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Span and cospan representations of weak double categories

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Dedicated to Bernhard Banaschewski on the occasion of his 90th birthday

Abstract. We prove that many important weak double categories can be 'represented' by spans, using the basic higher limit of the theory: the tabulator. Dually, representations by cospans via cotabulators are also frequent.

1 Introduction

Strict double categories were introduced and studied by C. Ehresmann [2, 3], the weak notion in our series [GP1 - GP4]. The strict case extends the more usual (if historically subsequent) notion of 2-category, while the weak one extends bicategories, priorly established by Bénabou [1]. The extension is made clear in Section 4.

This note is about weak double categories and the (horizontal) *tabulator* of a vertical arrow. The latter is the 'basic' higher limit of the theory; in fact the main result of [4] says that a weak double category has all (horizontal)

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double limits if and only if it has: double products, double equalisers and tabulators.

We prove here that the existence of tabulators in a weak double category $\mathbb{A}$ produces, under suitable hypotheses, a lax functor $S\colon \mathbb{A} \to \mathbb{S}\mathrm{pan}(\mathbf{C})$ with values in the weak double category *of spans* over the category $\mathbf{C}$ of horizontal arrows of $\mathbb{A}$ (Theorem 7). We say that $\mathbb{A}$ is *span representable* when this functor S is horizontally faithful.

Many important weak double categories can be represented in this sense, by spans or - dually - by cospans, via cotabulators.

Outline. We begin by a brief review of basic notions on weak double categories, from [4, 5], including the weak double categories of spans and cospans, and the (co)tabulator of a vertical arrow.

Sections 7 and 8 give the main definitions and results recalled above, about (co)span representability. Various weak (or strict) double categories are examined in Sections 9 - 13, proving that many of them are both span and cospan representable. Yet the weak double category $\mathbb{S}\mathrm{pan}\mathbf{Set}$, which is trivially span representable, is *not* cospan representable (Section 9), and $\mathbb{C}\mathrm{osp}\mathbf{Set}$ behaves in a dual way.

Finally, some common patterns in the previous proofs of representability are analysed in Section 14.

2 Definition

A (strict) *double category* $\mathbb{A}$ consists of the following structure.

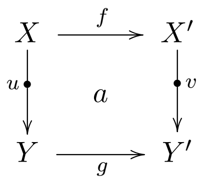
(a) A set $\mathrm{Ob}\mathbb{A}$ of *objects* of $\mathbb{A}$.

(b) *Horizontal morphisms* $f\colon X \to X'$ between the previous objects; they form the category $\mathrm{Hor}_0\mathbb{A}$ *of the objects and horizontal maps* of $\mathbb{A}$, with composition written as gf and identities $1_X\colon X \to X$.

(c) *Vertical morphisms* $u\colon X \to\!\!\!\bullet\, Y$ (often denoted by a dot-marked arrow) between the same objects; they form the category $\mathrm{Ver}_0\mathbb{A}$ *of the objects and vertical maps* of $\mathbb{A}$, with composition written as $v{\bullet}u$ (or $u \otimes v$, in diagrammatic order) and identities written as $e_X\colon X \to\!\!\!\bullet\, X$ or $1^\bullet_X$.

(d) *Double cells* $a\colon (u\ {}^{f}_{g}\ v)$ with a *boundary* formed of two vertical arrows

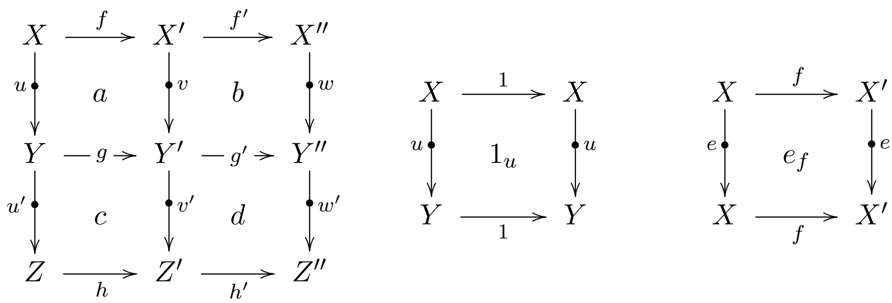
u, v and two horizontal arrows f, g



(2.1.1)

Writing $a\colon (X\ {}^{X}_{g}\ v)$ or $a\colon (e\ {}^{1}_{g}\ v)$ we mean that $f = 1_X$ and $u = e_X$. The cell a is also written as $a\colon u \to v$ (with respect to its *horizontal* domain and codomain, which are *vertical* arrows) or as $a\colon f \twoheadrightarrow g$ (with respect to its *vertical* domain and codomain, which are *horizontal* arrows).

We refer now to the following diagrams of cells, where the first is called a *consistent matrix* $\binom{a\ b}{c\ d}$ of cells



(2.1.2)

(e) Cells have a *horizontal composition*, consistent with the horizontal composition of arrows and written as $(a\,|\,b)\colon (u\ {}^{f'f}_{g'g}\ w)$, or $a|b$; this composition gives the category $\mathrm{Hor}_1\mathbb{A}$ *of vertical arrows and cells* $a\colon u \to v$ of $\mathbb{A}$, with identities $1_u\colon (u\ {}^{1}_{1}\ u)$.

(f) Cells have also a *vertical composition*, consistent with the vertical composition of arrows and written as $\left(\frac{a}{c}\right)\colon (u'{\bullet}u\ {}^{f}_{h}\ v'{\bullet}v)$, or $\frac{a}{c}$, or $a \otimes c$; this composition gives the category $\mathrm{Ver}_1\mathbb{A}$ *of horizontal arrows and cells* $a\colon f \twoheadrightarrow g$ of $\mathbb{A}$, with identities $e_f = 1^{\bullet}_f\colon (e\ {}^{f}_{f}\ e)$.

(g) The two compositions satisfy the *interchange laws* (for binary and zeroary compositions), which means that we have, in diagram (2.1.2):

$$\begin{aligned} &\left(\frac{a\,|\,b}{c\,|\,d}\right) = \left(\frac{a}{c}\;\middle|\;\frac{b}{d}\right), && \left(\frac{1_u}{1_{u'}}\right) = 1_{u' \bullet u}, \\ &(e_f \,|\, e_{f'}) = e_{f'f}, && 1_{e_X} = e_{1_X}. \end{aligned} \tag{2.1.3}$$

The first condition says that a consistent matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has a precise *pasting*; the last says that an object X has an *identity cell* $\Box_X = 1_{e_X} = e_{1_X}$. The expressions $(a \,|\, f')$ and $(f \,|\, b)$ will stand for $(a \,|\, e_{f'})$ and $(e_f \,|\, b)$, when this makes sense.

$\mathbb{A}$ is said to be *flat* if every double cell $a\colon (u \,{}^{f}_{g}\, v)$ is determined by its boundary - namely the arrows f, g, u, v. A standard example is the double category $\mathbb{R}$el**Set** of sets, mappings and relations, recalled below in Section 9(c).

