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A characterization of finitely generated multiplication modules

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Abstract. Let R be a commutative ring with identity and M be a finitely generated unital R-module. In this paper, first we give necessary and sufficient conditions that a finitely generated module to be a multiplication module. Moreover, we investigate some conditions which imply that the module M is the direct sum of some cyclic modules and free modules. Then some properties of Fitting ideals of modules which are the direct sum of finitely generated module and finitely generated multiplication module are shown. Finally, we study some properties of modules that are the direct sum of multiplication modules in terms of Fitting ideals.

1 Introduction and Preliminaries

Let R be a unitary commutative ring and M be a finitely generated R-module. Let $X = \{x_1, ..., x_n\}$ be a set of generators of M. A relation of M is a vector $(a_1, ..., a_n)$ in R^n such that $\sum_{i=1}^{n} a_i x_i = 0$. For a positive integer $k = 0, ..., n-1$, the k-th Fitting ideal of M is defined to be the ideal $\mathrm{Fitt}_k(M)$ generated by the determinants of all $(n-k) \times (n-k)$ subdeterminants of

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the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & \vdots & \vdots \end{pmatrix},$$

where the vectors $(a_{i1}, ..., a_{in})$ are the relations of M. If $k \geq n$, we define $\text{Fitt}_k(M) = R$. These ideals form an ascending sequence of invariant ideals for M, independently of the choice of X (see Fitting's lemma [4, Corollary 20.4]). The most important Fitting ideal of M is the first of the $\text{Fitt}_j(M)$ that is nonzero. We shall denote this Fitting ideal by $\text{I}(M)$. If P is a prime ideal of R, then $\text{Fitt}_j(M)_P = \text{Fitt}_j(M_P)$, for every j. Also, if $\text{I}(M)$ contains a nonzerodivisor, then $\text{I}(M_P) = \text{I}(M)_P$ for every prime ideal P of R.
An element of R is regular if it is a nonzerodivisor and an ideal of R is regular if it contains a regular element. Let M be a finitely generated R-module. $\text{T}(M)$, the torsion submodule of M, is the submodule of M consisting of all elements of M that are annihilated by a regular element of R. M is said to be a torsion module if $M = \text{T}(M)$ and a torsion-free module if $\text{T}(M) = 0$.
An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case, we can take $I = (N : M)$ [2]. P. Vamos in [10] shows that every Artinian finitely generated distributive module is cyclic. A. Barnard in [2], interestingly shows that an R-module M is distributive, if and only if, every finitely generated submodule is a multiplication module. Using this interesting fact, the above result of Vamos is generalized in [2], by proving that, in fact, every finitely generated Artinian multiplication module is cyclic. Since then, the concept of multiplication modules received attention by some authors, in particular, by those working on the theory of prime submodules, see [5]. In this article, we are also following the methods used in [2], to state and prove our main result.

2 Fitting ideals of multiplication modules

In this section, we study some properties of finitely generated multiplication modules and Fitting ideals of them. We will denote by $\mu(M)$ the minimal number of generators of M and define $\omega(M) = min\{k \mid \text{Fitt}_k(M) \neq 0\}$.

Proposition 2.1. *Let (R, P) be a local ring and M be a finitely generated R-module. Then M can be generated by n elements if and only if $Fitt_n(M) = R$.*

Proof. [4, Proposition 20.6]. $\square$

Theorem 2.2. *Let M be a finitely generated R-module. If $Fitt_r(M) = I(M)$ is a regular principal ideal and $Fitt_{r+1}(M) = R$, then $M = N \oplus R^r$, where N is a multiplication module.*

