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On descent for coalgebras and type transformations

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Abstract. We find a criterion for a morphism of coalgebras over a Barr-exact category to be effective descent and determine (effective) descent morphisms for coalgebras over toposes in some cases. Also, we study some exactness properties of endofunctors of arbitrary categories in connection with natural transformations between them as well as those of functors that these transformations induce between corresponding categories of coalgebras. As a result, we find conditions under which the induced functors preserve natural number objects as well as a criterion for them to be exact. Also this enable us to give a criterion for split epis in a category of coalgebras to be effective descent.

1 Introduction

Every natural transformation $\eta : F \to G$ between two endofunctors (or types) of a category C induces a functor between categories of coalgebras C_F and C_G. This was already observed by Rutten in [34] for the category Set of sets and maps between them and other facts about natural transformations

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and coalgebras were proved by Gumm in [11, 13]. Following [25], this paper, inter alia, aims at extending some of these facts to categories of coalgebras beyond Set and at studying the behavior of the induced functor with respect to some special morphisms and (co)limits.

Of a particular interest are (effective) descent morphisms. While a morphism in a finitely complete category is a descent morphism if and only if it is a pullback-stable regular epi [8], the characterization of effective descent morphisms is far more complicated, and in many concrete cases analyzing its meaning can be a challenging problem. Amidst some well-known, there are two known characterizations of effective descent morphisms of topological spaces, due to J. Reiterman and W. Tholen (see [32] and its reformulation due to M.M. Clementino and D. Hofmann, see [7] (and the references therein) where a description of effective descent morphisms of (ordinary) preorders can also be found. Moreover, Joyal and Tierney proved that open surjections are effective descent morphisms in the category of locales [22]. This is also the case in that of (Grothendieck) toposes [31]. In Barr-exact categories, the effective descent morphisms are precisely the regular epis [20]. This is also the case in locally cartesian closed ones [18, 28]. In the category of torsion-free abelian groups, every surjective homomorphism is an effective descent morphism [18]. Furthermore, a homomorphism $R \to S$ of commutative rings is an effective descent morphism if and only if it is a pure mono of $R-$modules [19]. A variety, i.e. a full subcategory of a category of structures for a first order (one sorted) language closed under substructures, products and homomorphic images, is a regular category not necessarily exact for which effective descent morphisms are exactly the regular epis (strong surjective homomorphisms) [33]. In this paper, we investigate (effective) descent morphisms in categories of coalgebras over endofunctors as well as their preservation and their reflection in connection with the functor induced by natural transformations between types. Following Section 2 which is devoted to preliminaries, we deal with descent for coalgebras in Section 3. We show that in the category of coalgebras of a pullback preserving endofunctor of a Barr-exact category morphisms which are regular epis therein are effective descent morphisms. For pullback preserving endofunctors generating cofree comonads over toposes, (effective) descent morphisms are precisely epis. In Section 4, we discuss exactness properties of the induced functor as well as its action on some special morphisms including (effective) descent morphisms. This enables us to find conditions for a category of coalgebras over a topos to be Barr-exact not necessarily a topos as well as those for every split epi in a category of coalgebras to be effective descent.

2 Preliminaries

This section is devoted to the presentation of basic tools needed in this paper.

2.1 Some special natural transformations, morphisms and functors

Recall (for example, from [9]) that a morphism $e : A \to B$ in C is called a *strong epi* provided that for all morphisms $f : A \to C$, $g : B \to D$ and all mono $m : C \to D$ such that $m \circ f = g \circ e$, there exists a unique morphism $d : B \to C$ such that $d \circ e = f$ and $m \circ d = g$. The dual concept is that of *strong mono.* It is shown in the above reference that a morphism is an iso iff it is a strong epi and a mono, the class of strong epis is closed under composition, u is a strong epi when so are $v \circ u$ and v and that, in case C has binary products, every strong epi is an epi and every regular epi is a strong epi. In [3], the notion of strong epi is defined as above but by assuming that e is an epi. We shall adopt this last definition, unless we expressly mention the contrary.

Given two functors $F, G : \mathsf{C} \to \mathsf{D}$, a *transformation* $\eta : F \to G$ is a class of morphisms $(\eta_X : F(X) \to G(X))_{X \in \mathsf{C}}$ in D called *components* of the transformation. Saying *(strong) mono*, *epi* or *retraction* about a transformation, we mean a component-wise such. Every morphism $\varphi : X \to Y$ in C gives rise to a square in D called the *transformation square* for φ and η is called *natural* in case it is commutative, and *cartesian* when it is a pullback, for all φ [11, 13]. Of course cartesian implies natural and to check that natural isos are mono and cartesian is routine.

Example 2.1. For $\mathsf{C} = \mathsf{Set} = \mathsf{D}$, consider F and G defined by $F(X) = B \times X^A$ and $G(X) = (B \times X)^A$, then the natural transformation $\eta_X : F(X) \to G(X)$, defined for every $b \in B$, $\phi \in X^A$ and $a \in A$ by $\eta_X \langle b, \phi \rangle (a) = \langle b, \phi(a) \rangle$, changes a Moore-automaton $\sigma : X \to B \times X^A$ with input alphabet A and output alphabet B, into a Mealy-automaton $\eta_X \circ \sigma : X \to (B \times X)^A$, for any sets A and B [34]. η is mono unless $A = \emptyset \neq B$, and cartesian [25].

Proof. To check that η is natural and η_X is an injective map for every set X unless $A = \emptyset \neq B$ is routine. Let's show that the square below is a pullback for any map $f : X \to Y$. Since η is natural, it is commuting. Let $(B \times X)^A \xleftarrow{\pi_1} Z \xrightarrow{\pi_2} B \times Y^A$ be a span such that $\eta_Y \circ \pi_2 = (id_B \times f)^A \circ \pi_1$. Let's find a unique map $\zeta : Z \to B \times X^A$

$$\begin{tikzcd}
B \times X^A \arrow[r, "id_B \times f^A"] \arrow[d, "\eta_X"'] & B \times Y^A \arrow[d, "\eta_Y"] \\
(B \times X)^A \arrow[r, "(id_B \times f)^A"'] & (B \times Y)^A
\end{tikzcd}$$

such that $\eta_X \circ \zeta = \pi_1$ and $(id_B \times f^A) \circ \zeta = \pi_2$. In case ζ exists, it is unique since η is mono. For any $z \in Z$, set $\pi_1(z) := \langle b_z^1, \phi_z^1 \rangle : A \to B \times X : a \mapsto (b_z^1, x_z^a)$ and $\pi_2(z) := \langle b_z^2, \phi_z^2 \rangle$ where $b_z^2 \in B$ and $\phi_z^2 : A \to Y$. Then $\eta_Y \circ \pi_2(z) : A \to B \times Y$ is defined by $(\eta_Y \circ \pi_2(z))(a) = (b_z^2, \phi_z^2(a))$ and $(id_B \times f)^A \circ \pi_1(z) : A \to B \times Y$ is defined by $((id_B \times f)^A \circ \pi_1(z))(a) = ((id_B \times f)^A(\langle b_z^1, \phi_z^1 \rangle(a)) = (b_z^1, f(x_z^a))$. Thus $\eta_Y \circ \pi_2 = (id_B \times f)^A \circ \pi_1$ implies $b_z^1 = b_z^2$ and, for every $a \in A$, $\phi_z^2(a) = f(x_z^a)$. Consider $\varphi : Z \to B \times X^A$ defined by $\varphi(z) = \langle b_z^1, \phi_z \rangle$ where $\phi_z : A \to X : a \mapsto x_z^a$. Clearly $\eta_X \circ \varphi = \pi_1$. Likewise, $((id_B \times f^A) \circ \varphi)(z) = (id_B \times f^A)(\varphi(z)) = (id_B \times f^A)(\langle b_z^1, \phi_z \rangle) = \langle b_z^1, f \circ \phi_z \rangle = \langle b_z^2, f \circ \phi_z \rangle$. Now $f \circ \phi_z : A \to Y$ is defined by $(f \circ \phi_z)(a) = f(\phi_z(a)) = f(x_z^a) = \phi_z^2(a)$. Thus $(id_B \times f^A) \circ \varphi = \pi_2$. Set $\zeta = \varphi$. $\square$

It is well known that the domain of a mono and cartesian transformation preserves pullbacks whenever its codomain does ([25], Lemma 2.2). In order to extend this to arbitrary limits, we prove the following:

Lemma 2.2. *If in a category the illustrated square with u monic is a pullback for each $i \in I$, $(u_i)_{i \in I}$ is a natural transformation and the bottom line defines a limit diagram, then so does the top line.*

$$\begin{tikzcd}
L \arrow[r, "l_i"] \arrow[d, "u"'] & A_i \arrow[d, "u_i"] \\
L' \arrow[r, "l'_i"'] & A'_i
\end{tikzcd}$$

Proof. Let $(l''_i : L'' \to A_i)_{i \in I}$ be a natural source in C. Then for the induced natural source $(u_i \circ l''_i : L'' \to A'_i)_{i \in I}$, the universal property of the limit yields a unique morphism $c : L'' \to L'$ such that, for each $i \in I$, $l'_i \circ c = u_i \circ l''_i$. For each i, the universal property of the pullback yields a unique morphism $a_i : L'' \to L$ such that $l_i \circ a_i = l_i$" and $u \circ a_i = c$. Since u is monic, all the a_i's are identical to a morphism a. Let $b : L'' \to L$ be a morphism such that $l_i \circ b = l''_i$ for each $i \in I$. Then $u_i \circ l_i \circ b = u_i \circ l''_i$; i.e, $u_i \circ l_i \circ b = u_i \circ l_i \circ a$ so that, $l'_i \circ u \circ b = l'_i \circ u \circ a$, for each $i \in I$. Therefore, since u is monic and the l'_i's are jointly monic, it follows that $b = a$. $\square$

The next result justifies ([25], Lemma 2.2) and gives an instance of Lemma 2.2.

