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On finitely generated modules whose first nonzero Fitting ideals are regular

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Abstract. A finitely generated R-module is said to be a module of type (F_r) if its $(r-1)$-th Fitting ideal is the zero ideal and its r-th Fitting ideal is a regular ideal. Let R be a commutative ring and N be a submodule of R^n which is generated by columns of a matrix $A = (a_{ij})$ with $a_{ij} \in R$ for all $1 \leq i \leq n$, $j \in \Lambda$, where Λ is a (possibly infinite) index set. Let $M = R^n/N$ be a module of type (F_{n-1}) and $\mathbf{T}(M)$ be the submodule of M consisting of all elements of M that are annihilated by a regular element of R. For $\lambda \in \Lambda$, put $M_\lambda = R^n/ < (a_{1\lambda}, ..., a_{n\lambda})^t >$. The main result of this paper asserts that if M_λ is a regular R-module, for some $\lambda \in \Lambda$, then $M/\mathbf{T}(M) \cong M_\lambda/\mathbf{T}(M_\lambda)$. Also it is shown that if M_λ is a regular torsionfree R-module, for some $\lambda \in \Lambda$, then $M \cong M_\lambda$. As a consequence we characterize all non-torsionfree modules over a regular ring, whose first nonzero Fitting ideals are maximal.

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1 Introduction and Preliminaries

Let R be a commutative ring with identity and M be a finitely generated R-module. For a set $\{x_1, \ldots, x_n\}$ of generators of M there is an exact sequence

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0\ , \tag{1}$$

where R^n is a free R-module with the set $\{e_1, \ldots, e_n\}$ of basis, the R-homomorphism φ is defined by $\varphi(e_j) = x_j$ and N is the kernel of φ. Let N be generated by $u_\lambda = a_{1\lambda}e_1 + \ldots + a_{n\lambda}e_n$, with λ in some index set Λ. Assume that A be the following matrix:

$$\begin{pmatrix} \cdots & a_{1\lambda} & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & a_{n\lambda} & \cdots \end{pmatrix}.$$

(We call A the matrix presentation of the sequence (1)). Let $\mathrm{Fitt}_i(M)$ be an ideal of R generated by the minors of size $n - i$ of matrix A. For $i \geq n$, $\mathrm{Fitt}_i(M)$ is defined R and for $i < 0$, $\mathrm{Fitt}_i(M)$ is defined as the zero ideal. It is known that $\mathrm{Fitt}_i(M)$ is the invariant ideal determined by M, that is, it is determined uniquely by M and it does not depend on the choice of the set of generators of M [4]. The ideal $\mathrm{Fitt}_i(M)$ will be called the i-th Fitting ideal of the module M. It follows from the definition that $\mathrm{Fitt}_i(M) \subseteq \mathrm{Fitt}_{i+1}(M)$, for every i. The most important Fitting ideal of M is the first of the $\mathrm{Fitt}_i(M)$ that is nonzero. We shall denote this Fitting ideal by $\mathrm{I}(M)$.

An element of R is regular if it is a nonzerodivisor and an ideal of R is regular if it contains a regular element. Assume that $\mathbf{T}(M)$, the torsion submodule of M, be the submodule of M consisting of all elements of M that are annihilated by a regular element of R. An R-module M is a torsion module if $M = \mathbf{T}(M)$ and is a torsionfree R-module if $\mathbf{T}(M) = 0$.

One of the most interesting question for modules is, "Can we characterize modules according to the first nonzero Fitting ideal of them?" D.A. Buchsbaum and D. Eisenbud have shown in [2] that if R is a Noetherian ring then, M is a finitely generated projective R-module of constant rank if and only if $\mathrm{I}(M) = R$. Also a lemma of J. Lipman asserts that if R is a quasilocal ring and $M = R^n/K$, where K is a submodule of R^n, then $\mathrm{I}(M)$ is regular principal if and only if K is finitely generated free and $M/T(M)$ is free of rank $n-q$ ([6]) and J. Ohm generalized this result to global case [7]. At this point, a natural question arises: if $\mathrm{I}(M)$ is any ideal, how much can we say about the structure of M? In this paper this question is partly answered.

