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Uniform Lipschitz-connectedness and metric convexity

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Dedicated to Themba Dube on his 65^{th} birthday

Abstract. In this paper we continue with our study of uniformly Lipschitz-connected metric spaces. We obtain further properties of uniformly Lipschitz-connected metric spaces and then obtain a generalisation of a result due to Edelstein. In addition, we show that for a proper Lipschitz-connected metric space, $L_d = 1$ precisely when X is convex, which leads us to conjecture that L_d is a kind of measure of convexity in a proper Lipschitz-connected metric space. We provide some examples to corroborate our conjecture.

1 Introduction

The concept of uniformly Lipschitz-connected sets in metric spaces was introduced by J. M. Borwein in 1983 ([2]). In [1], we investigated uniformly Lipschitz-connected metric spaces and one of our principal results there was that this class of spaces with Lipschitz maps is coreflective in the category

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of Lipschitz-connected metric spaces and Lipschitz maps. We also showed amongst other things that a metric space (X, d) is uniformly Lipschitz-connected precisely when the value $L_d < \infty$. In the present paper we obtain different ways of looking at the value L_d for a uniformly Lipschitz-connected metric space. In particular, we show that for a proper Lipschitz-connected metric space X, X is convex precisely when $L_d = 1$, which leads us to conjecture that in a proper Lipschitz-connected metric space, L_d is a measure of convexity. In the final section we provide a number of examples where we calculate L_d for each of them. This confirms two things, firstly, that all the spaces cited are uniformly Lipschitz-connected (since L_d is finite for each), and secondly, that our conjecture appears to be correct.

In the present paper we show that L_d is some kind of measure of convexity and we provide a few examples to illustrate this.

2 Preliminaries

All spaces considered in this paper are metric. It is understood that all intervals $[a, b]$ satisfy $a < b$.

A *path* from a point x_0 to a point x_1 in X is a continuous map $g : [a, b] \longrightarrow X$ such that $g(a) = x_0$ and $g(b) = x_1$. A subset C of X is *pathwise connected* if any two points in C can be joined by a path that lies entirely in C.

A map $g : [a, b] \longrightarrow X$ is called a *Lipschitz map* if there exists $L > 0$ such that for all $s, t \in [a, b]$ we have $d(g(s), g(t)) \leq L|s - t|$. L is called a *Lipschitz constant*. We note that such maps are continuous and hence are Lipschitz paths from $g(a)$ to $g(b)$. A subset C of X is then said to be *Lipschitz-connected* if any two points in C can be joined by a Lipschitz path that lies entirely in C. A subset C of a metric space (X, d) is *uniformly Lipschitz-connected* if there exists a positive constant L such that for any two points x_0 and x_1 in C we can find a path $g : [0, 1] \longrightarrow C$ with $g(0) = x_0$ and $g(1) = x_1$ such that $d(g(s), g(t)) \leq L|s - t|d(x_0, x_1)$ for all $s, t \in [0, 1]$ ([2]). We say that L is a *universal Lipschitz constant* for (X, d).

Below is Menger's ([5]) definition of convexity.

Definition 2.1. A metric space (X, d) is said to be *(metrically) convex* if for any two points $x, y \in X$, $x \neq y$, there exists $z \in X$ such that $d(x, y) =$

$d(x,z)+d(z,y)$.

3 Different ways of looking at L_d

Let (X,d) be a Lipschitz-connected metric space. For any two points $x_0, x_1 \in X$ there is then a Lipschitz-path $g : [a,b] \longrightarrow X$ connecting x_0 to x_1. Hence there exists $L > 0$ such that $d(g(s), g(t)) \leq L|s-t|$ for all $s, t \in [a,b]$. Thus the dilatation of g

$$\operatorname{dil}(g) = \sup\left\{\frac{d(g(s),g(t))}{|s-t|} : s,t \in [a,b], s \neq t\right\} < \infty$$

and is also a Lipschitz constant. Of course, the dilatation can be defined for any path f in any metric space. If $\operatorname{dil}(f)$ is finite, then f would be a Lipschitz path. It is also clear that $d(x_0,x_1) \leq \operatorname{dil}(g)$.

We note that if the path g is reparametrized, the length of the path does not change, however the dilatation does. Hence we only consider paths on $[0,1]$.