Proposition 2.3. *If in the illustrated commutative cube in a category the front face is a pullback and ρ is monic, then the back face is a pullback if the left face is a pullback and σ is monic. In particular, this is the case if left and top faces are pullbacks.*

$$\begin{tikzcd}
A \arrow[rr, "\varphi"] \arrow[dr, "\rho"'] \arrow[dd, "u"'] & & B \arrow[dr, "\sigma"] \arrow[dd, "v"] & \\
& E \arrow[rr, "\nu"] \arrow[dd, "k"] & & F \arrow[dd, "\omega"] \\
D \arrow[rr, "\psi"] \arrow[dr, "\delta"'] & & C \arrow[dr, "\tau"'] & \\
& H \arrow[rr, "\mu"'] & & G
\end{tikzcd}$$

Proof. Assume that the left face is a pullback and the edge σ is a mono and let (P, α, β) be a source such that $v \circ \alpha = \psi \circ \beta$. Then $\tau \circ v \circ \alpha = \tau \circ \psi \circ \beta$, that is, $\omega \circ \sigma \circ \alpha = \mu \circ \delta \circ \beta$. Now the front face is a pullback. Thus there exists a unique morphism $\lambda : P \to E$ such that $\nu \circ \lambda = \sigma \circ \alpha$ and $k \circ \lambda = \delta \circ \beta$. Since the left face is also a pullback, it follows from the last equality that there exists a unique morphism $\theta : P \to A$ such that $u \circ \theta = \beta$ and $\rho \circ \theta = \lambda$. Thus $\sigma \circ \alpha = \nu \circ \lambda = \nu \circ \rho \circ \theta = \sigma \circ \varphi \circ \theta$. Therefore, σ mono implies $\alpha = \varphi \circ \theta$. If $\varepsilon : P \to A$ is a morphism in C such that $\varphi \circ \varepsilon = \alpha$ and $u \circ \varepsilon = \beta$. Then $\sigma \circ \varphi \circ \varepsilon = \sigma \circ \alpha$ and $\delta \circ u \circ \varepsilon = \delta \circ \beta$, that is, $\nu \circ \rho \circ \varepsilon = \nu \circ \lambda$ and $k \circ \rho \circ \varepsilon = k \circ \lambda$.

Now ν and k are jointly monic. Thus $\rho \circ \varepsilon = \lambda$ and since ρ is monic, one deduces that $\varepsilon = \theta$. The "in particular" statement follows from Lemma 2.2 or from the fact that if the top face is a pullback too, then σ is monic. $\square$

The following inter alia extends ([25], Lemma 2.2):

Lemma 2.4. *If the codomain of a mono and cartesian transformation preserves limits of diagrams over a given scheme, then so does its domain. Likewise, the domain of a strong mono natural transformation preserves strong monos if its codomain does.*

Proof. The first statement follows straightforwardly from Lemma 2.2. Now assume that η is natural, strong mono and G preserves strong monos. Let $\varphi : A \to B$ be a strong mono in C. We claim that $F(\varphi)$ is a mono in D: Indeed, let $\theta, \lambda : E \to F(A)$ be morphisms such that $F(\varphi) \circ \theta = F(\varphi) \circ \lambda$. Then $\eta_B \circ F(\varphi) \circ \theta = \eta_B \circ F(\varphi) \circ \lambda$; i.e., $G(\varphi) \circ \eta_A \circ \theta = G(\varphi) \circ \eta_A \circ \lambda$. Now φ is a strong mono, G preserves strong monos and η is strong mono. Thus $G(\varphi) \circ \eta_A$ is a composite of strong monos. Therefore it is a strong mono, and hence a mono. Thus, canceling it from the left, one has $\theta = \lambda$. Now let $e : C \to D$ be an epi, $u : C \to F(A)$ and $v : D \to F(B)$ be morphisms in D such that $F(\varphi) \circ u = v \circ e$. Then $\eta_B \circ v \circ e = \eta_B \circ F(\varphi) \circ u$; i.e., $\eta_B \circ v \circ e = G(\varphi) \circ \eta_A \circ u$. Therefore, since $G(\varphi)$ is a strong mono there exists a unique morphism $\delta : D \to G(A)$ such that $G(\varphi) \circ \delta = \eta_B \circ v$ and $\delta \circ e = \eta_A \circ u$. Now η_A is a strong mono in D. Thus from the second equality, it follows that there exists a unique $\mu : D \to F(A)$ such that $\mu \circ e = u$ and $\eta_A \circ \mu = \delta$. But then, one has $F(\varphi) \circ \mu \circ e = F(\varphi) \circ u = v \circ e$ and since e is an epi, it holds $F(\varphi) \circ \mu = v$. $\square$

Remark 2.5. In case the category C has binary coproducts, the second part of Lemma 2.4 still holds when we consider strong mono in the sense of [9].

Recall the following definitions from, for example, [26]. By a *conservative* functor is meant one which reflects isos. Moreover, a morphism is an *extremal epi* when it does not factor through any proper subobject of its codomain and a *pullback-stable* morphism is one belonging to a class of morphisms which is stable under pullback. It is not hard to check that every strong epi in the sense of [9] is an extremal epi and that (as in [3] where an extremal epi is already an epi) a morphism is an iso if and only if it is both an extremal epi and a mono; moreover, in case C has epi-mono factorizations or has equalizers, every extremal epi must actually be an epi. For later use, we record the following which summarizes some results of ([26], p.2):

Lemma 2.6. *Let $V : \mathsf{C} \to \mathsf{D}$ be a conservative functor. If V preserves monos, then it reflects extremal epis. If in addition C and D have and V preserves pullbacks, then V reflects pullback-stable extremal epis.*

2.2 (Effective)descent morphisms

Given an object $B \in \mathsf{C}$, the *slice category* $(\mathsf{C} \downarrow B)$ is the one whose objects are pairs (A, φ), where $\varphi : A \to B$ is a morphism in C, and whose morphisms are $\psi : (A, \varphi) \to (A', \varphi')$, where $\psi : A \to A'$ is a morphism in C such that $\varphi' \circ \psi = \varphi$. When C has pullbacks, every morphism $p : E \to B$ in C induces a functor $p! : (\mathsf{C} \downarrow E) \to (\mathsf{C} \downarrow B)$ sending $\varphi : A \to E$ to $p \circ \varphi : A \to B$. This functor has a right adjoint $p^* : (\mathsf{C} \downarrow B) \to (\mathsf{C} \downarrow E)$ (known as *change-of-base* functor) given by pulling back along p. Denoting by $\mathsf{Des}(p)$ the Eilenberg-Moore category of algebras of the monad induced by the adjunction $p! \dashv p^*$, we recall (for example, from [18]):

Definition 2.7. In a category C with pullbacks, a morphism $p : E \to B$ is said to be a *descent* (respectively, an *effective descent*) morphism if the pullback functor $p^* : (\mathsf{C} \downarrow B) \to (\mathsf{C} \downarrow E)$ is premonadic (respectively, monadic), i.e. if the comparison functor $\Phi^p : (\mathsf{C} \downarrow B) \to \mathsf{Des}(p)$ is fully faithful (respectively, an equivalence of categories).

By definition, the effective descent morphisms $p : E \to B$ are those morphisms which facilitate an algebraic description of $(\mathsf{C} \downarrow B)$ by means of $(\mathsf{C} \downarrow E)$ [20]. A sufficient condition for a morphism to be effective descent is given by:

Theorem 2.8. ([26], Theorem 7) *Let C and D be categories with pullbacks and $V : \mathsf{C} \to \mathsf{D}$ a conservative functor that preserves pullbacks. Assume further that D has coequalizers and C has and V preserves coequalizers of V-split pairs. Then, if the image under V of a morphism $f : A \to B$ in C is a split epi, then f is an effective descent morphism in C.*

In a category with finite limits, the *power object* of an object A (if it exists) is an object $\mathbf{P}A$ which represents $Sub(-\times A)$, so that $Hom(-, \mathbf{P}A) \cong Sub(- \times A)$ naturally and such a category is a *topos* if every object has a power object [6].