3 Hints at weak double categories

More generally, in a *weak double category* $\mathbb{A}$ the horizontal composition behaves categorically (and we still have ordinary categories $\mathrm{Hor}_0\mathbb{A}$ and $\mathrm{Hor}_1\mathbb{A}$), while the composition of vertical arrows is categorical up to *comparison cells*:

- for a vertical arrow $u\colon X \to\!\!\!\!\bullet\; Y$ we have a *left unitor* and a *right unitor*

$$\lambda u\colon e_X \otimes u \to u, \qquad \rho u\colon u \otimes e_Y \to u,$$

- for three consecutive vertical arrows $u\colon X \to\!\!\!\!\bullet\; Y$, $v\colon Y \to\!\!\!\!\bullet\; Z$ and $w\colon Z \to\!\!\!\!\bullet\; T$ we have an *associator*

$$\kappa(u, v, w)\colon u \otimes (v \otimes w) \to (u \otimes v) \otimes w.$$

Interchange holds strictly, as above. The comparison cells are *special* (which means that their horizontal arrows are identities) and horizontally invertible. Moreover they are assumed to be *natural* and *coherent*, in a sense made precise in [4], Section 7; after stating naturality with respect to double cells, the coherence axioms are similar to those of bicategories.

$\mathbb{A}$ is said to be *unitary* if the unitors are identities, so that the vertical identities behave as strict units - a constraint which in concrete cases can often be easily met. The terminology of the strict case is extended to the present one, as far as possible.

A *lax* (*double*) *functor* $F\colon \mathbb{X} \to \mathbb{A}$ between weak double categories amounts to assigning:

(a) two functors $\mathrm{Hor}_0 F$ and $\mathrm{Hor}_1 F$, consistent with domain and codomain

$$\begin{array}{ccc} \mathrm{Hor}_1\mathbb{X} & \xrightarrow{\mathrm{Hor}_1 F} & \mathrm{Hor}_1\mathbb{A} \\ {\scriptstyle \mathrm{Dom}}\downarrow & & \downarrow{\scriptstyle \mathrm{Dom}} \\ \mathrm{Hor}_0\mathbb{X} & \xrightarrow[\mathrm{Hor}_0 F]{} & \mathrm{Hor}_0\mathbb{A} \end{array} \qquad \begin{array}{ccc} \mathrm{Hor}_1\mathbb{X} & \xrightarrow{\mathrm{Hor}_1 F} & \mathrm{Hor}_1\mathbb{A} \\ {\scriptstyle \mathrm{Cod}}\downarrow & & \downarrow{\scriptstyle \mathrm{Cod}} \\ \mathrm{Hor}_0\mathbb{X} & \xrightarrow[\mathrm{Hor}_0 F]{} & \mathrm{Hor}_0\mathbb{A} \end{array} \qquad (3.1.1)$$

(b) for any object X in $\mathbb{X}$, a special cell, the *identity comparison* of F

$$\underline{F}(X)\colon e_{FX} \to Fe_X\colon FX \mathrel{\to\!\!\!\!\!\!\bullet\;\,} FX,$$

(c) for any vertical composite $u \otimes v\colon X \mathrel{\to\!\!\!\!\!\!\bullet\;\,} Y \mathrel{\to\!\!\!\!\!\!\bullet\;\,} Z$ in $\mathbb{X}$, a special cell, the *composition comparison* of F

$$\underline{F}(u,v)\colon Fu \otimes Fv \to F(u \otimes v)\colon FX \mathrel{\to\!\!\!\!\!\!\bullet\;\,} FZ.$$

Again, these comparisons must satisfy axioms of naturality and coherence with the comparisons of $\mathbb{X}$ and $\mathbb{A}$ [5].

4 Dualities

A weak double category has a *horizontal opposite* $\mathbb{A}^h$ (reversing the horizontal direction) and a *vertical opposite* $\mathbb{A}^v$ (reversing the vertical direction); a strict structure also has a *transpose* $\mathbb{A}^t$ (interchanging the horizontal and vertical issues).

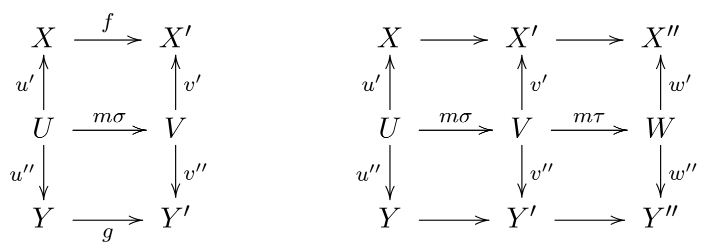
The prefix 'co', as in *colimit*, *coequaliser* or *colax double functor*, refers to horizontal duality, the main one. Let us note that a weak double category whose horizontal arrows are identities is the same as a *bicategory written in vertical*, that is, with arrows and weak composition in the vertical direction and strict composition in the horizontal one. This is why *the* op*lax functors of bicategories correspond here to* co*lax double functors.* (Transposing the theory of double categories, as is done in some papers, would avoid this conflict of terminology, but would produce other conflicts at a more basic level: for instance, colimits in **Set** would become ‘op-limits’ in $\mathbb{R}\mathrm{el}$**Set** and $\mathbb{S}\mathrm{pan}$**Set**.)

5 Spans and cospans

For a category **C** with (a fixed choice of) pullbacks there is a weak double category $\mathbb{S}\mathrm{pan}(\mathbf{C})$ *of spans* over **C**, which will play here an important role.

Objects, horizontal arrows and their composition come from **C**, so that $\mathrm{Hor}_0(\mathbb{S}\mathrm{pan}\mathbf{C}) = \mathbf{C}$.

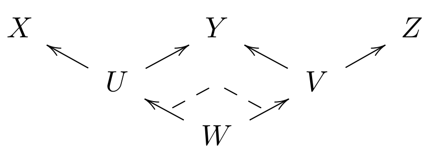
A vertical arrow $u\colon X \to\!\!\!\bullet\, Y$ is a span $u = (u', u'')$, that is, a diagram $X \leftarrow U \rightarrow Y$ in **C**, or equivalently a functor $u\colon \vee \to \mathbf{C}$ defined on the formal-span category $\bullet \leftarrow \bullet \rightarrow \bullet$. A vertical identity is a pair $e_X = (1_X, 1_X)$. A cell $\sigma\colon (u\ {}^{f}_{g}\ v)$ is a natural transformation $u \to v$ of such functors and amounts to the left commutative diagram below


(5.1.1)

We say that the cell σ is *represented* by its middle arrow $m\sigma\colon U \to V$, which determines it together with the boundary (the present structure is not flat).