For a Lipschitz-connected metric space (X,d), define (see [1]) for any two points
$x_0, x_1 \in X$ the number $w(x_0,x_1)$ given by

$$w(x_0,x_1) = \inf\{\operatorname{dil}(g) : g : [0,1] \to (X,d) \text{ is a Lipschitz path connecting } x_0 \text{ to } x_1\}.$$

Then w is a metric on X and $d(x_0,x_1) \leq w(x_0,x_1)$. Furthermore a constant L_d (which could take the value ∞) was defined for such X by the formula

$$L_d = \sup\left\{\frac{w(x_0,x_1)}{d(x_0,x_1)} : x_0,x_1 \in X, x_0 \neq x_1\right\}.$$

It is always true that $L_d \geq 1$. It was proved in [1] that for a Lipschitz-connected metric space (X,d), X is uniformly Lipschitz-connected precisely when $L_d < \infty$.

There is also another description of the constant L_d given by the following proposition.

Proposition 3.1. *Let (X, d) be uniformly Lipschitz-connected. Then $L_d = \inf\{L : L$ is a universal Lipschitz constant$\}$.*

Proof. Take any L, a universal Lipschitz constant. Take any $x_0, x_1 \in X$, $x_0 \neq x_1$. Find a path $g : [0, 1] \longrightarrow X$ connecting x_0 to x_1 such that

$$d(g(s), g(t)) \leq L|s - t|d(x_0, x_1) \text{ for all } s, t \in [0, 1].$$

Then

$$w(x_0, x_1) \leq \mathrm{dil(g)} \leq L\, d(x_0, x_1)$$

and so

$$\frac{w(x_0, x_1)}{d(x_0, x_1)} \leq L.$$

Thus $L_d \leq L$. Since this is true for any such L, we have

$$L_d \leq \inf\{L : L \text{ is a universal Lipschitz constant}\}.$$

To show equality, assume $L_d < \inf\{L\}$. Find $\varepsilon > 0$ such that $L_d + \varepsilon < \inf\{L\}$. We shall show that $L_d + \varepsilon$ is a universal Lipschitz constant: Take any $x_0, x_1 \in X, x_0 \neq x_1$. We have $w(x_0, x_1) \leq L_d\, d(x_0, x_1)$. Since $\varepsilon > 0$, we can find a Lipschitz path g with domain $[0, 1]$ from x_0 to x_1 such that

$$w(x_0, x_1) \leq \mathrm{dil}(g) \leq w(x_0, x_1) + \varepsilon d(x_0, x_1).$$

Thus, for all $s, t \in [0, 1]$, we have

$$\begin{aligned} d(g(s), g(t)) &\leq \mathrm{dil}(g)\ |s - t| \\ &\leq (w(x_0, x_1) + \varepsilon d(x_0, x_1))\ |s - t| \\ &\leq (L_d\, d(x_0, x_1) + \varepsilon d(x_0, x_1))\ |s - t| \\ &= (L_d + \varepsilon)\ |s - t|\ d(x_0, x_1). \end{aligned}$$

Thus $L_d + \varepsilon$ is a universal Lipschitz constant. But then this means that

$$\inf\{L\} \leq L_d + \varepsilon < \inf\{L\},$$

which is a contradiction. □

The following is the well known definition of the length of a path in a metric space (see [3]).

Definition 3.2. A *partition* of $[a,b]$ is a finite collection of points $Y = \{y_0, y_1, ..., y_n\}$ such that $a = y_0 < y_1 < \ldots < y_n = b$. The *length* of a path $\gamma : [a,b] \to (X,d)$, written $L(\gamma)$, is defined as

$$L(\gamma) = \sup\left\{\sum_{i=1}^{n} d(\gamma(y_{i-1}), \gamma(y_i)) : \{y_0, y_1, \ldots, y_n\} \text{ is a partition of } [a,b]\right\}.$$

Proposition 3.3. *If $g : [a,b] \longrightarrow (X,d)$ is a Lipschitz path, then $L(g) \leq \text{dil}(g)|b-a|$. In particular, if (X,d) is uniformly Lipschitz-connected with a universal Lipschitz constant L and $x, y \in X$, then for any Lipschitz path g with domain $[0,1]$ from x to y satisfying the condition $d(g(s), g(t)) \leq L|s-t|d(x,y)$, we have $L(g) \leq L\, d(x,y)$.*

Proof. Let $P = \{y_i\}$ be any partition of $[a,b]$. Then

$$\begin{aligned} \sum_i d(g(y_i), g(y_{i+1})) &\leq \sum_i \text{dil}(g)\, |y_i - y_{i+1}| \\ &= \text{dil}(g) \sum_i |y_i - y_{i+1}| \\ &= \text{dil}(g)|b-a|. \end{aligned}$$

Hence $L(g) \leq \text{dil}(g)|b-a|$.
The next part is true, since $L\, d(x,y)$ is a Lipschitz constant of g. □

Recall that a metric space is called *proper* if all its closed bounded sets are compact. It is known that a proper metric space is complete. We show first that under the assumption that X is a proper metric space, L_d is also a universal Lipschitz constant.

