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Basic notions and properties of ordered semihyperrings

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Abstract. In this paper, we introduce the concept of semihyperring $(R, +, \cdot)$ together with a suitable partial order $\leq$. Moreover, we introduce and study hyperideals in ordered semihyperrings. Simple ordered semihyperrings are defined and its characterizations are obtained. Finally, we study some properties of quasi-simple and B-simple ordered semihyperrings.

1 Introduction

Hyperrings extend the classical notion of rings, substituting both or only one of the binary operations of addition and multiplication by hyperoperations. Hyperrings were introduced by several authors in different ways. If only the addition is a hyperoperation and the multiplication is a binary operation, then we say that R is a Krasner hyperring [31]. Davvaz [11] has defined some relations in hyperrings and prove isomorphism theorems. For a more comprehensive introduction about hyperrings, we refer to [15].

As a generalization of a ring, semiring was first introduced by Vandiver [40] in 1934. A *semiring* is a structure $(R, +, \cdot, 0)$ with two binary operations $+$ and $\cdot$ such that $(R, +, 0)$ is a commutative semigroup, $(R, \cdot)$

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is a semigroup, multiplication is distributive from both sides over addition and $0 \cdot x = 0 = x \cdot 0$ for all $x \in R$. Semiring theory has many applications to other branches. Semirings are studied in relations with applications in [20]. In [41], Vougiouklis generalizes the notion of hyperring and named it as semihyperring, where both the addition and multiplication are hyperoperation. Semihyperrings are a generalization of Krasner hyperrings. Note that a semiring with zero is a semihyperring. Davvaz in [12] studied the notion of semihyperrings in a general form. Ameri and Hedayati defined k-hyperideals in semihyperrings in [2].

Hyperstructures, in particular hypergroups, were introduced in 1934 by Marty [33] at the eighth congress of Scandinavian Mathematicians. The notion of algebraic hyperstructure has been developed in the following decades and nowadays by many authors, especially Corsini [9, 10], Davvaz [13–15], Mittas [34], Spartalis [37], Stratigopoulos [39] and Vougiouklis [42]. Basic definitions and notions concerning hyperstructure theory can be found in [9].

In [22], Heidari and Davvaz studied a semihypergroup $(H, \circ)$ together with a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone condition. Indeed, an *ordered semihypergroup* $(H, \circ, \leq)$ is a semihypergroup $(H, \circ)$ together with a partial order $\leq$ such that satisfies the monotone condition as follows:

$$x \leq y \Rightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z, \text{ for all } x, y, z \in H.$$

Here, $z \circ x \leq z \circ y$ means for any $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly. Indeed, the concept of ordered semihypergroups is a generalization of the concept of ordered semigroups. The concept of ordering hypergroups introduced by Chvalina [6] as a special class of hypergroups and studied by many authors, for example, see [5, 7, 16, 23, 24]. In [4], polygroups which are partially ordered are introduced and some properties and related results are given.

As we know, partially ordered rings play an important role in the abstract algebra. $(R, +, \cdot)$ is a *partially ordered ring* [19] if R has a partial order $\leq$ satisfying the following conditions: (1) $a \leq b$ implies $a + c \leq b + c$ for each $c \in R$; (2) $0 \leq a$ and $0 \leq b$ imply $0 \leq a \cdot b$. It is an easy consequence of (1) and (2) of the above definition, if $a, b, c \in R$ with $a \leq b$ and $0 \leq c$, then $a \cdot c \leq b \cdot c$. A semigroup $(S, \cdot)$ is a *partially ordered semigroup*, if there is defined on S a partial order $\leq$, which is compatible with the operation

$\cdot$ in S. Ordered semigroups have been extensively investigated by many authors, for example, see [1, 8, 28, 29, 36]. There is a classical book by Fuchs [17] where ordered algebraic structures are discussed.

The present paper is organized as follows: In section 2, we recall some concepts of semihyperrings. In section 3, we introduce the notion of ordered semihyperrings and present several examples of them. In section 4, we intend to concentrate our efforts on the characterizations of simple, quasi-simple and B-simple ordered semihyperrings in terms of hyperideals, quasi-hyperideals and bi-hyperideals.

2 Basic terminology

This section explains some basic notions and definitions that have been used in this paper. In what follows, we summarize some basic definitions about semihypergroups and semihyperrings.