Proof. $I(M)$ is a regular principal ideal of R, thus by [7, Theorem 6.2], $M = T(M) \oplus R^r$. So, $\mathrm{Fitt}_0(T(M)) = \mathrm{Fitt}_r(M)$ and $\mathrm{Fitt}_1(T(M)) = \mathrm{Fitt}_{r+1}(M) = R$. Let P be a prime ideal of R. We have $\mathrm{Fitt}_0((T(M))_P) = (\mathrm{Fitt}_0(T(M))_P$ and $(\mathrm{Fitt}_1(T(M)))_P = \mathrm{Fitt}_1((T(M))_P) = R_P$. By Proposition 2.1, $(T(M))_P$ is a cyclic module. Therefore, by [2, Proposition 5], $T(M)$ is a multiplication module. $\square$

Corollary 2.3. *Let M be a finitely generated torsion R-module. If $I(M) = Fitt_r(M)$ is a regular multiplication ideal and $Fitt_{r+1}(M) = R$, then M is a multiplication module.*

Proof. Let P be a maximal ideal of R. Since $I(M)$ is a regular multiplication ideal, $(I(M))_P$ is a regular principal ideal. By Theorem 4.4, $M_P \cong N \oplus R^r$, for some nonnegative integer r and some multiplication module N. But M_P is a torsion module, so $M_P \cong N$. Since R_P is a local ring, by [2, Proposition 4], N is a cyclic module. Hence by [2, Proposition 5], M is a multiplication module. $\square$

Lemma 2.4. *Let M be a finitely generated R-module. Suppose that $\mu(M) = r$ and $\omega(M) = s$. If the Fitting ideals of M are regular principal ideals, then there exist $\alpha_1, ..., \alpha_{r-s} \in R$ such that $(\alpha_{i+1}) \subseteq (\alpha_i)$ and $Fitt_k(M) = (\alpha_1 ... \alpha_{r-k})$ for all $s \leq k \leq r - 1$.*

Proof. Let P be a maximal ideal of R. By [1, Theorem 2.6], we have $M_P \cong \oplus_{i=1}^{r-s}(R_P/(a_i/1)) \oplus R^s$, where $(a_{i+1}/1) \subseteq (a_i/1)$. So, by [1, Proposition 2.5], $\mathrm{Fitt}_k(M_P) = (a_1 ... a_{r-k})$. Assume that $\mathrm{Fitt}_k(M) = (\beta_{r-k})$, $s \leq k \leq r - 1$. Since $\mathrm{Fitt}_{r-i-1}(M) \subseteq \mathrm{Fitt}_{r-i}(M)$, $(\beta_{i+1}) \subseteq (\beta_i)$. Hence there exists

$\gamma_{i+1} \in R$ such that $\beta_{i+1} = \gamma_{i+1}\beta_i$. Therefore, $\mathrm{Fitt}_{r-1}(M_P) = (a_1/1) = (\mathrm{Fitt}_{r-1}(M))_P = (\beta_1)_P = (\beta_1/1)$. Put $\alpha_1 = \beta_1$. Since

$$\begin{aligned}\mathrm{Fitt}_{r-2}(M_P) &= (a_1/1)(a_2/1) = (\alpha_1/1)(a_2/1)\\ &= (\mathrm{Fitt}_{r-2}(M))_P = (\beta_2)_P = (\beta_2/1) = (\gamma_2/1)(\alpha_1/1),\end{aligned}$$

$(a_2/1) = (\gamma_2/1)$. Put $\alpha_i = \gamma_i$. So, $\mathrm{Fitt}_k(M_P) = (\alpha_1 ... \alpha_{r-k})_P = (\mathrm{Fitt}_k(M))_P$. This implies that $(\alpha_1 ... \alpha_{r-k}) = \mathrm{Fitt}_k(M)$.