The horizontal composition $\sigma|\tau$ of σ with a second cell $\tau\colon v \to w$ is a composition of natural transformations, as in the right diagram above; it gives the category $\mathrm{Hor}_1(\mathbb{S}\mathrm{pan}\mathbf{C}) = \mathbf{Cat}(\vee, \mathbf{C})$.

The vertical composition $u \otimes v$ of spans is computed by (chosen) pullbacks in **C**



$$W = U \times_Y V. \qquad (5.1.2)$$

This is extended to double cells, in the obvious way. For the sake of simplicity *we make* $\mathbb{S}\mathrm{pan}(\mathbf{C})$ *unitary*, by adopting the 'unit constraint' for pullbacks: the chosen pullback of an identity along any morphism is an identity. The associator κ is determined by the universal property of pullbacks.

Dually, for a category $\mathbf{C}$ with (a fixed choice of) pushouts there is a unitary weak double category $\mathbb{C}\mathrm{osp}(\mathbf{C})$ *of cospans* over $\mathbf{C}$, that is horizontally dual to $\mathbb{S}\mathrm{pan}(\mathbf{C}^{\mathrm{op}})$. We have now

$$\mathrm{Hor}_0(\mathbb{C}\mathrm{osp}\mathbf{C}) = \mathbf{C}, \qquad \mathrm{Hor}_1(\mathbb{C}\mathrm{osp}\mathbf{C}) = \mathbf{Cat}(\wedge, \mathbf{C}), \tag{5.1.3}$$

where $\wedge = \vee^{\mathrm{op}}$ is the formal-cospan category $\bullet \to \bullet \leftarrow \bullet$.

A vertical arrow $u = (u', u'')\colon \wedge \to \mathbf{C}$ is now a cospan, that is, a diagram $X \to U \leftarrow Y$ in $\mathbf{C}$, and a cell $\sigma\colon u \to v$ is a natural transformation of such functors. Their vertical composition is computed with pushouts in $\mathbf{C}$; again, we generally follow the 'unit constraint' for pushouts.

6 Tabulators

The (horizontal) *tabulator* of a vertical arrow $u\colon X \to\!\!\!\!\bullet\; Y$ in the weak double category $\mathbb{A}$ is an object $T = \top u$ equipped with a double cell $t_u\colon e_T \to u$

$$\begin{array}{ccc} T & \xrightarrow{p} & X \\ {\scriptstyle e_T}\downarrow & t_u & \downarrow{\scriptstyle u} \\ T & \xrightarrow[q]{} & Y \end{array} \qquad\qquad \begin{array}{ccccc} H & \xrightarrow{f} & T & \xrightarrow{p} & X \\ {\scriptstyle e}\downarrow & e_f & {\scriptstyle e}\downarrow & t_u & \downarrow{\scriptstyle u} \\ H & \xrightarrow[f]{} & T & \xrightarrow[q]{} & Y \end{array} \quad = \quad h, \tag{6.1.1}$$

such that the pair $(T, t_u\colon e_T \to u)$ is a universal arrow from the functor $e\colon \mathrm{Hor}_0\mathbb{A} \to \mathrm{Hor}_1\mathbb{A}$ to the object u of $\mathrm{Hor}_1\mathbb{A}$. Explicitly, this means that for every object H and every cell $h\colon e_H \to u$ there is a unique horizontal map $f\colon H \to T$ such that $(e_f \,|\, t_u) = h$, as in the right diagram above.

(In [5] we also considered a higher dimensional universal property, which was dropped in later papers and is not used here.) We say that $\mathbb{A}$ *has tabulators* if all of them exist, or equivalently if the degeneracy functor $e\colon \mathrm{Hor}_0\mathbb{A} \to \mathrm{Hor}_1\mathbb{A}$ has a right adjoint

$$\top\colon \mathrm{Hor}_1\mathbb{A} \to \mathrm{Hor}_0\mathbb{A}, \qquad e \dashv \top. \tag{6.1.2}$$

In this situation *one can try to represent* $\mathbb{A}$ *as a weak double category of spans*, as we shall see below.

Dually $\mathbb{A}$ *has cotabulators* if the degeneracy functor has a left adjoint

$$\bot\colon \mathrm{Hor}_1\mathbb{A} \to \mathrm{Hor}_0\mathbb{A}, \qquad \bot \dashv e, \tag{6.1.3}$$

so that every vertical arrow $u\colon X \to\!\!\!\bullet\; Y$ has a cotabulator-object $\bot u$, equipped with two horizontal morphisms $i\colon X \to \bot u$, $j\colon Y \to \bot u$ and a universal cell $\iota\colon (u\, {}^{i}_{j}\, e)$. This *may* allow representing $\mathbb{A}$ as a weak double category of cospans.

For a category $\mathbf{C}$ with pullbacks, the tabulator in $\mathbb{S}\mathrm{pan}(\mathbf{C})$ of a span $u = (u', u'') = (X \leftarrow U \to Y)$ is its central object U, with projections u', u'' and the obvious cell $t_u\colon e_U \to u$. The cotabulator is the pushout of the span in $\mathbf{C}$, provided it exists. All this cannot be formulated within the bicategory $\mathbf{Span}(\mathbf{C})$ (vertically embedded in $\mathbb{S}\mathrm{pan}(\mathbf{C})$ as specified in Section 4).

7 Theorem and Definition (Span representation)

We suppose that: (a) *the weak double category* $\mathbb{A}$ *has tabulators,*

(b) *the ordinary category* $\mathbf{C} = \mathrm{Hor}_0(\mathbb{A})$ *of objects and horizontal arrows has pullbacks.*

There is then a canonical lax functor, which is trivial in degree zero

$$S\colon \mathbb{A} \to \mathbb{S}\mathrm{pan}(\mathbf{C}), \qquad \mathrm{Hor}_0(S) = \mathrm{id}\mathbf{C}, \tag{7.1.1}$$

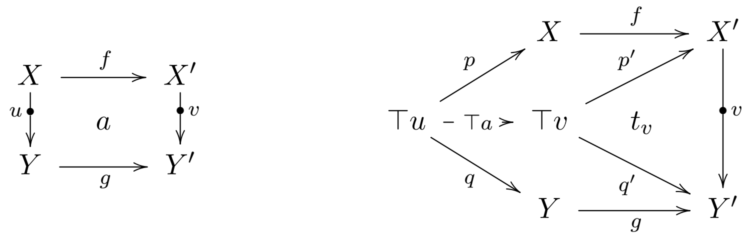
and takes a vertical arrow $u\colon X \to\!\!\!\bullet\; Y$ *of* $\mathbb{A}$ *to the span* $Su = (p, q)\colon X \to\!\!\!\bullet\; Y$ *determined by the tabulator* $\top u$ *and its projections* $p\colon \top u \to X$, $q\colon \top u \to Y$.

The lax functor S *will be called the* span representation *of* $\mathbb{A}$.

Note. Related results can be found in Niefield [8], for weak double categories with vertical companions and adjoints.