We use the definition of uniform equicontinuity and the Arzelá-Ascoli theorem in Proposition 3.6 and so we state them below.

Definition 3.4. ([6]) Let X and Y be metric spaces. A sequence of maps $(f_n)_{n \geq 0}$ from X to Y, is said to be *uniformly equicontinuous* if for every $\epsilon > 0$, there exists $\eta > 0$ such that for every integer $n \geq 0$ and for every $x, y \in X$, we have

$$d(x,y) < \eta \implies d(f_n(x), f_n(y)) < \epsilon.$$

Theorem 3.5. (*Arzelá-Ascoli Theorem*)([6]) *Let Y be a separable metric space, let X be a proper metric space and let $(f_n)_{n\geq 0}$ be a uniformly equicontinuous sequence of maps from Y to X such that for each $y \in Y$, the sequence $(f_n(y))_{n\geq 0}$ is bounded. Then there exists a subsequence of (f_n) that converges uniformly on every compact subset of Y to a map $f : Y \longrightarrow X$ and the limit map f is uniformly continuous.*

Proposition 3.6. *Let (X, d) be a proper metric space which is uniformly Lipschitz-connected. Then*

$$L_d = \inf\{L : L \textit{ is a universal Lipschitz constant for } (X, d)\}$$

is also a universal Lipschitz constant for (X, d).

Proof. Let L be a universal Lipschitz constant for (X, d). Now for each n, there exists a universal Lipschitz constant L_n such that $L_n < L_d + \frac{1}{n}$. (We may assume that $L_n \leq L$). Now fix $x, y \in X$. Then there exists $g_n : [0, 1] \longrightarrow (X, d)$ from x to y such that $d(g_n(s), g_n(t)) \leq L_n|s-t|d(x, y) \leq L|s-t|d(x, y)$ for all $s, t \in [0, 1]$. Thus g_n is $(Ld(x, y))$-Lipschitz for all n, and hence the sequence (g_n) is uniformly equicontinuous. Also $d(g_n(s), g_n(0)) \leq L\, s\; d(x, y) \leq Ld(x, y)$ and hence the sequence $(g_n(s))$ is bounded for each $s \in [0, 1]$. Thus by the Arzelá-Ascoli theorem (Theorem 3.5) there exists a subsequence, call it (h_n), such that $(h_n(s))$ is convergent for all $s \in [0, 1]$. Let $\lim_{n\longrightarrow\infty} h_n(s) = g(s)$ for all $s \in [0, 1]$. Then $g(0) = x$ and $g(1) = y$. We will show that g satisfies

$$d(g(s), g(t)) \leq L_d|s - t|d(x, y) \text{ for all } s, t \in [0, 1].$$

Now $d(h_n(s), h_n(t)) \leq L_n|s-t|d(x, y) \leq (L_d+\frac{1}{n})|s-t|d(x, y)$. Taking limits as $n \longrightarrow \infty$, we obtain $d(g(s), g(t)) \leq L_d|s - t|d(x, y)$ for all $s, t \in [0, 1]$ showing that L_d is a universal Lipschitz constant for (X, d).

□

Further, under the assumption that X is a proper metric space, we obtain a characterisation of metric convexity. The following lemma, which is proved in [6], is required for this.

Lemma 3.7. ([6]) *Let (X,d) be a proper metric space, let $x,y \in X$ and suppose that there is a path of finite length in X connecting x to y. Then there exists a path whose length is equal to the infimum of the lengths of all paths that join x to y.*

We note that for a complete and convex space, $L_d = 1$ (see [1]).

Proposition 3.8. *Let (X,d) be a Lipschitz-connected proper metric space. Then*

$$L_d = 1 \iff X \textit{ is metrically convex.}$$

Proof. For the necessity, let $x,y \in X$. Since $1 = L_d = \sup\{\frac{w(x,y)}{d(x,y)} : x,y \in X, x \neq y\}$, $w(x,y) \leq d(x,y)$. But it is always true that $d(x,y) \leq w(x,y)$, so $d(x,y) = w(x,y)$.

