noetherian then  $\mathcal{F}^\uparrow P$  fulfils the ACC.

*Proof.* Clearly,  $\mathbf{NL}$  implies  $\mathbf{WL}$ . The equivalence of the statements  $\mathbf{C}_{<\omega}^\omega$ ,  $\mathbf{U}_{<\omega}^\omega$ ,  $\mathbf{D}_{<\omega}^\omega$ ,  $\mathbf{KL}$ ,  $\mathbf{KT}$  and  $\mathbf{KT}'$  can be found, more or less explicitly, in the literature on weak choice axioms (see, for example, Felscher [21], Herrlich [26], Howard and Rubin [28]). For  $\mathbf{C}_{<\omega}^\omega \Leftrightarrow \mathbf{T}_{<\omega}^\omega$ , see Krom [35] or Herrlich [26]. Let us verify the more difficult implications in the following "bicyclic" diagram:

$$\begin{array}{ccccccc}
 \mathbf{WL} & \Leftarrow & \mathbf{CL} & \Leftarrow & \mathbf{U}_{<\omega}^\omega & \Rightarrow & \mathbf{KL} \\
 & & \Downarrow & & \Uparrow & & \Downarrow \\
 \mathbf{KT} & \Rightarrow & \mathbf{KT}' & \Rightarrow & \mathbf{C}_{<\omega}^\omega & \Leftarrow & \mathbf{D}_{<\omega}^\omega
 \end{array}$$

$\mathbf{U}_{<\omega}^\omega \Rightarrow \mathbf{CL}$ : Let  $Y_0 = \uparrow F_0 \supseteq Y_1 = \uparrow F_1 \supseteq \dots$  be a properly ascending chain in  $\mathcal{S} = \mathcal{F}^\uparrow P$  (ordered by  $\supseteq$ ), with finite sets  $F_i = \text{Min } Y_i$  of minimal elements. Enumerate the union  $U = \bigcup \{F_n : n \in \omega\}$ . We construct recursively a properly ascending sequence in  $P$ , using the same notation and argument as in the proof of (0)  $\Rightarrow$  (3') in Proposition 4.1. Take any  $x_0$  in  $F_0$  such that there are infinitely many  $Y_n^0 = Y_n(x_0)$ . Assume for some  $m \in \omega \setminus \{0\}$  and all  $i < m$  we have found elements  $x_i$  in  $P$  and sequences  $(Y_n^i) = (\uparrow F_n^i)$  in  $\mathcal{S}$  with  $x_0 < \dots < x_{m-1}$  so that  $\{Y_n^i : n \in \omega\}$  is an infinite chain in  $\mathcal{S}$  with  $\uparrow x_i = Y_0^i$ ,  $Y_n^i \supseteq Y_{n+1}^i$  and  $F_n^i = \text{Min } Y_n^i \subseteq F_n$ . This has been achieved for  $m=1$ . Suppose  $m > 1$ . Then there is a first  $n_m$  such that  $Y_{n_m}^{m-1}$  is properly contained in  $Y_0^{m-1} = \uparrow x_{m-1}$ . Put  $Z_n^m = Y_{n+n_m}^{m-1}$ . Pick the first element  $x = x_m$  of the finite set  $\text{Min } Z_0^m = F_{n_m}^{m-1}$  in the enumeration of  $U$  such that the set  $\{Z_n^m(x) : n \in \omega\}$  is infinite. Put  $Y_n^m = Z_n^m(x_m)$ . Then  $x_{m-1} < x_m$  (as  $x_m \in \uparrow x_m = Y_0^m \subseteq Y_{n_m}^{m-1} \subset Y_0^{m-1} = \uparrow x_{m-1}$ ), and

$\{Y_n^m : n \in \omega\}$  is an infinite chain in  $\mathcal{S}$  with  $Y_n^m \supseteq Y_{n+1}^m$ . This gives a properly ascending sequence  $(x_n)$  in  $P$ .

**WL**  $\Rightarrow$  **KT**: Let  $T$  be an  $\omega$ -tree and consider an element  $x \in T$  with  $F_n = xR^n \neq \emptyset$  for all  $n$ , where  $R$  denotes the covering relation (whence all  $yR$  are finite). Passing to the subtree  $\uparrow x$ , we may assume that  $x$  is the least element of  $T$ . By induction, one verifies that the non-empty sets  $F_n$  are finite and disjoint; thus,  $(\uparrow F_n : n \in \omega)$  is a properly ascending sequence in  $\mathcal{F}^\uparrow T$  (indeed, we have the inclusion  $F_{n+1} = \bigcup \{yR : y \in xR^n\} \subseteq \uparrow F_n$ , and assuming  $z \in F_n \subseteq \uparrow F_{n+1}$ , we would find a  $y \in F_{n+1}$  with  $y \leq z$ ; but, since  $T$  is concatenated,  $y \leq z$  would force  $z$  to lie in some  $F_m$  with  $m > n$ ). Now, by contraposition, **WL** yields an ideal  $D$  of  $T$ , hence an  $\omega$ -chain, with no greatest element; then  $D$  may be refined to a properly ascending sequence  $x_0 R x_1 R x_2 \dots$  with  $x_0 = x$  (no choice required, since in an  $\omega$ -tree, the inserted elements to obtain non-refinable chains are unique).