Proof. As in Section 6 we write $t_u\colon (e\, {}^{p}_{q}\, u)$ the universal cell of the tabulator $\top u$.

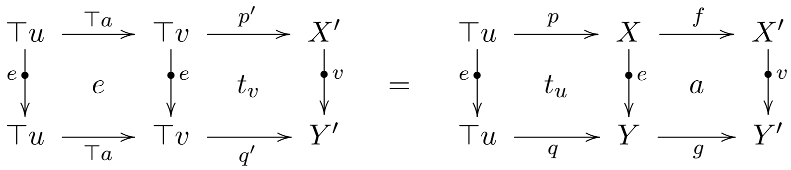
The action of S on a cell a of $\mathbb{A}$ is described by the following diagram



(7.1.2)

where the cell $Sa\colon Su \to Sv$ (a morphism of spans) is represented by the coherent morphism $\top a\colon \top u \to \top v$. The latter is determined by the universal property of the universal cell t_v of the tabulator $\top v$

$$(\top a \,|\, t_v) = (t_u \,|\, a) \qquad (p'.\top a = fp,\ q'.\top a = gq), \tag{7.1.3}$$



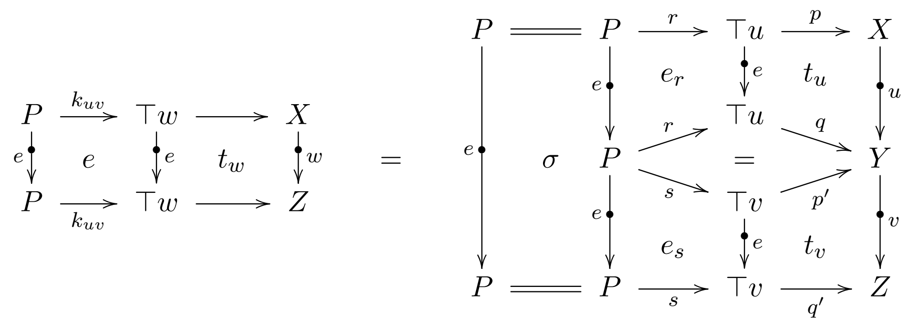
(In the composition $(\top a \,|\, t_v)$ we write $\top a$ for $e_{\top a}$, as already warned at the end of Section 2.)

To define the laxity comparisons, an object X of $\mathbb{A}$ gives a special cell $\underline{S}(X)\colon e_X \to S(e_X)$ represented by the morphism

$$k_X\colon X \to \top e_X, \qquad (k_X \,|\, t_{eX}) = \Box_X. \tag{7.1.4}$$

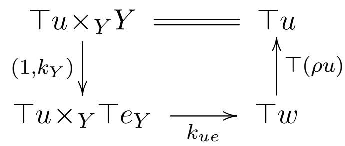
For a vertical composite $w = u \otimes v\colon X \to\!\!\!\!\!\bullet\;\; Y \to\!\!\!\!\!\bullet\;\; Z$, the comparison $\underline{S}(u, v)\colon Su \otimes Sv \to Sw$ is represented by the morphism k_{uv} defined below, where $P = \top u \times_Y \top v$ is a pullback and $\sigma = \lambda(e_P)^{-1} = \rho(e_P)^{-1}$

$$k_{uv}\colon P \to \top w, \qquad (k_{uv} \,|\, t_w) = \left(\sigma \,\Big|\, \frac{r \,|\, t_u}{s \,|\, t_v}\right), \tag{7.1.5}$$



(Note that one can not apply interchange to $(r \,|\, t_u) \otimes (s \,|\, t_v)$.) Finally we have to verify the coherence conditions of the comparisons of S (see [5], Section 2.1), and we only check axiom (iii) for the right unitor.

For a vertical map $u\colon X \to\!\!\!\bullet\; Y$ and $w = u \otimes e_Y$ we have to verify that the following diagram of morphisms of $\mathbf{C}$ commutes



(7.1.6)

where the pullback $\top u \times_Y Y$ is realised as $\top u$, by the unit constraint, and the morphism $\top(\rho u)$ is defined by: $(\top(\rho u) \,|\, t_u) = (t_w \,|\, \rho u)$.

Equivalently, by applying the (cancellable) universal cell t_u and the isocell $\rho = \rho(e_{\top u})$, we show that

$$(\rho \,|\, (1, k_Y) \,|\, k_{ue} \,|\, \top(\rho u) \,|\, t_u) = (\rho \,|\, t_u).$$

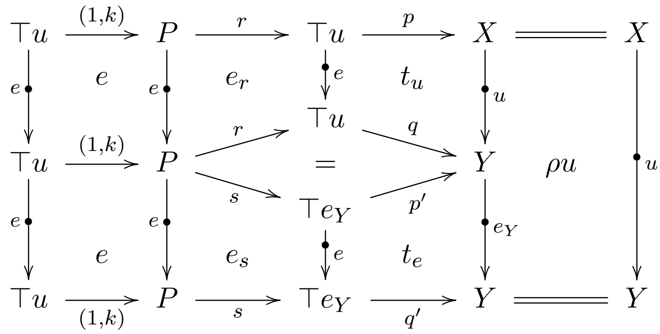
In fact we have

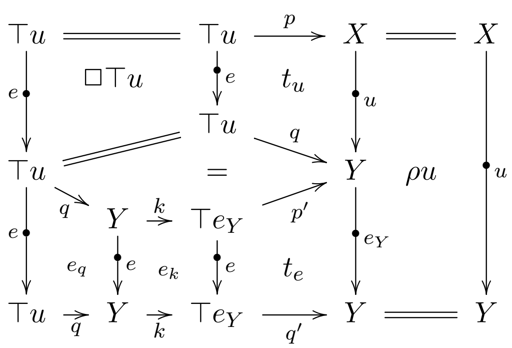
$$(\rho \,|\, (1, k_Y) \,|\, k_{ue} \,|\, \top(\rho u) \,|\, t_u) = (\rho \,|\, (1, k_Y) \,|\, k_{ue} \,|\, t_w \,|\, \rho u)$$

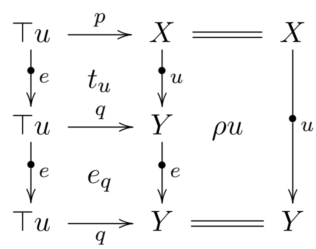
$$= \left(\rho \,|\, (1, k_Y) \,|\, \sigma \,|\, \frac{r \,|\, t_u}{s \,|\, t_e} \,|\, \rho u\right) = \left(\rho \,|\, \rho^{-1} \,|\, \frac{(1, k_Y) \,|\, r \,|\, t_u}{(1, k_Y) \,|\, s \,|\, t_e} \,|\, \rho u\right)$$

$$= \left(\frac{\square \top u \,|\, t_u}{e_q \,|\, e_k \,|\, t_e} \,|\, \rho u\right) = \left(\frac{\square \top u \,|\, t_u}{e_q \,|\, \square Y} \;|\; \rho u\right) = \left(\frac{t_u}{e_q} \,|\, \rho u\right) = (\rho(e_{\top u}) \,|\, t_u). \tag{7.1.7}$$

The fourth, fifth and seventh terms of these computations are represented below, with $P = \top u \times_Y \top e_Y$ and $k = k_Y$.