For $n \in \mathbb{N}$ there exists a Lipschitz path g_n connecting x to y, such that $d(x,y) + \frac{1}{n} > \mathrm{dil}(g_n)$. Thus $d(x,y) = d(g_n(0), g_n(1)) \leq \mathrm{dil}(g_n) < d(x,y) + \frac{1}{n}$. Hence $\lim_{n\to\infty} \mathrm{dil}(g_n) = d(x,y)$. It is true that $d(x,y) \leq L(g_n) \leq \mathrm{dil}(g_n)$, by Proposition 3.3. So $d(x,y) = \lim_{n\to\infty} L(g_n) = \inf_n L(g_n)$ (since $d(x,y) \leq L(g_n) < d(x,y) + \frac{1}{n}$). Using Lemma 3.7 we can find a path g from x to y such that $L(g) = \inf\{L(h) : h \text{ is a path from } x \text{ to } y\} \leq \inf\limits_n L(g_n) = d(x,y)$. If $z = g(\frac{1}{2})$, then $d(x,y) \leq d(x,z) + d(z,y) = d(g(0), g(\frac{1}{2})) + d(g(\frac{1}{2}), g(1)) \leq L(g) = d(x,y)$, proving that X is convex.

For the converse, we note that a proper space is complete and together with convexity we get $L_d = 1$ (see [1]). □

The above Proposition leads us to conjecture that L_d is a measure of convexity. The more convex a space, the closer to 1 is L_d. We will attempt to confirm this in Section 6, where we calculate L_d for some examples.

4 An extension of Edelstein's Lemma

The following two definitions were introduced by Edelstein in the paper [4].

Definition 4.1. A map $f : X \longrightarrow X$ of a metric space into itself is *locally contractive* if for every $x \in X$ there exist $\varepsilon > 0$ and $\lambda,\ 0 \leq \lambda < 1$ (depending on x) such that $p, q \in S(x, \varepsilon)$ (the open ball of radius ε centred at x) $\Longrightarrow\ d(f(p), f(q)) \leq \lambda\, d(p, q)$.

Definition 4.2. A map $f : X \longrightarrow X$ of a metric space into itself is (ε, λ)-*uniformly locally contractive* if for every $x \in X,\ p, q \in S(x, \varepsilon) \Longrightarrow d(f(p), f(q)) \leq \lambda\, d(p, q)$.
(so here ε and λ do not depend on x)

Remark 4.3. A (globally) contractive map as mentioned in [4] is precisely a (∞, λ)-uniformly locally contractive map for some $\lambda,\ 0 \leq \lambda < 1$.

Edelstein showed in [4] that in the case of complete, convex metric spaces, an (ε, λ)-uniformly locally contractive map is also globally contractive with the same λ. The following result generalizes Proposition 4.3 of [1] (every complete metrically convex metric space is uniformly Lipschitz-connected) as seen in the corollary that follows.

Proposition 4.4. *Let (X, d) be uniformly Lipschitz-connected with L_d the unique constant associated with this metric space. Then every (ε, λ)-uniformly locally contractive map $f : X \longrightarrow X$ is Lipschitz, and satisfies the condition*

$$d(f(p), f(q)) \leq \lambda L_d\, d(p, q) \textit{ for all } p, q \in X.$$

Proof. Take any $L > 0$ be a universal Lipschitz constant. Take any $p, q \in X,\ p \neq q$. Find a path g such that $g(0) = p,\ g(1) = q$ and $d(g(s), g(t)) \leq L\ |s - t|\ d(p, q)$ for all $s, t \in [0, 1]$. Partition $[0, 1]$ into equal intervals of length $\frac{1}{n}$ with n chosen such that $\frac{1}{n} < \frac{\varepsilon}{Ld(p,q)}$ and let $x_i = g(\frac{i}{n})$ for all $i = 0, 1, 2, ..., n$. Thus $x_0 = p$ and $x_n = q$. Now, for each $i = 1, 2, \ldots, n$ we have

$$\begin{aligned} d(x_i, x_{i-1}) &= d\left(g\left(\frac{i}{n}\right), g\left(\frac{i-1}{n}\right)\right) \\ &\leq L \left|\frac{i}{n} - \frac{i-1}{n}\right| d(p,q) \\ &= L\frac{1}{n}\, d(p,q) \\ &< \varepsilon, \end{aligned}$$

