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A history of selected topics in categorical algebra I: From Galois theory to abstract commutators and internal groupoids

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Abstract. This paper is a chronological survey, with no proofs, of a direction in categorical algebra, which is based on categorical Galois theory and involves generalized central extensions, commutators, and internal groupoids in Barr exact Mal'tsev and more general categories. Galois theory proposes a notion of central extension, and motivates the study of internal groupoids, which is then used as an additional motivation for developing commutator theory. On the other hand, commutator theory suggests: (a) another notion of central extension that turns out to be equivalent to the Galois-theoretic one under surprisingly mild additional conditions; (b) a way to describe internal groupoids in 'nice' categories. This is essentially a 20 year story (with only a couple of new observations), from introducing categorical Galois theory in 1984 by the author, to obtaining and publishing final forms of results (a) and (b) in 2004 by M. Gran and by D. Bourn and M. Gran, respectively.

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Introduction

On the last page of [40], S. Eilenberg and S. Mac Lane say: "An inspection of the concept of a functor and of a natural equivalence shows that they may be applied not only to groups and their homomorphisms, but also to...", and then "...These and similar applications can be embodied in a suitable axiomatic theory. The resulting much wider concept of naturality, as an equivalence between functors, will be studied in a subsequent paper". The subsequent paper [41], which initiated a fundamental new step in the development of the whole structure of mathematics, was published in 1945, although it was "Presented to the Society", whatever that means, in September 8, 1942 (even earlier than [40] was "communicated").

Hence, accidentally or not, the category of groups was the first category to appear. On the other hand, the appearance of the category of groups at the foundation of categorical algebra in [122] and [123] was certainly not an accident, but a clear intention of the author to develop a categorical approach to group theory, with a particular emphasis at pairs of concepts dual to each other. This intention is much less visible in [124], where the title *Categorical algebra* expresses what is today called *Category theory* – and not *Categorical approach to algebra*, which is the way we use this expression in the present paper. And using it that way we can say:

Mac Lane's paper [123] created the first great wave in the development of categorical algebra, whose 'main stream' soon after that becomes the theory of abelian categories, although the non-abelian part was also quite successful.

Let us add here a general remark about the early history of category theory. The above-mentioned first wave was followed by the domination of topos theory with impressive motivations from geometry and logic, and by creation of new ideas and directions in general category theory, from those described in [125] to categorical logic and to categorical topology (and many others, at the time and later), which itself created various gaps between *abstract* and *concrete*, and between *thinking categorically* and the set-theory-based mathematics. Moreover, this created a kind of conflict between those who followed the development of category theory and those who never used categorical notions and results beyond the very first ones and what was introduced by 'non-category-theorists' specifically for the purposes of their narrow areas. The term *exact category* provides a typical example:

for category-theorists it usually means *Barr exact category*, making all varieties of universal algebras exact, while nobody else would agree that, say, semigroups form an exact category!

Attributing, only very vaguely of course, the first wave of categorical algebra to 1950-1985, and the formation of what was described in the general remark above to 1960-1990, I would say that the second wave of categorical algebra begins soon after 1990 with extensive use of several concepts and 'theories' that would not be associated with this part of category theory before. The list of such new ingredients, put in essentially-chronological order, includes (regular and) Barr exact categories [3], internal categorical structures, Smith commutator theory [141], categorical Galois theory in the author's sense, and Bourn protomodular categories [6], among many others. I would also say, in spite of being one of the authors of [106], that that paper made an important link between the two waves, often referred to simply as *Old = New*. At this stage, however, category theory expands enormously, and what we call categorical algebra here becomes only a small part of it.

The present paper, which will hopefully become a part of a much greater project, is devoted to one direction of the second-wave categorical algebra, which is based on categorical Galois theory and involves central extensions, commutators, and internal groupoids. As the Abstract says, Galois theory proposes a notion of central extension, and motivates the study of internal groupoids, which is then used as an additional motivation for developing commutator theory. On the other hand, commutator theory suggests: (a) another notion of central extension that turns out to be equivalent to the Galois-theoretic one under surprisingly mild additional conditions; (b) a way to describe internal groupoids in 'nice' categories. This is essentially a 20 year story (except Example 12.5), from introducing categorical Galois theory in 1984 by the author [88], to obtaining and publishing final forms of results (a) and (b) in 2004 by M. Gran and by D. Bourn [14] and M. Gran [72], respectively.

The paper has 12 sections:

4. Generalized central extensions
5. Internal categories in Mal'tsev varieties
6. Mal'tsev categories and commutators
7. Congruence modularity and Kiss difference term
8. Internal groupoids in congruence modular varieties
9. From pregroupoids to pseudogroupoids
10. Explicit presentations of commutators via limits and colimits in Mal'tsev categories
11. Back to central extensions
12. Back to internal groupoids

(apart from this Introduction), organized as follows:

Section 1 recalls the passage from the fundamental theorem of classical Galois theory (Theorem 1.1, formulated via Galois connections) to its Grothendieck's form (Theorem 1.3), and mentions its generalization due to A. R. Magid [126]. Section 2 is a simplified short survey of categorical Galois theory. Section 3 shows that applying Galois theory to the abelianization reflection of groups into abelian groups makes central extensions of groups covering morphisms in the sense of Galois theory. There are two conclusions: (a) categorical Galois theory has important examples far away from the contexts considered by A. Grothendieck; (b) there is a natural way to generalize central extensions, from groups to, say, other algebraic structures. Moreover, as shown in Section 4 (Theorem 4.1), in the case of groups with additional algebraic structure (only requiring *pointedness*) such generalized central extensions will be the same as those introduced by A. Fröhlich's school. Section 5 moves from varieties of groups with additional structure to arbitrary varieties of universal algebras admitting a Mal'tsev term (see (5.1); however its only purpose is to describe internal categories/groupoids in such varieties, postponing central extensions to Section 11. Section 6 recalls the passage from the results presented in Section 5 to their categorical counterparts, and in fact shows how this passage leads to purely-categorical motivation for introducing a general notion of commutator. The next algebraic generalization, from Mal'tsev (=congruence permutable) to congruence modular is considered in Section 8 after recalling the notion of *Kiss difference term* in Section 7. While a categorical approach to commutators in Mal'tsev categories can be based on *Kock pregroupoids* (mentioned already in Section 6), the congruence modular context requires replacing them with so-called *pseudogroupoids*, as explained in Section 9. Section 10 gives explicit presentations of the commutator of two internal equivalence relations in a Barr exact Mal'tsev category with finite colimits (actually each of the two uses a slightly more general context) due to M. C. Pedicchio and to D. Bourn. Section 11 compares Galois-theoretic and commutator-theoretic definition of central extensions, which we refer to as *categorically central extensions* and *algebraically central extensions*, respectively. The most general level for this comparison is the level of a *factor permutable category* in the sense of M. Gran. Section 12 describes internal groupoids in regular categories satisfying the *shifting property* in the sense of D. Bourn and M. Gran. It also contains a new example showing that the shifting property does not imply congruence modularity in the case of (varieties of) infinitary algebras.

The greater project (if it will be ever completed, possibly with co-authors, and maybe even without the present author) should include papers devoted to:

- Aspects of categorical algebra developed by D. Bourn, partly with co-authors, which is based on what Bourn calls the fibration of points and his notion of protomodularity.

- Further (in comparison with this paper) results on commutators and central extensions, higher central extensions, and (co)homology in semi-abelian and more general categories.

- Categorical universal algebra, including a systematic study of categorical counterparts of classes of varieties of universal algebras determined by the so-called Mal'tsev conditions.

- The full story around semi-abelian categories in the sense of [106] and the above-mentioned "Old = New".

- Internal categorical structures at various levels of generality (including those considered in this paper, but with many more results, and with a special attention to the semi-abelian context, where internal crossed modules [97] can be used.
- Radical and torsion theories in semi-abelian and more general categories.
- Applications to internal and topological algebras, to monoids, to quandles, and to topos theory.

These papers would probably refer to the work of D. Arias, F. Borceux, D. Bourn, R. Brown, A. Carboni, J. M. Casas, A. S. Cigoli, M. M. Clementino, T. Datuashvili, D. Dikranjan, G. Donadze, M. Duckerts, V. Even, T. Everaert, J. Goedecke, M. Gran, M. Grandis, J. A. R. Gray, M. Hartl, H. Inassaridze, N. Inassaridze, E. Inyangala, Z. Janelidze, T. Janelidze-Gray, M. Jibladze, P. T. Johnstone, G. M. Kelly, E. Khmaladze, R. Kieboom, M. Ladra González, S. Lack, J. Lambek, B. Loiseau, S. Mantovani, N. Martins-Ferreira, L. Márki, G. Metere, A. Montoli, M. C. Pedicchio, A. Patchkoria, G. Peschke, T. Pirashvili, T. Porter, D. Rodelo, A. H. Roque, J. Rosický, V. Rossi, M. Sobral, L. Sousa, W. Tholen, A. Ursini, T. Van der Linden, E. M. Vitale... and to many old authors, starting from S. Mac Lane. This list is certainly incomplete, and I apologize to those whose names are not mentioned.

Remark 0.1.
(a) Since our reference list is so large, I would like to list here the authors whose contributions were particularly important for developments described in this paper. Ordered alphabetically they are:

- D. Bourn, who found a new categorical construction of the Smith–Pedicchio commutator described in Section 10, and was first to consider it in regular Mal'tsev categories instead of Barr exact Mal'tsev categories. In addition to that, in his joint work with M. Gran, he extended the description of internal groupoids known for Barr exact Mal'tsev categories and for congruence modular varieties to the context of regular categories satisfying the *shifting property* (see Theorem 12.6).

- A. Carboni, whose joint work with M. C. Pedicchio and N. Pirovano is briefly described at the beginning of Section 6. His interest in Mal'tsev categories played an important stimulating role.

- M. Gran, who made a considerable progress in understanding internal categorical structures and central extensions, not only in the categorical context, but also in the universal-algebraic context (see Remark 8.2, Theorem 11.3, Theorem 11.6, and the above-mentioned Theorem 12.6).