A mapping $\circ : H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the family of all non-empty subsets of H, is called a *hyperoperation* on H. The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we denote

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b,\ A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for every $x, y, z \in H$, $x \circ (y \circ z) = (x \circ y) \circ z$, that is

$$\bigcup_{u\in y\circ z} x \circ u = \bigcup_{v\in x\circ y} v \circ z.$$

A *hypergroup* is a semihypergroup $(H, \circ)$ such that $H \circ x = x \circ H = H$ for all $x \in H$, which is called reproduction axiom. A *canonical hypergroup* [34] is a non-empty set H endowed with an additive hyperoperation $+ : H \times H \to \mathcal{P}^*(H)$, satisfying the follwing properties: (1) $x + (y + z) = (x + y) + z$ for any $x, y, z \in H$; (2) $x + y = y + x$ for any $x, y \in H$; (3) There exists $0 \in H$ such that $0 + x = x + 0 = x$, for any $x \in H$; (4) For every $x \in H$, there exists one and only one $x' \in H$, such that $0 \in x + x'$; (we shall write $-x$ for x' and we call it the opposite of x.) (5) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$, that is $(H, +)$ is reversible. A *Krasner hyperring* [31] is an algebraic hypersructure $(R, +, \cdot)$ which satisfies the following axioms: (1) $(R, +)$ is a canonical hypergroup; (2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, that is, $x \cdot 0 = 0 \cdot x = 0$; (3) The multiplication $\cdot$ is distributive with respect to the hyperoperation $+$.

Definition 2.1. [41] A *semihyperring* is an algebraic hypersructure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a commutative semihypergroup with a zero element 0 satisfying $x + 0 = 0 + x = \{x\}$, that is, (i) For all $x, y, z \in R, x + (y + z) = (x + y) + z$, (ii) For all $x, y \in R, x + y = y + x$, (iii) There exists $0 \in R$ such that $x + 0 = 0 + x = \{x\}$ for all $x \in R$;

(2) $(R, \cdot)$ is a semihypergroup;

(3) The multiplication $\cdot$ is distributive with respect to the hyperoperation $+$, that is, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$;

(4) The element $0 \in R$ is an absorbing element, that is, $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$.

A semihyperring R is called *commutative* if $(R, \cdot)$ is a commutative semihypergroup. A non-empty subset A of a semihyperring $(R, +, \cdot)$ is called a *subsemihyperring* of R if for all $x, y \in A$, $x + y \subseteq A$ and $x \cdot y \subseteq A$. A non-empty subset I of a semihyperring $(R, +, \cdot)$ is called a *left* (respectively *right*) *hyperideal* of $(R, +, \cdot)$ if for all $x, y \in I$, $x + y \subseteq I$ and $r \cdot x \subseteq I$ for all $x \in I$ and $r \in R$ (respectively $x \cdot r \subseteq I$). A non-empty subset I of R is called a *hyperideal* of R if it is both left and right hyperideal of R, that is, $x + y \subseteq I$, for all $x, y \in I$ and $x \cdot r, r \cdot x \subseteq I$, for all $x \in I$ and $r \in R$.

3 Ordered semihyperrings

First, we introduce the notion of ordered semihyperrings and present some examples of them.

Definition 3.1. An *ordered semihyperring* $(R, +, \cdot, \leq)$ is a semihyperring equipped with a partial order relation $\leq$ such that for all $a, b, c \in R$, we have

(1) $a \leq b$ implies $a + c \leq b + c$, meaning that for any $x \in a + c$, there exists $y \in b + c$ such that $x \leq y$.

(2) $a \leq b$ and $0 \leq c$ imply $a \cdot c \leq b \cdot c$, meaning that for any $x \in a \cdot c$, there exists $y \in b \cdot c$ such that $x \leq y$. The case $c \cdot a \leq c \cdot b$ is defined similarly.

Semihyperrings are viewed as ordered semihyperrings under the equality order relation. Indeed: Let $(R, +, \cdot)$ be a semihyperring. Define the order on R by $\leq := \{(a, b) : a = b\}$. Then $(R, +, \cdot, \leq)$ is an ordered semihyperring. An ordered semihyperring $(R, +, \cdot, \leq)$ is an ordered subsemihyperring of $(T, +, \cdot, \leq)$ if R is a subsemihyperring of T and the order on R is the restriction to R of the order on T. Let $(R, +, \cdot)$ and $(T, \oplus, \odot)$ be semihyperrings. A mapping $\varphi : R \to T$ is said to be homomorphism if $\varphi(x+y) \subseteq \varphi(x) \oplus \varphi(y)$ and $\varphi(x \cdot y) \subseteq \varphi(x) \odot \varphi(y)$. A *homomorphism* of ordered semihyperrings $\varphi : (R, +, \cdot, \leq) \to (T, \oplus, \odot, \preceq)$ is a semihyperring homomorphism such that for all $a, b \in R$, $a \leq b$ implies $\varphi(a) \preceq \varphi(b)$. The kernel of φ, kerφ, is defined by ker$\varphi = \{x \in R : \varphi(x) = 0\}$.

In the following we present several examples of ordered semihyperrings with different covering relations.