i.e. $x_i, x_{i-1} \in S(x_i, \varepsilon)$. Now take any map $f : X \longrightarrow X$ which is (ε, λ)-uniformly locally contractive. Thus by hypothesis, $d(f(x_i), f(x_{i-1})) \leq \lambda d(x_i, x_{i-1})$. Hence

$$\begin{aligned} d(f(p), f(q)) &\leq \sum_{i=1}^{n} d(f(x_i), f(x_{i-1})) \\ &\leq \sum_{i=1}^{n} \lambda d(x_i, x_{i-1}) \\ &= \lambda \sum_{i=1}^{n} d(x_i, x_{i-1}) \\ &\leq \lambda \sum_{i=1}^{n} L d(p,q) \frac{1}{n} \\ &\leq \lambda\, L\, d(p,q). \end{aligned}$$

Now since this inequality is true for all such L we can pass to the infimum to get

$$\begin{aligned} d(f(p), f(q)) &\leq \inf_L \{\lambda\, L\, d(p,q)\} \\ &= \lambda\, d(p,q) \inf_L \{L\} \\ &= \lambda\, L_d\, d(p,q). \end{aligned}$$

□

Corollary 4.5. ([4]) *If X is a complete convex metric space, then every mapping*

$f : X \longrightarrow X$ which is (ε, λ)-uniformly locally contractive is also globally contractive with the same λ.

Proof. For such X, (X, d) is uniformly Lipschitz-connected using Proposition 4.3 of [1] and hence $L_d = 1$ using Example 4.12 of [1] (for every complete metrically convex metric space, $L_d = 1$). Thus for any $p, q \in X$ we have $d(f(p), f(q)) \leq \lambda\, d(p, q)$. □

5 Some remarks about the dilatation of a Lipschitz map

In this section we give some properties of the dilatations of Lipschitz paths which we use in Section 6, where we calculate the value of L_d for various spaces.

Let (X, d) be a metric space and let $[a, b]$ have the usual (absolute value) metric. We note that $[a, b]$ is a Lipschitz-connected metric space. For $a, b \in \mathbb{R}, a < b$, we will call the map $\phi : [0, 1] \to [a, b]$ defined by $\phi(s) = (1 - s)a + sb = a + s(b - a)$ *the linear reparametrization on* $[a, b]$.

Lemma 5.1. *If ϕ is the linear reparametrization map on $[a, b]$, then* $\mathrm{dil}(\phi) = |b - a|$.

Proof.

$$\begin{aligned} \mathrm{dil}(\phi) &= \sup\left\{ \tfrac{\mathrm{d}(\phi(s),\phi(t))}{|s-t|} : s, t \in [0, 1], s \neq t \right\} \\ &= \sup\left\{ \tfrac{|s-t||b-a|}{|s-t|} : s, t \in [0, 1], s \neq t \right\} \\ &= |b - a|. \end{aligned}$$

□

In the following lemma, we show that for the linear reparametrization map ϕ, $\mathrm{dil}(\phi)$ has the smallest dilatation amongst all the reparametrization maps, which are essentially the onto and strictly increasing maps.

Lemma 5.2. *If $\psi : [0, 1] \to [a, b]$ is any onto, strictly increasing map, then* $\mathrm{dil}(\phi) \leq \mathrm{dil}(\psi)$, *where ϕ is the linear reparametrization on $[a, b]$.*

Proof. Since ψ is onto and increasing, $\psi(0) = a$ and $\psi(1) = b$.
So $\mathrm{dil}(\psi) \geq \frac{\mathrm{d}(\psi(0),\psi(1))}{|0-1|} = |a - b| = \mathrm{dil}(\phi)$. □

The following result is well known.

Lemma 5.3. ([6]) *If $f : X \to Y$, $g : Y \to Z$ are Lipschitz maps with X, Y, Z metric spaces, then* $\operatorname{dil}(g \circ f) \leq \operatorname{dil}(g) \operatorname{dil}(f)$.

Theorem 5.4. *If $\phi : [0, 1] \to [a, b]$ is the linear reparametrization map on $[a, b]$ and $g : [a, b] \to X$ a Lipschitz map, then* $\operatorname{dil}(g \circ \phi) = \operatorname{dil}(g)|b - a|$.