- H. P. Gumm, who introduced factor permutable varieties of universal algebras and the algebraic shifting property, and showed that every congruence modular variety is factor permutable and that a variety has the shifting property if and only if it is congruence modular. His results are not used explicitly in this paper, but they inspired introducing the categorical counterparts of the factor permutability and the shifting property (see Remark 11.5 and Remark 12.4, respectively).

- J. Hagemann and C. Hermann, whose generalization of Smith's commutator theory from Mal'tsev varieties to congruence modular varieties was also certainly a good source of inspiration (together with further results of P. Gumm, E. Kiss, and R. Freese and R. McKenzie; see Remark 7.1).

- G. M. Kelly, who convinced me to replace universal-algebraic context in the study central extensions to the context of Barr exact categories, and made a great contribution in this study. He also greatly contributed in the study of Mal'tsev and Goursat categories.

- J. Lambek, whose interest in Mal'tsev operations and their role in homological algebra brought attention to what is now called Mal'tsev and Goursat categories.

- M. C. Pedicchio, who played the main role in extending the Smith commutator theory to categorical context; her name is mentioned many times in this paper.

- J. D. H. Smith, who is the father of the abstract theory of commutators. He introduced them for Mal'tsev varieties, which seemed to be a natural next step of generalization after the groups with multiple operators in the sense of P. J. Higgins. However, his approach was a fundamental new step of abstraction opening big doors for further generalizations.

(b) If the list above were to include mathematicians of previous generations, A. Mal'tsev would certainly be among them. Note that his surname appears in references also as Malcev, Mal'cev, and Maltsev; the correct form, as accepted today, is Mal'tsev, which corresponds to Russian Мальцев, with sound "ts" corresponding to Russian "ц".

(c) A morphism $f : A \to B$ in a category with pullbacks determines an internal equivalence relation $A \times_B A \rightrightarrows A$ on A, the kernel pair of f. Unfortunately no widely accepted symbol was ever introduced for this basic concept; for instance D. Bourn often writes $R[f]$ (which I strongly disagree with), while some papers of various authors that do not need to use ordinary kernels (including [27] and [26]) write $Ker(f)$ (or $Kerf$ or $kerf$). I shall write $Eq(f)$, as in [98] and elsewhere.

Apart from standard terminology of category theory (as, say, in Mac Lane's [125]) and of general algebra, this paper freely uses various more recent notions of categorical algebra, most of which can be found e.g. in the book [4] of F. Borceux and D. Bourn. Note also that:

- Defining internal structures in a category we follow [107] rather than [4].

- Internal Kock pregroupoids, called just internal pregroupoids (although they have nothing to do with pregroupoids in the sense used in Theorem 2.11), are defined e.g. in [133] (originally in [119]).

- Goursat categories are defined and studied in detail in [26].

- A few more special algebraic notions are occasionally used in the first four sections – but ignoring them will not prevent the reader from understanding the rest of the paper.

1 Grothendieck's and Magid's Galois theories

Given a group G and a G-set E, there is a standard Galois connection

$$P(E) \underset{\{e\in E \mid g\in H \Rightarrow ge=e\} = H^* \leftarrow\!\shortmid H}{\overset{S \mapsto S_* = \{g\in G \mid e\in S \Rightarrow ge=e\}}{\rightleftarrows}} P(G) \tag{1.1}$$

between the power sets of E and of G, which, as every Galois connection, induces a bijection

$$\{S \in P(E) \mid S = (S_*)^*\} \approx \{H \in P(G) \mid H = (H^*)_*\} \tag{1.2}$$

between the sets of Galois closed subsets of E and of G. Describing these closed subsets, one immediately observes that:

- if H is a closed subset of G, then it is a subgroup of G;
- if E is equipped with any kind of algebraic structure, G acts on E via automorphisms of that structure, and S is a Galois closed subset of E, then S is a subalgebra of E.

In addition to these observations, the *fundamental theorem of classical Galois theory* says:

Theorem 1.1. *If $F \subseteq E$ is a finite Galois field extension and G is its Galois group, then, for the Galois connection above, we have:*

(a) *a subset H of G is Galois closed if and only if it is a subgroup of G;*

(b) *a subset S of E is Galois closed if and only if it is a subfield of E containing F.*

In fact, according to many textbooks, the fundamental theorem of classical Galois theory should also include information about normal subextensions and lifting of homomorphisms between subextensions of the given Galois extensions. The resulting 'one page theorem' is actually a simple corollary of the following short one:

Theorem 1.2. *If $E \supseteq F$ is a finite Galois field extension and G is its Galois group, then the opposite category of subextensions of $E \supseteq F$ is equivalent to the category of transitive G-sets.*

The Grothendieck form of the fundamental theorem of classical Galois theory is even more elegant:

Theorem 1.3. *If $E \supseteq F$ is a finite Galois field extension and G is its Galois group, then the opposite category of F-algebras A with $E \otimes_F A \approx E^n$ for some natural n is equivalent to the category of finite G-sets.*

In order to deduce Theorem 1.2 from Theorem 1.3 one just needs to observe that the isomorphism $E \otimes_F A \approx E^n$ holds if and only if A is isomorphic to a finite product of subextensions of $F \subseteq E$ (considered as F-algebras). The algebras of that kind are said to be split over $E \supseteq F$, and their opposite category is denoted by $Spl(E \supseteq F)$ (or $Spl(E/F)$).

Theorem 1.3 has several known generalizations and counterparts, by A. Grothendieck himself and by other authors. For instance in the case of commutative rings, its most general version is due to A. R. Magid [126]. In fact it is more explicitly formulated in [127] as Theorem 6.1; see also [89] and [25]. Instead of considering the details let us only mention that the counterparts of separable algebras of the Galois theory of fields are:

- Étale coverings in the Grothendieck's Galois theory of schemes [82].

- Ordinary covering maps of 'good' topological spaces in the classical theory of covering spaces that goes back to H. Poincaré.

- Componentially locally strongly separable algebras in Magid's Galois theory of commutative rings. Recall that: (a) for a commutative ring R (with 1), a commutative R-algebra A (with 1) is said to be separable, if $A \otimes_R A$ is a projective A-module; (b) a separable R-algebra A is said to be strongly separable, if it is also a projective R-module; (c) an R-algebra A is said to be locally strongly separable, if every finite subset of it is contained in a strongly separable R-subalgebra; (d) an R-algebra A is said to be componentially locally strongly separable, if, for every maximal ideal x of the Boolean algebra of idempotents of R, the Boolean localization $A_x = A \otimes_R (R/Rx) = A \otimes_R R_x$ is a locally strongly separable R_x-algebra.

2 Categorical Galois theory

As it was shown in [88], the main theorem of A. R. Magid's separable Galois theory of commutative rings [126] can be obtained a special case of an abstract theorem in a purely categorical context, far more general than what is usually considered as the context of Grothendieck's Galois theory. The resulting *categorical Galois theory*, whose four slightly different versions were first presented in [88], [90], [92], and [95], respectively, can be briefly described as follows.

Definition 2.1. (Definition 1.4 of [98]) A Galois structure on a category $\mathbf{C}$ with finite limits consists of an adjunction

$$(I, H, \eta, \varepsilon) : \mathbf{C} \to \mathbf{X}. \tag{2.1}$$

together and two classes F and Φ of morphisms in $\mathbf{C}$ and $\mathbf{X}$ respectively, whose elements are called fibrations; the following conditions on fibrations are required:

(a) all pullbacks along fibrations exist, and the classes of fibrations are pullback stable;

(b) the classes of fibrations are closed under composition and contain all isomorphisms;

(c) the functors I and H preserve fibrations.

Such a Galois structure is said to be finitely complete, absolute, admissible, or closed, if it satisfies the following conditions respectively:

(d) the categories $\mathbf{C}$ and $\mathbf{X}$ have all finite limits;

(e) all morphisms in $\mathbf{C}$ and in $\mathbf{X}$ are fibrations;

(f) for every object C in $\mathbf{C}$ and every fibration $\varphi : X \to I(C)$ in $\mathbf{X}$, the composite of canonical morphisms $I(C \times_{HI(C)} H(X)) \to IH(X) \to X$ is an isomorphism;

(g) for every object A in $\mathbf{C}$, the morphism $\eta_A : A \to HI(A)$ is a fibration, and for every object $X \in \mathbf{X}$, the morphism $\varepsilon_X : IH(X) \to X$ is an isomorphism.

Let us repeat the examples listed in [98] in the same order:

Example 2.2. Let $\mathbf{A}$ be a category having a terminal object 1, and $\mathbf{C} = Fam(\mathbf{A})$ the category of families $A = (A_i)_{i\in I(A)}$ of objects in A, where a morphism $f : A \to B$ in $Fam(\mathbf{A})$ is a map $I(f) : I(A) \to I(B)$ together with a family $(f_i : A_i \to B_{I(f)(i)})_{i\in I(A)}$ of morphisms in $\mathbf{A}$. This defines $Fam(A)$ and a functor $I : Fam(A) \to \mathbf{Sets}$ simultaneously. The right adjoint H of I sends sets to families of terminal objects indexed by them. Taking F and Φ the classes of all morphisms in $\mathbf{C}$ and $\mathbf{X}$, respectively, we obtain a Galois structure that is absolute, admissible, and closed.

Example 2.3. The category $\mathbf{C}$ of connected locally connected topological spaces has an admissible Galois structure defined as in Example 2.2 but with F being the class of local homeomorphisms.

Example 2.4. Any adjunction $(I, H, \eta, \varepsilon) : \mathbf{C} \to \mathbf{X}$ becomes a Galois structure if we take F and Φ to be the classes of isomorphisms in $\mathbf{C}$ and $\mathbf{X}$ respectively. It is always admissible, and closed if and only if it is a category equivalence.

Example 2.5. Any pair $(\mathbf{C}, F)$ consisting of a category $\mathbf{C}$ and a class F of morphisms in $\mathbf{C}$ satisfying the conditions 2.1(a) and 2.1(b), determine an *identity* Galois structure; it consists of the identity adjunction $(1, 1, 1, 1) : \mathbf{C} \to \mathbf{C}$ with $F = \Phi$, and it is always admissible and closed.

Example 2.6. There is an obvious *finite version* of Example 2.2. It uses finite families of objects in $\mathbf{A}$ instead of arbitrary ones, and **Finite Sets** instead of **Sets**. In particular $\mathbf{C}$ could be the opposite category of finite-dimensional commutative algebras (with 1) over a field.