□

Theorem 2.5. *Let M be a finitely generated module over an integral domain R. Suppose that the Fitting ideals of M are principal. If $I(M)$ is a prime ideal of R, then M is a direct sum of a multiplication module and a free module.*

Proof. Let $\mu(M) = r$ and $\omega(M) = s$. By Lemma 2.4, there exist $\alpha_1, ..., \alpha_{r-s} \in R$ such that $I(M) = \langle \alpha_1 ... \alpha_{r-s} \rangle$. Since $I(M)$ is a prime ideal, there exists i, $1 \leq i \leq r-s$, such that $\alpha_i \in I(M)$. If $i = r-s$, then there exists $r \in R$ such that $\alpha_{r-s} = r\alpha_1 ... \alpha_{r-s}$. Therefore, $1_R = r\alpha_1 ... \alpha_{r-s-1}$. This means that $\mathrm{Fitt}_{r-i}(M) = R$ for all i, $1 \leq i \leq r-s-1$. So by Theorem 4.4, M is a direct sum of a multiplication module and a free module. If $r - s > i$, then there exists $r \in R$ such that $\alpha_i = r\alpha_1 ... \alpha_{i-1}\alpha_{i+1} ... \alpha_{r-s}$. So, $1_R = r\alpha_1 ... \alpha_{i-1}\alpha_{i+1} ... \alpha_{r-s}$. Hence $I(M) = R$. It is a contradiction by $I(M)$ is a prime ideal. □

Theorem 2.6. *Let M be a finitely generated torsion-free module over an integral domain R. Then, the Fitting ideals of M are multiplication ideals if and only if M is projective.*

Proof. ($\Longrightarrow$) Let $r = \omega(M)$ and P be a maximal ideal of R. Since M is a torsion-free R-module, M_P is a torsion-free R_P-module. $\mathrm{Fitt}_i(M_P)$ is a principal ideal because $\mathrm{Fitt}_i(M)$ is a multiplication ideal. Thus, by [1, Theorem 2.6], $M_P \cong \oplus_{i=1}^{n}(R_P/(a_i)) \oplus R_P^r$. Since $T(M_P) = 0$, $M_P \cong (R_P)^r$. By [6, Corrollary 1, p.58], M is a projective module.
($\Longleftarrow$) It is clear. □

In what follows we are characterizing, intrinsically, finitely generated modules which are multiplication.

Theorem 2.7. *Let M be a finitely generated R-module. Then $Fitt_1(M) = R$ if and only if M is a multiplication module.*

Proof. ($\Longrightarrow$) Let P be a maximal ideal of R. Since $\mathrm{Fitt}_1(M_P) = \mathrm{Fitt}_1(M)_P = R_P$, by Proposition 2.1, M_P is a cyclic module. By [2, Proposition 5], M is multiplication.
($\Longleftarrow$) Conversely, assume that M be a multiplication module and P be a maximal ideal of R. By [2, Proposition 5], M_P is a cyclic R_P-module. Hence $\mathrm{Fitt}_1(M)_P = \mathrm{Fitt}_1(M_P) = R_P$. So, $\mathrm{Fitt}_1(M) = R$.

□

Corollary 2.8. *Let M be a finitely generated R-module. If $Fitt_1(M) = R$, then $Fitt_0(M) = Ann_R(M)$.*

Proof. By [4, Proposition 20.7] and Theorem 2.7, it is clear. □

Theorem 2.9. *Let M be a finitely generated torsion R-module and P be a maximal ideal of R. Suppose that the Fitting ideals of M are regular principal ideals. If $I(M)$ is a P-primary ideal, then $M \cong \oplus_{i=1}^n R/(a_i)$, where $(a_{i+1}) \subseteq (a_i)$ for every $1 \leq i \leq n-1$.*