2 The main results

A finitely generated module is said to be a module of type (F_r) if its $(r-1)$-th Fitting ideal is the zero ideal and its r-th Fitting ideal is a regular ideal.

Let M be a finitely generated R-module. Then there exist some $n \in \mathbb{N}$ and a submodule N of R^n such that $M \cong R^n/N$. The purpose of this paper is to study the module of type (F_{n-1}).

Lemma 2.1. *Let R be a Noetherian ring and M be a finitely generated R-module of type (F_r). Then exactly one of the following holds:*

(1) *M is projective of constant rank r.*

(2) *M can not be generated by r elements.*

Proof. Let M can be generated by r elements. So, by definition we have $\mathrm{Fitt}_r(M) = R$. Since M is a module of type (F_r), so $\mathrm{Fitt}_{r-1}(M) = 0$. Hence by [2, Lemma 1], M is projective of constant rank r. □

Let (R, m) be a local ring and M be a finitely generated R-module. It is known that all minimal generator sets of M have the same cardinal. We will denote the minimal number of generators of M by $\mu(M)$.

Example 2.2. Let (R, m) be a local ring and m be a finitely generated regular ideal of R. Let M be a finitely generated module with $\mu(M) = n$ and $\mathrm{I}(M) = m$. Then M is a module of type (F_{n-1}). Because let $X = \{x_1, \cdots, x_n\}$ be a minimal generator set of M and

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$$

be an exact sequence, where R^n is a free R-module with the set $\{e_1, \ldots, e_n\}$ of basis, the R-homomorphism φ is defined by $\varphi(e_j) = x_j$ and N is the kernel of φ. Assume that

$$A = \begin{pmatrix} \cdots & a_{1\lambda} & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & a_{n\lambda} & \cdots \end{pmatrix}$$

is the matrix presentation of this sequence, with λ in some index set Λ. Since X is a minimal generator set of M, it is easily seen that $a_{i\lambda} \in m$, for every i, $1 \le i \le n$ and every $\lambda \in \Lambda$. Let $m = \mathrm{I}(M) = \mathrm{Fitt}_i(M)$. Since $a_{i\lambda} \in m$, hence $m = \mathrm{Fitt}_i(M) \subseteq m^{n-i}$. If $n = i$, then $m = \mathrm{Fitt}_n(M) = R$, which is a contradiction. Hence by Nakayama's Lemma $i = n-1$. So M is a module of type (F_{n-1}).

An R-module M is called a regular module if $\mathrm{I}(M)$ is a regular ideal.

Theorem 2.3. *Let R be a commutative ring and N be a submodule of R^n consisting of elements $A_\lambda = (a_{1\lambda}, ..., a_{n\lambda})^t$ with λ in some index set Λ. Let $M \cong R^n/N$ be an R-module of type (F_{n-1}). For $\lambda \in \Lambda$, put $M_\lambda = R^n/ < (a_{1\lambda}, ..., a_{n\lambda})^t >$. If M_λ is a regular module, for some $\lambda \in \Lambda$, then $M/\mathbf{T}(M) \cong M_\lambda/\mathbf{T}(M_\lambda)$.*

Proof. Clearly

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$$

is an exact sequence and

$$A = \begin{pmatrix} \cdots & a_{1\lambda} & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & a_{n\lambda} & \cdots \end{pmatrix}$$

is the matrix presentation of this sequence. Let M_l be a regular R-module, for some $l \in \Lambda$. This means that $\mathrm{I}(M_l)$ is a regular ideal.

Put $M_l = R^n/ < A_l >$. Set $x_i = e_i + N$ and $y_i = e_i + < A_l >$ for all $1 \le i \le n$. Then $M =< x_1, ..., x_n >$ and $M_l =< y_1, ..., y_n >$. Define

$$f : M/\mathbf{T}(M) \longrightarrow M_l/\mathbf{T}(M_l); f(\sum_{i=1}^{n} a_i x_i + \mathbf{T}(M)) = \sum_{i=1}^{n} a_i y_i + \mathbf{T}(M_l).$$