Example 2.7. There is also a *profinite version* of Example 2.2, with $\mathbf{X} =$ **Profinite Spaces** $=$ **Stone Spaces**. It is in fact described in [23]. Let us restrict ourselves here to the following two special cases:

(a) $\mathbf{C}$ is the opposite category of commutative rings (with 1). In this case the functors I and H have the following description: $I(A)$ is the Boolean spectrum (=Pierce spectrum) of the ring A, that is, it is defined either as the Stone space (=the space of maximal ideals) of the Boolean algebra of idempotents in A, or as the space of connected components of the Zariski spectrum of A; $H(X)$ is the ring of continuous maps from the space X to the ring of integers equipped with the discrete topology.

(b) $\mathbf{C}$ is the category of compact Hausdorff spaces, I carries spaces to the spaces of their connected components, and H is the inclusion functor.

In both cases all the additional conditions of Definition 2.1 hold. In case (a) the admissibility was used in [88] (see also [90]) in order to show that the Galois theory of [126] extends to general categories. Case (b) was used in [24] to give a categorical description of the monotone-light factorization of continuous maps of compact spaces.

Example 2.8. Let $\mathbf{C}$ be a variety of universal algebras, $\mathbf{X}$ a subvariety in $\mathbf{C}$, $(I, H, \eta, \varepsilon) : \mathbf{C} \to \mathbf{X}$ the reflection-inclusion adjunction, and F and Φ the classes of regular epimorphisms (=surjections) in $\mathbf{C}$ and $\mathbf{X}$ respectively. This Galois structure is finitely complete and closed. Moreover, as shown in [101] (in fact in a more general context, recalled here later: see Subsection 11.1), it is admissible whenever $\mathbf{C}$ is congruence modular, that is, whenever the lattices of congruences of its objects are modular. As explained in Section 4, this Galois structure and its categorical counterpart are needed to develop a generalized theory of central extensions.

Example 2.9. Let $\mathbf{C}$ be the category of simplicial sets, $\mathbf{X}$ the category of (small) groupoids, $(I, H, \eta, \varepsilon) : \mathbf{C} \to \mathbf{X}$ the fundamental groupoid-nerve adjunction, and F and Φ the classes of Kan fibrations in $\mathbf{C}$ and $\mathbf{X}$ respectively. The admissibility of this Galois structure is used in [18], and the same can be done for many other Quillen homotopy structures.

A number of other examples and specific further properties and results that hold here and there can be found, explicitly or not, in [1, 2, 5, 10, 11, 15–20, 22–25, 29–31, 33–39, 42–60, 65, 68–81, 86, 90–96, 99–105, 111, 113, 115, 116, 130, 135–140, 143-149]; further generalizations are considered in [109], [110], and [112].

The main part of the following definition, namely 2.10(b), is a special case of a definition from [95], while its absolute version with (E, p) being normal is already in [88]:

Definition 2.10. Given an admissible Galois structure as in Definition 2.1, an object (A, f) of $F(B)$, that is, a fibration $f : A \to B$ in $\mathbf{C}$, is said to be:

(a) a trivial covering of B, if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HI(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle HI(f)} \\ B & \xrightarrow[\eta_B]{} & HI(B) \end{array} \tag{2.2}$$

is a pullback;

(b) split over $(E, p) \in F(B)$, if its pullback along p is a trivial covering of E;

(c) a monadic extension of B, if the pullback functor $f^* : F(B) \to F(A)$ is monadic;

(d) a covering of B, if there exists a monadic extension (E, p) of B such that (A, f) is split over (E, p); the full subcategory of $F(B)$ with objects all such (A, f) will be denoted by $Spl(E, p)$;

(e) a normal extension of B, if it satisfies the condition of (d) with $p = f$.

By analogy with Theorem 1.3, which is its very special case, (a simplified form of) the fundamental theorem of categorical Galois theory can be formulated as:

Theorem 2.11. [95] *Given an admissible Galois structure as above and a monadic extension (E, p) of B, there is a canonical category equivalence*

$$Spl(E, p) \sim \mathbf{X}^{Gal(E,p)} \cap \Phi, \tag{2.3}$$

in which:

(a) *$Gal(E, p)$ is the I-image of the kernel pair of p considered as an internal precategory (or a pregroupoid) in $\mathbf{X}$;*

(b) $\mathbf{X}^{Gal(E,p)} \cap \Phi$ *is the category of internal actions* $F = (F_0, \pi_F, \xi_F)$ *of* $Gal(E,p)$ *in* $\mathbf{X}$ *with* π_F *in* Φ*; such an internal action can be displayed in the form of commutative diagrams*

$$\begin{array}{ccc} I(E \times_B E) \times_{I(E)} F_0 & \xrightarrow{\xi_F} & F_0 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_F} \\ I(E \times_B E) & \xrightarrow[I(\pi_2)]{} & I(E) \end{array} \tag{2.4}$$

$$\begin{array}{ccccccc} I(E \times_B E \times_B E) \times_{I(E)} F_0 & \xrightarrow{\Theta} & I(E \times_B E) \times_{I(E)} I(E \times_B E) \times_{I(E)} F_0 & \xrightarrow{1 \times \xi} & I(E \times_B E) \times_{I(E)} F_0 & \xleftarrow{\langle I(\langle 1,1 \rangle)\pi_F, 1\rangle} & F_0 \\ \downarrow{\scriptstyle I(\langle \pi_1, \pi_3 \rangle) \times 1} & & & & \downarrow{\scriptstyle \xi_F} & \swarrow\!\!= & \\ I(E \times_B E) \times_{I(E)} F_0 & & \xrightarrow{\qquad\qquad\xi_F\qquad\qquad} & & F_0 & & \end{array} \tag{2.5}$$

where $\pi_i (i = 1, 2, 3)$ *denote various suitable pullback projections,* $\Theta = \langle I(\langle \pi_1, \pi_2 \rangle),\ I(\langle \pi_2, \pi_3 \rangle) \rangle \times 1$*, and the existence of the displayed pullbacks follows from the fact that* π_F *and* p *are fibrations.*

Applying this theorem to various examples of Galois structures we obtain many ‘concrete’ Galois theories, as explained in various above-mentioned papers. Recalling all of them would be a long story only one very special part/direction of which is relevant for the purposes of the present paper. That special direction is what the next two sections are devoted to.

3 Central extensions of groups

In order to present Grothendieck’s Galois theory as a special case of the categorical one it suffices to consider a Galois structure Γ satisfying *all* additional conditions. That is, we can assume Γ to be finitely complete, absolute, admissible, and closed, which reduces it to a semi-left-exact reflection $I : \mathbf{C} \to \mathbf{X}$ in the sense of C. Cassidy, M. Hébert, and G. M. Kelly [32] (see also [24] and [102]). Moreover, in the Grothendieck’s case, the category $\mathbf{X}$

involved is 'almost' the category of sets, and this makes the Galois groups $Gal(E,p)$ 'almost' coincide with the corresponding automorphism groups $Aut(E,p)$.

Therefore, back in 1984, to ask whether there exists an interesting non-Grothendieck Galois theory was to ask whether there exists a reflection $I : \mathbf{C} \to \mathbf{X}$ with an interesting Galois theory and $\mathbf{X}$ being very different from the category of sets, which will make $Gal(E,p)$ very different from $Aut(E,p)$. The readers might agree with me that the reflection $I : \mathbf{Groups} \to \mathbf{AbGroups}$ from the category of groups to the category of abelian groups would be 'the most classical' first example to try. Note that three areas of abstract algebra have their own descriptions of this reflection:

- Universal algebra sees it as the abelianization functor: for a group C, $I(C) = C/\sim$, where $\sim$ is the smallest congruence on C making the corresponding quotient group abelian.
- Homological algebra sees it as the first homology: $I(C) = H_1(C,\mathbb{Z})$.
- Group theory sees it via the commutators: $I(C) = C/[C,C]$.

And, examining Galois theory of this reflection we obtain:

Theorem 3.1. (Essentially from [90]) *For the reflection* $I : \mathbf{Groups} \to \mathbf{AbGroups}$ *above, we have:*

(a) *a group C is admissible, that is, for every morphism $\varphi : X \to I(C)$ in* **AbGroups**, *the composite $I(C \times_{HI(C)} H(X)) \to IH(X) \to X$ of condition* 2.1(f) (*or, equivalently, the morphism* $I(C \times_{HI(C)} H(X)) \to IH(X)$) *is an isomorphism, if and only if C has a perfect commutator, that is, if and only if* $[[C,C],[C,C]] = [C,C]$*;*

(b) (E,p) *is a monadic extension of a group B if and only if $p : E \to B$ is surjective;*

(c) *if $p : E \to B$ is a surjective homomorphism of perfect groups, then* (E,p) *is a normal extension of B if and only if it is central extension of B* (*that is, the kernel of p is contained in the centre of E; using the standard group-theoretic symbols,* $Ker(p) \subseteq Z(E)$)*;*

(d) *if* (E,p) *satisfies the equivalent conditions of* (c)*, then* $Gal(E,p) \approx Ker(p)$*, where* $Ker(p)$ *is considered as an internal group in* **AbGroups***; in particular, if* (E,p) *is a universal central extension of* B*, then* $Gal(E,p) \approx H_2(B,\mathbb{Z})$*.*

Although even more is proved in [90], having admissibility only in the case of a perfect commutator was 'bad'... Fortunately that problem was solved in [92] simply by using the Galois structure whose adjunction is same reflection and whose fibrations are (not all but just) the surjective group homomorphisms:

Theorem 3.2. (Essentially from [92]) *With respect to the Galois structure whose underlying adjunction is the reflection* I : **Groups** $\to$ **AbGroups** *above and whose fibrations are all surjective homomorphisms, of groups and of abelian groups respectively, we have:*

(a) *this Galois structure is complete, admissible, and closed;*

(b) *every fibration in it* (*that is, in the category of groups*) *is a monadic extension;*

(c) *the following conditions on* $(A,f) \in F(B)$ (*in the notation of* 2.10)*, are equivalent:*

  (c1) (A,f) *is a covering of* B*;*

  (c2) (A,f) *is a normal extension of* B*;*

  (c3) (A,f) *is a central extension of* B (*that is,* $Ker(p) \subseteq Z(E)$)*.*

(d) *if* (A,f) *satisfies the equivalent conditions of* (c)*, and* A *is perfect, then* $Gal(A,f) \approx Ker(f)$*.*

4 Generalized central extensions

As the results recalled in the previous section suggested, categorical Galois theory can also be considered as a generalized theory of central extensions. Moreover, it was a nice surprise for me that this generalization contains the theory of generalized central extensions due to A. Fröhlich, A. S.-T. Lue, and J. Furtado-Coelho as a special case (amongst several of their papers that include this topic we refer to [64], [120], [121], [66], and [67]). Let us recall:

- Let $\mathbf{C}$ be a variety of Ω-groups (=groups with multiple operators in the sense of [87]), $\mathbf{X}$ a subvariety of $\mathbf{C}$, $(I, H, \eta, \varepsilon) : \mathbf{C} \to \mathbf{X}$ the canonical reflection, and $R : \mathbf{C} \to \mathbf{C}$ the functor defined by $R(C) = Ker(\eta_C)$. For a surjective homomorphism $f : A \to B$, let us call (A, f) a *Fröhlich central extension* of B if, for every pair of morphisms $g, h : C \to A$ in $\mathbf{C}$ with $fg = fh$, we have $R(g) = R(h)$.