Proof. From Lemmas 5.3 and 5.1, we have $\operatorname{dil}(g \circ \phi) \leq \operatorname{dil}(g)|b - a|$. Now for $s, t \in [a, b], s \neq t$, it is clear that $s = \phi(\frac{s-a}{b-a})$ and $t = \phi(\frac{t-a}{b-a})$. Thus

$$\begin{aligned}
\frac{\mathrm{d}(g(s),g(t))}{|s-t|} &= \frac{d(g(\phi(\frac{s-a}{b-a})),g(\phi(\frac{t-a}{b-a})))}{|s-t|} \\
&\leq \frac{\operatorname{dil}(g\circ\phi)\, d(\frac{s-a}{b-a},\frac{t-a}{b-a})}{|s-t|} \\
&= \frac{\operatorname{dil}(g\circ\phi)\,|\frac{s-a}{b-a}-\frac{t-a}{b-a}|}{|s-t|} \\
&= \frac{\operatorname{dil}(g\circ\phi)\,|\frac{s-t}{b-a}|}{|s-t|} \\
&= \frac{\operatorname{dil}(g\circ\phi)}{|b-a|}.
\end{aligned}$$

Hence $\frac{\operatorname{dil}(g\circ\phi)}{|b-a|} \geq \sup\{\frac{\mathrm{d}(g(s),g(t))}{|s-t|} : s, t \in [a, b], s \neq t\} = \operatorname{dil}(g)$, proving that $\operatorname{dil}(g \circ \phi) \geq \operatorname{dil}(g)|b - a|$. Thus we obtain the result. $\square$

Proposition 5.5. *Let $f, g : [0, 1] \to (X, d)$ be any two Lipschitz paths and assume that whenever $s < t$ in $[0, 1]$, we have $L(f|_{[s,t]}) \leq L(g|_{[s,t]})$. Then* $\operatorname{dil}(f) \leq \operatorname{dil}(g)$.

Proof. Take any $s < t$ in $[0, 1]$. If $\{t_i\}$ is any partition of $[s, t]$, then

$$d(g(t_{i-1}), g(t_i)) \leq \operatorname{dil}(g)|t_{i-1} - t_i|,$$

and so

$$\begin{aligned}
\Sigma_{i=1}^n d(g(t_{i-1}), g(t_i)) &\leq \operatorname{dil}(g)\Sigma_{i=1}^n |t_{i-1} - t_i| \\
&= \operatorname{dil}(g)|s - t|.
\end{aligned}$$

This gives

$$L(g|_{[s,t]}) = \sup\{\Sigma_{i=1}^n d(g(t_{i-1}), g(t_i)) : \{t_i\} \text{ is any partition of } [s, t]\} \leq \operatorname{dil}(g)|s-t|.$$

So

$$\begin{aligned} d(f(s),f(t)) &\le L(f|_{[s,t]}) \\ &\le L(g|_{[s,t]}) \\ &\le \mathrm{dil}(g)|s-t|, \end{aligned}$$

and hence

$$\mathrm{dil}(f) \le \mathrm{dil}(g).$$

□

6 Calculation of L_d for certain spaces

In this section we calculate L_d for certain spaces to confirm our conjecture that L_d is a measure of convexity for a metric space.

We recall the definitions of dil(g), w and L_d, all of which we will use in our calculations below:

$$\mathrm{dil}(g) = \sup\left\{\frac{d(g(s),g(t))}{|s-t|} : s,t \in [a,b], s \neq t\right\},$$

$$w(x_0,x_1) = \inf\{\mathrm{dil}(g) : g : [0,1] \longrightarrow (X,d) \text{ is a Lipschitz path connecting } x_0 \text{ to } x_1\},$$

and

$$L_d = \sup\left\{\frac{w(x_0,x_1)}{d(x_0,x_1)} : x_0,x_1 \in X, x_0 \neq x_1\right\}.$$

Example 6.1. Calculating L_d for $X = S^1 = \{(\cos s, \sin s) : s \in [0,2\pi]\}$ with the usual metric:
Let $x_0 = (\cos s_0, \sin s_0)$ and $x_1 = (\cos s_1, \sin s_1)$, for $s_0, s_1 \in [0,2\pi]$, with $s_0 < s_1$.
By Proposition 5.5, we may assume that the angle between x_0 and x_1, $s_1 - s_0$, is less than or equal to π. (We can make the assumption that the angle is less than or equal to π, because by Proposition 5.5 the dilatation of the larger arc of the circle is larger than the dilatation of the smaller arc of the circle, and since we want to calculate w, it suffices to use the smaller dilatation.)