Let $x = \sum_{i=1}^n a_i x_i \in \mathrm{T}(M)$. Hence there exists a regular element q in R such that $qx = \sum_{i=1}^n q a_i x_i = 0$. So there exist some elements $r_k \in R$, $1 \le k \le m$ such that $q a_i = \sum_{k=1}^m r_k a_{i\lambda_k}, 1 \le i \le n, \lambda_k \in \Lambda$. For every $i, j, 1 \le i, j \le n$ and every $\lambda \in \Lambda$, we have $q a_i a_{j\lambda} = \sum_{k=1}^m r_k a_{i\lambda_k} a_{j\lambda}$. Thus $q(a_i a_{j\lambda} - a_j a_{i\lambda}) = \sum_{k=1}^m r_k (a_{i\lambda_k} a_{j\lambda} - a_{j\lambda_k} a_{i\lambda})$. Since M is a module of type

(F_{n-1}), so $\mathrm{Fitt}_{n-2}(M) = 0$. Hence $a_{i\lambda_k}a_{j\lambda} - a_{j\lambda_k}a_{i\lambda} = 0$ and so we have $q(a_i a_{j\lambda} - a_j a_{i\lambda}) = 0$. Since q is regular, hence

$$a_i a_{j\lambda} - a_j a_{i\lambda} = 0. \tag{1}$$

Now put $y = \sum_{i=1}^{n} a_i y_i$. Since $M_l = R^n / < A_l > = < y_1, ..., y_n >$, we have $a_{1l}y_1 + \cdots + a_{nl}y_n = 0$. Let $s, 1 \leq s \leq n$ be arbitrary. We have $a_{sl}y_s = -a_{1l}y_1 - \cdots - a_{nl}y_n$. Therefore $a_{sl}y = \sum_{i=1}^{n} a_{sl}a_i y_i = a_{sl}a_s y_s + \sum_{s\neq i=1}^{n} a_{sl}a_i y_i = \sum_{s\neq i=1}^{n} -a_s a_{il} y_i + \sum_{s\neq i=1}^{n} a_{sl}a_i y_i = \sum_{s\neq i=1}^{n}(a_{sl}a_i - a_{il}a_s)y_i$. Therefore by (2), we have $a_{sl}y = 0$. Since a_{sl} is arbitrary, so $\mathrm{I}(M_l)y = 0$. On the other hand, by hypothesis, $\mathrm{I}(M_l)$ is a regular ideal, hence $y \in \mathbf{T}(M_l)$. Therefore f is welldefined. It is clear that f is onto. Now let $y = \sum_{i=1}^{n} a_i y_i \in \mathbf{T}(M_l)$. So there exists a regular element $p \in R$ such that $p(a_1, ..., a_n)^t \in < A_l > \subseteq N$. Thus $\sum_{i=1}^{n} a_i x_i \in \mathbf{T}(M)$. Therefore f is an isomorphism. $\square$

Proposition 2.4. *Let $M \cong R^n/N$ be an R-module of type (F_{n-1}). If M_λ is a regular torsionfree module, for some $\lambda \in \Lambda$, then $M = M_\lambda = R^n/ < (a_{1\lambda}, ..., a_{n\lambda})^t >$.*

Proof. Let M_λ be a regular torsionfree R-module, for some $\lambda \in \Lambda$. Since M_λ is a regular module, hence by the proof of Theorem 2.3, $x = \sum_{i=1}^{n} a_i x_i \in \mathrm{T}(M)$ if and only if $\sum_{i=1}^{n} a_i y_i \in \mathrm{T}(M_\lambda) = 0$. For every $l \in \Lambda$, we have $\sum_{i=1}^{n} a_{il} x_i = 0 \in \mathrm{T}(M)$. Hence $\sum_{i=1}^{n} a_{il} y_i \in \mathrm{T}(M_\lambda) = 0$. Thus $(a_{1l}, ..., a_{nl})^t \in < A_\lambda >$, for every $l \in \Lambda$. So $N \subseteq < A_\lambda >$. This means that $M = M_\lambda$. $\square$

Corollary 2.5. *Let $M \cong R^n/N$ be a finitely generated R-module of type (F_{n-1}). If M_λ is a regular torsionfree module, for some $\lambda \in \Lambda$, then $pd_R(M) = 1$.*

Proof. By Proposition 2.4, we have $M \cong R^n/ < (a_{1\lambda}, \ldots, a_{n\lambda})^t > = M_\lambda$. Since M_λ is a regular module, hence $0 \longrightarrow R \longrightarrow R^n \longrightarrow M \longrightarrow 0$, is a free resolution of M. So $pd_R(M) = 1$. $\square$