Proof. Let $P \neq Q$ be a maximal ideal of R. Since

$$\mathrm{Fitt}_i(M_Q) = (\mathrm{Fitt}_i(M))_Q = R_Q$$

by [4, Proposition 20.8] and by [8, Theoerm 4.58], M_Q is free. But M_Q is torsion, so $M_Q = 0$. Since $\mathrm{Fitt}_i(M_P) = (\mathrm{Fitt}_i(M))_P$ are regular principal ideals and M_P is torsion, by [1, Theorem 2.6], $M_P \cong \oplus_{i=1}^n R_P/(a_i/1)$, where $(a_{i+1}/1) \subseteq (a_i/1)$. By [1, Proposition 2.5], $\mathrm{Fitt}_{n-1}(M_P) = (a_1/1) = (\mathrm{Fitt}_{n-1}(M))_P = (\beta_1)_P$, so $(a_1/1) = (\beta_1/1)$. We have $(\alpha_2) = \mathrm{Fitt}_{n-2}(M) \subseteq \mathrm{Fitt}_{n-1}(M) = (\beta_1)$, so there exists $\beta_2 \in R$ such that $\alpha_2 = \beta_2\beta_1$. Hence $\mathrm{Fitt}_{n-2}(M_P) = (a_1/1)(a_2/1) = (\mathrm{Fitt}_{n-2}(M))_P = (\alpha_2)_P$. So, $(a_1/1)(a_2/1) = (\beta_1/1)(a_2/1) = (\beta_1/1)(\beta_2/1)$. Therefore, $(a_2/1) = (\beta_2/1) \subseteq (\beta_1/1)$. Hence $M_P \cong (R_P/(\beta_1/1)) \oplus (R_P/(\beta_2/1)) \oplus ... \oplus (R_P/(\beta_n/1))$. We show that (β_i) is a P-primary ideal for all i, $1 \leq i \leq n$. Suppose that q be a prime ideal of R and $(\beta_i) \subseteq q$, then $I(M) \subseteq \mathrm{Fitt}_{n-i}(M) = (\beta_1 ... \beta_i) \subseteq q$. So $q = P$ and (β_i) is P-primary because $I(M)$ is P-primary. Thus, there exist $m_1, ..., m_n \in M$ such that $M_P = (m_1/1) \oplus ... \oplus (m_n/1)$. Thus, $\mathrm{Ann}_{R_P}(m_i/1) = (\beta_i/1) = (\beta_i)_P = (\mathrm{Ann}_R(m_i))_P$. Since $(\mathrm{Ann}_R(m_i))_Q =$

$R_Q = (\beta_i)_Q$, for every maximal ideal $Q \neq P$ of R, hence $\mathrm{Ann}_R(m_i) = (\beta_i)$. If $x \in Rm_l \cap \Sigma_{i=1,i\neq l}^n Rm_i$, then there exists $t \in R$ such that $x = tm_l$. So, $x/1 = tm_l/1 \in R_P(m_l/1) \cap \Sigma_{i=1,i\neq l}^n R_P(m_i/1)$. Therefore, $tm_l/1 = 0$. Hence there exists $s \in R - P$ such that $stm_l = 0$. So, $st \in \mathrm{Ann}_R(m_l) = (\beta_l)$ and $s \notin P$. This implies that $t \in (\beta_l) = \mathrm{Ann}_R(m_l)$ and $x = 0$. Put $N = (m_1) \oplus ... \oplus (m_n)$. Then $N_Q = M_Q$ for all maximal ideal Q of R, hence $M \cong R/(\beta_1) \oplus ... \oplus R/(\beta_n)$. □

Proposition 2.10. *Let M be a finitely generated R-module and P be a maximal ideal of R. Suppose that the Fitting ideals of M are regular principal. If $I(M) = Fitt_r(M)$ is a P-primary ideal, then $M \cong \oplus_{i=1}^n R/(a_i) \oplus R^r$, where $(a_{i+1}) \subseteq (a_i)$ for $1 \leq i \leq n-1$.*

Proof. By [7, Theorem 6.2], $M = T(M) \oplus R^r$. So, $\mathrm{Fitt}_0(T(M)) = \mathrm{Fitt}_r(M)$ and $\mathrm{Fitt}_i(T(M)) = \mathrm{Fitt}_{r+i}(M)$. Since the Fitting ideals of $T(M)$ are regular principal ideals of R, thus by Theorem 2.9, $T(M) = \oplus_{i=1}^n R/(a_i)$, where $(a_{i+1}) \subseteq (a_i)$. □

3 Fitting ideals of direct sum of finitely generated modules

In this section, we exhibit some properties of Fitting ideals of direct sum of a finitely generated module and a finitely generated multiplication module. We define $\lambda(M) = \max\{k : \mathrm{Fitt}_k(M) \neq R\}$, where M is an R-module.