- When $\mathbf{C}$ is a fixed category of associative algebras, "Fröhlich central" is the same "$\mathbf{X}$-central" in [64], with the formulation above given in [120] (different symbols are used), with a reference to [64]. The context of arbitrary Ω-groups is explicitly used only by Furtado-Coelho, although both Fröhlich and Lue also mention that it could be used.

- The fact that Theorem 3.2 extends to Fröhlich central extensions is hidden in a brief remark in [92], and a special case of it is already in [90] (it was known to me at least in 1986, when the Russian version of [90] was submitted for publication). However, a clear formulation with a full proof is only presented later as Theorem 5.2 in [101], a part of which repeated below as Theorem 4.1, in a style similar to Theorem 3.2.

- The theory of Fröhlich central extensions was originally based on Fröhlich's approach to the theory of Baer invariants, which suggests to consider the theory of Baer invariants from the Galois-theoretic viewpoint. An important first step in this direction was made by T. Everaert and T. Van der Linden [57].

Theorem 4.1. (Essentially from [101]) *With respect to the Galois structure, whose underlying adjunction is the reflection above from a variety* $\mathbf{C}$ *of* Ω*-groups to a subvariety* $\mathbf{X}$ *of it, and whose fibrations are all surjective homomorphisms in* $\mathbf{C}$ *and in* $\mathbf{X}$*, we have:*

(a) *this Galois structure is complete, admissible, and closed;*

(b) *every fibration in it* (*that is, in its categories* $\mathbf{C}$ *and* $\mathbf{X}$) *is a monadic extension;*

(c) *the following conditions on* $(A, f) \in F(B)$ (*in the notation of* 2.10), *are equivalent:*

  (c1) (A, f) *is a covering of* B*;*

  (c2) (A, f) *is a normal extension of* B*;*

  (c3) (A, f) *is a Fröhlich central extension of* B*.*

(*d*) *if* (A, f) *satisfies the equivalent conditions of* (*c*)*, and* A *has zero image in* $\mathbf{X}$*, then* $Gal(A, f) \approx Ker(f)$*.*

Remark 4.2. Theorem 4.1(c) is a part of Theorem 5.2 in [101], which has eight equivalent conditions. The equivalence of these eight conditions was extended to the *quasi-pointed* Barr exact (Bourn) protomodular categories by D. Bourn and M. Gran (see Theorem 2.1 in [11]).

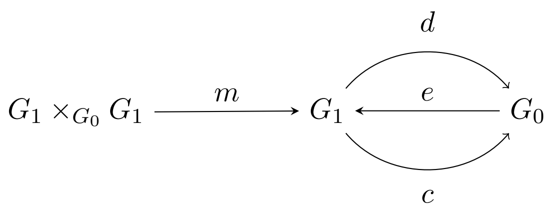
5 Internal categories in Mal'tsev varieties

As first observed in [90], Galois groupoids of Fröhlich central extensions admit a kind of simplified description. Soon after that I have learned about crossed modules from R. Brown, and about every reflexive homomorphic relation in a Mal'tsev variety (=congruence permutable variety) being a congruence from J. Lambek. This suggested to me that the existence of a Mal'tsev term, that is, a ternary term p with

$$p(x, y, y) = x = p(y, y, x), \tag{5.1}$$

in a variety $\mathbf{X}$ of universal algebras, should provide a simplified description of internal categories in $\mathbf{X}$. And indeed, the following definition and two theorems were introduced/proved in [93]:

Definition 5.1. An internal multiplicative graph $G = (G_0, G_1, d, c, e, m)$ in a category $\mathbf{X}$ with pullbacks is a diagram



(5.2)

in $\mathbf{X}$, in which $de = 1_{G_0} = ce, G_1 \times_{G_0} G_1 = G_1 \times_{(d,c)} G_1$ is the pullback of d and c, and the diagram

$$\begin{array}{ccccc} G_1 & \xrightarrow{\langle ec, 1_{G_1}\rangle} & G_1 \times_{G_0} G_1 & \xleftarrow{\langle 1_{G_1}, ed\rangle} & G_1 \\ & \searrow^{=} & \downarrow m & {}^{=}\swarrow & \\ & & G_1 & & \end{array} \tag{5.3}$$

commutes. In particular, when $\mathbf{X}$ is a variety of universal algebras, such a G is a an internal reflexive graph (G_0, G_1, d, c, e) in $\mathbf{C}$ equipped with a multiplication homomorphism $m : G_1 \times_{G_0} G_1 \to G_1$ written as $m(f, g) = fg$, and satisfying

$$f = f1_y \quad \text{and} \quad 1_y g = g \tag{5.4}$$

whenever $d(f) = y = c(g)$, where $1_y = e(y)$. We shall also write $f : y \to x$ when $d(f) = y$ and $c(f) = x$.

Theorem 5.2. *If $G = (G_0, G_1, d, c, e, m)$ is an internal multiplicative graph in a Mal'tsev variety $\mathbf{X}$ with Mal'tsev term p, then:*

(a) *for f and g in G_1 with $d(f) = y = c(g)$, we have*

$$p(f, 1_y, g) = fg = p(g, 1_y, f); \tag{5.5}$$

(b) *G is an internal groupoid in $\mathbf{X}$, that is, $d(fg) = d(g)$, $c(fg) = c(f)$, $f(gh) = (fg)h$ whenever $d(f) = c(g)$ and $d(g) = c(h)$, and every $f : y \to x$ in G_1 is invertible with respect to the multiplication m, with the inverse given by*

$$f^{-1} = p(1_x, f, 1_y) = p(1_y, f, 1_x). \tag{5.6}$$

Theorem 5.3. *The following conditions on an internal reflexive graph $G = (G_0, G_1, d, c, e)$ in a Mal'tsev variety $\mathbf{X}$ with Mal'tsev term p are equivalent:*

(a) *there exists (a unique) $m : G_1 \times_{G_0} G_1 \to G_1$ making (G_0, G_1, d, c, e, m) a multiplicative graph in $\mathbf{X}$;*

(b) *the map* $m : G_1 \times_{G_0} G_1 \to G_1$ *defined by the first equation of* (5.5) *is a morphism in* $\mathbf{X}$*;*

(c) *the map* $m : G_1 \times_{G_0} G_1 \to G_1$ *defined by the second equation of* (5.5) *is a morphism in* $\mathbf{X}$*.*

Remark 5.4. As mentioned in [93], everything we said in this section is 'Yoneda invariant'. That is, using Yoneda embedding, one can repeat it in any category of the form $\mathbf{S}^T$, where $\mathbf{S}$ is a category with finite limits and T an algebraic theory for which $\mathbf{Sets}^T$ is a Mal'tsev variety. Note that, in particular, when $\mathbf{S}$ is a naturally Mal'tsev category in the sense of [117], $\mathbf{S}^T \approx \mathbf{S}$ canonically for a suitable 'naturally Mal'tsev' T, and the important implications (i)⇒(ii) and (i)⇒(iii) in the main theorem of [117] follow from the results of [93]. However, although it is an important special case, [93] does not refer to [117] simply because I have not seen [117] then. Let us also mention the paper [131] whose context could be used to generalize the results of [93].

6 Mal'tsev categories and commutators

One of my discussions with A. Carboni in Milan, 1990, was devoted to Mal'tsev categories. He was insisting that the notion of Mal'tsev category considered in his paper with M. C. Pedicchio and J. Lambek is the right categorical counterpart of the notion of Mal'tsev variety, good for generalizing 'all results'. I asked him then if one can also generalize the results of [93]. The answer 'almost yes' was given in a few months in [28], where the authors kindly mention that the paper was inspired by [93]. However, these authors, A. Carboni, M. C. Pedicchio, and N. Pirovano, did much more:

- As far as I know, for the first time Mal'tsev categories were studied in [28] in full generality, that is, under no colimit assumption. Saying this, let us recall that the notion of Mal'tsev category has emerged slowly in several older papers that assumed one or more of the following conditions: existence of pushouts and/or (some) coequalizers, regularity, Barr exactness. The list of relevant papers includes [128], [129], [61], [62], [27], and [26].

- While conditions (b) and (c) of Theorem 5.3 could not be copied in the categorical context, what was done in [28] is even better in a sense. The known counterpart of (any of) those conditions in the case of the category of groups is $[Ker(d), Ker(c)] = 0$, which, translated into the language of universal algebra via the J. D. H. Smith commutator theory [141], would say: the congruences on G_1 determined by d and by c centralize each other. And exactly this formulation, with the right categorical notion of centralization developed, was obtained in [28].

The categorical notion of centralization immediately gives the categorical counterpart of the Smith commutator: for internal equivalence relations R and S on an object A of a Mal'tsev category, the commutator $[R, S]$ should be defined as the smallest equivalence relation T on A, such that the equivalence relations on A/T induced by R and S centralize each other. After that one should just analyse what is needed for such induced equivalence relations and the smallest T to exist. A slightly different approach, more closely related to what was done in universal algebra before, is used in [132] and yet another one in [133]; however, as Theorem 3.9 of [132] shows, the result is the same. I would say, due to the importance of the papers [132] and [133], the Mal'tsev-categorical version of Smith commutator should now be called the Smith–Pedicchio commutator.