In this paper, we adopt the definition of equivalence relation from p.17 of [9], where the category is assumed to have finite limits. Recall (for example, from [9, 21, 29]) that an equivalence relation in a category is *effective* if it is the kernel pair of some morphism, a *regular* category is a finitely complete category with coequalizers of kernel pairs in which regular epis are stable under pullback, and an *exact* or a *Barr-exact* category is one which is regular and every equivalence relation is effective. Examples and counterexamples of Barr-exact categories may be found in [9, 33]. We have:

Lemma 2.9. *Every topos is Barr-exact. In a finitely complete category, a descent morphism is the same as a pullback-stable regular epi. In a topos (Barr-exact category), a descent morphism is the same as an effective descent one, i.e., an epi (a regular epi).*

Proof. Any topos has finite (co)limits and epis therein are stable under pullback and are regular [6]. Thus it is a regular category. See also [27]. Moreover, in a topos every equivalence relation is effective [6, 28]. The second assertion is Lemma 1.2.1 in [8] and together with the well-known fact that in a topos (Barr-exact category) an effective descent morphism is the same as an epi [28] (a regular epi [20]) yield the last one. □

3 Coalgebras and (effective) descent morphisms

In this section, we investigate (effective) descent morphisms in categories of coalgebras. Here and throughtout, $F : \mathsf{C} \to \mathsf{C}$ is an *endofunctor*, also called a *type* [15].

3.1 Some facts about coalgebras

A *coalgebra* is a pair $\mathcal{A} = (A, \alpha : A \to F(A))$. A morphism from (A, α) to (B, β) is a morphism $\varphi : A \to B$ such that $F(\varphi) \circ \alpha = \beta \circ \varphi$. With morphisms between them, coalgebras form a category [1, 4]. Denote it by C_F and by $U_F : \mathsf{C}_F \to \mathsf{C}$ the forgetful functor that sends every coalgebra to its carrier. It will simply be denoted by U if it is clear from the context. When U has a right adjoint (or, equivalently, F generates a cofree comonad, or, U is comonadic (see for example [21])), F is called a *covarietor*, see for example [1, 4] where it is shown that C_F has all colimits that exist in C as well as all limits that are preserved by F. In fact, they are created by the forgetful functor. As an immediate consequence thereof, epis (isos) in C_F are precisely morphisms which are epis (isos) in C so that U reflects epis (isos). Since U creates existing colimits it creates existing coequalizers in particular. Therefore, it reflects them. However, generally, it does not reflect regular epis (take for example, the morphism $!_{\mathcal{A}}$ in the proof of Example 4.16 below). In case C is a topos it does preserve them simply because every epi in C is regular. In cases where either C is the category of sets or has epi-strong mono factorizations with F preserving strong monos, strong monos in C_F are morphisms which are strong monos in C and C_F has also epi-strong mono factorizations created by U, see [1]. Thus, Lemma 2.4 immediately yields:

Proposition 3.1. *If the codomain of a mono and cartesian transformation between endofunctors of a complete category preserves limits, then the corresponding categories of coalgebras are both complete. Moreover, the category of coalgebras of the domain of a strong mono natural transformation whose codomain preserves strong monos has epi-strong mono factorizations if the base category has.*

Recall (for example from [24]) that a coalgebra $\mathcal{A}$ is called *extensional* provided that it is domain of monos only. See also [12, 17]. A characterization thereof is given by:

Lemma 3.2. (p. 300, [24]) *If the category C has coequalizers or a final coalgebra exists, then for every coalgebra $\mathcal{A}$, the following are equivalent:*

(1) *$\mathcal{A}$ is extensional;*

(2) *for all coalgebra $\mathcal{B}$, $|Hom(\mathcal{B}, \mathcal{A})| \leq 1$;*

(3) *$\mathcal{A} \cong \mathcal{A} \times \mathcal{A}$.*

Below is a result on the structure of categories of coalgebras:

Lemma 3.3. ([21], p.102) *If C is a topos and F is a pullback preserving functor which generates a cofree comonad (for example a partial product functor), then C_F is a topos.*

3.2 (Effective) descent morphisms for coalgebras

Lemmata 3.3 and 2.9 yield:

Proposition 3.4. *For every pullback preserving covarietor over a topos, a morphism of coalgebras is a* (*an effective*) *descent morphism iff it is an epi.*

The category Set is a topos and examples of covarietors over it can be found in [1, 4]. For each of them the category of coalgebras is complete [1]. Thus putting together Lemma 2.9 and Proposition 3.4, we obtain:

Corollary 3.5. *For any covarietor over* Set*, the class of all descent morphisms coincides with the class of all pullback-stable regular epis and, for a pullback preserving one, the class of all* (*effective*) *descent morphisms coincides with the class of epis, in the corresponding category of coalgebras.*

Recall (for example from [3]) that by a *strongly complete* category is meant one that is complete and has intersections. If the category C is a (strongly) complete topos, then for each object A, one has the functorial adjunction $A \times - \dashv (-)^A$ thanks to the fact that every topos is cartesian closed [3, 6]. Thus $(-)^A$ preserves all limits. Also, if C is any Boolean topos (and not a general topos), the finite-power object functor $\mathbf{K}$ (*Kuratowski functor*) (i.e., the subfunctor of the covariant power object functor $\mathbf{P}$ given by the construction of power objects such that $\mathbf{K}(A)$ may be thought of as 'the object of finite subobjects of A') does preserve weak pullbacks (see [21], p. 93). In case $\mathsf{C} = \mathsf{Set}$, $\mathbf{K}$ is the finite-powerset functor $\mathcal{P}_\omega$ and it is a covarietor (see [1, 4, 21]) as well as $(-)^A$, for as a polynomial functor, the latter preserves weak pullbacks (or equivalently, it does weakly preserve pullbacks) [12, 14, 34], and is a partial product functor [21]. Moreover, for every weak kernel preserving endofunctor over Set, every epi (mono) is regular [14, 15]. Thus Corollary 3.5 yields:

Example 3.6. For every weak kernel preserving covarietor over Set, the class of descent morphisms coincides with the class of pullback-stable epis. In particular, for any set A, in $\mathsf{Set}_{(-)^A}$, the class of (effective) descent morphisms coincides with that of epis.

Since the category of coalgebras has all limits which are preserved by the type, Lemma 2.9 straightforwardly yields:

Proposition 3.7. *For every finite limit preserving endofunctor on a finitely complete category, a descent morphism in the corresponding category of coalgebras is the same as a pullback-stable regular epi.*

The forgetful functor $U_F : \mathsf{C}_F \to \mathsf{C}$ doesn't reflect extremal epis in general (take again for example, the morphism $!_{\mathcal{A}}$ in the proof of Example 4.16 below) but we have:

Theorem 3.8. *The forgetful functor U_F reflects extremal monos. For a pullback preserving endofunctor on a category with pullbacks, it reflects effective descent morphisms. In particular, so do forgetful functors U_F and U_G when there exists a mono and cartesian transformation $F \to G$ with G preserving pullbacks.*

Proof. For the first two statements, use the dual of Lemma 2.6 and Theorem 15 of [26]. By invoking moreover Lemma 2.4, the "in particular" statement follows. □

From (the proof of) Lemma 2.9 and Theorem 3.8, we deduce:

Proposition 3.9. *For every pullback preserving endofunctor on a topos (Barr-exact category), morphisms of coalgebras carried by epis (regular epis) are effective descent morphisms.*

Example 3.10. The subfunctor $T : \mathsf{Set} \to \mathsf{Set}$ of the exponential functor $(-)^{\mathbb{N}}$ defined by $T(X) = \{f \in X^{\mathbb{N}} : (\exists m)(\forall n \geq m)(\forall n' \geq m) f(n) = f(n')\}$ preserves all finite limits but does not preserve the infinite product $P = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times ...$ ([36], Example 2.2). Thus by Lemma 2.4 there does not exist any mono and cartesian transformation between T and $(-)^{\mathbb{N}}$. In particular, the trivial mono transformation $T \to (-)^{\mathbb{N}}$ given by set-inclusion is not cartesian. Since T preserves pullbacks, it does weakly preserve kernels so that every epi in Set_T is regular. Therefore, by Proposition 3.7 descent morphisms in Set_T are precisely pullback-stable epis and, by Proposition 3.9, every epi is an effective descent morphism. Therefore, in Set_T, the class of (effective) descent morphisms coincides with that of epis, as in $\mathsf{Set}_{(-)^{\mathbb{N}}}$ (see Example 3.6).

Regular epis may fail to be effective descent (see [18], Examples 2.7, pp: 261-262).

4 Functors induced by type transformations

In this section, we confine attention to some exactness properties of functors induced by natural transformations between endofunctors of categories and study their behavior with respect to some special morphisms including (effective) descent ones.