Proposition 3.1. *Let M be a finitely generated R-module such that $M = M_1 \oplus M_2$, for some R-modules M_1, M_2, where $n_2 = \omega(M_1)$ and $n_1 = \lambda(M_1)$. If M_2 is a multiplication module, then*

$$\mathrm{Fitt}_k(M) = \begin{cases} R & \text{if } k > n_1 + 1 \\ \mathrm{Ann}_R(M_2) + \mathrm{Fitt}_{n_1}(M_1) & \text{if } k = n_1 + 1 \\ \mathrm{Fitt}_k(M_1)\mathrm{Ann}_R(M_2) + \mathrm{Fitt}_{k-1}(M_1) & \text{if } n_2 < k \leq n_1 \\ \mathrm{Fitt}_k(M_1)\mathrm{Ann}_R(M_2) & \text{if } k = n_2 \\ 0 & \text{if } k < n_2. \end{cases}$$

Proof. By [3, p. 174], $\mathrm{Fitt}_k(M) = \Sigma_{i+j=k}\mathrm{Fitt}_i(M_1)\mathrm{Fitt}_j(M_2)$. Let k be a positive integer. We consider the following cases:
Case i: $k > n_1 + 1$. In this case both $\mathrm{Fitt}_{n_1+1}(M_1)$ and $\mathrm{Fitt}_k(M_1)$ are R.

So, $\mathrm{Fitt}_k(M) = R$.
Case *ii*: $k = n_1 + 1$. In this case

$$\mathrm{Fitt}_k(M) = \mathrm{Fitt}_{n_1+1}(M_1)\mathrm{Fitt}_0(M_2) + \mathrm{Fitt}_{n_1}(M_1)\mathrm{Fitt}_1(M_2) + \cdots$$

$$+\mathrm{Fitt}_0(M_1)\mathrm{Fitt}_{n_1+1}(M_2) = \mathrm{Ann}_R(M_2) + \mathrm{Fitt}_{n_1}(M_1).$$

Case *iii*: $n_2 < k < n_1$. We have

$$\mathrm{Fitt}_k(M) = \mathrm{Fitt}_k(M_1)\mathrm{Fitt}_0(M_2) + \mathrm{Fitt}_{k-1}(M_1)\mathrm{Fitt}_1(M_2) + \cdots +$$

$$\mathrm{Fitt}_0(M_1)\mathrm{Fitt}_k(M_2) = \mathrm{Fitt}_k(M_1)\mathrm{Ann}_R(M_2) + \mathrm{Fitt}_{k-1}(M_1).$$

The other cases are obvious. □

Corollary 3.2. *Let M be a finitely generated R-module such that $M = M_1 \oplus M_2$, where $n_1 = \lambda(M_1) =$ and $n_2 = \omega(M_1)$. If M_2 is a multiplication module and $Ann_R(M_2) \subseteq Ann_R(M_1)$, then*

$$Fitt_k(M) = \begin{cases} R & \text{if } k > n_1 + 1 \\ Fitt_{k-1}(M_1) & \text{if } n_2 < k \leq n_1 + 1 \\ 0 & \text{if } k \leq n_2. \end{cases}$$

Proof. By [4, Proposition 20.7], $\mathrm{Ann}_R(M_1)\mathrm{Fitt}_k(M_1) \subseteq \mathrm{Fitt}_{k-1}(M_1)$. By hypothesis, $\mathrm{Ann}_R(M_2) \subseteq \mathrm{Ann}_R(M_1)$. This implies that $\mathrm{Ann}_R(M_2)\mathrm{Fitt}_k(M_1) \subseteq \mathrm{Fitt}_{k-1}(M_1)$. By Proposition 2.1, we have

$$\mathrm{Fitt}_{n_1+1}(M) = \mathrm{Ann}_R(M_2) + \mathrm{Fitt}_{n_1}(M_1).$$

Since $\mathrm{Fitt}_{n_1+1}(M_1) = R$, hence

$$\mathrm{Fitt}_{n_1+1}(M) = \mathrm{Ann}_R(M_2) + \mathrm{Fitt}_{n_1}(M_1) = \mathrm{Fitt}_{n_1}(M_1).$$