$\square$

8 Span and cospan representability

Let $\mathbb{A}$ be a weak double category. (a) We say that $\mathbb{A}$ is (horizontally) *span representable* if:

- it has tabulators,
- the ordinary category $\mathbf{C} = \mathrm{Hor}_0(\mathbb{A})$ has pullbacks,
- the span-representation lax functor $S\colon \mathbb{A} \to \mathbb{S}\mathrm{pan}(\mathbf{C})$ of (7.1.1) is *horizontally faithful.*

The last condition means that the ordinary functors $\mathrm{Hor}_0 S$ and $\mathrm{Hor}_1 S$ are faithful. This is trivially true for $\mathrm{Hor}_0(S) = \mathrm{id}\mathbf{C}$, and also for $\mathrm{Hor}_1 S$ when $\mathbb{A}$ is flat.

(b) By horizontal duality, if $\mathbb{A}$ has cotabulators and $\mathbf{C} = \mathrm{Hor}_0(\mathbb{A})$ has pushouts we form a colax functor of *cospan representation*

$$C\colon \mathbb{A} \to \mathbb{C}\mathrm{osp}(\mathbf{C}), \qquad\qquad \mathrm{Hor}_0(C) = \mathrm{id}\mathbf{C}, \qquad\qquad (8.1.1)$$

that takes a vertical arrow $u\colon X \to\!\!\!\!\bullet\;\, Y$ of $\mathbb{A}$ to the cospan $Cu = (i, j)\colon X \to\!\!\!\!\bullet\;\, Y$ formed by the cotabulator $\bot u$ and its 'injections' $i\colon A \to \bot u$, $j\colon Y \to \bot u$.

In this situation we say that $\mathbb{A}$ is *cospan representable* if this colax functor is horizontally faithful.

9 Some basic cases

(a) For a category $\mathbf{C}$ with pullbacks, the weak double category $\mathbb{S}\mathrm{pan}(\mathbf{C})$ is span representable, in a strict sense: the functor $S\colon \mathbb{S}\mathrm{pan}(\mathbf{C}) \to \mathbb{S}\mathrm{pan}(\mathbf{C})$ is an isomorphism, and even the identity for the natural choice of the tabulator of a span, namely its central object. Dually, for every category $\mathbf{C}$ with pushouts, $\mathbb{C}\mathrm{osp}(\mathbf{C})$ is 'strictly' cospan representable.

(b) On the other hand it is easy to see that $\mathbb{S}\mathrm{pan}\mathbf{Set}$ is *not* cospan representable, while $\mathbb{C}\mathrm{osp}\mathbf{Set}$ is *not* span representable. For the first fact we consider a morphism of spans $\sigma\colon u \to u$ represented in the left diagram below, where the objects are cardinal sets ($0 = \varnothing, 1 = \{0\}, 2 = \{0, 1\}$).

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ \uparrow & & \uparrow \\ 2 & \xrightarrow{m\sigma} & 2 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 \end{array} \qquad\qquad\qquad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 2 & \xrightarrow{m\sigma} & 2 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array} \qquad\qquad (9.1.1)$$

All the arrows to 1 are determined but the mapping $m\sigma\colon 2 \to 2$ is arbitrary; the cotabulator pushout is $\bot u = 1$ and $\bot\sigma$ does not determine σ.

The second counterexample is shown in the right diagram above, where again $m\sigma\colon 2 \to 2$ is arbitrary, the tabulator pullback is $\top u = 0$ and $\top\sigma$ does not detect σ.

Similar counterexamples can be given for any category $\mathbf{C}$ with finite limits (or colimits) and some object with at least two endomorphisms.

(c) The (strict) double category $\mathbb{A} = \mathbb{R}\mathrm{el}\mathbf{Set}$ *of sets, mappings and relations* [4] has $\mathrm{Hor}_0(\mathbb{A}) = \mathbf{Set}$, relations for vertical arrows and (flat) double cells given by an inequality in the ordered category of relations

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & X' \\ {\scriptstyle u}\big\downarrow\!\bullet & \leqslant & \bullet\!\big\downarrow{\scriptstyle v} \\ Y & \xrightarrow[\;g\;]{} & Y' \end{array} \qquad\qquad gu \leqslant vf. \tag{9.1.2}$$

Tabulators and cotabulators exist: $\top u \subset X{\times}Y$ is the relation itself, as a subset of $X{\times}Y$, while $\bot u$ is the pushout of the span $Su = (X \leftarrow \top u \to Y)$, or of any span representing the relation.

Since $\mathbb{R}\mathrm{el}\mathbf{Set}$ is flat it is automatically span and cospan representable. The same holds replacing $\mathbf{Set}$ with any regular category with pushouts.

In the examples below we examine other strict or weak double categories, referring to their definition in [4, 5], briefly reviewed here.

10 Representing profunctors

The weak double category $\mathbb{C}\mathrm{at}$ *of categories, functors and profunctors* was introduced in [4], Section 3.1. Objects are small categories, a horizontal arrow is a functor and a vertical arrow is a profunctor $u\colon X \mathrel{\bullet\!\!\to} Y$, defined as a functor $u\colon X^{\mathrm{op}}{\times}Y \to \mathbf{Set}$. A cell $a\colon (u\,{}^{f}_{g}\,v)$ is a natural transformation $a\colon u \to v(f^{\mathrm{op}}\times g)\colon X^{\mathrm{op}}\times Y \to \mathbf{Set}$. Compositions and comparisons are known or easily defined.