Example 3.2. Let $R = \{0, a, b, c\}$ be a set with two hyperoperations $\oplus$ and $\odot$ as follows:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	$\{0, a, b\}$	$\{0, a, c\}$
b	b	$\{0, a, b\}$	$\{0, b\}$	$\{0, b, c\}$
c	c	$\{0, a, c\}$	$\{0, b, c\}$	$\{0, c\}$

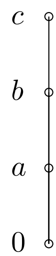
$\odot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	$\{0, a\}$
c	0	0	$\{0, a\}$	$\{0, b\}$

Then, $(R, \oplus, \odot)$ is a semihyperring [25]. We have $(R, \oplus, \odot, \leq)$ is an ordered semihyperring where the order relation $\leq$ is defined by:

$$\leq \; := \{(0, 0), (a, a), (b, b), (c, c), (0, a), (0, b), (0, c), (a, b), (a, c), (b, c)\}.$$

The covering relation and the figure of R are given by:

$$\prec \; = \{(0, a), (a, b), (b, c)\}.$$



Example 3.3. Consider the semihyperring $R = \{0, a, b\}$ with the hyperaddition $\oplus$ and the hypermultiplication $\odot$ defined as follows:

$\oplus$	0	a	b
0	0	a	b
a	a	$\{0, a\}$	R
b	b	R	$\{0, b\}$

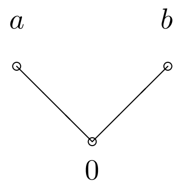
$\odot$	0	a	b
0	0	0	0
a	0	$\{0, a\}$	$\{0, b\}$
b	0	$\{0, b\}$	$\{0, a\}$

Then, $(R, \oplus, \odot)$ is a semihyperring. We have $(R, \oplus, \odot, \leq)$ is an ordered semihyperring, where the order relation $\leq$ is defined by:

$$\leq := \{(0, 0), (a, a), (b, b), (0, a), (0, b)\}.$$

The covering relation and the figure of R are given by:

$$\prec = \{(0, a), (0, b)\}.$$



Example 3.4. Let $R = \{0, a, b, c\}$ be a set with the hyperaddition $\oplus$ and the multiplication $\odot$ defined as follows:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	$\{a, b\}$	b	c
b	b	b	$\{0, b\}$	c
c	c	c	c	$\{0, c\}$

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

Then, $(R, \oplus, \odot)$ is a semihyperring. We have $(R, \oplus, \odot, \leq)$ is an ordered semihyperring where the order relation $\leq$ is defined by:

$$\leq := \{(0,0), (a,a), (b,b), (c,c), (0,a), (0,b), (0,c), (a,b), (a,c), (b,c)\}.$$

The covering relation and the figure of R are given by:

$$\prec = \{(0,a), (a,b), (b,c)\}.$$

Hyperideals of semihyperrings play an important role in the structure theory of ordered semihyperrings. In the following, we define hyperideals in ordered semihyperrings and study some of their related properties.

Definition 3.5. Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. A non-empty subset I of R is called a *left hyperideal* of R if it satisfies the following conditions:

(1) $x + y \subseteq I$ for all $x, y \in I$;

(2) $r \cdot x \subseteq I$ for all $x \in I$ and $r \in R$;

(3) When $x \in I$ and $r \in R$ such that $r \leq x$, imply that $r \in I$.

A right hyperideal of an ordered semihyperring R is defined in a similar way. I is called a *hyperideal* of R if it is both left and right hyperideal of R. It is clear that $\{0\}$ and R are hyperideals of R. A left, right or hyperideal I of an ordered semihyperring R is called proper if $I \neq \{0\}$ and $I \neq R$. A proper hyperideal I of R is called *minimal* if there is no proper hyperideal K of R such that $K \subseteq I$. Equivalently, if for any hyperideal K of R such that $K \subseteq I$, then we have $K = \{0\}$ or $K = I$.

Example 3.6. Let $(R, \oplus, \odot, \leq)$ be the ordered semihyperring defined as in Example 3.4. It is easy to see that $I = \{0, a, b\}$ is a right hyperideal of R, but it is not a left hyperideal of R.

Lemma 3.7. *Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring and $\{I_k : k \in \Lambda\}$ be a family of hyperideals of R. Then, $\bigcap\limits_{k\in\Lambda} I_k$ is a hyperideal of R.*

Proof. Let $x, y \in \bigcap\limits_{k\in\Lambda} I_k$. Then $x, y \in I_k$ for each $k \in \Lambda$. Since each I_k is a hyperideal of R, it follows that $x + y \subseteq I_k$ for all $k \in \Lambda$. Thus we have $x + y \subseteq \bigcap\limits_{k\in\Lambda} I_k$. Now, let $x \in R$ and $a \in \bigcap\limits_{k\in\Lambda} I_k$. Since $a \in \bigcap\limits_{k\in\Lambda} I_k$, we have $a \in I_k$ for all $k \in \Lambda$. Since each I_k is a hyperideal of R, it follows that $x \cdot a \subseteq I_k$ for all $k \in \Lambda$. So, we have $x \cdot a \subseteq \bigcap\limits_{k\in\Lambda} I_k$. If $x \in \bigcap\limits_{k\in\Lambda} I_k$, $y \in R$ and $y \leq x$, then $x \in I_k$ for each $k \in \Lambda$. Since I_k is a hyperideal of R, we obtain $y \in I_k$ for all $k \in \Lambda$. Thus $y \in \bigcap\limits_{k\in\Lambda} I_k$. This completes the proof. □