Lemma 2.6. *Let $M \cong R^n/N$ be a finitely generated non-torsionfree R-module of type (F_{n-1}) . Then $M/\mathbf{T}(M)$ is free of rank $n-1$ if there exists $\lambda \in \Lambda$ such that $< a_{1\lambda}, ..., a_{n\lambda} >$ is a principal regular ideal.*

Proof. By [7, Theorem 6.2], $< a_{1\lambda}, ..., a_{n\lambda} >$ is a principal regular ideal if and only if $M_\lambda/\mathbf{T}(M_\lambda)$ is free of rank $n-1$. (Note that if $< a_{1\lambda}, ..., a_{n\lambda} >$ is a regular ideal, then $< (a_{1\lambda}, ..., a_{n\lambda})^t >$ is a free R-module). Therefore by Theorem 2.3, $M/\mathbf{T}(M)$ is free of rank $n-1$ if there exists some $\lambda \in \Lambda$ such that $< a_{1\lambda}, ..., a_{n\lambda} >$ is a principal regular ideal. □

Theorem 2.7. *Let (R, m) be a Noetherian local ring and $M \cong R^n/N$ be an R-module of type (F_{n-1}). Assume that there exists an element $(a_{1l}, ..., a_{nl})^t \in N$ such that a_{sl} is a regular prime element of R, for some s, $1 \leq s \leq n$. Then exactly one of the following holds:*
(1) $M \cong R/\operatorname{I}(M) \oplus R^{n-1}$,
(2) $M \cong R^n/ < (a_{1l}, ..., a_{nl})^t >$.

Proof. Clearly
$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$$
is an exact sequence. Assume that
$$A = \begin{pmatrix} \cdots & a_{1\lambda} & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & a_{n\lambda} & \cdots \end{pmatrix}$$
be the matrix presentation of this sequence. Put $p = a_{sl}$ and $M_l = R^n/ < (a_{1l}, ..., a_{nl})^t >$. We consider two cases:

Case 1) Let $p = a_{sl} \mid a_{il}$, for every i, $1 \leq i \leq n$. Thus $\operatorname{I}(M_l) =< a_{1l}, ..., a_{nl} >= (p)$ is a regular principal ideal. So by Lemma 2.6, $M/\mathbf{T}(M)$ is free of rank $n-1$. Hence $M \cong \mathbf{T}(M) \oplus R^{n-1}$. If $\operatorname{T}(M) = 0$, then $M \cong R^{n-1}$. Let $\mathbf{T}(M) \neq 0$. Since M is a module of type (F_{n-1}), so by Lemma 2.1, $\mu(M) = n$. Because R is a local ring, it is easily seen that $n = \mu(M) = \mu(\mathbf{T}(M)) + \mu(R^{n-1}) = \mu(\mathbf{T}(M)) + n - 1$. Hence $\mu(\mathbf{T}(M)) = 1$ and so $\mathbf{T}(M)$ is a cyclic R-module. On the other hand by [1, page 174], $\operatorname{I}(M) = \operatorname{Fitt}_0(\mathbf{T}(M))$. Since $\mathbf{T}(M)$ is a cyclic module, hence $\mathbf{T}(M) \cong R/\operatorname{Fitt}_0(\mathbf{T}(M))$ and so $M \cong R/\operatorname{I}(M) \oplus R^{n-1}$.

Case 2) Now let there exist some t, $1 \leq t \leq n$ such that $p = a_{sl} \nmid a_{tl}$. We claim that $M \cong M_l$, in this case. First we show that $\operatorname{T}(M_l) = 0$. Assume that there exists some regular element $q \in R$ and an element $s' \in R$ such that $q(a_1, ..., a_n)^t = s'(a_{1l}, ..., a_{nl})^t$, for some $(a_1, ..., a_n)^t \in R^n$. Therefore for $1 \leq i \leq n$, we have
$$qa_i = s'a_{il}. \tag{2}$$

So $qa_s = s'a_{sl} = s'p$. Thus $p \mid qa_s$. Since p is a prime element, then $p \mid q$ or $p \mid a_s$.