4.1 The induced functor and some of its basic properties

We introduce the functor induced by a natural transformation between types and give some basic properties thereof. Recall (from for example [3]) that a functor $\mathsf{C} \to \mathsf{D}$ is an *embedding* if it is injective on morphisms. This amounts to saying that it is faithful and injective on objects. If in addition, it is full, then C is said to be *fully embeddable* into D, i.e., C is isomorphic to a full subcategory of D.

Theorem 4.1. *Let F, G and H be C-endofunctors. Every natural transformation $\eta : H \circ F \to G \circ H$ induces a functor $H_\eta : \mathsf{C}_F \to \mathsf{C}_G$ defined as $H_\eta(A, \alpha) = (H(A), \eta_A \circ H(\alpha))$ and maps every homomorphism $\varphi : (A, \alpha) \to (B, \beta)$ into $H(\varphi) : H(A) \to H(B)$. If C has and H preserves binary products as well as monos, then H_η sends F-bisimulations (R, ρ) to G-bisimulations $(H(R), \eta_R \circ H(\rho))$. Moreover, if H is faithful, then so is H_η and, if in addition η is mono, then for all (A, α) and (B, β) in C_F and every morphism $\varphi : A \to B$ in C such that $H(\varphi) : (H(A), \eta_A \circ H(\alpha)) \to (H(B), \eta_B \circ H(\beta))$ is a homomorphism in C_G, φ is a homomorphism in C_F from (A, α) to (B, β). Also, if H is conservative, then so is H_η. It is an embedding if so is H and η is mono.*

Proof. We prove the last statement. For the remainder, see [34] and [25], Theorem 2.3. Assume that H is an embedding and η is mono. Then, H is faithful. Thus, to show that H_η is an embedding too, it suffices to show that it is injective on objects. Let $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ be coalgebras such that $H_\eta(\mathcal{A}) = H_\eta(\mathcal{B})$. Then $H(A) = H(B)$ and $\eta_A \circ H(\alpha) = \eta_B \circ H(\beta)$. Now H is injective on objects. Thus, $A = B$ and, since η is mono, one has $H(\alpha) = H(\beta)$ so that $\alpha = \beta$ for H is faithful. $\square$

Here and throughout, we call H_η the *functor induced by* η and $\widetilde{\eta} := Id_\eta$.

Corollary 4.2. *For every mono-natural transformation $\eta : F \to G$ in C and all F-coalgebras (A, α) and (B, β), a morphism $\varphi : A \to B$ is a morphism in C_F from (A, α) to (B, β) iff φ is a morphism in C_G from $(A, \eta_A \circ \alpha)$ to $(B, \eta_B \circ \beta)$. That is, the category C_F is fully embeddable into C_G: it is isomorphic to its image under $\widetilde{\eta}$.*

The following result will be frequently used:

Lemma 4.3. *The following statements are true:*

(1) *If η is strong mono, then the class $\widetilde{\eta}(\mathsf{C}_F)$ is closed under codomains of epis.*

(2) *If η is cartesian, then for every homomorphism $\varphi : (A, \alpha') \to \widetilde{\eta}(\mathcal{B})$ in C_G, there exists a unique morphism $\alpha : A \to F(A)$ such that $\alpha' = \eta_A \circ \alpha$ and $\varphi : (A, \alpha) \to \mathcal{B}$ is a homomorphism in C_F. In particular, $\widetilde{\eta}(\mathsf{C}_F)$ is closed under domains of homomorphisms.*

(3) *If η is retraction, then C_G and $\widetilde{\eta}(\mathsf{C}_F)$ have the same objects.*

Proof. For (1) and (2) see [25], Lemma 2.4. In case η is retraction and (A, α') is a coalgebra in C_G, $(A, \alpha') = \widetilde{\eta}((A, \theta_A \circ \alpha'))$, where θ_A is any right inverse of η_A. □

In case $\mathsf{C} = \mathsf{Set}$, every endofunctor gives rise to an epi and hence retraction transformation and to a (strong) mono one (see for example, [13], 4.1). Moreover, one easily checks that Lemma 4.3 (1) still holds when we consider strong mono in the sense of [9].

4.2 Behavior of the induced functor with respect to (co)limits and some special morphisms

For the notions of preservation, reflection, creation and (unique) lifting of limits, see for example [3], pp. 223-227, where it is shown (see Proposition 13.25) that a functor creates limits if and only if it lifts them uniquely and reflects them as well; but it does not necessarily preserve them. For the induced functor, we have:

Theorem 4.4. *Let η be as in Theorem 4.1. Then, H_η preserves all colimits of diagrams over a given scheme that C has and H preserves; and $H_\eta(\mathsf{C}_F)$*

is closed under these colimits. In particular, H_η preserves (reflects) epis whenever H does. Moreover, if H is faithful, then H_η reflects monos.

Proof. Assume that C has and H preserves colimits of diagrams over a scheme I. Then C_F and C_G have colimits of diagrams over the same scheme and they are created by the forgetful functor (see Section 3). Let $D : \mathsf{I} \to \mathsf{C}_F$ with $D(i) = \mathcal{A}_i := (A_i, \alpha_i)$ be a diagram in C_F and $(e_i : \mathcal{A}_i \to \mathcal{A})_{i\in\mathsf{I}}$ with $\mathcal{A} := (A, \alpha)$ a colimit thereof. That is, $(e_i : A_i \to A)_{i\in\mathsf{I}}$ is a colimit of the underlying diagram in C and $\alpha : A \to F(A)$ is the (unique) morphism in C making each e_i a morphism in C_F. We want to show that $(H(e_i) : H_\eta(\mathcal{A}_i) \to H_\eta(\mathcal{A}))_{i\in I}$ is a colimit of the diagram $H_\eta \circ D : I \to \mathsf{C}_G$. Let $(e'_i : (H(A_i), \eta_{A_i} \circ H(\alpha_i)) \to (A', \alpha'))_{i\in I}$ be a colimit of the $H_\eta(\mathcal{A}_i)$'s in C_G; that is $(e'_i : H(A_i) \to A')_{i\in I}$ is a colimit of the $H(A_i)$'s in C and $\alpha' : A' \to G(A')$ is a (unique) morphism in C making each e'_i a morphism in C_G. We want to show that $(H(A), \eta_A \circ H(\alpha)) \cong (A', \alpha')$. Since H preserves all colimits of diagrams over the scheme I in C, $(H(e_i) : H(A_i) \to H(A))_{i\in I}$ is also a colimit of the $H(A_i)$'s in C. Thus, the universal property of the colimit in C yields a unique $\lambda : H(A) \to A'$ such that, for all $i \in I$, $\lambda \circ H(e_i) = e'_i$. Likewise, by the universal property of the colimit in C_G, there exists a unique morphism $\theta : (A', \alpha') \to (H(A), \eta_A \circ H(\alpha))$ such that, for all $i \in I$, $H(e_i) = \theta \circ e'_i$. In particular, these equalities hold in C. Therefore, since the $H(e_i)$'s are jointly epic in C as well as the e'_i's, it follows from the two families of equalities that $\theta\circ\lambda = 1_{H(A)}$ and $\lambda\circ\theta = 1_{A'}$. Hence $\theta : A' \to H(A)$ is an iso in C and so is $\theta : (A', \alpha') \to (H(A), \eta_A \circ H(\alpha))$ in C_G thanks to the conservativeness of the forgetful functor U_G. To show that $H_\eta(\mathsf{C}_F)$ is closed under these colimits, let $T : I \to \mathsf{C}_G$ be a diagram in C_G with $T(i) := (A'_i, \alpha'_i) \in H_\eta(\mathsf{C}_F)$ for each $i \in \mathsf{I}$. Then for each $i \in \mathsf{I}$, there exists a morphism $\alpha_i : A_i \to F(A_i)$ in C such that $(A'_i, \alpha'_i) = (H(A_i), \eta_{A_i} \circ H(\alpha_i))$. Let $(v_i : (A'_i, \alpha'_i) \to (D', \delta'))_{i\in\mathsf{I}}$ be the colimit of T in C_G and $(f_i : (A_i, \alpha_i) \to (D, \delta))_{i\in\mathsf{I}}$ that of the diagram $\overline{T} : I \to \mathsf{C}_F$ in C_F with $\overline{T}(i) = (A_i, \alpha_i)$. Then from the above, $(D', \delta') \cong H_\eta((D, \delta))$ so that $(D', \delta') \in H_\eta(\mathsf{C}_F)$. In particular, H_η preserves (reflects) epis for epis in C_F and C_G are carried by epis in C. If H is faithful, then, by Theorem 4.1, so is H_η and it is well known that every faithful functor reflects monos. $\square$

It is not hard to check that in case H is full and faithful and η is mono, then the subcategory $H_\eta(\mathsf{C}_F)$ of C_G is full. As in Theorem 4.4, we give below some exactness properties of the induced functor.