Theorem 3.8. *Let φ be a homomorphism from an ordered semihyperring $(R, +, \cdot, \leq)$ into an ordered semihyperring $(T, \oplus, \odot, \preceq)$. Then, ker φ is a hyperideal of R.*

Proof. Let $x \in ker\varphi$. Then we have $\varphi(x) = 0$. Since φ is a homomorphism, it follows that $\{\varphi(0)\} = \varphi(0) + 0 = \varphi(0) \oplus \varphi(x) \supseteq \varphi(0 + x) = \varphi(\{x\}) = \{\varphi(x)\}$. This implies that $\varphi(0) = \varphi(x) = 0$. Thus we have $0 \in ker\varphi$. Since $0 \in ker\varphi$, it follows that $ker\varphi \neq \emptyset$.

Let $x_1, x_2 \in ker\varphi$. Then $\varphi(x_1) = 0 = \varphi(x_2)$. Since φ is a homomorphism, it follows that $\varphi(x_1 + x_2) \subseteq \varphi(x_1) \oplus \varphi(x_2) = 0 \oplus 0 = \{0\}$. Hence $x_1 + x_2 \subseteq ker\varphi$. Now, let $r \in R$ and $x \in ker\varphi$. Then we have $\varphi(x) = 0$. Since φ is a homomorphism, it follows that $\varphi(r \cdot x) \subseteq \varphi(r) \odot \varphi(x) = \varphi(r) \odot 0 = 0$. So, $r \cdot x \subseteq ker\varphi$. Similarly, we have $x \cdot r \subseteq ker\varphi$. Now, let $x \in ker\varphi$, $r \in R$ and $r \leq x$. Since φ is a homomorphism, it follows that $\varphi(r) \leq \varphi(x) = 0$. Thus $\varphi(r) = 0$. So, $r \in ker\varphi$. This completes the proof. □

Theorem 3.9. *Let φ be a homomorphism from an ordered semihyperring $(R, +, \cdot, \leq)$ into an ordered semihyperring $(T, \oplus, \odot, \preceq)$. If I is a hyperideal of T, then $\varphi^{-1}(I) = \{a \in R : \varphi(a) \in I\}$ is a hyperideal of R containing $ker\varphi$.*

Proof. Since $0 \in \varphi^{-1}(I)$, it follows that $\varphi^{-1}(I) \neq \emptyset$. Let $a, b \in \varphi^{-1}(I)$. Then $\varphi(a), \varphi(b) \in I$. Since I is a hyperideal of T, we have $\varphi(a+b) \subseteq \varphi(a) \oplus \varphi(b) \subseteq I$. Hence $a + b \subseteq \varphi^{-1}(I)$. Let $x \in R$ and $a \in \varphi^{-1}(I)$. Then $\varphi(a) \in I$. Since φ is a homomorphism, it follows that $\varphi(x \cdot a) \subseteq \varphi(x) \odot \varphi(a) \subseteq I$. Thus $x \cdot a \in \varphi^{-1}(I)$. Similarly, $a \cdot x \in \varphi^{-1}(I)$. Now, suppose that $a \in \varphi^{-1}(I)$ and $r \in R$ such that $r \leq a$. Then $\varphi(a) \in I$. Since $r \leq a$ and φ is a homomorphism, we have $\varphi(r) \preceq \varphi(a)$. Since I is a hyperideal of T, it follows that $\varphi(r) \in I$. So, $r \in \varphi^{-1}(I)$. This proves that $\varphi^{-1}(I)$ is a hyperideal of R, as desired. Moreover, if $x \in ker\varphi$, then $\varphi(x) = 0 \in I$. Hence $x \in \varphi^{-1}(I)$. Therefore, $ker\varphi \subseteq \varphi^{-1}(I)$. □

4 Main results

Let A be a non-empty subset of an ordered semihyperring $(R, +, \cdot, \leq)$. Then the subset $\{x \in R : x \leq a \text{ for some } a \in A\}$ is denoted by $(A]$. For $A = \{a\}$, we write $(a]$ instead of $(\{a\}]$. If A and B are non-empty subsets of R, then we have

(1) $A \subseteq (A]$;

(2) $((A]] = (A]$;

(3) $(A] \cdot (B] \subseteq (A \cdot B]$;

(4) $((A] \cdot (B]] = (A \cdot B]$;

(5) If $A \subseteq B$, then $(A] \subseteq (B]$.

It can be easily verified that the condition (3) in Definition 3.5 is equivalent to $(I] \subseteq I$.

Let A be a non-empty subset of an ordered semihyperring R. The intersection of all hyperideals of R containing A, is the hyperideal of R generated by A. A hyperideal generated by a non-empty subset A of R will be denoted by $\langle A \rangle$. So, $\bigcap\{I \mid A \subseteq I \text{ and } I \text{ is a hyperideal of } R\} = \langle A \rangle$.