The cotabulator $\bot u = X +_u Y$ of a profunctor $u\colon X \mathrel{\bullet\!\!\to} Y$ is the gluing, or collage, of X and Y along u, with new maps given by $(\bot u)(x, y) = u(x, y)$ and no maps 'backwards'; the composition of the new maps with the old ones is defined by the action of u. The inclusions $i\colon X \to \bot u$ and $j\colon Y \to \bot u$ are obvious, as well as the structural cell $\iota\colon (u\,{}^{i}_{j}\,e)$

$$\iota\colon u \to e_{\bot u}(i^{\mathrm{op}}{\times}j)\colon X^{\mathrm{op}}{\times}Y \to \mathbf{Set}, \qquad\qquad \iota(x, y)\colon u(x, y) = \bot u(x, y). \tag{10.1.1}$$

The tabulator $\top u$ is the *category of elements* of u, or Grothendieck construction. It has objects (x, y, λ) with $x \in \mathrm{Ob}X$, $y \in \mathrm{Ob}Y$, $\lambda \in u(x, y)$ and maps (f, g) of $X \times Y$ which form a commutative square in the collage

$X +_u Y$

$$(f,g)\colon (x,y,\lambda) \to (x',y',\lambda') \qquad\qquad (f\colon x \to x',\ g\colon y \to y'),$$
$$g\lambda = \lambda' f \qquad\qquad (u(1_x,g)(\lambda) = u(f,1_y)(\lambda') \in u(x,y')). \tag{10.1.2}$$

The functors p, q are obvious, and the structural cell $\tau = t_u\colon e_{\top u} \to u$ is the natural transformation

$$\tau\colon e_{\top u} \to u(p^{\mathrm{op}} \times q)\colon (\top u)^{\mathrm{op}} \times \top u \to \mathbf{Set},$$
$$\tau(x,y,\lambda;x',y',\lambda')\colon \top u(x,y,\lambda;x',y',\lambda') \to u(x,y'), \qquad (f,g) \mapsto g\lambda = \lambda' f. \tag{10.1.3}$$

$\mathbb{C}$at is easily seen to be span and cospan representable. Indeed, for a cell $a\colon (u\ {}^{f}_{g}\ v)$, both the functors $\top a$ and $\bot a$ determine every component $a_{xy}\colon u(x,y) \to v(fx,gy)$ of the natural transformation $a\colon u \to v(f^{\mathrm{op}} \times g)\colon X^{\mathrm{op}} \times Y \to \mathbf{Set}$

$$\top a\colon \top u \to \top v, \quad \top a(x,y,\lambda) = (fx, gy, a_{xy}(\lambda)),$$
$$\bot a\colon \bot u \to \bot v, \quad \bot a(\lambda\colon x \to y) = a_{xy}(\lambda)\colon fx \to gy \qquad (\lambda \in u(x,y)). \tag{10.1.4}$$

11 Representing adjoints

We prove now that the double category $\mathbb{A}\mathrm{dj}\mathbf{Cat}$ of (small) *categories, functors and adjunctions*, introduced in [4], Section 3.5, is also span and cospan representable.

Again $\mathrm{Hor}_0(\mathbb{A}\mathrm{dj}\mathbf{Cat}) = \mathbf{Cat}$. A vertical arrow is now an ordinary adjunction, conventionally directed as the *left* adjoint

$$\begin{gathered} u = (u_\bullet, u^\bullet, \eta, \varepsilon)\colon X \to\!\!\!\!\bullet\ Y, \quad (u_\bullet\colon X \to Y) \dashv (u^\bullet\colon Y \to X), \\ \eta\colon 1_X \to u^\bullet u_\bullet, \qquad\qquad \varepsilon\colon u_\bullet u^\bullet \to 1_Y. \end{gathered} \tag{11.1.1}$$

A double cell $a = (a_\bullet, a^\bullet)\colon u \to v$ is a pair of mate natural transformations, each of them determining the other via the units and counits of the two adjunctions

$$\begin{gathered} a_\bullet\colon v_\bullet f \to g u_\bullet, \qquad a^\bullet\colon f u^\bullet \to v^\bullet g, \\ a^\bullet = (f u^\bullet \to v^\bullet v_\bullet f u^\bullet \to v^\bullet g u_\bullet u^\bullet \to v^\bullet g), \\ a_\bullet = (v_\bullet f \to v_\bullet f u^\bullet u_\bullet \to v_\bullet v^\bullet g u_\bullet \to g u_\bullet). \end{gathered} \tag{11.1.2}$$

(a) In $\mathbb{A}\mathrm{dj}\mathbf{Cat}$ the tabulator $\top u$ of an adjunction $u = (u_\bullet, u^\bullet)\colon X \to\!\!\bullet\!\!\to Y$ is the 'graph' of the adjunction, namely the following comma category, equipped with the comma-projections p, q and an obvious cell $\tau = t_u\colon (e \, {}^p_q \, u)$

$$\begin{aligned} &\top u = (u_\bullet {\downarrow} Y) \;\cong\; (X {\downarrow} u^\bullet), \quad (x, y; c\colon u_\bullet x \to y) \leftrightarrow (x, y; c'\colon x \to u^\bullet y), \\ &\qquad p\colon \top u \to X, \qquad\qquad\qquad q\colon \top u \to Y, \\ &\quad \tau_\bullet\colon u_\bullet p \to q\colon \top u \to Y, \qquad\qquad \tau_\bullet(x, y; c) = c\colon u_\bullet x \to y. \end{aligned} \tag{11.1.3}$$

The tabulator of a cell $a\colon (u \, {}^f_g \, v)$, with components $a_\bullet x\colon v_\bullet f x \to g u_\bullet x$, is the following functor

$$\top a\colon \top u \to \top v, \qquad \top a(x, y; c\colon u_\bullet x \to y) = (fx, gy; g(c).a_\bullet x\colon v_\bullet f x \to gy). \tag{11.1.4}$$

This proves that $\mathbb{A}\mathrm{dj}\mathbf{Cat}$ *is span representable*: in fact the component $a_\bullet x\colon v_\bullet f x \to g u_\bullet x$ is determined by $\top a(x, u_\bullet x; 1\colon u_\bullet x \to u_\bullet x) = (fx, g u_\bullet x; a_\bullet x\colon v_\bullet f x \to g u_\bullet x)$.

(b) In $\mathbb{A}\mathrm{dj}\mathbf{Cat}$ the cotabulator $C = \bot u = X +_u Y$ is the category consisting of the disjoint union $X + Y$, together with new maps $\hat{c}_x = (x, y; c\colon u_\bullet x \to y)\hat{} \in C(x, y)$ from objects of X to objects of Y that are 'represented' by objects $(x, y; c\colon u_\bullet x \to y)$ of $\top u = (u_\bullet \downarrow Y)$; the composition of the new maps with old maps $\varphi \in X(x', x)$, $\psi \in Y(y, y')$ is defined in the obvious way

$$\psi.\hat{c}_x.\varphi = (x', y'; \psi.c.u_\bullet(\varphi)\colon u_\bullet x' \to u_\bullet x \to y \to y')\hat{}. \tag{11.1.5}$$

The universal cell $\iota_\bullet\colon i \to j u_\bullet\colon X \to \bot u$ is given by $\iota_\bullet x = (1_{u_\bullet x})\hat{} \in C(x, u_\bullet x)$.

The cotabulator of a cell $a\colon (u \, {}^f_g \, v)$, with components $a_\bullet x\colon v_\bullet f x \to g u_\bullet x$, works as f and g on the old objects and arrows, as $\top a$ on the new arrows

$$\bot a\colon \bot u \to \bot v, \qquad \bot a(x, y; h\colon u_\bullet x \to y)\hat{} = (fx, gy; g(h).a_\bullet x\colon v_\bullet f x \to gy)\hat{}. \tag{11.1.6}$$

This determines $a_\bullet x$ as above.