**U** $_{<\omega}^\omega$   $\Rightarrow$  **KL**: Given a locally finite relation  $R$ , consider the set  $Y$  of all  $x$  such that  $xR^n \neq \emptyset$  for all  $n$ . By induction, the sets  $xR^n$  are finite, so their union  $U_x = \bigcup \{xR^n : n \in \omega\}$  is countable by **U** $_{<\omega}^\omega$ . Fix an  $x \in Y$  and an enumeration of  $U_x$ , and put  $Y_x = Y \cap U_x$ . For each  $y \in Y_x$ , the set  $yR$  is contained in  $U_x$  (since  $y \in xR^n$  implies  $yR \subseteq xR^{n+1}$ ), and so  $Y \cap yR$  is contained in  $Y_x$ . Further, the finiteness of  $yR$  assures that the set  $Y \cap yR$  is non-empty: assuming, on the contrary, that for each  $z \in yR$  there is a (least)  $n_z$  with  $zR^{n_z} = \emptyset$ , one would obtain for  $n = \max\{n_z : yRz\}$  that  $zR^n$  is empty for all  $z \in yR$ , hence  $yR^{n+1} = \emptyset$ , in contrast to  $y \in Y$ . Now, one may define a map  $\varphi : Y_x \rightarrow Y_x$  by taking for  $\varphi(y)$  the first element of  $Y \cap yR$  with respect to the enumeration of  $U_x$ . Then  $x_n = \varphi^n(x)$  defines a sequence with the desired property  $x_n R x_{n+1}$  (cf. [21]).  $\square$

We do not know whether **NL** is equivalent to **CL**, hence to **C** $_{<\omega}^\omega$ . But:

**Proposition 5.3.** ***AN** is strictly stronger than **NL**. Hence, if all posets with ACC are noetherian then so are the free unital semilattices over them.*

*Proof.* By Theorem 3.2, **AN** implies **DS**, hence **C** $_{<\omega}^\omega$ , which is equivalent to **CL**, by Theorem 5.2; and clearly **CL**+**AN** entails **NL**. If **NL** would imply **AN** then, by Theorem 4.3, **UP**  $\Rightarrow$  **NL**  $\Rightarrow$  **AN**  $\Rightarrow$  **DS**, whereas it is known that **UP** does not imply **DS** (see [28]).  $\square$

A unital meet-semilattice  $S$  is *distributive* if for  $x, y, z \in S$  with  $x \wedge y \leq z$  there are  $x_0 \geq x$  and  $y_0 \geq y$  with  $x_0 \wedge y_0 = z$ . This together with co-well-foundedness of  $S$  (or the ACC plus DC) makes the set  $P$  of primes  $\wedge$ -dense in  $S$ , hence  $S$  is isomorphic to  $\mathcal{F}^\uparrow P$ .

**Corollary 5.4.** *In  $\mathbf{ZF} + \mathbf{DC}$ , the category of posets with ACC is equivalent to the category of distributive unital meet-semilattices with ACC.*

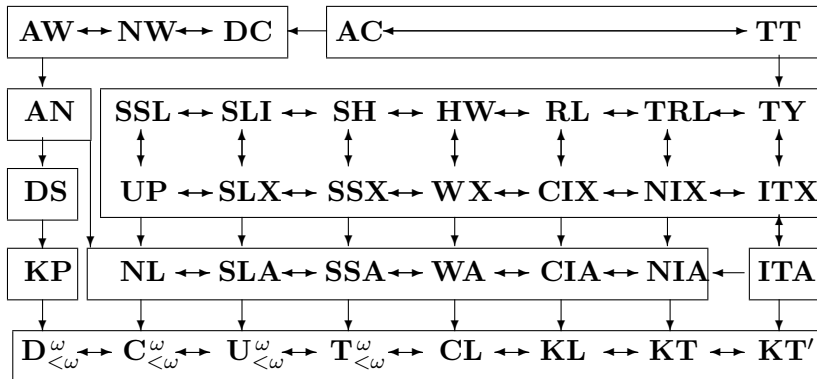
Another related result is also attributed to König (cf. [22, 5–2.5]):

**KP** *König’s Lemma for Posets:*  
*infinite posets satisfying the ACC and the DCC contain infinite antichains.*

Since this is provable for denumerable posets in  $\mathbf{ZF}$  (see [22]), **KP** follows from **DS**. In [15] it is shown that **DS** is equivalent to the conjunction of  $\mathbf{C}_{<\omega}^\omega$  with

**DF** *Distributive lattices satisfying the ACC and the DCC are finite.*

**Diagram of Choice Principles**



**TT** is Tychonoff’s Theorem: products of compact spaces are compact  
**TY** is Tychonoff’s Theorem for a class of sober spaces containing  $(2, \mathcal{P}2)$   
**X** denotes any class of spaces that is stable under forming closed subspaces and contains all powers of a space  $\mathbb{V}$  with specialization poset  $\heartsuit$ .

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