Theorem 4.5. *If η is cartesian, then $\widetilde{\eta}$ preserves all existing weak equalizers, and equalizers whose domains have extensional images under it. If η is mono, then $\widetilde{\eta}$ reflects all existing limits and colimits. Moreover, if η is both mono and cartesian, then $\widetilde{\eta}$ preserves and creates all existing limits.*

Proof. Assume that η is cartesian and let $\epsilon : \mathcal{E} \to \mathcal{A}$ be a weak equalizer of $\varphi, \psi : \mathcal{A} \to \mathcal{B}$ in C_F and $\tau : (C, \gamma') \to \widetilde{\eta}(\mathcal{A})$ a morphism in C_G such that $\varphi \circ \tau = \psi \circ \tau$. We want to find a morphism $\mu : (C, \gamma') \to \widetilde{\eta}(\mathcal{E})$ in C_G such that $\tau = \epsilon \circ \mu$. Since η is cartesian, Lemma 4.3 yields a unique morphism $\gamma : C \to F(C)$ such that $\gamma' = \eta_C \circ \gamma$ and $\tau : (C, \gamma) \to \mathcal{A}$ is a morphism in C_F. But then, by the universal property of the equalizer, there exists a unique morphism $\rho : (C, \gamma) \to \mathcal{E}$ in C_F such that $\tau = \epsilon \circ \rho$. Now, by Theorem 4.1, ρ is a morphism in C_G from (C, γ') to $\widetilde{\eta}(\mathcal{E})$ too. Hence just set $\mu := \rho$. If moreover $\widetilde{\eta}(\mathcal{E})$ is extensional, then ϵ is a mono in C_G so that ρ is unique

Assume that η is mono. Then by Corollary 4.2, $\widetilde{\eta}$ is fully faithful. Thus it does trivially reflect all existing limits and colimits. If moreover η is cartesian, then $\widetilde{\eta}$ preserves all existing limits easily follows from the facts that it is fully faithful and $\widetilde{\eta}(\mathsf{C}_F)$ is closed under the domains of morphisms by Lemma 4.3. We now prove the creation of existing limits. Let $D : \mathsf{I} \to \mathsf{C}_F$ be a diagram with $D(i) = (D_i, \delta_i)$ and $\mathcal{L} = (u_i : (L, l') \to (D_i, \eta_{D_i} \circ \delta_i))_{i \in \mathsf{I}}$ a limit of $\widetilde{\eta} \circ D$ in C_G. Since η is cartesian, then Lemma 4.3 yields that for every i there exists a unique morphism $l_i : L \to F(L)$ such that $l' = \eta_L \circ l_i$ and $u_i : (L, l_i) \to (D_i, \delta_i)$ is a morphism in C_F. Now in addition η is mono. Thus, for all $i, j \in \mathsf{I}$, $l_i = l_j$. Set $l := l_i$ for all i. Then $\mathcal{S} = (u_i : (L, l) \to (D_i, \delta_i))_{i \in \mathsf{I}}$ is a natural source for D such that $\widetilde{\eta}(\mathcal{S}) = \mathcal{L}$ and it is unique as such thanks to the fact that η is mono. To end the proof, we show that $\mathcal{S}$ is a limit of D. Let $(v_i : (A, \alpha) \to (D_i, \delta_i))_{i \in \mathsf{I}}$ be an arbitrary natural source for D. Then its image $(v_i : (A, \eta_A \circ \alpha) \to (D_i, \eta_{D_i} \circ \delta_i)_{i \in \mathsf{I}}$ under $\widetilde{\eta}$ is obviously a natural source for $\widetilde{\eta} \circ D$. Therefore, there exists a unique morphism $v : (A, \eta_A \circ \alpha) \to (L, \eta_L \circ l)$ in C_G such that $v_i = u_i \circ v$ for all i. By Corollary 4.2 $v : (A, \alpha) \to (L, l)$ is also a morphism in C_F satisfying the above equations and its uniqueness as such is straightforward. $\square$

Every strongly complete category has epi-strong mono factorizations [3], and for a strong mono preserving covarietor on it, the category of coalgebras is complete (Theorem 4.14, [1]). Moreover, if a functor lifts limits and its codomain is (strongly) complete, then so is its domain and this functor preserves all small limits (and arbitrary intersections) (Proposition 3.19, [3]). Thus since creation implies lifting of limits and by Lemma 2.4 and Proposition 3.1 the categories of coalgebras in item (3) below are complete, Theorem 4.5 yields:

Corollary 4.6. *Assume that η is mono and cartesian. Then the following hold:*

(1) *if C is strongly complete and F (G) is a covarietor preserving strong monos, then $\widetilde{\eta}$ preserves (creates) all limits.*

(2) *if the category C_G is (strongly) complete, then so is C_F and $\widetilde{\eta}$ preserves all small limits (and arbitrary intersections).*

(3) *if C is complete and G (F) preserves limits, then $\widetilde{\eta}$ preserves and creates all (existing) limits.*

(4) *if C_F has a final coalgebra Ω_F, then $\widetilde{\eta}(\Omega_F)$ is the final coalgebra in C_G. Conversely, if a final coalgebra Ω_G exists in C_G, then a final coalgebra Ω_F exists in C_F and $\widetilde{\eta}(\Omega_F) \cong \Omega_G$.*

Corollary 4.7. *If F and G are such that in contrast to G a final coalgebra exists for F, then there does not exist any mono and cartesian $F \to G$ or $G \to F$.*

Example 4.8. (1) For $A, B \neq \emptyset$, consider $\eta : B \times (-)^A \to (B \times -)^A$ from Example 2.1. As polynomial functors, $B \times (-)^A$ and $(B \times -)^A$ are covarietors [4, 21]. Thus by (Corollary 4.15, [1]) $\mathsf{Set}_{B\times(-)^A}$ and $\mathsf{Set}_{(B\times -)^A}$ are complete. Thus by Theorem 4.5 $\widetilde{\eta}$ creates and preserves limits. In particular, denoting by A^* the set of words (finite lists) of elements of A, by ε the empty word, and by $a.\omega$ the word obtained by prefixing a to ω for $a \in A$ and $\omega \in A^*$, (B^{A^*}, π), where $\pi(\phi) = \langle \phi(\varepsilon), \psi \rangle$ with $\psi(a)(\omega) = \phi(a.\omega)$ is final in $\mathsf{Set}_{B\times(-)^A}$ [13, 34]. Thus, so is $(B^{A^*}, \eta_{B^{A^*}} \circ \pi)$ in $\mathsf{Set}_{(B\times -)^A}$.

(2) In case C is a non-degenerate topos (i.e. with $\mathbf{0} \neq \mathbf{1}$), there does not exist any mono and cartesian $F \to \mathbf{P}$ or $\mathbf{P} \to F$, for F such that a final coalgebra exists and $\mathbf{P}$ the covariant power object functor (by Corollary 4.7 and (Example 2.2. [21])).

For the definition of natural number object (NNO, for short), see [6]. We have:

Theorem 4.9. *If C is a topos, F and G are covarietors with G preserving pullbacks and η is mono and cartesian, then the functor $\widetilde{\eta}$ preserves NNO's.*

Proof. By Lemmata 2.4 and 3.3 C_F and C_G are toposes. Thus they have finite (co)limits [6], and, by Theorems 4.5 and 4.4 $\widetilde{\eta}$ preserves them. Hence Corollary 5.9, Chapter 7, [6] yields the desired result. □

As far as the exactness properties of the induced functor are concerned, the creation of colimits has not been explored yet. Below is an attempt at doing it:

Theorem 4.10. *If η is strong mono, then $\widetilde{\eta}$ creates all existing coequalizers. If moreover it is cartesian, then $\widetilde{\eta}$ creates (and hence reflects) all existing colimits with epic canonical injections for diagrams over a given scheme.*