Definition 4.1. An element a in an ordered semihyperring R is called *regular* if there exists an element $x \in R$ such that $a \leq axa$. An ordered semihyperring R is called regular if each element of R is regular.

Equivalent definitions:

(1) $a \in (aRa]$, $\forall a \in R$.

(2) $A \subseteq (ARA]$, $\forall A \subseteq R$.

Example 4.2. Let $(R, \oplus, \odot, \leq)$ be the ordered semihyperring defined as in Example 3.3. Now, it is easy to see that R is a regular ordered semihyperring.

Theorem 4.3. *Let I be a hyperideal of an ordered semihyperring $(R, +, \cdot, \leq)$. Then $(I]$ is a hyperideal of R generated by I.*

Proof. Since $I \subseteq (I]$, it follows that $\emptyset \neq (I]$. Assume that $x \in R$ and $a, b \in (I]$. Then there exist $r, s \in I$ such that $a \leq r$ and $b \leq s$. Since R is an ordered semihyperring, we obtain $a + b \leq r + b$ and $r + b \leq r + s$. Since I is a hyperideal of R, we have $a + b \leq r + s \subseteq I$. Hence, for any $u \in a + b$, there exists $v \in I$ such that $u \leq v$. Thus we have $u \in (I]$. So, $a + b \subseteq (I]$. Also, we have $x \cdot a \leq x \cdot r \subseteq I$ and $a \cdot x \leq r \cdot x \subseteq I$. This implies that $x \cdot a \subseteq (I]$ and $a \cdot x \subseteq (I]$. Since I is a hyperideal of R, we have $(I] \subseteq I$. So, $((I]] \subseteq (I]$. Hence $(I]$ is a hyperideal of R. If A is a hyperideal of R such that $I \subseteq A$, then $(I] \subseteq (A] \subseteq A$. So, $(I] \subseteq A$. This completes the proof. $\square$

We now prove the following lemma which is the crucial lemma in the establishment of our main theorems.

Lemma 4.4. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. Then, the following propositions are true:*

(1) *$(Ra]$ is a left hyperideal of R for all $a \in R$.*

(2) *$(aR]$ is a right hyperideal of R for all $a \in R$.*

Proof. (1): Let $x, y \in (Ra]$. Then $x \leq u$ and $y \leq v$ for some $u, v \in Ra$. Since $u, v \in Ra$, we have $u \in r \cdot a$ and $v \in s \cdot a$ where $r, s \in R$. Since R is an ordered semihyperring, we obtain $x + y \leq u + y$ and $u + y \leq u + v$. So, $x + y \leq u + v \subseteq r \cdot a + s \cdot a = (r + s) \cdot a$. Hence, for any $w \in x + y$, there exists $w' \in (r + s) \cdot a$ such that $w \leq w' \in Ra$. Thus we have $x + y \subseteq (Ra]$. So, the first condition of the definition of left hyperideal is verified.

Let $x \in (Ra]$ and $r \in R$. Since $x \in (Ra]$, it follows that $x \leq b$ for some $b \in Ra$. Since $b \in Ra$, we have $b \in r' \cdot a$ where $r' \in R$. Since R is an ordered semihyperring and $x \leq b$, we obtain $r \cdot x \leq r \cdot b$. So, we have

$r \cdot x \leq r \cdot b \subseteq r \cdot (r' \cdot a) = (r \cdot r') \cdot a \subseteq Ra$. Hence, for any $c \in r \cdot x$, there exists $t \in Ra$ such that $c \leq t$. This means that $r \cdot x \subseteq (Ra]$, and so the second condition of the definition of left hyperideal is verified.

Now, suppose that $x \in (Ra]$ and $y \in R$ such that $y \leq x$. Since $x \in (Ra]$, it follows that $x \leq s$ for some $s \in Ra$. Since $y \leq x$ and $x \leq s$, we obtain $y \leq s$. Since $y \in R$, $y \leq s$ and $s \in Ra$, we have $y \in (Ra]$. Hence $(Ra]$ is a left hyperideal of R, as desired.

(2): This proof is straightforward. □

Definition 4.5. An ordered semihyperring $(R, +, \cdot, \leq)$ is said to be *left simple* if the following conditions hold:

(1) $\{0\}$ and R are the only left hyperideals of R;

(2) $R \cdot R \neq \{0\}$.

In the same way, we can define a right simple ordered semihyperring.

Example 4.6. Let $(R, \oplus, \odot, \leq)$ be the ordered semihyperring defined as in Example 3.4. We can see that R is a left simple ordered semihyperring.

Theorem 4.7. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. Then, the following assertions hold:*

(1) *R is left simple if and only if $(Ra] = R$ for all $a \in R \setminus \{0\}$.*

(2) *R is right simple if and only if $(aR] = R$ for all $a \in R \setminus \{0\}$.*

Proof. (1): Assume that R is a left simple ordered semihyperring and $0 \neq a \in R$. By (1) of Lemma 4.4, $(Ra]$ is a left hyperideal of R. Since R is a left simple ordered semihyperring, we have $(Ra] = R$.