Let $p \mid q$. Thus there exists an element $t' \in R$ such that $q = pt'$. So $pt'a_s = s'p$. Since p is regular, one gets that

$$s' = t'a_s. \tag{3}$$

By (3) and (4) we have $pt'a_i = t'a_s a_{il}$, for every i, $1 \leq i \leq n$. Since t' is a regular element, so, for every i,

$$pa_i = a_s a_{il}. \tag{4}$$

Since $p \nmid a_{tl}$, hence $p \mid a_s$. Therefore there exists $t'' \in R$ such that $a_s = pt''$. By (5), we have $pa_i = pt''a_{il}$. So $a_i = t''a_{il}$, for every i, $1 \leq i \leq n$. Hence $(a_1, ..., a_n)^t = t''(a_{1l}, ..., a_{nl})^t$. Thus $\mathbf{T}(M_l) = 0$.

Now, let $p \nmid q$. Then $p \mid a_s$. Thus there exists some $t'' \in R$ such that $a_s = t''p$. From (3), we have $qa_s = s'p$. Thus $qt''p = s'p$. So $s' = qt''$. Again by (3), $qa_i = qt''a_{il}$. Hence $(a_1, ..., a_n)^t = t''(a_{1l}, ..., a_{nl})^t$ and so $\mathbf{T}(M_l) = 0$. By Proposition 2.4, we have $M = M_l = R^n / < (a_{1l}, ..., a_{nl})^t >$. $\square$

3 Modules over regular rings

The Krull dimension of R is the supremum of all lengths of chains of prime ideals of R. Let R be a Noetherian local ring with maximal ideal m and Krull dimension d. Recall that R is called a regular local ring if m has a generating set with d elements. The generating set of d elements for m is called a regular system of parameters of R. The ring R is called a regular ring if R_P is regular local ring, for every prime ideal P of R.

Proposition 3.1. *Let (R, m) be a regular local ring and M be a finitely generated R-module. If $\mathrm{I}(M)$ contains a part of a regular system of parameters, then exactly one of the following holds:*

(1) $M \cong R/\,\mathrm{I}(M) \oplus R^{n-1}$,

(2) $M \cong R^n / < (a_1, ..., a_n)^t >$, *for some $n \in \mathbb{N}$ and $a_1, ..., a_n \in R$.*

Proof. Let $\mu(M) = n$ and

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$$

be an exact sequence. Let $A = (a_{ij})_{n\times m}$ be the matrix presentation of this sequence. (Note that since every regular local ring is Noetherian, hence $\ker(\varphi)$ is finitely generated, and so A has finitely many columns). Since $\mu(M) = n$, it is easily seen that $a_{ij} \in m$, for all i, j. We have $\mathrm{Fitt}_i(M) \subseteq m^{n-i}$. Since $\mathrm{I}(M)$ contains a part of a system of parameters, so by [5, Theorem 9.1.1], $\mathrm{I}(M) \nsubseteq m^2$. Thus $\mathrm{I}(M) = \mathrm{Fitt}_{n-1}(M)$. This means that M is a module of type (F_{n-1}). Since $\mathrm{I}(M) \nsubseteq m^2$, hence there exist some i, j, $1 \leq i \leq n$ and $1 \leq j \leq m$ such that $a_{ij} \notin m^2$. Because every regular local ring is a UFD ([3, Theorem 19.19]), hence a_{ij} is an irreducible element. It is easily seen that every irreducible element in a UFD is prime. So by Theorem 2.7, $M \cong R/\mathrm{I}(M) \oplus R^{n-1}$ or $M \cong R^n/ < (a_1, ..., a_n)^t >$, for some $n \in \mathbb{N}$ and $a_i \in R, 1 \leq i \leq n$. $\square$

Corollary 3.2. *Let (R, m) be a regular local ring and M be a finitely generated R-module. If $\mathrm{I}(M) = m$, then*

(1) *$M \cong R/m \oplus R^{n-1}$, if M is not torsionfree,*

(2) *$M \cong R^n/ < (a_{1l}, ..., a_{nl})^t >$, if M is torsionfree.*

Proof. Since m is generated by a regular system of parameters, so by Proposition 3.1, we are done. $\square$

Now, we generalize Corollary 3.2, to global case.