What I called here "yet another" approach of [133] fully agreed with my own idea that came out directly from the analysis of the group case, independently of [141] and of [28]. According to it, one should define the commutator $[R, S]$ of two equivalence relations R and S on an object A in a category $\mathbf{X}$ as follows:

- Under mild conditions on $\mathbf{X}$, the forgetful functor

  $$U : Cat(\mathbf{X}) \to ReflGraph(\mathbf{X}) \tag{6.1}$$

  from the category $ReflGraph(\mathbf{X})$ of internal reflexive graphs in $\mathbf{X}$ to the category $Cat(\mathbf{X})$ of internal categories in $\mathbf{X}$ has a left adjoint F. If $\mathbf{X}$ is a Mal'tsev category (satisfying those mild conditions), then, for every $G = (G_0, G_1, d, c, e)$ in $ReflGraph(\mathbf{X})$, the canonical morphism $(\eta_G)_1 : G_1 \to UF(G)_1$ is a regular epimorphism; for example this is the case when $\mathbf{X}$ is any Mal'tsev variety.

- If, given A, R, and S as above, we can find $G = (G_0, G_1, d, c, e)$ in $ReflGraph(\mathbf{X})$ with $A = G_1$ and R and S being the kernel pairs of d and c, respectively, then we define the commutator $[R, S]$ as the kernel pair of $(\eta_G)_1$. We shall briefly write

$$[R, S] = Eq((\eta_G)_1). \tag{6.2}$$

- In order to define $[R, S]$ for arbitrary R and S, one should modify the structures involved in (6.1) in such a way that a counterpart of a reflexive graph needed for R and S can always be found. As follows from the results of [133] such a modification does exist: we should replace (6.1) with the forgetful functor

$$U' : Pregroupoid(\mathbf{X}) \to Span(\mathbf{X}) \tag{6.3}$$

to the category $Span(\mathbf{X})$ of spans in $\mathbf{X}$ from the category $Pregroupoid(\mathbf{X})$ of internal pregroupoids in $\mathbf{X}$ in the sense of A. Kock (see e.g. [119]; as already mentioned, they have nothing to do with pregroupoids in the sense of categorical Galois theory, mentioned in Theorem 2.11), also called herdoids. And, whenever $\mathbf{X}$ is a Barr exact Mal'tsev category, we always have

$$[R, S] = Eq((\eta'_G)_1), \tag{6.4}$$

for G being the span $A/R \leftarrow A \to A/S$ and $(\eta'_G)_1$ being the canonical morphism similar to $(\eta_G)_1$.

All this leads to what I would call "the most practical" purely-algebraic definition of the Smith commutator, which well agrees with Theorem 5.3 and which was certainly known in universal algebra, although, as far as I know, was never mentioned explicitly before [104]:

Definition 6.1. (Definition 2.2 in [104]) Let $\mathbf{X}$ be a Mal'tsev variety with a Mal'tsev term p, A an object in $\mathbf{X}$, and R and S congruences on A. The Smith commutator $[R, S]$ is the smallest congruence T on A such that the map

$$\{(a, b, c) \in A^3 \mid (a, b) \in R \ \&\ (b, c) \in S)\} \to A/T, \tag{6.5}$$

sending (a, b, c) to the T-class of $p(a, b, c)$ is a homomorphism of algebras.

7 Congruence modularity and Kiss difference term

When I mentioned the categorical approach to commutator theory to L. Márki, his reply was that the universal-algebraic commutator theory already changed its main context from congruence permutable (=Mal'tsev) to congruence modular, which is far more general. Then, since M. C. Pedicchio discovered the relevance of Kock pregroupoids in commutator theory, I asked her whether the pregroupoid approach can be extended to the congruence modular case. This discussion eventually led to two joint papers: [107], whose 'second' main result is recalled in the next section, and [108], where a general definition of a commutator (recalled in Section 9) was given. In order to speak about these papers, we need to recall a purely-algebraic result, due to E. Kiss, from his paper [118]; it says that every congruence modular variety admits what the author calls a 4-difference term; we call it a Kiss difference term. Let us also recall:

- a Kiss difference term is defined as a 4-ary term q with

$$q(x, y, x, y) = x = q(y, y, x, x), \tag{7.1}$$

  and such that, for every two congruences R and S on the same algebra, one has

$$\begin{aligned}&((a, b), (c, d), (c', d) \in R \;\&\; (a, c), (a, c'), (b, d) \in S)\\ &\Rightarrow (q(a, b, c, d), q(a, b, c', d)) \in [R, S].\end{aligned} \tag{7.2}$$

- a Mal'tsev term p (when it exists) immediately gives a Kiss difference term q via

$$q(x, y, t, z) = p(x, y, z); \tag{7.3}$$

- in any congruence distributive variety, we always have $[R, S] = R \cap S$, which allows us to define a Kiss term q via

$$q(x, y, t, z) = t. \tag{7.4}$$

As (7.3) and (7.4) show, Kiss difference terms nicely provide an alternative way of seeing Mal'tsev and congruence distributive varieties as special cases of congruence modular varieties.

Remark 7.1. In (7.2), since the ground variety is not Mal'tsev, the commutator $[R, S]$ cannot be defined as in the previous section, and a more sophisticated definition from universal algebra (see [85], [83], [84], [63], where several definitions equivalent in the congruence modular case, are compared, and [118]) is needed. The more recent definition from [108] given in Section 9 (see (9.7)) is such that in the congruence modular case, it is:

(a) equivalent to all other known definitions;

(b) suggested, in a sense, by using a Kiss difference term instead of a Mal'tsev term.

In more general cases its relationship with other commutators was studied by A. Szendrei, published nowhere (as far as I know) except the three-line conference talk abstract [142], where the main result is not formulated.

8 Internal groupoids in congruence modular varieties

The main purpose of [107] was to extend the results of [93] (repeated in Section 5 of the present paper) from Mal'tsev to congruence modular varieties, using a Kiss difference term instead of a Mal'tsev term. Accordingly, the results of [107] include the following:

Theorem 8.1. (Corollary 4.3 in [107]) *Let* $G = (G_0, G_1, d, c, e)$ *be an internal reflexive graph in a congruence modular variety* $\mathbf{X}$ *with Kiss difference term* q. *The following conditions are equivalent:*

(a) *there exists an internal groupoid in* $\mathbf{X}$ *whose underlying internal reflexive graph is* G*;*

(b) *there exists a unique internal groupoid in* $\mathbf{X}$ *whose underlying internal reflexive graph is* G*;*

(c) *the commutator* $[Eq(d), Eq(c)]$ *is trivial (that is, it is the internal equality relation* Δ_{G_1}*), and* (G_1, d, c) *induces a symmetric and transitive relation on* G_0.

If these conditions hold, then, for each $f : y \to x$, $g : z \to y$, $h : t \to z$ *in* G_1, *we have*

$$fg = q(f, 1_y, u, g), \tag{8.1}$$

$$f^{-1} = q(1_x, f, v, 1_y), \tag{8.2}$$

for every u and v in G_1 with $d(u) = z$, $c(u) = x$, $d(v) = x$, and $c(v) = y$ (such u and v always do exist by the transitivity and symmetry in condition (c)*.*

Remark 8.2. (a) As follows from (7.3), the formulas (8.1) and (8.2) naturally generalize the (first parts of the) formulas (5.5) and (5.6), respectively.

(b) As follows from (7.4) and (8.1) ((8.2) is not needed in this case), every internal groupoid in a congruence distributive variety is an equivalence relation. This fact, however, was observed before (8.1), originally by M. C. Pedicchio, who showed that this property even characterizes the congruence distributive varieties (published later in [134]; as mentioned in [107] she actually did it in the paper “Internal groupoids and pregroupoids in regular categories, to appear”, hence before the submission of [107]).

(c) A very different proof of (a)⇔(c) of Theorem 8.1 is given in [134] among other results.

(d) An almost immediate but beautiful reformulation of condition (c) of Theorem 8.1 is due to M. Gran (see Remark 1.1 in [71]). It replaced the second requirement of 8.1(c) with (Eq(d))(Eq(c)) = (Eq(c))(Eq(d)). In fact [71] has many more interesting results, one of which is mentioned in Section 11.

9 From pregroupoids to pseudogroupoids

Theorem 9.1 below combines some results of [133] (cf. Section 6). In fact the equivalence of its conditions can be used as definitions ‘by each other’ of at least two out of the three notions involved, which is perfectly shown in [133]. Having this in mind, we do not recall the notion “centralizing each other” used in 9.1(a).

Theorem 9.1. *Let* **X** *be a Barr exact Mal’tsev category, A an object in* **X**, *and R and S internal equivalence relations on A. The following conditions are equivalent:*

(a) *R and S centralize each other;*

(b) $[R, S] = \Delta_A$ *(the internal equality relation on A);*

(c) *there exists an internal Kock pregroupoid in* $\mathbf{X}$ *whose underlying span is*

$$A/R \leftarrow A \rightarrow A/S;$$

(d) *there exists a unique internal Kock pregroupoid in* $\mathbf{X}$ *whose underlying span is*

$$A/R \leftarrow A \rightarrow A/S.$$

In particular, these conditions hold whenever $R \wedge S = \Delta_A$.

Remark 9.2. The equality $R \wedge S = \Delta_A$ above makes the span $A/R \leftarrow A \rightarrow A/S$ an internal relation, and conversely, every internal relation $B \leftarrow A \rightarrow C$, in which $A \rightarrow B$ and $A \rightarrow C$ are regular epimorphisms, is of this form. On the other hand, for an arbitrary span $B \leftarrow A \rightarrow C$, using the (regular epi, mono) factorizations $A \rightarrow B' \rightarrow B$ and $A \rightarrow C' \rightarrow C$, we obtain a closely related span, namely $B' \leftarrow A \rightarrow C'$, of regular epimorphisms. We then observe:

(a) $B \leftarrow A \rightarrow C$ admits an internal Kock pregroupoid structure if and only if $B' \leftarrow A \rightarrow C'$ does;

(b) therefore to say that $R \wedge S = \Delta_A$ implies (c) of Theorem 9.1, is the same as to say that every internal relation in $\mathbf{X}$ admits an internal Kock pregroupoid structure;

(c) an internal relation admits a (necessarily unique) internal Kock pregroupoid structure if and only if it is difunctional;

(d) a category is a Mal'tsev category if and only if every internal relation in it is difunctional;

(e) therefore the last assertion of Theorem 9.1 actually characterizes Barr exact Mal'tsev categories among the Barr exact categories.