Conversely, suppose that $(Ra] = R$ for all $0 \neq a \in R$. Let A be a left hyperideal of R and $0 \neq a \in A$. By assumption, $(Ra] = R$. If $r \in R$, then $r \in (Ra]$. So, $r \leq w$ for some $w \in Ra$. Thus $w \in x \cdot a$ where $x \in R$. Since A is a left hyperideal of R, we have $r \leq w \in x \cdot a \subseteq A$. So, $r \in A$. Thus $R \subseteq A$ and so $A = R$. Therefore, R is a left simple ordered semihyperring.

(2): This proof is straightforward. □

In the following, we introduce the notion of simple ordered semihyperrings. Also, we characterize this type of ordered semihyperrings in terms of hyperideals.

Definition 4.8. An ordered semihyperring $(R, +, \cdot, \leq)$ is said to be *simple* if the following conditions hold:

(1) $\{0\}$ and R are the only hyperideals of R;

(2) $R \cdot R \neq \{0\}$.

Theorem 4.9. *If R is a left (respectively right) simple ordered semihyperring, then R is a simple ordered semihyperring.*

Proof. Assume that R is a left simple ordered semihyperring. Then $\{0\}$ and R are the only left hyperideals of R. If A is a hyperideal of R, then A is a left hyperideal of R. So, we have $A = \{0\}$ or $A = R$. Therefore, R is a simple ordered semihyperring. Similarly, if R is a right simple ordered semihyperring, then R is a simple ordered semihyperring. □

A subsemihyperring T of R is called simple if $T \cdot T \neq \{0\}$ and for every non-zero hyperideal I of T, we have $I = T$. In fact, a subsemihyperring T of R is called simple if the ordered semihyperring $(T, +, \cdot, \leq)$ is simple.

Lemma 4.10. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. Then, the following statements hold:*

(1) *If I is a left hyperideal of R and T is a left simple subsemihyperring of R such that $T \cap I \neq \{0\}$, then we have $T \subseteq I$.*

(2) *If R is left and right simple, then it is regular.*

Proof. (1): Suppose that $0 \neq x \in I \cap T$. Since T is left simple and $0 \neq x \in T$, so by Theorem 4.7, we have $(Tx] = T$. Therefore, we have $T = (Tx] \subseteq (TI] \subseteq (RI] \subseteq (I] = I$.

(2): Let $a \in R$. By Theorem 4.7, we have $a \in R = (aR] = (Ra]$. So, $a \in R = (aR] = (a \cdot (R \cdot a]] \subseteq (a \cdot R \cdot a]$. Thus R is a regular ordered semihyperring. □

Lajos in [32] studied the notion of left (respectively right) duo rings. In the following, we define left (respectively. right) duo ordered semihyperrings.

Definition 4.11. An ordered semihyperring $(R, +, \cdot, \leq)$ is said to be *left* (respectively *right*) *duo* if every left (respectively right) hyperideal of R is a hyperideal of R. An ordered semihyperring R is duo if it is both left and right duo.

Obviously, every left (respectively right) simple ordered semihyperring R is a left (respectively right) duo ordered semihyperring, but the converse is not true in general.

Example 4.12. Let $R = \{0, a, b, c\}$ be a set with two hyperoperations $\oplus$ and $\odot$ as follows:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	a	b
b	b	a	$\{0, b\}$	$\{0, b, c\}$
c	c	a	$\{0, b, c\}$	$\{0, c\}$

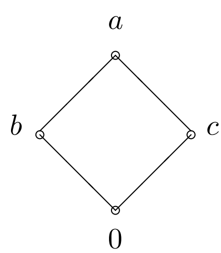
$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	$\{0, b\}$	0
b	0	0	0	0
c	0	$\{0, c\}$	0	0

Then, $(R, \oplus, \odot)$ is a semihyperring [25]. We have $(R, \oplus, \odot, \leq)$ is an ordered semihyperring where the order relation $\leq$ is defined by:

$$\leq \; := \{(0, 0), (a, a), (b, b), (c, c), (0, a), (0, b), (0, c), (b, a), (c, a)\}.$$

The covering relation and the figure of R are given by:

$$\prec \; = \{(0, b), (0, c), (b, a), (c, a)\}.$$



$\{0\}$, $\{0, b\}$, $\{0, c\}$, $\{0, b, c\}$ and R are left hyperideals of R. It is easy to see that R is a left duo ordered semihyperring, but it is not a left simple ordered semihyperring.

An *ordered semiring* is a semiring R equipped with a partial order $\leq$ such that the operations are monotonic and 0 is the least element of R. In [18], Gan and Jiang studied some properties of ideals in ordered semirings. Note that every ordered semiring with zero is an ordered semihyperring. In [26], Iseki introduced the notion of quasi-ideal for a semiring without zero. By a *quasi-ideal* of a semiring R we mean an additive subsemigroup Q of R such that $RQ \cap QR \subseteq Q$. A comprehensive review of the theory of quasi-ideals appears in [38]. Our aim in the following is to introduce and study the notion of a quasi-hyperideal of ordered semihyperrings.