Proof. Since η is strong mono, it follows from Lemma 4.3 (1) that $\widetilde{\eta}(\mathsf{C}_F)$ is closed under codomains of epis and the creation of coequalizers is straightforwardly checked. Assume that in addition η is cartesian and let $T : \mathsf{I} \to \mathsf{C}_F$ with $T(i) = (A_i, \alpha_i)$ be a diagram and $(e_i : (A_i, \eta_{A_i} \circ \alpha_i) \to (D, \delta))_{i \in \mathsf{I}}$ a colimit in C_G of the diagram $\widetilde{\eta} \circ T$ with e_i epic for all $i \in \mathsf{I}$. We need to find a natural sink $\mathcal{S} = (e'_i : (A_i, \alpha_i) \to (D', \delta'))_{i \in \mathsf{I}}$ in C_F such that $\widetilde{\eta}(\mathcal{S}) = (e_i : (A_i, \eta_{A_i} \circ \alpha_i) \to (D, \delta))_{i \in \mathsf{I}}$ and $\mathcal{S}$ is a colimit of T with e'_i epic for all $i \in \mathsf{I}$. Since η is strong mono and e_i is epic for all $i \in \mathsf{I}$, it follows again from Lemma 4.3 (1) that for all $i \in \mathsf{I}$, there exists a morphism $\delta^1_i : D \to F(D)$ in C such that $e_i : (A_i, \alpha_i) \to (D, \delta^1_i)$ is a morphism in C_F and $\delta = \eta_D \circ \delta^1_i$. Since η_D is (strongly) monic, the δ^1_i's are identical to a fixed morphism $\delta^1 : D \to F(D)$ in C. We claim that the natural sink $S^1 := (e_i : (A_i, \alpha_i) \to (D, \delta^1))_{i \in \mathsf{I}}$ is a good candidate for $\mathcal{S}$: Indeed, clearly $\widetilde{\eta}(\mathcal{S}^1) = (e_i : (A_i, \eta_{A_i} \circ \alpha_i) \to (D, \delta))_{i \in \mathsf{I}}$ and $\mathcal{S}^1$ is unique as such for η is (strong) mono. To show that $\mathcal{S}^1$ is a colimit of T, let $(u_i : (A_i, \alpha_i) \to (C, \gamma))_{i \in \mathsf{I}}$ be a natural sink in C_F. Then for its image $(u_i : (A_i, \eta_{A_i} \circ \alpha_i) \to (C, \eta_C \circ \gamma))_{i \in \mathsf{I}}$ under $\widetilde{\eta}$, the universal property of the colimit yields a unique $\theta : (D, \delta) \to (C, \eta_C \circ \gamma)$ in C_G such that $\theta \circ e_i = u_i$ for all $i \in \mathsf{I}$. Since η is cartesian, Lemma 4.3 (2) yields a unique morphism $\delta" : D \to F(D)$ in C such that $\delta = \eta_D \circ \delta"$ and $\theta : (D, \delta") \to (C, \gamma)$ is a morphism in C_F. But then again because η is (strong) mono, $\delta" = \delta^1$. □

Typical examples of colimits with epic injections arise from sums in the category Rng of rings with identity and ring homomorphisms preserving them (for example the coproduct injection $i_A : A \to A + \mathbb{Z}_k$ is epic, for any ring A and any integer $k \in \mathbb{N}$ where $\mathbb{Z}_k$ denotes the ring of integers modulo k : Indeed, $\mathbb{Z}$ is the initial object and if $i_{\mathbb{Z}_k} : \mathbb{Z}_k \to A + \mathbb{Z}_k$ denotes the other coproduct injection, then the pushout $!'_{\mathbb{Z}_k} : A \to A +_{\mathbb{Z}} \mathbb{Z}_k$ of the regular epimorphism $!_{\mathbb{Z}_k} : \mathbb{Z} \to \mathbb{Z}_k$ along $!_A : \mathbb{Z} \to A$ factors as $!'_{\mathbb{Z}_k} = [i_A, i_{\mathbb{Z}_k}] \circ i_A$ where, $[i_A, i_{\mathbb{Z}_k}] : A + \mathbb{Z}_k \to A +_{\mathbb{Z}} \mathbb{Z}_k$ which is the unique moprhism given by the universal property of the pushout, is a regular epi (even an iso) since it is the coequalizer of $i_A \circ !_A$ with $i_{\mathbb{Z}_k} \circ !_{\mathbb{Z}_k}$ (which are identical)).

An *exact sequence* is a diagram $\bullet \rightrightarrows \bullet \longrightarrow \bullet$ which is both a pullback and a coequalizer. In regular categories, such sequences are stable under pullback [30]. Recall that any regular epi $f : A \to B$ in a category with kernel pairs is the coequalizer of its kernel pair. Thus the resulting diagram $R(f) \rightrightarrows A \xrightarrow{f} B$ is an exact sequence which is an effective equivalence relation if the category has finite limits.

Theorem 4.11. *If η is mono, then $\widetilde{\eta}$ reflects exact sequences. If moreover it is cartesian, then it preserves them. It creates them if η is strong mono and cartesian.*

Proof. The first assertion follows from Theorem 4.5, the second from 4.4 and 4.5 and the last one from 4.10 and 4.5. □

Recalling (for example from [30]) that an *exact functor* between regular categories is one which preserves finite limits and exact sequences, we have:

Theorem 4.12. *If C, F, G and η are as in Theorem* 4.9*, then the functor $\widetilde{\eta}$ is exact and preserves the calculus of relations.*

Proof. C_F and C_G are toposes (proof of Theorem 4.9). Therefore they are regular so that, by Theorems 4.5 and 4.11, $\widetilde{\eta}$ is exact. Moreover, by Lemma 2.9 both categories are Barr-exact. Thus by Proposition 1.2 of [5], $\widetilde{\eta}$ preserves the calculus of relations. □

The preservation of coequalizers obviously implies that of regular epis. However, the converse does not hold (see for example, [3], Remark 7.73). For $\widetilde{\eta}$, it holds:

Theorem 4.13. *$\widetilde{\eta}$ reflects extremal monos, and if η is strong mono, then it preserves them and regular epis as well. Moreover, if η is both mono and cartesian, then $\widetilde{\eta}$ preserves and reflects extremal epis, reflects regular epis and strong ones, and preserves strong monos.*

Proof. By Theorem 4.4, $\widetilde{\eta}$ preserves epis. Moreover it is conservative by virtue of Theorem 4.1. Thus, by the dual of Lemma 2.6, it reflects extremal monos.

Assume that η is strong mono and let $\varphi : \mathcal{A} \to \mathcal{B}$ be an extremal mono in C_F so that $\varphi : \widetilde{\eta}(\mathcal{A}) \to \widetilde{\eta}(\mathcal{B})$ factors in C_G as $\varphi = \mu \circ \theta$ where $\theta : \widetilde{\eta}(\mathcal{A}) \to (C, \gamma')$ is an epi. Since η is strong mono, Lemma 4.3 yields a unique $\gamma : C \to F(C)$ such that $\gamma' = \eta_C \circ \gamma$ and $\theta : \mathcal{A} \to (C, \gamma)$ is an epi in C_F. Therefore, $\mu : \widetilde{\eta}(\mathcal{C}) \to \widetilde{\eta}(\mathcal{B})$ is a morphism in C_G. Now η strong mono implies η mono. Thus by Corollary 4.2, $\mu : (C, \gamma) \to \mathcal{B}$ is a morphism in C_F and since φ is an extremal mono therein, θ must be an iso in C_F and hence in C_G. By Theorem 4.4, $\widetilde{\eta}$ preserves coequalizers and therefore regular epis.

Assume that η is both mono and cartesian and let $\varphi : \mathcal{A} \to \mathcal{B}$ be an extremal epi in C_F. Assume that φ factors in C_G in a homomorphism followed by a mono as $\widetilde{\eta}(\mathcal{A}) \xrightarrow{\psi} \mathcal{C} \rightarrowtail^{\mu} \widetilde{\eta}(\mathcal{B})$. We want to show that μ is necessarily an iso. Since η is cartesian, Lemma 4.3 yields a unique coalgebra $\mathcal{D}$ in C_F such that $\widetilde{\eta}(\mathcal{D}) = \mathcal{C}$ and $\mu : \mathcal{D} \to \mathcal{B}$ is a homomorphism in C_F. But then, since η is mono too, it follows from Corollary 4.2 that φ also factors in C_F as $\mathcal{A} \xrightarrow{\psi} \mathcal{D} \xrightarrow{\mu} \mathcal{B}$. On the other hand, by Theorem 4.4, the functor $\widetilde{\eta}$ reflects monos. Thus, μ is a mono in C_F. Therefore, since φ is an extremal epi, μ must be an iso in C_F, and hence in C_G.

We now show the reflection of extremal epis. By Theorem 4.5, $\widetilde{\eta}$ preserves limits. Therefore, it preserves pullbacks and hence monos in particular. Now again by Theorem 4.1 it is conservative. Thus by Lemma 2.6, extremal epis are reflected. For the reflection of strong epis, let $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism in C_F such that $\varphi : \widetilde{\eta}(\mathcal{A}) \to \widetilde{\eta}(\mathcal{B})$ is a strong epi in C_G and let $\mathcal{A} \xrightarrow{\lambda} \mathcal{C} \rightarrowtail^{\mu} \mathcal{D} \xleftarrow{\rho} \mathcal{B}$ be morphisms in C_F where μ is a mono, such that $\rho \circ \varphi = \mu \circ \lambda$. We want to show that φ is an epi and find a unique diagonal fill-in $\delta : \mathcal{B} \to \mathcal{C}$ in C_F such that $\mu \circ \delta = \rho$ and $\delta \circ \varphi = \lambda$. By hypothesis $\widetilde{\eta}(\varphi)$ is a strong epi in C_G. Thus it is an epi in C and hence in C_F. On the other hand, again by Theorem 4.5, μ mono in C_F implies μ mono in C_G from $\widetilde{\eta}(\mathcal{C})$ to $\widetilde{\eta}(\mathcal{D})$. Moreover, the equation $\rho \circ \varphi = \mu \circ \lambda$ also obviously holds in C_G. Thus, since the epi φ is strong in C_G, there exists a unique diagonal fill-in $\chi : \widetilde{\eta}(\mathcal{B}) \to \widetilde{\eta}(\mathcal{C})$ in C_G such that $\chi \circ \varphi = \lambda$ and $\mu \circ \chi = \rho$. But then since η is mono, it follows from Corollary 4.2 that $\chi : \mathcal{B} \to \mathcal{C}$ is also a morphism in C_F and obviously the last two equations also hold in C_F. As for the uniqueness of χ satisfying them, this follows directly from the fact that μ is a mono. Thus it suffices to set $\delta := \chi$.