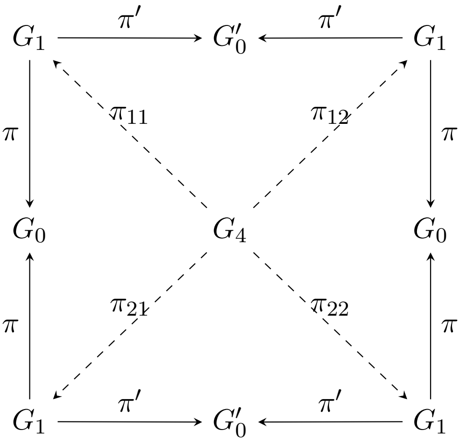
As this remark suggests, in order to define the notion of commutator in a general (or, say, Barr exact) category we have to replace the Kock pregroupoid structure with a new one that exists on every internal relation. Such a structure, called *pseudogroupoid* was introduced in [108] (the preprint version was published in February 1998 as Trieste University Preprint in Mathematics 423). As explained in [108], this new structure was "almost" introduced in [141], and then again in [118] (but in a very different way). Let us recall it:

Definition 9.3.

(a) Given a span

$$G = (G_0 \xleftarrow{\ \pi\ } G_1 \xrightarrow{\ \pi'\ } G_0') \tag{9.1}$$

in a category **X** with finite limits, the system $(G_4, \pi_{11}, \pi_{12}, \pi_{21}, \pi_{22})$ is defined as the limiting cone of the diagram



(9.2)

of solid arrows, as displayed. When $\mathbf{X} = \mathbf{Sets}$, shall write (f, g, k, h) for the unique element u in G_4 with $\pi_{11}(u) = f$, $\pi_{12}(u) = k$, $\pi_{21}(u) =$

g, and $\pi_{22}(u) = h$, and display this element as

$$\begin{array}{ccc} & x & \\ {\scriptstyle f}\nearrow & & \nwarrow{\scriptstyle k} \\ y & & z \\ {\scriptstyle g}\searrow & & \swarrow{\scriptstyle h} \\ & t & \end{array} \tag{9.3}$$

where $\pi'(f) = x = \pi'(k)$, $\pi(f) = y = \pi(g)$, $\pi(k) = z = \pi(h)$, and $\pi'(g) = t = \pi'(h)$.

(b) A pseudogroupoid is a pair (G, m) in which G is a span (9.1) in **Sets**, such that, for every element (9.3) in G_4, we have:

- $m(f, g, k, h)$ is parallel to k, that is, $\pi m(f, g, k, h) = z$ and $\pi' m(f, g, k, h) = x$;
- $m(f, g, k, h)$ does not depend on k, that is, $m(f, g, k, h) = m(f, g, k', h)$ whenever (f, g, k', h) also belongs to G_4;
- if $f = g$, then $m(f, g, k, h) = h$;
- if $g = h$, then $m(f, g, k, h) = f$;
- $m(m(f, g, k, h), i, n, j)) = m(f, g, n, m(h, i, l, j))$ whenever (h, i, l, j) and (f, g, l, n) are in G_4.

(c) An internal psudogroupoid in a category $\mathbf{X}$ is defined by (b) using the Yoneda embedding $\mathbf{X} \to \mathbf{Sets}^{\mathbf{X}^{op}}$.

Remark 9.4. The object G^4 in Definition 9.3(a) forms an internal double equivalence relation on G_1 (in the sense of [108]) denoted by $Eq(G)$ in [108]. Using the symbol $\square$ of [28], also used in [8] and elsewhere, we have

$$Eq(G) = Eq(\pi) \square Eq(\pi'), \tag{9.4}$$

up to permutations of factors (this is not mentioned in in [108]).

Together with the forgetful functors (6.1) and (6.3), let us consider the forgetful functor

$$U'' : Pseudogroupoid(\mathbf{X}) \to Span(\mathbf{X}) \tag{9.5}$$

(we denote it by U'' since we used U and U' for (6.1) and (6.3), respectively). Given internal equivalence relations R and S on an object A in a category $\mathbf{X}$ with finite limits, it is natural now to define the commutator $[R, S]$ as

$$[R, S] = Eq((\eta''_G)_1), \tag{9.6}$$

for G being the span $A/R \leftarrow A \to A/S$ and $(\eta''_G)_1$ being the canonical morphism similar to $(\eta_G)_1$ and $(\eta'_G)_1$. This is done in [108] for $\mathbf{X}$ being a variety of universal algebras. However, when $\mathbf{X}$ is an abstract category and we don't want to make the existence of a left adjoint for U'' an additional requirement on $\mathbf{X}$, we could say:

- If $\mathbf{X}$ is a Barr exact Mal'tsev category with coequalizers, then the existence of left adjoints of U and U' is proved in [132] and [133] (up to trivial reformulations).

- If $\mathbf{X}$ is a regular Mal'tsev category with finite colimits, U has a left adjoint by Proposition 3.4 in [8] (of course it is not just a passage from Barr exactness to regularity that took nine years from [132] to [8]: the whole new approach was developed in [8]); the fact that U' has a left adjoint can also be deduced from the results of [8].

- The above-mentioned results solve out problem in the regular Mal'tsev case, since, whenever $\mathbf{X}$ is a Mal'tsev category, the internal pseudogroupoid and internal pregroupoid structures in $\mathbf{X}$ are the same.

- For a general category $\mathbf{X}$ with finite limits, we only require $\mathbf{X}$ to have arbitrary intersections of subobjects, and then define $[R, S]$ (as it is done in [108]) as the intersection of all $Eq(\alpha_1)$, for all morphisms α from the span $A/R \leftarrow A \to A/S$ to an underlying span of any internal pseudogroupoid.

- What we still don't know, is a reasonable sufficient condition on $\mathbf{X}$ that holds in every regular Mal'tsev category and in every congruence modular variety of universal algebras, under which the morphisms of the form $(\eta''_G)_1$ (see (9.6)) are regular epimorphisms – which would imply that the left adjoint of U'' is completely determined by all commutators and, most importantly, allow one to generalize Theorem 9.1.

Let us now restrict ourselves again to the algebraic context, and for the rest of this section $\mathbf{X}$ will denote a variety of universal algebras. Let us also fix any object (=algebra) A in $\mathbf{X}$ and internal equivalence relations (=congruences) R and S on A.

As already mentioned in Section 7, when $\mathbf{X}$ is congruence modular, the commutator $[R, S]$ can be defined in several equivalent ways, which we did not recall since we prefer (9.6) here, but which are important for us since they support (9.6) by being equivalent to it. As we also mentioned in Section 7, the notion of Kiss difference suggests a simpler definition of a commutator, at least in a congruence modular variety, where it always exist. And indeed, as the congruence modular counterpart of Definition 6.1, we have

Theorem 9.5. (Follows from Theorem 5.5 in [108]) *Let $\mathbf{X}$ be a congruence modular variety with a Kiss difference term q, and A, R, and S be as above. The commutator $[R, S]$ is the smallest congruence T on A such that:*

(a) *the map*

$$\{(a,b,d,c) \in A^4 \mid ((a,b),(c,d) \in R) \ \&\ ((a,d),(b,c) \in S)]\} \to A/T, \tag{9.7}$$

sending (a,b,d,c) to the T-class of $q(a,b,d,c)$ is a homomorphism of algebras.

(b) $(q(a,b,d,c), q(a,b,d',c)) \in T$, *whenever both* (a,b,d,c) *and* $q(a,b,d',c)$ *belong to the domain of the map* (9.7).

Strictly speaking, this theorem cannot be used as a definition of commutator, since the commutators are already involved in the definition of a Kiss difference term; still, as soon as a Kiss difference term is fixed it is useful in calculating particular commutators in the same way as Definition 6.1 is in the Mal'tsev (=congruence permutable) case.

In fact consideration of commutators in [108] led to introducing there new notions of Kiss, Gumm, Lipparini, and abelianizable varieties, but we shall not consider them here since we know almost nothing about their categorical counterparts.

10 Explicit presentations of commutators via limits and colimits in Mal'tsev categories

There are two such presentations, due to M. C. Pedicchio [132] and due to D. Bourn [8], which assume the ground category **X** to be a Barr exact Mal'tsev category with coequalizers and a regular Mal'tsev category with finite colimits, respectively (although the requirement of having all finite colimits in [8] can also be weakened).

Pedicchio's construction of the commutator $[R, S]$, of two equivalence relations R and S on an object A in a Barr exact Mal'tsev category **X** with coequalizers, extends one of the constructions known for algebras, and it can be described in several steps as follows:

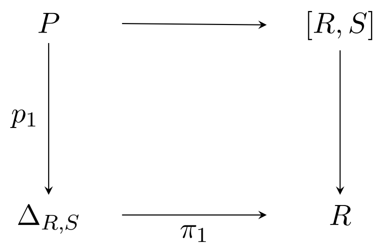
- Let us write (A, R, d_R, c_R, e_R) and (A, S, d_S, c_S, e_S) for R and S, respectively, considered as internal reflexive graphs in **X**. Take $q : R \to Q$ to be the coequalizer of morphisms $e_R d_S, e_R c_R : S \to R$.

- Define $\Delta_{R,S} = (\Delta_{R,S}, \pi_1, \pi_2)$ as the kernel pair of $R \to Q$.

- Construct the pullback

$$\begin{array}{ccc} P & \xrightarrow{p_2} & A \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle e_R} \\ \Delta_{R,S} & \xrightarrow[\pi_2]{} & R \end{array} \tag{10.1}$$

- Define $[R,S]$ via the (commutative) diagram



(10.2)

where the top horizontal arrow and the right-hand vertical arrow form the (regular epi, mono)-factorization of the composite $\pi_1 p_1$.