Now let $g : [0,1] \longrightarrow (X,d)$ be the path from x_0 and x_1 defined by $g(s) = (f \circ \phi)(s)$, where $f : [s_0, s_1] \longrightarrow (X,d)$ is given by $f(s) = (\cos s, \sin s), s \in [s_0, s_1]$ and $\phi : [0,1] \longrightarrow [s_0, s_1]$ is the linear reparametrization on $[s_0, s_1]$. Now for $s, t \in [s_0, s_1]$, it can be shown that $d(f(s), f(t)) = 2\sin(\frac{t-s}{2})$ since $t - s \leq \pi$, and hence

$$\begin{aligned} \mathrm{dil}(f) &= \sup\{\tfrac{d(f(s),f(t))}{|s-t|} : s,t \in [s_0,s_1], s<t\} \\ &= \sup\{\tfrac{\sin((t-s)/2)}{(t-s)/2} : s,t \in [s_0,s_1], s<t\}. \end{aligned}$$

Now $\frac{\sin x}{x}$ is a decreasing function on $(0, \frac{\pi}{2}]$, and since $0 < (t-s)/2 \leq \pi/2$, $\mathrm{dil}(f) = \lim\limits_{t-s \longrightarrow 0} \dfrac{\sin((t-s)/2)}{(t-s)/2} = 1$. By Theorem 5.4, $\mathrm{dil}(g) = \mathrm{dil}(f)\,|s_1 - s_0|$.

Thus $\mathrm{dil}(g) = \mathrm{dil}(f)|s_0 - s_1| = s_1 - s_0$ and so $w(x_0, x_1) = s_1 - s_0$, by Proposition 5.5, it suffices to use the shorter arc. We therefore obtain

$$\begin{aligned} A = L_d &= \sup\left\{\tfrac{w(x_0,x_1)}{d(x_0,x_1)} : x_0, x_1 \in X, x_0 \neq x_1\right\} \\ &= \sup\left\{\tfrac{s_1-s_0}{2\sin((s_1-s_0)/2)} : s_0, s_1 \in [0,\pi], s_0 < s_1\right\} \\ &= \sup\left\{\tfrac{(s_1-s_0)/2}{\sin((s_1-s_0)/2)} : s_0, s_1 \in [0,\pi], s_0 < s_1\right\} \\ &= \tfrac{\pi}{2} \\ &\approx 1,570796327, \end{aligned}$$

since $\frac{x}{\sin x}$ is an increasing function on $(0, \pi/2]$. $\square$

Example 6.2. Calculating L_d for the space $X = \{(s, s^2) : s \in [-1,1]\}$ with the usual metric:
Let $x_0 = (s_0, s_0^2)$ and $x_1 = (s_1, s_1^2)$, with $s_0 < s_1$. Define $f : [s_0, s_1] \longrightarrow (X,d)$ by $f(s) = (s, s^2)$, where $s \in [s_0, s_1]$. Then

$$\begin{aligned} \mathrm{dil}(f) &= \sup\left\{\frac{d(f(s),f(t))}{|s-t|} : s,t \in [s_0,s_1], s<t\right\} \\ &= \sup\{\sqrt{1+(s+t)^2} : s,t \in [s_0,s_1], s<t\}. \end{aligned}$$

We will have to consider the 4 cases for s_0 and s_1 to calculate $\mathrm{dil}(f)$.

Then $\mathrm{dil}(f) = \begin{cases} \sqrt{1+s_1^2} & ,-1 \le s_0 \le 0 \le s_1 \le 1 \text{ and } |s_0| \le |s_1| \\ \sqrt{1+s_0^2} & ,-1 \le s_0 \le 0 \le s_1 \le 1 \text{ and } |s_0| > |s_1| \\ \sqrt{1+4s_1^2} & ,0 \le s_0 < s_1 \le 1 \\ \sqrt{1+4s_0^2} & ,-1 \le s_0 < s_1 \le 0 \end{cases}$. If $g = f \circ \phi$, where ϕ is the linear reparametrization map on $[s_0, s_1]$, then by Theorem 5.4 (since $\mathrm{dil}(f) < \infty$, f is a Lipschitz path) we have

$$\mathrm{dil}(g) = \begin{cases} \sqrt{1+s_1^2}\,|s_0 - s_1| & ,-1 \le s_0 \le 0 \le s_1 \le 1 \text{ and } |s_0| \le |s_1| \\ \sqrt{1+s_0^2}\,|s_0 - s_1| & ,-1 \le s_0 \le 0 \le s_1 \le 1 \text{ and } |s_0| > |s_1| \\ \sqrt{1+4s_1^2}\,|s_0 - s_1| & ,0 \le s_0 < s_1 \le 1 \\ \sqrt{1+4s_0^2}\,|s_0 - s_1| & ,-1 \le s_0 < s_1 \le 0 \end{cases}.$$