By Proposition 2.1, we have

$$\mathrm{Fitt}_k(M) = \mathrm{Fitt}_k(M_1)\mathrm{Ann}_R(M_2) + \mathrm{Fitt}_{k-1}(M_1), n_2 < k \leq n_1.$$

Therefore, $\mathrm{Fitt}_k(M) = \mathrm{Fitt}_{k-1}(M_1)$. If $k = n_2$, then

$$\mathrm{Fitt}_{n_2}(M) = \mathrm{Fitt}_{n_2}(M_1)\mathrm{Fitt}_0(M_2) + \mathrm{Fitt}_{n_2-1}(M_1)\mathrm{Fitt}_1(M_2) + \cdots$$

$$= \mathrm{Fitt}_{n_2}(M_1)\mathrm{Ann}_R(M_2) \subseteq \mathrm{Fitt}_{n_2-1}(M_1) = 0.$$

□

Lemma 3.3. *Let $M = M_1 \oplus M_2$ be the direct sum of two finitely generated R-modules M_1 and M_2, where M_2 is a multiplication module. Then:*
(i) $Fitt_k(M_1) \subseteq Fitt_{k+1}(M) \subseteq Fitt_{k+1}(M_1)$ for all $k \geq 0$.
(ii) $\omega(M_1) + 1 \geq \omega(M) \geq \omega(M_1)$.
(iii) If R is an integral domain and $Ann_R(M_2) \neq 0$, then $\omega(M) = \omega(M_1)$.

Proof. (i) By Proposition 2.1, we have $\mathrm{Fitt}_k(M) = \mathrm{Fitt}_k(M_1)\mathrm{Ann}_R(M_2) + \mathrm{Fitt}_{k-1}(M_1)$. So, $\mathrm{Fitt}_k(M_1) \subseteq \mathrm{Fitt}_{k+1}(M) \subseteq \mathrm{Fitt}_{k+1}(M_1)$.
(ii) Let's put $n_2 = \omega(M_1)$. By Proposition 2.1,

$$\mathrm{Fitt}_{n_2+1}(M) = \mathrm{Fitt}_{n_2+1}(M_1)\mathrm{Ann}_R(M_2) + \mathrm{Fitt}_{n_2}(M_1).$$

Since $\mathrm{Fitt}_{n_2}(M_1) \neq 0$, $\mathrm{Fitt}_{n_2+1}(M) \neq 0$. So, $\omega(M_1)+1 \geq \omega(M)$. Therefore, by Proposition 2.1, $\mathrm{Fitt}_{n_2-1}(M) = 0$, $\omega(M) \geq \omega(M_1)$.
(iii) By Proposition 2.1, $\mathrm{Fitt}_{n_2}(M) = \mathrm{Fitt}_{n_2}(M_1)\mathrm{Ann}_R(M_2)$ and R is an integral domain, hence $\mathrm{Fitt}_{n_2}(M) \neq 0$. So, by (ii), $\omega(M) = \omega(M_1)$. □

Proposition 3.4. *Let R be a valuation ring and M be a finitely generated R-module. Suppose that $M = M_1 \oplus M_2$, for some R-modules M_1, M_2, where M_2 is a multiplication module.*
If $Ann_R(M_2)$ and the Fitting ideals of M_1 are principal and cancellation ideals, then the Fitting ideals of M are principal and cancellation.