12 Representing $\mathbb{D}\mathrm{bl}$

The strict double category $\mathbb{D}\mathrm{bl}$ of *weak double categories, lax functors and colax functors* is a crucial structure, on which the theory of double adjoints is based. We refer the reader to its introduction in [5], Section 2, where the non-obvious point of double cells is dealt with.

We prove now that $\mathbb{D}\mathrm{bl}$ is span representable, *horizontally* and *vertically*.

(a) First, every colax functor $U\colon \mathbb{A} \mathrel{-\!\!\bullet\!\!\to} \mathbb{B}$ has a *horizontal tabulator* (T, P, Q, τ).

The weak double category $\mathbb{T} = U \downarrow \mathbb{B}$ is a 'one-sided' double comma (see [5], Section 2.5), with strict projections P and Q, which can be used as horizontal *or* vertical arrows. Below the cell η is simply represented by the horizontal transformation $1_Q\colon Q \to Q$ and the tabulator cell $\tau = t_U$ is linked to the comma-cell π by the unit η and counit ε of the *companionship* of Q with 'itself' (see [5])

$$\begin{array}{ccccc} \mathbb{T} & \xrightarrow{1} & U\!\downarrow\!\mathbb{B} & \xrightarrow{P} & \mathbb{A} \\ {\scriptstyle e}\bullet\!\downarrow & \eta & \downarrow\!\bullet{\scriptstyle Q} & \pi & \downarrow\!\bullet{\scriptstyle U} \\ \mathbb{T} & \xrightarrow[Q]{} & \mathbb{B} & \xrightarrow[1]{} & \mathbb{B} \end{array} \qquad \tau = (\eta \,|\, \pi), \quad \pi = \left(\tfrac{\tau}{\varepsilon}\right). \tag{12.1.1}$$

To be more explicit, the tabulator $\mathbb{T}$ has objects

$$(A, B, b\colon UA \to B), \tag{12.1.2}$$

with A in $\mathbb{A}$ and b horizontal in $\mathbb{B}$. A horizontal arrow of $\mathbb{T}$

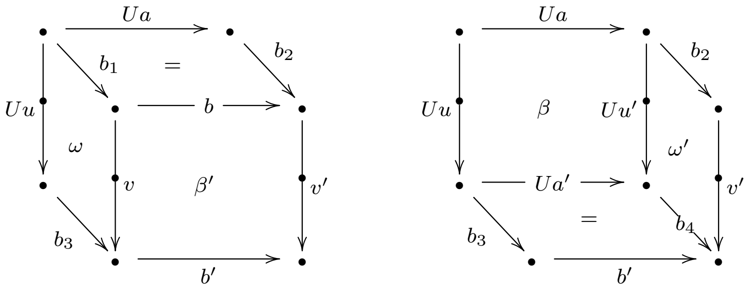
$$(a, b)\colon (A_1, B_1, b_1) \to (A_2, B_2, b_2), \tag{12.1.3}$$

'is' a commutative square in $\mathrm{Hor}_0\mathbb{B}$, as in the upper square of diagram (12.1.5), below (where the slanting direction must be viewed as horizontal). A vertical arrow of $\mathbb{T}$

$$(u, v, w)\colon (A_1, B_1, b_1) \to (A_3, B_3, b_3), \tag{12.1.4}$$

'is' a double cell in $\mathbb{B}$, as in the left square of diagram (12.1.5). A double cell (β, β') of $\mathbb{T}$ forms a commutative diagram of double cells of $\mathbb{B}$

$$(\beta, \beta')\colon ((u, v, \omega) \genfrac{}{}{0pt}{}{(a,b)}{(a',b')} (u', v', \omega')), \qquad (\omega \,|\, \beta') = (\beta \,|\, \omega), \tag{12.1.5}$$



The composition laws of $\mathbb{T}$ are obvious, as well as the (strict) double functors P, Q. The double cell τ has components

$$\tau(A, B, b) = b\colon UA \to B, \qquad\qquad \tau(u, v, \omega) = \omega\colon Uu \to v. \qquad (12.1.6)$$

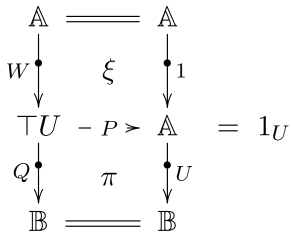
Its universal property follows trivially from that of the double comma, in [5], Theorem 2.6(a).

(b) We have thus a span representation

$$S : \mathbb{D}\mathrm{bl} \to \mathbb{S}\mathrm{pan}(\mathrm{Lx}\mathbf{Dbl}), \qquad (12.1.7)$$

where $\mathrm{Lx}\mathbf{Dbl} = \mathrm{Hor}_0\mathbb{D}\mathrm{bl}$ is the category of weak double categories and lax functors. (Note that, even though the projections P, Q of the double comma $\mathbb{T}$ are strict double functors, a cell $\varphi\colon (U\ {}^{F}_{G}\ V)$ in $\mathbb{D}\mathrm{bl}$ gives a lax functor $\top\varphi\colon \top U \to \top V$.)

To prove that $\mathbb{D}\mathrm{bl}$ is horizontally span representable, we use the *vertical* universal property of the double comma $\mathbb{T} = U{\downarrow}\mathbb{B}$, in [5], Theorem 2.6(b), and deduce the existence of a colax functor $W\colon \mathbb{A} \to \mathbb{T}$ and a cell ξ such that:



$$= 1_U \qquad\qquad (QW = U). \qquad (12.1.8)$$

Now a cell $\varphi\colon (U\ {}^{F}_{G}\ V)$ in $\mathbb{D}\mathrm{bl}$ can be recovered from the lax functor $\top\varphi\colon \top U \to \top V$ as follows

$$\varphi = (1_U \,|\, \varphi) = (\xi \otimes \pi \,|\, e_F \otimes \varphi) = (\xi \otimes t_U \otimes \varepsilon \,|\, e_F \otimes \varphi \otimes e_G)$$
$$= (\xi \,|\, e_F) \otimes (t_U \,|\, \varphi) \otimes (\varepsilon \,|\, e_G) = (\xi \,|\, e_F) \otimes (\top\varphi \,|\, t_V) \otimes (\varepsilon \,|\, e_G).$$

(c) Transpose duality leaves $\mathbb{D}\mathrm{bl}$ invariant up to isomorphism: sending an object $\mathbb{A}$ to the horizontal opposite $\mathbb{A}^h$ and transposing double cells we have an isomorphism $\mathbb{D}\mathrm{bl} \to \mathbb{D}\mathrm{bl}^t$. Therefore $\mathbb{D}\mathrm{bl}$ is also *vertically span representable*, which means that $\mathbb{D}\mathrm{bl}^t$ is span representable by a lax functor

$$S' : \mathbb{D}\mathrm{bl}^t \to \mathbb{S}\mathrm{pan}(\mathrm{Cx}\mathbf{Dbl}) \qquad (\mathrm{Cx}\mathbf{Dbl} = \mathrm{Hor}_0\mathbb{D}\mathrm{bl}^t = \mathrm{Ver}_0\mathbb{D}\mathrm{bl}). \quad (12.1.9)$$