That of regular epis follows from the fact that $\widetilde{\eta}(\mathsf{C}_F)$ is closed under domains of morphisms by Corollary 4.2 and the reflection of coequalizers given by Theorem 4.5.

To end the proof, we show the preservation of strong monos. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a strong mono in C_F. Then φ is a mono [3], so that, again by Theorem 4.5, $\varphi : \widetilde{\eta}(\mathcal{A}) \to \widetilde{\eta}(\mathcal{B})$ is a mono in C_G too. Let

$$\widetilde{\eta}(\mathcal{A}) \xleftarrow{\ \zeta\ } (C, \gamma') \overset{\psi}{\twoheadrightarrow} (D, \delta') \xrightarrow{\ \theta\ } \widetilde{\eta}(\mathcal{B})$$

be a diagram in C_G where ψ is an epi and $\theta \circ \psi = \varphi \circ \zeta$. We want to find a diagonal fill-in $\kappa : (D, \delta') \to \widetilde{\eta}(\mathcal{A})$ such that $\kappa \circ \psi = \zeta$ and $\varphi \circ \kappa = \theta$ in C_G. Now using the fact that η is cartesian, it follows from Lemma 4.3 that there is a unique morphism $\gamma : C \to F(C)$ and a unique morphism $\delta : D \to F(D)$ such that $\gamma' = \eta_C \circ \gamma$, $\delta' = \eta_D \circ \delta$, $\zeta : (C, \gamma) \to \mathcal{A}$ and $\theta : (D, \delta) \to \mathcal{B}$ are morphisms in C_F. But then, considering the equation $\theta \circ \psi = \varphi \circ \zeta$ in C_F, the fact that the mono φ is strong in C_F yields a unique diagonal fill-in $\pi : (D, \delta) \to \mathcal{A}$ such that $\pi \circ \psi = \zeta$ and $\varphi \circ \pi = \theta$. Obviously $\pi : (D, \delta') \to \widetilde{\eta}(\mathcal{A})$ is also a morphism in C_G and its uniqueness follows from the fact that ψ is an epi in C_G and hence right-cancelable. Thus it suffices to set $\kappa := \pi$. $\square$

Remark 4.14. The result about strong epis (respectively, monos) in Theorem 4.13 still holds when we consider strong epi (respectively, mono) in the sense of [9].

Remark 4.15. For η mono and non-cartesian, $\widetilde{\eta}$ fails to preserve monos and hence all limits. Also, its domain need not preserve limits when its codomain does.

This may be seen through the following which shows that most of the results involving cartesianness established so far can provide an effective means for showing that a given transformation is not cartesian.

Example 4.16. Consider the functor $F := (-)^3_2$ given on sets as $X^3_2 = \{(x_1, x_2, x_3) \in X^3 | x_i = x_j \text{ for some } i \neq j\}$ and on maps $f : X \to Y$ as $f^3_2(x_1, x_2, x_3) = (f(x_1), f(x_2), f(x_3))$ and set $G = (-)^3$. Then the trivial (strong) mono-transformation $\eta : (-)^3_2 \to (-)^3$ from [17] whose components $\eta_X : X^3_2 \to X^3$ are defined by $\eta_X((x_1, x_2, x_3)) = (x_1, x_2, x_3)$ is non-cartesian. Moreover, $\widetilde{\eta}$ preserves extremal monos and reflects coequalizers but does not reflect regular epis, strong epis and extremal ones, and does not preserve all monos. Furthermore, in contrast to $(-)^3_2$, $(-)^3$ preserves all limits.

Proof. The $(-)^3_2$-coalgebra $\mathcal{A} = (A, \alpha)$ defined by $A = \{a, b\}$ with $\alpha(a) = (a, a, b)$, $\alpha(b) = (a, b, a)$ is extensional and the final coalgebra $\Omega_{(-)^3_2}$ is carried by $\{*\}$ [23, 24]. $!_{\mathcal{A}} : \mathcal{A} \to \Omega_{(-)^3_2}$ is both monic and epic, and every epi (mono) in $\mathsf{Set}_{(-)^3}$ is regular as seen in Section 3. Thus $\widetilde{\eta}(!_{\mathcal{A}})$ is a regular epi in $\mathsf{Set}_{(-)^3}$ and hence strong and extremal one. However, $!_{\mathcal{A}}$ isn't an extremal epi in $\mathsf{Set}_{(-)^3_2}$, and hence is neither regular nor strong for, otherwise, since it is monic, it would be an iso, which is false. Moreover, η is mono. Thus although $\widetilde{\eta}$ does not reflect regular epis, by Theorem 4.13, it does reflect coequalizers. Hence, η is non-cartesian. Again by Theorem 4.13, it preserves extremal monos. It doesn't preserve all monos: indeed, $!_{\mathcal{A}}$ is a mono in $\mathsf{Set}_{(-)^3_2}$ but $\widetilde{\eta}(!_{\mathcal{A}})$ is not a mono in $\mathsf{Set}_{(-)^3}$ for, regular monos therein are injective maps [15]. Also, as seen earlier, $(-)^3$ preserves all limits in Set. However, $(-)^3_2$ does not preserve products for, otherwise, in the category $\mathsf{Set}_{(-)^3_2}$ the product $\mathcal{A} \times \mathcal{A}$ would be carried by $A \times A$. But this is not the case for $\mathcal{A} \times \mathcal{A} \cong \mathcal{A}$ by Lemma 3.2. □

From Example 4.16, item (1) in the following is deduced.

Remark 4.17. (1) A functor that reflects coequalizers does not necessarily reflect regular epis. Likewise, neither strong epis nor extremal ones are reflected.
(2) If η is not mono and cartesian, then $\widetilde{\eta}$ may fail to reflect regular epis.

Item (2) in the above shows the relevance of phrase "mono and cartesian" in Theorem 4.13. To illustrate it, one has the following which gives an example of transformation that satisfies none of the properties encountered so far.

Example 4.18. Let κ be any ordinal number with $\kappa > 2$ and defines the Set-endofunctor $\mathcal{P}_\kappa$ which sends every set A to the set $\mathcal{P}_\kappa(A) := \{X \mid X \subseteq$

$A, |X| < \kappa\}$ and acts on maps as the covariant powerset functor $\mathcal{P}$ and consider the functor $(-)^3_2$ from Example 4.16. Then the transformation $\delta^\kappa : (-)^3_2 \to \mathcal{P}_\kappa$ defined for all set X by $\delta^\kappa_X((x_1, x_2, x_3)) = \{x_1, x_2, x_3\}$ which is obviously natural (and extends to a natural transformation $(-)^3_2 \to \mathcal{P}$ which is not mono and cartesian (Example 4.8 (2))) is neither epi, mono nor cartesian. Moreover, $\widetilde{\delta^\kappa}$ does not reflect regular epis in general.

Proof. δ^κ is not epi is clear. Also it isn't mono because for any two-element set $X = \{x_1, x_2\}$, $\delta^\kappa_X((x_1, x_1, x_2)) = \delta^\kappa_X((x_1, x_2, x_2))$, and it isn't cartesian for the transformation square for $!_X : X \to \mathbf{1}$ where $\mathbf{1} = \{*\}$ is a weak pullback. Indeed, δ^κ_X sends (x_1, x_1, x_1) to $\{x_1\}$, (x_2, x_2, x_2) to $\{x_2\}$, the other elements to X and $\delta^\kappa_{\mathbf{1}} : \{(*, *, *)\} \to \{*\}$. Consider a set T with $|T| \geq 2$ and let $u : T \to \mathcal{P}_\kappa(X)$ and $v : T \to \mathbf{1}^3_2$ be maps such that $\mathcal{P}_\kappa(!_X) \circ u = \delta^\kappa_{\mathbf{1}} \circ v$. Then for all $t \in T$, $!_X(u(t)) = \{*\}$ so that $\emptyset \neq u(t) \subseteq X$. Therefore $w : T \to X^3_2$ defined by

$$w(t) = \begin{cases} (x_i, x_i, x_i), \; if \; u(t) = \{x_i\}, \; for \; some \; i \in \{1, 2\} \\ (x_i, x_j, x_i), \; if \; u(t) = X \; for \; some \; i, \; j \in \{1, 2\}, i \neq j \end{cases}$$

is a mediating morphism such that $u = \delta^\kappa_X \circ \omega$ and $v = (!_X)^3_2 \circ w$ and isn't unique. $\widetilde{\delta^\kappa}$ does not reflect regular epis in general: Set $\kappa := \omega$ and consider $!_\mathcal{A} : \mathcal{A} \to \Omega_{(-)^3_2}$; the epi $\widetilde{\delta^\omega}(!_\mathcal{A})$ is regular in $\mathsf{Set}_{\mathcal{P}_\omega}$ as is any epi therein, unlike $!_\mathcal{A}$ in $\mathsf{Set}_{(-)^3_2}$. □