Proof. By Proposition 2.1, $\mathrm{Fitt}_{n_1+1}(M) = \mathrm{Ann}_R(M_2) + \mathrm{Fitt}_{n_1}(M_1)$. So, $\mathrm{Fitt}_{n_1+1}(M) = \mathrm{Ann}_R(M_2)$ or $\mathrm{Fitt}_{n_1+1}(M) = \mathrm{Fitt}_{n_1}(M_1)$. Hence $\mathrm{Fitt}_{n_1+1}(M)$ is a principal and cancellation ideal. Let $n_2 < k \leq n_1$. By Proposition 2.1, $\mathrm{Fitt}_k(M) = \mathrm{Fitt}_k(M_1)\mathrm{Ann}_R(M_2) + \mathrm{Fitt}_{k-1}(M_1)$. We have two cases:
Case i: Let $\mathrm{Fitt}_k(M) = \mathrm{Fitt}_{k-1}(M_1)$. Hence $\mathrm{Fitt}_k(M)$ is a principal and cancellation ideal.
Case ii: Let $\mathrm{Fitt}_k(M) = \mathrm{Fitt}_k(M_1)\mathrm{Ann}_R(M_2)$. It's clear that $\mathrm{Fitt}_k(M)$ is a principal ideal. Let I and J be two ideals of R and $\mathrm{Fitt}_k(M)I = \mathrm{Fitt}_k(M)J$. Hence $\mathrm{Fitt}_k(M_1)\mathrm{Ann}_R(M_2)I = \mathrm{Fitt}_k(M_1)\mathrm{Ann}_R(M_2)J$. We obtain $I = J$. Hence $\mathrm{Fitt}_k(M)$ is a cancellation ideal.
Suppose that $\mathrm{Fitt}_{n_2}(M) \neq 0$. By Proposition 2.1,

$$\mathrm{Fitt}_{n_2}(M) = \mathrm{Fitt}_{n_2}(M_1)\mathrm{Ann}_R(M_2).$$

Hence $\mathrm{Fitt}_{n_2}(M)$ is a principal and cancellation ideal. □

Proposition 3.5. *Let $M_1, ..., M_n$ be finitely generated multiplication R-modules, where $Ann_R(M_1) \subseteq Ann_R(M_2) \subseteq ... \subseteq Ann_R(M_n)$. Set $M = \oplus_{i=1}^n M_i$ then*

$$\mathrm{Fitt}_k(M) = \begin{cases} R & \text{if } k \geq n \\ \mathrm{Ann}_R(M_{k+1})...\mathrm{Ann}_R(M_n) & \text{if } k = 0, 1, ..., n-1. \end{cases}$$

Proof. We prove by induction on n. If $n = 1$, then it's clear that $\mathrm{Fitt}_0(M) = \mathrm{Ann}_R(M_1)$ and $\mathrm{Fitt}_k(M) = R$, $k \geq 1$. Suppose that, the assertion is true for $k \leq n$. If $k = 0, 1, ..., n-2$, then

$$\mathrm{Fitt}_k(M) = \Sigma_{i+j=k}\mathrm{Fitt}_i(\oplus_{i=1}^{n-1} M_i)\mathrm{Fitt}_j(M_n) = \mathrm{Fitt}_k(\oplus_{i=1}^{n-1} M_i)\mathrm{Fitt}_0(M_n)$$

$$+ \cdots + \mathrm{Fitt}_0(\oplus_{i=1}^{n-1} M_i)\mathrm{Fitt}_k(M_n).$$

So, by induction hypothesis,

$$\mathrm{Fitt}_k(M) = \mathrm{Ann}_R(M_{k+1}) \cdots \mathrm{Ann}_R(M_{n-1})\mathrm{Ann}_R(M_n)$$

$$+ \cdots + \mathrm{Ann}_R(M_k) \cdots \mathrm{Ann}_R(M_{n-2})\mathrm{Ann}_R(M_{n-1}).$$