The latter sends a lax functor $F\colon \mathbb{A} \to \mathbb{B}$ to the span $S'(F) = (\mathbb{A} \leftarrow \mathbb{T}' \to \mathbb{B})$ associated to its *vertical tabulator* $(\mathbb{T}, P, Q, \tau)$, where the weak double category $\mathbb{T} = \mathbb{B} \downarrow F$ has objects $(A, B, b\colon B \to FA)$, and the cell τ is vertically universal

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{1} & \mathbb{T} \\ {\scriptstyle P}\big\downarrow\!\bullet & \tau & \bullet\!\big\downarrow{\scriptstyle Q} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array} \qquad (12.1.10)$$

13 Theorem (Representing quintets)

The 2-category $\mathbf{C}$ *is 2-complete if and only if the associated double category* $\mathbb{Q}\mathbf{C}$ *of quintets has all double limits. In this case the double category* $\mathbb{Q}\mathbf{C}$ *is span representable.*

Proof. Let us recall that the double category $\mathbb{Q}\mathbf{C}$ of quintets (introduced by C. Ehresmann) has for horizontal and vertical maps the morphisms of $\mathbf{C}$, while its double cells are defined by 2-cells of $\mathbf{C}$

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle u}\big\downarrow & \swarrow\varphi & \big\downarrow{\scriptstyle v} \\ Y & \xrightarrow[g]{} & Y' \end{array} \qquad \varphi\colon vf \to gu\colon X \to Y'. \qquad (13.1.1)$$

It is known that $\mathbf{C}$ is 2-complete if and only if it has 2-products, 2-equalisers and cotensors by the arrow-category $\mathbf{2}$ [9]. First, it is easy to see that 2-products (respectively, 2-equalisers) in $\mathbf{C}$ are 'the same' as double products (respectively, double equalisers) in $\mathbb{Q}\mathbf{C}$. Second, the cotensor $\mathbf{2} * X$ can be obtained as the tabulator of the vertical identity of X: they are defined by the same universal property.

Conversely, if the **C**-morphism $u\colon X \to Y$ is viewed as vertical in $\mathbb{Q}\mathbf{C}$, its tabulator $(\top u; p, q; \tau)$ can be constructed as the following inserter $(\top u; i, \tau)$

$$\top u \xrightarrow{\;i\;} X \times Y \underset{p''}{\overset{up'}{\rightrightarrows}} Y, \qquad\qquad \tau\colon up'i \to p''i\colon \top u \to Y, \qquad (13.1.2)$$

letting $p = p'i\colon \top u \to X$, $q = p''i\colon \top u \to Y$ and viewing τ as a double cell with boundary $(1\,{}^{p}_{q}\,u)$.

If **C** is 2-complete, $\mathbb{Q}\mathbf{C}$ is span representable because the lax span representation $S\colon \mathbb{Q}\mathbf{C} \to \mathbb{S}\mathrm{pan}(\mathbf{C})$ operates on a double cell $a\colon vf \to gu$ of $\mathbb{Q}\mathbf{C}$ producing a morphism of spans $Sa\colon Su \to Sv$ whose central map $\top a\colon \top u \to \top v$ is defined as follows

$$\begin{aligned} &t_u\colon up'i \to p''i\colon \top u \to Y, && t_v\colon vq'j \to q''j\colon \top v \to Y', \\ &j.\top a = (f \times g)i, && t_v.\top a = gt_u.ap'i\colon vfp'i \to gp''i. \end{aligned} \qquad (13.1.3)$$

$$\begin{array}{ccccc} \top u & \xrightarrow{\;i\;} & X \times Y & \underset{p''}{\overset{up'}{\rightrightarrows}} & Y \\ {\scriptstyle \top a}\downarrow & & {\scriptstyle f\times g}\downarrow & & \downarrow{\scriptstyle g} \\ \top v & \xrightarrow[\;j\;]{} & X' \times Y' & \underset{q''}{\overset{vq'}{\rightrightarrows}} & Y' \end{array}$$

Now f and g are determined as the vertical faces of the morphism Sa. To recover the 2-cell $a\colon vf \to gu$ of **C** from the morphism $\top a$, one uses the map $h\colon X \to \top u$ determined by the conditions $ih = (1, u)\colon X \to X \times Y$ and $t_u.h = 1_u$, so that

$$t_v.\top a.h = (gt_u.ap'i)h = gt_u h.ap'ih = ap'(1, u) = a.$$

□

14 Splitting tabulators

In order to 'explain' how so many double categories are span representable, we observe that the proof for the non-obvious cases above follows a pattern of the following type (as in Section 13), or a vertical version of the same (as in Section 12). However the argument is rather complicated, and - in the examples above - we preferred to give a direct proof, following this guideline.

We are in a weak double category $\mathbb{A}$ with tabulators, and the category $\mathbf{C} = \mathrm{Hor}_0(\mathbb{A})$ has pullbacks. In order that $\mathbb{A}$ be span representable it is sufficient that, for every vertical arrow u, there exist two cells s_u and ε satisfying the following condition:

$$\begin{array}{ccccccc} X & \xrightarrow{h} & \top u & \xrightarrow{p} & X & = = = & X \\ & & {\scriptstyle e}\bullet\downarrow & t_u & {\scriptstyle u}\bullet\downarrow & & \\ {\scriptstyle u}\bullet\downarrow & s_u & \top u & \xrightarrow{q} & Y & \rho u & \bullet\downarrow{\scriptstyle u} \\ & & {\scriptstyle q_*}\bullet\downarrow & \varepsilon & \bullet\downarrow{\scriptstyle e} & & \\ Y & = = = & Y & = = = & Y & = = = & Y \end{array} \quad = \quad 1_u. \qquad (14.1.1)$$

(Typically, q_* is the vertical companion of q and $\varepsilon = \varepsilon_q$ its counit, but this is not needed in the proof. In the strict case ρu is trivial.)

In fact one can recover a cell $a\colon (u \,{}^{f}_{g}\, v)$ from $\top a$ (and Sa), as follows

$$a = (s_u \,|\, \frac{t_u}{\varepsilon} \,|\, \rho u \,|\, a) = (s_u \,|\, \frac{t_u}{\varepsilon} \,|\, \frac{a}{e_g} \,|\, \rho v) = (s_u \,|\, \frac{t_u \,|\, a}{\varepsilon \,\,|\, e_g} \,|\, \rho v) = (s_u \,|\, \frac{\top a \,|\, t_v}{\varepsilon \,\,|\, e_g} \,|\, \rho v).$$

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