As can be seen from previous results, in some cases the category of coalgebras over the domain of a natural transformation inherits some properties from that of the coalgebras over its codomain. Another illustration is given by:

Theorem 4.19. *If C is a topos, G is a pullback preserving covarietor and η is strong mono and cartesian, then the category C_F is Barr-exact.*

Proof. By Lemma 3.3, the category C_G is a topos. Moreover, by Lemma 2.4 F preserves pullbacks. Thus the category C_F has pullbacks. On the other hand, as a topos the category C_G has a final object. Thus by Corollary 4.6 so does C_F so that C_F is finitely complete. Because it is a topos, C has finite colimits [6, 28]; hence so does C_F which in particular has coequalizers of kernel pairs. Still because C_G is a topos, it is a regular category so that regular epis are stable under pullback and, by Theorem 4.5, $\widetilde{\eta}$ preserves and reflects existing limits and by Theorem 4.13 regular epis. Thus in C_F regular epis are stable under pullback. Hence C_F is a regular category. Now, we show that every equivalence relation in C_F is effective. Let $\mathcal{R}$ be an equivalence relation on a coalgebra $\mathcal{A}$ in C_F with projections π_1 and π_2. The functoriality of $\widetilde{\eta}$ yields that $\widetilde{\eta}(\mathcal{R})$ is reflexive and symmetric, and transitive too, since in addition $\widetilde{\eta}$ preserves finite limits and pullbacks in particular. Therefore it is an equivalence relation which is effective because C_G is a topos. Thus, there exists a morphism $\varepsilon : \widetilde{\eta}(\mathcal{A}) \to \mathcal{C}'$ for which π_1, π_2 is the kernel pair. Now as a topos, C_G has regular epi-mono factorizations [6, 9]. Thereofore ε factors as $\varepsilon = \mu \circ \gamma$ where $\gamma : \widetilde{\eta}(\mathcal{A}) \to \mathcal{D}'$ is a regular epi and μ a mono. Since η is strong mono and cartesian, Lemma 4.3 yields a morphism $\gamma_1 : \mathcal{A} \to \mathcal{D}$ in C_F such that $\widetilde{\eta}(\gamma_1) = \gamma$. Clearly π_1, π_2 is also the kernel pair of γ so that the three morphisms form an exact sequence in C_G which is the image under $\widetilde{\eta}$ of the diagram formed by π_1, π_2 and γ_1 in C_F. Thus by Theorem 4.11 the latter is an exact sequence too. $\square$

Theorem 4.20. *If C has and G preserves pullbacks and η is mono and cartesian, then $\widetilde{\eta}$ reflects pullback-stable extremal monos, reflects and preserves pullback-stable extremal epis, preserves pullback-stable strong monos and reflects both pullback-stable regular epis and pullback-stable strong epis.*

Proof. By Lemma 2.4, the functor F preserves pullbacks too, so that both categories C_F and C_G have pullbacks. Moreover, by Theorem 4.5, $\widetilde{\eta}$ preserves pullbacks. Thus, by Lemma 2.6, pullback-stable extremal epis are reflected. Now let φ be a morphism such that $\widetilde{\eta}(\varphi)$ is a pullback-stable extremal mono. Still because $\widetilde{\eta}$ preserves pullbacks, the image of the pullback of φ along any morphism under $\widetilde{\eta}$ is a pullback of $\widetilde{\eta}(\varphi)$, which is an extremal mono by our assumption on φ. Now by Theorem 4.13, $\widetilde{\eta}$ reflects extremal monos; so the pullback of φ is an extremal mono too. Hence φ is a pullback-stable extremal mono. Still by Theorem 4.13, $\widetilde{\eta}$ reflects regular epis (strong epis); thus the proof of the reflection of pullback-stable regular epis (pullback-stable strong epis) is obtained by just replacing 'extremal mono' in the proof of reflection of extremal monos with 'regular epi' ('strong epi').

To prove the preservation of pullback-stable extremal epis, let $\varphi : \mathcal{A} \to \mathcal{B}$ be a pullback-stable extremal epi in C_F. Then, by Theorem 4.13, $\varphi : \widetilde{\eta}(\mathcal{A}) \to$

$\widetilde{\eta}(\mathcal{B})$ is an extremal epi in C_G. Let $\psi : \mathcal{C}' \to \widetilde{\eta}(\mathcal{B})$ be a morphism in C_G and $(\mathcal{D}', \varphi', \psi')$ the pullback of φ with ψ in C_G with $\psi \circ \varphi' = \varphi \circ \psi'$. Then, by Lemma 4.3, there are coalgebras $\mathcal{C}$ and $\mathcal{D}$ in C_F such that $\varphi' : \mathcal{D} \to \mathcal{C}$ and $\psi' : \mathcal{D} \to \mathcal{A}$ are morphisms in C_F with $\widetilde{\eta}(\mathcal{D}) = \mathcal{D}'$ and $\widetilde{\eta}(\mathcal{C}) = \mathcal{C}'$. Now, by Theorem 4.5, $\widetilde{\eta}$ reflects pullbacks. Thus the square whose image under $\widetilde{\eta}$ is the above pullback of $\widetilde{\eta}(\varphi)$ along $\widetilde{\eta}(\psi)$ in C_G is the pullback of φ along ψ in C_F. Therefore, since $\widetilde{\eta}$ preserves pullbacks again by Theorem 4.5, φ' is the pullback of φ in C_G along ψ. The remainder is obtained the same way by just replacing 'extremal epi' with 'strong mono' in what precedes. □

4.3 Induced functor and (effective) descent morphisms

We recall:

Theorem 4.21. ([25]) *Assume that C has and G preserves pullbacks and η is mono and cartesian. Then $\widetilde{\eta}$ reflects and preserves (effective) descent morphisms.*

But, the following whose 'may not' part can be depicted by Example 3.10 holds:

Remark 4.22. In the absence of cartesianess in Theorem 4.21, $\widetilde{\eta}$ may fail to reflect and preserve (effective) descent morphisms or not.

To illustrate the 'may' part, we have the following which gives a case where $\widetilde{\eta}$ preserves descent morphisms but does not reflect them:

Example 4.23. Equalizers always exists for coalgebras over Set without any requirement on the endofunctor and the functor $(-)^3_2$ preserves mono sources so that binary products exists in the category $\mathsf{Set}_{(-)^3_2}$ ([17], pp. 172-180). Thus this category is finitely complete (although $(-)^3_2$ does not preserve finite limits !). Therefore, by Lemma 2.9, a descent morphism in $\mathsf{Set}_{(-)^3_2}$ is the same as a pullback-stable regular epi. Moreover, Example 3.6 yields that the class of (effective) descent morphisms coincides with that of epis in $\mathsf{Set}_{(-)^3}$. Thus for η from Example 4.16, the morphism $!_\mathcal{A}$ is such that $\widetilde{\eta}(!_\mathcal{A})$ is a descent morphism in $\mathsf{Set}_{(-)^3}$ but $!_\mathcal{A}$ isn't a descent morphism in $\mathsf{Set}_{(-)^3_2}$ for it isn't even a regular epi. However, since a descent morphism in $\mathsf{Set}_{(-)^3_2}$ is a pullback-stable regular epi, it is an epi so that by Lemma 4.4 its image under $\widetilde{\eta}$ is an epi in $\mathsf{Set}_{(-)^3}$ and hence is descent.

Theorem 4.24. *Assume that* C *has coequalizers, has and* G *preserves pullbacks and* η *is mono and cartesian. If a morphism* φ *in* C_F *is such that* $\widetilde{\eta}(\varphi)$ *is a split epi in* C_G*, then* φ *and* $\widetilde{\eta}(\varphi)$ *are effective descent morphisms.*

Proof. Let φ, C, G and η be as assumed. Then by Lemma 2.4, F preserves pullbacks too. Thus, both categories C_F and C_G have pullbacks and coequalizers (see Section 3). Moreover, $\widetilde{\eta}$ is conservative by Theorem 4.1, preserves coequalizers by Theorem 4.4 and pullbacks by Theorem 4.5. Thus, by Theorem 2.8, φ is an effective descent morphism in C_F and, by Theorem 4.21, so is its image under $\widetilde{\eta}$. □

Since every functor preserves split epis, we deduce:

Corollary 4.25. *If* C*,* G *and* η *are as in Theorem* 4.24*, then every split epi in* C_F *and its image under* $\widetilde{\eta}$ *are effective descent morphisms.*

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