Definition 4.13. A non-empty subset Q of an ordered semihyperring $(R, +, \cdot, \leq)$ is called a *quasi-hyperideal* of R if the following conditions hold:

(1) $Q + Q \subseteq Q$;

(2) $(Q \cdot R) \cap (R \cdot Q) \subseteq Q$;

(3) When $x \in Q$ and $y \in R$ such that $y \leq x$, imply that $y \in Q$.

A quasi-hyperideal Q of R is said to be *minimal* if it contains no non-zero proper quasi-hyperideal of R. Every left and right hyperideal of an ordered semihyperring R is a quasi-hyperideal of R. The converse is not true, in general, that is, a quasi-hyperideal may not be a left or a right hyperideal of R.

Example 4.14. Let $R = \{0, a, b\}$ be a set with two hyperoperations $\oplus$ and $\odot$ as follows:

$\oplus$	0	a	b
0	0	a	b
a	a	a	$\{a, b\}$
b	b	$\{a, b\}$	b

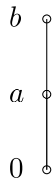
$\odot$	0	a	b
0	0	0	0
a	0	$\{0, a\}$	$\{0, a\}$
b	0	$\{0, b\}$	$\{0, b\}$

Then, $(R, \oplus, \odot)$ is a semihyperring [3]. We have $(R, \oplus, \odot, \leq)$ is an ordered semihyperring where the order relation $\leq$ is defined by:

$$\leq := \{(0, 0), (a, a), (b, b), (0, a), (0, b), (a, b)\}.$$

The covering relation and the figure of R are given by:

$$\prec = \{(0, a), (a, b)\}.$$



Now, it is easy to see that $Q = \{0, a\}$ is a quasi-hyperideal of R, but it is not a left hyperideal of R.

In the following, we define quasi-simple ordered semihyperrings and investigate some of their related properties.

Definition 4.15. An ordered semihyperring $(R, +, \cdot, \leq)$ is said to be *quasi-simple* if R has no non-zero proper quasi-hyperideal.

Theorem 4.16. *Let Q be a quasi-hyperideal of an ordered semihyperring R. If Q is a quasi-simple ordered semihyperring, then Q is a minimal quasi-hyperideal of R.*

Proof. Let Q be a quasi-hyperideal of R. Then, $Q \cdot Q \subseteq R \cdot Q$ and $Q \cdot Q \subseteq Q \cdot R$. Since Q is a quasi-hyperideal of R, it follows that $Q \cdot Q \subseteq R \cdot Q \cap Q \cdot R \subseteq Q$. Hence Q is a subhyperring of R. Let Q' be a non-zero quasi-hyperideal of R such that $Q' \subseteq Q$. Then $(Q \cdot Q') \cap (Q' \cdot Q) \subseteq (R \cdot Q') \cap (Q' \cdot R) \subseteq Q'$. Hence Q' is a non-zero quasi-hyperideal of Q. Since Q is a quasi-simple ordered semihyperring, it follows that $Q' = Q$. Therefore, Q is a minimal quasi-hyperideal of R. $\square$

Theorem 4.17. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. Then R is a quasi-simple ordered semihyperring if and only if $(Ra] \cap (aR] = R$ for all $a \in R \setminus \{0\}$.*

Proof. Assume that R is a quasi-simple ordered semihyperring. By (1) of Lemma 4.4, $(Ra]$ is a left hyperideal of R. Similarly, $(aR]$ is a right hyperideal of R. It can be easily verified that $(Ra] \cap (aR]$ is a quasi-hyperideal of R. Also, $(Ra] \subseteq (R] = R$ and $(aR] \subseteq (R] = R$ imply $(Ra] \cap (aR] \subseteq R$. Since R is quasi-simple, we obtain $R = (Ra] \cap (aR]$.

Conversely, suppose that $(Ra] \cap (aR] = R$ for all $a \in R \setminus \{0\}$. Let Q be a non-zero quasi-hyperideal of R. For any $0 \neq q \in Q$, we have $R = (Rq] \cap (qR] \subseteq (RQ] \cap (QR] \subseteq Q$ So, $R \subseteq Q$. Thus $R = Q$. Therefore, R is a quasi-simple ordered semihyperring. $\square$

Good and Hughes [21] introduced the notion of bi-ideals of a semigroup as early as 1952. Later, bi-ideals of ordered semigroups were studied by many authors, for example, see [27, 30, 43]. By a *bi-ideal* we mean a sub-semigroup A of a semigroup $(S, \cdot)$ such that $A \cdot S \cdot A \subseteq A$. A subset A of a ring $(R, +, \cdot)$ is called a bi-ideal [35] of R if: (1) A is a subring of R; (2) $A \cdot R \cdot A \subseteq A$. In the following, we introduce the notion of bi-hyperideals of ordered semihyperrings and provide some related results.