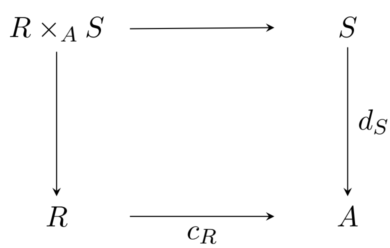
Briefly

$$[R,S] = \pi_1(\pi_2^{-1}(A)), \tag{10.3}$$

as it is written in [132] (with X instead of A).

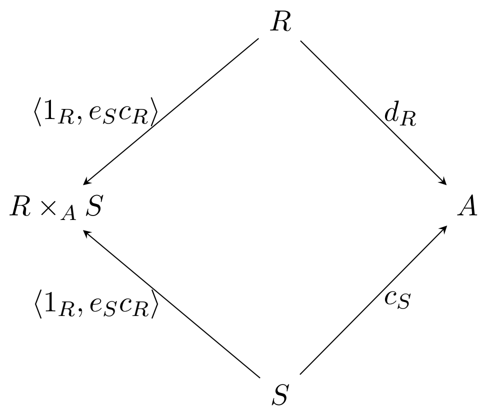
While the advantage of Pedicchio's approach is in showing that the construction of $[R,S]$ can be copied from the algebraic context, the advantage of Bourn's approach is in its simplicity. According to [8], we define $[R,S]$ as follows:

- Again we write (A,R,d_R,c_R,e_R) and (A,S,d_S,c_S,e_S) for R and S, respectively, considered as internal reflexive graphs in $\mathbf{X}$. But now we begin with the pullback



(10.4)

- After that we take M to be the colimit of the diagram



(10.5)

- And define $[R, S]$ as the kernel pair of the canonical morphism $A \to M$.

Remark 10.1.

(a) Further analysis shows that while Pedicchio's construction is a categorical counterpart of a construction in universal algebra (called there the commutator defined via "the term condition"), Bourn's construction is the direct realization of (6.4), since the canonical morphism $A \to M$ is in fact the same as the canonical morphism $(\eta'_G)_1$ used in (6.4), where G is the span $A/R \leftarrow A \to A/S$.

(b) Recall that, instead of internal Kock pregroupoids, Bourn is actually working with slightly different structures introduced in his joint papers with Gran [12] and [13] and called *connectors* there. Omitting various elegant results of those two papers on centrality in regular Mal'tsev categories, let us only mention that an internal Kock pregroupoid in a category $\mathbf{X}$, whose underlying span is $A/R \leftarrow A \to A/S$, is the same as Bourn–Gran connector on the pair (R, S) satisfying suitable associativity condition; however, that condition is automatically satisfied when $\mathbf{X}$ is a Mal'tsev category. Still, as soon as $\mathbf{X}$ has non-effective equivalence relations even associative connectors become more general than internal Kock pregroupoids.

(c) The problem with commutators of (internal) non-effective equivalence relations is that the commutators themselves are always effective by the definition. In particular, when R and/or S are non-effective, the fundamental property $[R, S] \leq R \wedge S$ could fail to hold.

(d) As follows from the presentation (1.5) in [108], the notion of internal pseudogroupoid can be generalized in the same way as the notion of connector generalizes the notion of internal Kock pregroupoid. We are omitting details since we are not going to use this generalization here.

11 Back to central extensions

While the classical definition of central extension of groups is very straightforward, it has several well-motivated generalizations, out of which we will consider two, in Subsections 11.1 and 11.2, respectively; we shall compare them in Subsection 11.3. In fact there are at least two more:

- One of them is already considered, in the special case of Ω-groups.
- The other one defines central extensions as certain *torsors*; it can be easily compared with the one from Subsection 11.1 using the relationship between Galois theory and torsors in general, however that would take us far away from the main topics of this paper.

11.1 Central extensions categorically, via Galois theory

The context of [101] uses a Barr exact category $\mathbf{C}$ and a Birkhoff subcategory $\mathbf{X}$ of $\mathbf{C}$, which is defined as full reflective subcategory of $\mathbf{C}$ closed under subobjects and quotient objects. This determines a Galois structure Γ consisting of the reflection-inclusion adjunction $(I, H, \eta, \varepsilon) : \mathbf{C} \to \mathbf{X}$ and fibrations in $\mathbf{C}$ and in $\mathbf{X}$ being regular epimorphisms. And, exactly as in the special case considered in Example 2.8, this Galois structure is admissible whenever the semilattices of internal equivalence relations on objects in $\mathbf{C}$ are modular lattices (see Theorem 3.4 in [101]). And when this Galois structure Γ is admissible, a central extension of an object B in $\mathbf{C}$ is defined in as a covering of B with respect to Γ. That is, (A, f) is a central extension of B, if and only if $f : A \to B$ is a regular epimorphism and there exists a regular epimorphism $p : E \to B$ for which the diagram

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\eta_{E\times_B A}} & HI(E \times_B A) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle HI(\pi_1)} \\ E & \xrightarrow[\eta_E]{} & HI(E) \end{array} \tag{11.1}$$

where $\pi_1 : E \times_B A \to E$ is the suitable projection, is a pullback (here we could omit H since it is an inclusion functor. This definition is formulated directly for $(\mathbf{C}, \mathbf{X})$ in [101], since no other Galois structure is used there. Furthermore, as proved in [101], when $\mathbf{C}$ is a Goursat category, every central extension is normal – which means that, in the Goursat case, we could equivalently take (E, p) to be (A, f) itself. Let us also mention a new proof of this result by M. Gran and D. Rodelo [76], and that V. Rossi [139] proved a similar result for *almost Barr exact* Mal'tsev categories, but only under the assumptions on $\mathbf{X}$ we will use below in this section.

11.2 Central extensions via the commutator theory

Let $f : A \to B$ be a regular epimorphism in a Barr exact category (we shall not consider here the more general context of a regular category) $\mathbf{C}$. When there is a chosen notion of commutator in $\mathbf{C}$, one might call (A, f) a central extension of B if

$$[\nabla_A, Eq(f)] = \Delta_A, \tag{11.2}$$

that is, when the commutator of the largest internal equivalence relation $\nabla_A = A \times A$ and the kernel pair of f is trivial. However, there is another possibility, which would be the same in when $\mathbf{C}$ is either a Mal'tsev category or a congruence modular variety of universal algebras (or just a Gumm variety in the sense of [108]). For, we observe:

- If the commutators are defined via (9.6), (11.2) would mean that

$(\eta''_G)_1$, for

$$G = (1 \longleftarrow A \xrightarrow{\quad f \quad} B) \tag{11.3}$$

is a monomorphism, or, equivalently that G is a subspan of the underlying span of an internal pseudogroupoid.

- Or, we could require instead the span (11.3) itself to be the underlying span of an internal pseudogroupoid – or of an internal Kock pregroupoid, which is equivalent since 1 is involved.

We shall choose this second option, and, if the span (11.3) itself is indeed the underlying span of an internal Kock pregroupoid, we shall say that (A, f) is an *algebraically central* extension of B. In fact this choice was made in [72].

11.3 The comparison results

The obvious distinction between the central extensions defined via (11.1), which we will call now categorically central, and the algebraically central extensions is that the categorical ones depend on a specified Birkhoff subcategory $\mathbf{X}$ of $\mathbf{C}$, while the algebraic ones do not. Therefore, and following the classical case of groups, let us assume that:

- $\mathbf{C}$ is a Barr exact category;
- $\mathbf{X}$ is the full subcategory of $\mathbf{C}$ formed by all affine objects, where an object A in $\mathbf{C}$ is said to be affine if the span $1 \leftarrow A \rightarrow 1$ admits an internal Kock pregroupoid structure;
- $\mathbf{X}$ is a Birkhoff subcategory of $\mathbf{C}$, the assumption to which we shall refer as *Birkhoff subcategory assumption*;
- The Galois structure described in Subsection 11.1 must be admissible; we shall refer to this as *admissibity assumption*.

The first comparison results (after immediate corollaries of the results mentioned in Sections 3 and 4), where the Birkhoff subcategory assumption follows from known properties of commutators (see e.g. [63]) and the admissibility follows from the already mentioned Theorem 3.4 in [101], were:

Theorem 11.1. [103]

(a) *If* **C** *is a congruence modular variety of universal algebras, then every categorically central extension in* **C** *is algebraically central.*

(b) *If* **C** *is a variety of* Ω*-groups, then a pair* (A, f) *is a categorically central extension if and only if it is algebraically central.*

Theorem 11.2. (Theorem 4.3(b) of [104]) *If* **C** *is a Mal'tsev variety of universal algebras, then a pair* (A, f) *is a categorically central extension in* **C** *if and only if it is algebraically central.*

Theorem 11.3. [71] *If* **C** *is a congruence modular variety of universal algebras, then a pair* (A, f) *is a categorically central extension in* **C** *if and only if it is algebraically central.*

Here Theorem 11.2 implies Theorem 11.1(b), and Gran's Theorem 11.3 implies both Theorem 11.1 and Theorem 11.2 of course. Moreover, Gran's Theorem 11.3 is a kind of final result, which allows us to claim that categorical Galois theory and commutator theory in universal algebra lead to the same notion of centrality. On the other hand, since commutator theory related techniques can be developed in a categorical context, a purely-categorical generalization of Theorem 11.3 would be a natural next step. Gran's Theorem 11.6 below is an important result of such kind. In order to formulate it we need part (a) of the following:

Definition 11.4. [72] A regular category **C** is said to be factor permutable, if, for every product diagram

$$K \xleftarrow{\ \pi_1\ } K \times L \xrightarrow{\ \pi_2\ } L \tag{11.4}$$

and every internal equivalence relation R on $K \times L$, we have

$$(Eq(\pi_1))R = R(Eq(\pi_1)), Eq(\pi_2))R = R(Eq(\pi_2)) \tag{11.5}$$

(it suffices to require, say, only the first of these equalities of course).

Remark 11.5.

(a) Factor permutable categories were introduced as the precise categorical counterpart of factor permutable varieties in the sense of H. P. Gumm [84], and, according to Corollary 4.5 in [84], every congruence modular variety of universal algebras is factor permutable.