Theorem 3.3. *Let Q be a maximal ideal of a regular ring R. Let M be a finitely generated R-module such that the R_Q-module M_Q is not torsionfree. Then $I(M) = Q$ if and only if $M \cong P \oplus R/Q$, where P is a projective R-module.*

Proof. Let $\mathrm{I}(M) = Q \in Max(R)$. Then $\mathrm{I}(M_Q) = QR_Q$ and $\mathrm{I}(M_q) = R_q$ for every maximal ideal $Q \neq q$. By [2, Lemma 1], for every $Q \neq q \in Max(R)$, there exists some positive integer m such that $M_q \cong R_q^m$. Since M_Q is not a torsionfree R_Q-module, Corollary 3.2 yields that $M_Q \cong R_Q^n \oplus R_Q/QR_Q$, for some positive integer n. So $(M/\mathbf{T}(M))_q$ is free for every maximal ideal q of R. Therefore $M/\mathbf{T}(M)$ is a projective R-module. On the other hand, $\mathbf{T}(M_Q) \cong R_Q/QR_Q$. Thus $\mathbf{T}(M_Q)$ is a simple R_Q-module. Put $\mathcal{A} = \{ann_R(y) : \mathbf{T}(M_Q) =< \frac{y}{1} >\}$. Let $\mathbf{T}(M_Q) =< \frac{x}{1} >$ such that $ann_R(x)$ is maximal in $\mathcal{A}$. It is easily seen that $ann_R(x) = Q$. Define $f : R/Q \longrightarrow \mathbf{T}(M)$; $f(r + Q) = rx$. It is clear that f_q is an isomorphism for every maximal ideal q of R. Hence $\mathbf{T}(M) \cong R/Q$. Therefore $M \cong M/\mathbf{T}(M) \oplus \mathbf{T}(M) \cong P \oplus R/Q$, for some projective R-module P. Conversely, let $M \cong P \oplus R/Q$, for some projective R-module P and maximal ideal Q. Then it is clear that $\mathrm{I}(M) = Q$. $\square$

Corollary 3.4. *Let R be a Noetherian regular ring and Q be a maximal ideal of R. Let M be a finitely generated R-module with $\mathrm{I}(M) = Q$. Then*
(1) *If the R_Q-module M_Q is torsionfree, then $pd_R(M) = 1$,*
(2) *If the R_Q-module M_Q is not torsionfree, then $pd_R(M) = gldim(R_Q)$.*

Proof. Let $Q \neq q$ be a maximal ideal of R. So $\mathrm{I}(M_q) = \mathrm{I}(M)_q = R_q$. Thus by [2, Lemma 1], M_q is a free R_q-module. Also we have $\mathrm{I}(M_Q) = QR_Q$. If the R_Q-module M_Q is torsionfree, then by Corollary 3.2, $M_Q \cong R_Q^n / < (a_1, ..., a_n)^t >$, for some $a_1, ..., a_n \in R_Q$. So $pd_R(M) = \sup_q pd(M_q) = 1$.

Next, assume that the R_Q-module M_Q is not torsionfree. So by Corollary 3.2, $M_Q \cong R_Q/QR_Q \oplus R_Q^{n-1}$. Hence $pd_R(M) = \sup_q pd(M_q) = gldim(R_Q)$. $\square$

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References

[1] Brown, W.C., "Matrices over Commutative Rings", Pure Appl. Math. 169, Marcel Dekker Inc., 1993.

[2] Buchsbaum D.A. and Eisenbud, D., *What makes a complex exact?*, J. Algebra 25 (1973), 259-268.

[3] Eisenbud, D., "Commutative Algebra with a view toward Algebraic Geometry", Springer-Verlag, 1995.

[4] Fitting, H., "Die Determinantenideale eines Moduls", Jahresbericht d. Deutschen Math.-Vereinigung, 46 (1936), 195-228.

[5] Gopalakrishnan, N.S., "Commutative Algebra", Oxonian Press, 1984.

[6] Lipman, J., *On the Jacobian ideal of the module of differentials*, Proc. Amer. Math. Soc. 21 (1969) 423-426.

[7] Ohm, J., *On the first nonzero Fitting ideal of a module*, J. Algebra 320 (2008), 417-425.

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