Definition 4.18. Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. A non-empty subset A of R is called a *bi-hyperideal* of R if it satisfies:

(1) $A + A \subseteq A$ and $A \cdot A \subseteq A$;

(2) $A \cdot R \cdot A \subseteq A$;

(3) When $x \in A$ and $y \in R$ such that $y \leq x$, imply that $y \in A$.

Definition 4.19. Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. A non-zero bi-hyperideal A of R is called a *minimal bi-hyperideal* of R if A does not properly contain any non-zero bi-hyperideal.

The remainder of this paper focusses on B-simple ordered semihyperrings.

Definition 4.20. An ordered semihyperring $(R, +, \cdot, \leq)$ is said to be B-*simple* if the following conditions hold:

(1) R has no non-zero proper bi-hyperideals;

(2) $R \cdot R \neq \{0\}$.

Theorem 4.21. *Let A be a bi-hyperideal of an ordered semihyperring $(R, +, \cdot, \leq)$. Then, $(uAv]$ is a bi-hyperideal of R for every $u, v \in R$. In particular, $(uRv]$ is a bi-hyperideal of R for every $u, v \in R$.*

Proof. Suppose that $x, y \in (uAv]$ and $r \in R$. Then $x \leq s$ and $y \leq t$ for some $s, t \in uAv$. So, there exist $p, q \in A$ such that $s \in upv$ and $t \in uqv$. Since R is an ordered semihyperring and A is a bi-hyperideal of R, we have $x \cdot y \leq s \cdot t \subseteq (upv) \cdot (uqv) = u(p(v \cdot u)q)v \subseteq uAv$. So, for any $k \in x \cdot y$, there exists $l \in uAv$ such that $k \leq l$. This means that $x \cdot y \subseteq (uAv]$. Similarly, $x + y \leq s + t \subseteq upv + uqv = u(p + q)v \subseteq uAv$. Thus $x + y \subseteq (uAv]$. So, the first condition of the definition of bi-hyperideal is verified. Also, $x \cdot r \cdot y \leq s \cdot r \cdot t \subseteq (upv) \cdot r \cdot (uqv) = u((pv) \cdot r \cdot (uq))v = u(p \cdot (vru) \cdot q)v$. Since A is a bi-hyperideal of R, we get $p \cdot (vru) \cdot q \subseteq A \cdot R \cdot A \subseteq A$. Hence, for any $a \in x \cdot r \cdot y$, there exists $b \in uAv$ such that $a \leq b$. Thus $x \cdot r \cdot y \subseteq (uAv]$. This means that $(uAv] \cdot R \cdot (uAv] \subseteq (uAv]$, and so the second condition of the definition of bi-hyperideal is verified. Now, suppose that $x \in (uAv]$, $y \in R$ and $y \leq x$. Then, $x \leq w$ for some $w \in uAv$. Since $\leq$ is transitive, it follows that $y \leq w$ for some $w \in uAv$. This implies that $y \in (uAv]$. Therefore, $(uAv]$ is a bi-hyperideal of R. Since R is a bi-hyperideal of itself, $(uRv]$ is a bi-hyperideal of R. □

Corollary 4.22. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. Then, the following conditions are equivalent:*

(1) *R is B-simple.*

(2) *$(uRu] = R$ for all $u \in R \setminus \{0\}$.*

(3) *R is a left and right simple ordered semihyperring.*

Proof. (1) $\Rightarrow$ (2): Assume that (1) holds. By Theorem 4.21, $(uRu]$ is a bi-hyperideal of R for all $u \in R \setminus \{0\}$. Since R is B-simple, we obtain $(uRu] = R$ for all $u \in R \setminus \{0\}$.

(2) $\Rightarrow$ (3): It is easy to see that $(uRu] \subseteq (Ru] \subseteq (R] = R$ and $(uRu] \subseteq (uR] \subseteq (R] = R$. By assumption, we get $(Ru] = R$ and $(uR] = R$ for all $u \in R \setminus \{0\}$. By Theorem 4.7, R is left and right simple.

(3) $\Rightarrow$ (1): Assume that (3) holds. Let A be a non-zero bi-hyperideal of R. By assumption, we have $(aR] = R$ and $(R] = R$ for all $a \in A \setminus \{0\}$. Since A is a bi-hyperideal of R, we obtain $R = (aR] = (a(Ra]] \subseteq ((a](Ra]] = (aRa] \subseteq (ARA] \subseteq (A] \subseteq A$. Thus we have $R = A$. This completes the proof. □

Corollary 4.23. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. If I is a minimal bi-hyperideal of R and J a bi-hyperideal of R, then $I = (uJv]$ for every $u, v \in I$.*

Proof. By Theorem 4.21, $(uJv]$ is a bi-hyperideal of R. Since I is a minimal bi-hyperideal of R and $(uJv] \subseteq (IJI] \subseteq (IRI] \subseteq (I] \subseteq I$, we obtain $(uJv] = I$. □

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