(b) Every Mal'tsev category and every strongly unital category in the sense of Bourn [7] is factor permutable.

Theorem 11.6. (See Corollaries 3.15 and 3.17, and Theorems 5.2 and 6.1 in [72]) *If* $\mathbf{C}$ *is a Barr exact factor permutable category whose category of affine objects is reflective in it, then the Birkhoff subcategory assumption and the admissibility assumption hold, and a pair* (A, f) *is a categorically central extension in* $\mathbf{C}$ *if and only if it is algebraically central.*

Note also that assuming the shifting property (see Definition 12.3 in the next section) instead of factor permutabity, Rossi proved the admissibility assumption in [139].

12 Back to internal groupoids

Let us begin by comparing the descriptions of internal groupoids in a Mal'tsev variety in Section 5 (originally from [93]) and in a congruence modular variety in Section 8 (originally from [107]). The first of them can be deduced from the second one using the implication (b)⇒(a) of the following theorem to the kernel pairs of the domain and codomain morphisms of an internal reflexive graph:

Theorem 12.1. *Let* R *and* S *be internal equivalence relations (=congruences) on an object* A *in a congruence modular variety* $\mathbf{X}$ *with a Kiss difference term* q*, and let* G *be the span* $A/R \leftarrow A \rightarrow A/S$ *in* $\mathbf{X}$*. The following conditions are equivalent:*

(a) $[R, S] = \Delta_A$;

(b) *the map* $m : G_4 \rightarrow G_1 = A$ *defined by* $m(a, b, d, c) = q(a, b, d, c)$ *is a morphism in* $\mathbf{X}$*, and* $q(a, b, d, c) = q(a, b, d', c)$ *whenever* (a, b, d, c) *and* (a, b, d', c) *are in* G_4*.*

However, this theorem, which is just a simplified version of Theorem 9.5, is identical to Theorem 3.8(iii) in [118], up to terminology and notation of course.

In order to change our context from varieties of algebras to abstract categories, we will need to replace congruence modularity with the *shifting property* formulated in Definition 12.3 below, which itself comes from universal algebra and is equivalent to congruence modularity in case of varieties (see Remark 12.4).

Theorem 12.2. *Let* $\mathbf{X}$ *be a category with finite limits,* A *an object in* $\mathbf{X}$*, and* R*,* S*, and* T *internal equivalence relations on* A*. Then the following conditions are equivalent:*

(a) *if* $R \wedge S \leq T$*, then for every object* X *in* $\mathbf{X}$ *and morphisms* $a, b, c, d : X \to A$*, the morphism* $\langle c, d\rangle : X \to A \times A$ *factors through* (*the canonical morphism*) $T \to A \times A$ *whenever* $\langle a, b\rangle$ *and* $\langle c, d\rangle$ *factors through* $R \to A \times A$*,* $\langle c, a\rangle$ *and* $\langle b, d\rangle : X \to A \times A$ *factor through* $S \to A \times A$*, and* $\langle a, b\rangle$ *factors through* $T \to A \times A$*;*

(b) *condition (a) under the additional assumption* $T \leq R$*;*

(c) *for every internal equivalence relation* T *on* A *with* $R \wedge S \leq T \leq R$*, the diagram*

$$\begin{array}{ccc} T\square S & \rightrightarrows & T\square R \\ \downarrow & & \downarrow \\ T & \rightrightarrows & R \end{array} \qquad (12.1)$$

of canonical morphisms is a discrete fibration of internal equivalence relations, that is, the square involving the first and the third (*or, equivalently, the second and the forth*) *horizontal arrow in it is a pullback.*

Definition 12.3. The category $\mathbf{X}$ is said to satisfy the shifting property if it satisfies the equivalent conditions of Theorem 12.2.

Remark 12.4. (a) The shifting property formulated as 12.2(a) is a straightforward generalization of its universal-algebraic version introduced earlier by H. P. Gumm [84], where the roles of a, b, c, d were played by elements of A. It was proved in [84] that a variety of universal algebras satisfies it if and only if it is congruence modular ("The Shifting Lemma"). The categorical version of shifting property, condition 12.2(c), and its equivalence to 12.2(a) are due to D. Bourn and M. Gran [14]. The implication (a)⇒(b) is trivial, while the implication (a)⇒(b) is almost so: just replace T with its meet with R.
(b) When the ground category **X** is regular, conditions 12.2(a) and 12.2(b) can be formulated, respectively, as follows:

$$R \wedge S \leq T \Rightarrow (S(R \wedge T)S) \wedge R \leq T, \tag{12.2}$$

$$R \wedge S \leq T \leq R \Rightarrow (STS) \wedge R \leq T. \tag{12.3}$$

Suppose now **X** is not only regular but also *modular*, in the sense that the semilattices of internal equivalence relations on objects in **C** are modular lattices (this condition, different from *modularity in the sense of A.* Carboni [21], was already mentioned in Subsection 11.1 and before). In this case (12.3) becomes trivial:

$$(STS) \wedge R \leq (S \vee T) \wedge R \leq (R \wedge S) \vee T = T,$$

which, in the case of algebras, was Gumm's observation. That is, every modular regular category has the shifting property. This observation was known to D. Bourn and M. Gran when they wrote [14], as follows from the explanation in their Example 2.4.2; some further comparison results are also made by D. Bourn in [9]. However, as the Example 12.5 below shows, the shifting property does not imply modularity even in the case of varieties of infinitary universal algebras. Unfortunately no such clarification has been made for factor permutability: we only know that:

- not every factor permutable exact category is modular – since this was known already for varieties of universal algebras;
- adding infinitary operations to the signature of a modular variety does not make it non-factor-permutable – since neither shifting condition nor factor permutability will fail after adding infinitary operations;

- the shifting condition restricted to *effective* equivalence relations does not imply factor permutability (even for effective equivalence relations), as a simple example in the quasi-variety of commutative monoids with cancellation constructed by Z. Janelidze [114] shows.

Example 12.5. Let **X** be the variety of distributive lattices equipped with an additional operation λ of countable arity. We do not assume the existence of 0 and/or 1, but assuming their existence would require only a minor modification of what we are saying. We take:

- $\mathbb{N}$ to be the set $\{0, 1, 2, ...\}$ of natural numbers with its usual order, considered as a lattice;

- A to be the sublattice set of $\mathbb{N} \times \mathbb{N}$ consisting of all pairs (n, m) of natural numbers with $|n - m| \leq 1$, with the operation λ defined by

$$\lambda(a_1, a_2, a_3, ...) = \begin{cases} (0,0), \text{if the distances } d(a_n, a_{n+1}) \ (n \in \mathbb{N}) \text{ are bounded;} \\ (0,1), \text{if the distances } d(a_n, a_{n+1}) \ (n \in \mathbb{N}) \text{ are unbounded;} \end{cases}$$

- $R = \{((n, m), (n', m')) \in A \times A \mid m = m'\}$;

- $S = \{((n, m), (n', m')) \in A \times A \mid n = n'\}$;

- T has equivalence classes $\{(0, 0)\}$, $\{(0, 1)\}$, $\{(1, 0)\}$, $\{(1, 1), (1, 2)\}$, and all other equivalence classes as in R.

It is then easy to check that R, S, and T are congruences on A with $R \wedge S = T \wedge S = \Delta$ (=the equality relation) and $R \vee S = T \vee S = \nabla_A = A \times A$ (and $T \subseteq R$) which shows that the lattice of congruences on A is not congruence modular. On the other hand, **X** satisfies the shifting condition since the variety of distributive lattices does, and since the shifting condition involves only constructions (namely the intersection and composition of congruences) that do not depend on λ. Note that the underlying lattices of A, R, and S here are the same as in Remark 3.4 of [26].

The most general known categorical version of Theorem 8.1 (together with Gran's 8.2(d)) is

Theorem 12.6. (Corollaries 5.2 and 5.3 in [14], formulated slightly differently) *Let $G = (G_0, G_1, d, c, e)$ be an internal reflexive graph in a regular category* **X** *satisfying the shifting property. The following conditions are equivalent:*

(a) *there exists an internal groupoid in* **X** *whose underlying internal reflexive graph is G;*

(b) *there exists a unique internal groupoid in* **X** *whose underlying internal reflexive graph is G;*

(c) *there exists an internal Kock pregroupoid in* **X** *whose underlying span is*

$$G_0 \xleftarrow{\quad d \quad} G_1 \xrightarrow{\quad c \quad} G_0$$

(d) *there exists a unique internal Kock pregroupoid in* **X** *whose underlying span is as in (c);*

(e) *there exists an internal pseudogroupoid in* **X** *whose underlying span is as in (c), and (G_1, d, c) induces a symmetric and transitive relation on G_0;*

(*f*) *there exists a unique internal pseudogroupoid in* **X** *whose underlying span is as in (c), and (G_1, d, c) induces a symmetric and transitive relation on G_0;*

(g) *there exists an internal pseudogroupoid in* **X** *whose underlying span is as in (c), and*

$$(Eq(d))(Eq(c)) = (Eq(c))(Eq(d));$$

(h) *there exists a unique internal pseudogroupoid in* **X** *whose underlying span is as in (c), and*

$$(Eq(d))(Eq(c)) = (Eq(c))(Eq(d)).$$

Furthermore, if these conditions hold, then $[Eq(d), Eq(c)] = \Delta_{G_1}$, *where the commutator* $[Eq(d), Eq(c)]$ *is defined as the intersection of all* $Eq(\alpha_1)$, *for all morphisms* α *from the span formed by* d *and* c *to an underlying span of an internal pseudogroupoid; in particular, this commutator does exist.*

Remark 12.7.

(a) We do not know whether the $[Eq(d), Eq(c)] = \Delta_{G_1}$ implies the existence of an internal pseudogroupoid structure on the span

$$G_0 \xleftarrow{\;d\;} G_1 \xrightarrow{\;c\;} G_0$$

otherwise we could formulate Theorem 12.6 similarly to Theorem 8.1 (since all other conditions would obviously be equivalent to the listed ones).

(b) The advantage of using regular categories instead of Barr exact ones is apparent even in the purely universal-algebraic context – since all quasi-varieties of algebras are regular, and a quasi-variety is Barr exact if and only if it is a variety.

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