Since $\mathrm{Ann}_R(M_i) \subseteq \mathrm{Ann}_R(M_{i+1})$,
$\mathrm{Fitt}_k(M) = \mathrm{Ann}_R(M_{k+1}) \cdots \mathrm{Ann}_R(M_n)$. Let $k = n-1$. We have

$$\mathrm{Fitt}_{n-1}(M) = \Sigma_{i+j=k}\mathrm{Fitt}_i(\oplus_{i=1}^{n-1} M_i)\mathrm{Fitt}_j(M_n)$$

$$= \mathrm{Fitt}_{n-1}(\oplus_{i=1}^{n-1} M_i)\mathrm{Fitt}_0(M_n) + \mathrm{Fitt}_{n-2}(\oplus_{i=1}^{n-1} M_i).$$

By induction hypothesis,

$$\mathrm{Fitt}_{n-1}(M) = \mathrm{Fitt}_0(M_n) + \mathrm{Ann}_R(M_{n-1}) = \mathrm{Ann}_R(M_n) + \mathrm{Ann}_R(M_{n-1})$$

$$= \mathrm{Ann}_R(M_n).$$

Let $k \geq n$, then

$$\mathrm{Fitt}_k(M) = \Sigma_{i+j=k}\mathrm{Fitt}_i(\oplus_{i=1}^{n-1} M_i)\mathrm{Fitt}_j(M_n) = R.$$

□

Theorem 3.6. *Let $M_1, ..., M_n$ be finitely generated multiplication R-modules, where $Ann_R(M_1) \subseteq Ann_R(M_2) \subseteq ... \subseteq Ann_R(M_n)$. Set $M = \oplus_{i=1}^n M_i$. The Fitting ideals of M are principal and cancellation if and only if $Ann_R(M_1)$, $Ann_R(M_2), ..., Ann_R(M_n)$ are principal and cancellation.*

Proof. ($\Longrightarrow$) By Proposition 3.5, $\mathrm{Fitt}_{n-1}(M) = \mathrm{Ann}_R(M_n)$, thus $\mathrm{Ann}_R(M_n)$ is principal and cancellation. Again by Proposition 3.5,

$$\mathrm{Fitt}_{n-2}(M) = \mathrm{Ann}_R(M_{n-1})\mathrm{Ann}_R(M_n).$$

Suppose that $\mathrm{Fitt}_{n-2}(M_{n-1}) = \langle\alpha\rangle$ and $\mathrm{Ann}_R(M_n) = \langle\beta\rangle$. So, there exists $x \in \mathrm{Ann}_R(M_{n-1})$ such that $\alpha = x\beta$. Let $y \in \mathrm{Ann}_R(M_{n-1})$. Since $\mathrm{Ann}_R(M_{n-1}) \subseteq \mathrm{Ann}_R(M_n)$, there exists $t \in R$ such that $y = t\beta$. We have $y\beta \in \mathrm{Fitt}_{n-2}(M)$, hence there exists $r \in R$ such that $y\beta = r\alpha = rx\beta$. So, $\langle y\rangle\langle\beta\rangle = \langle rx\rangle\langle\beta\rangle$. Therefore, $\langle y\rangle\mathrm{Ann}_R(M_n) = \langle rx\rangle\mathrm{Ann}_R(M_n)$. Therefore, $y \in \langle x\rangle$. So, $\mathrm{Ann}_R(M_{n-1}) = \langle x\rangle$. Now, suppose that I, J be two ideals of R and $\mathrm{Ann}_R(M_{n-1})I = \mathrm{Ann}_R(M_{n-1})J$. Hence $\mathrm{Ann}_R(M_n)\mathrm{Ann}_R(M_{n-1})I = \mathrm{Ann}_R(M_n)\mathrm{Ann}_R(M_{n-1})J$. This implies that $\mathrm{Fitt}_{n-2}(M)I = \mathrm{Fitt}_{n-2}(M)J$. Hence $I = J$. So, $\mathrm{Ann}_R(M_{n-1})$ is cancellation. Similarly

$$\mathrm{Ann}_R(M_1), ..., \mathrm{Ann}_R(M_{n-2})$$

are principal and cancellation.
($\Longleftarrow$) It is clear. $\square$

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