By Proposition 5.5 it suffices to use the shortest path from x_0 to x_1 as this will give the smallest dilatation. Hence, $w(x_0, x_1) = \mathrm{dil}(g)$ as g is the shortest path from x_0 to x_1.

$$\text{Thus, } w(x_0, x_1) = \begin{cases} \sqrt{1+s_1^2}\,|s_0 - s_1| & ,-1 \le s_0 \le 0 \le s_1 \le 1 \text{ and } |s_0| \le |s_1| \\ \sqrt{1+s_0^2}\,|s_0 - s_1| & ,-1 \le s_0 \le 0 \le s_1 \le 1 \text{ and } |s_0| > |s_1| \\ \sqrt{1+4s_1^2}\,|s_0 - s_1| & ,0 \le s_0 < s_1 \le 1 \\ \sqrt{1+4s_0^2}\,|s_0 - s_1| & ,-1 \le s_0 < s_1 \le 0 \end{cases}.$$

Now $d(x_0, x_1) = \sqrt{1+(s_0+s_1)^2}|s_0 - s_1|$. In cases 1 and 2 above $d(x_0, x_1)$ is minimized when $s_0 = -s_1$, and in cases 3 and 4 we note that $f(s) = \frac{\sqrt{1+4s^2}}{\sqrt{1+s^2}}$ is increasing on $[0,1]$ and decreasing on $[-1,0]$. Hence, we get

$B = L_d = \max\{\sqrt{2}, \sqrt{2}, \sqrt{5/2}, \sqrt{5/2}\} = \sqrt{5/2} \approx 1,58113883.$ □

For the following examples the calculation of the L_d is similar to Example 6.2 but simpler.

Example 6.3. Calculating L_d for the space $X = \{(s, s^2) : s \in [0,1]\}$ with the usual metric:

As in Example 6.2,

$$\begin{aligned} C = L_d &= \sup\left\{\tfrac{w(x_0,x_1)}{d(x_0,x_1)} : x_0, x_1 \in X, x_0 \neq x_1\right\} \\ &= \sup\left\{\frac{\sqrt{1+4s_1^2}}{\sqrt{1+(s_0+s_1)^2}} : s_0, s_1 \in [0,1], s_0 < s_1\right\} \\ &= \sup\left\{\frac{\sqrt{1+4s_1^2}}{\sqrt{1+s_1^2}} : s_1 \in [0,1]\right\} \\ &= \tfrac{\sqrt{5}}{\sqrt{2}} \\ &\approx 1,58113883, \end{aligned}$$

since the function $\frac{\sqrt{1+4s^2}}{\sqrt{1+s^2}}$ is an increasing function on $[0,1]$. □

Example 6.4. Calculating L_d for the space $X = \{(s, s^2) : s \in [0, \frac{1}{2}]\}$ with the usual metric:
As in Example 6.3,

$$\begin{aligned} D = L_d &= \sup\left\{\tfrac{w(x_0,x_1)}{d(x_0,x_1)} : x_0, x_1 \in X, x_0 \neq x_1\right\} \\ &= \sup\left\{\frac{\sqrt{1+4s_1^2}}{\sqrt{1+(s_0+s_1)^2}} : s_0, s_1 \in [0,\tfrac{1}{2}], s_0 < s_1\right\} \\ &= \sup\left\{\frac{\sqrt{1+4s_1^2}}{\sqrt{1+s_1^2}} : s_1 \in [0,\tfrac{1}{2}]\right\} \\ &= \tfrac{2\sqrt{2}}{\sqrt{5}} \\ &\approx 1,264911064, \end{aligned}$$

since the function $\frac{\sqrt{1+4s^2}}{\sqrt{1+s^2}}$ is an increasing function on $[0, \frac{1}{2}]$. □

Example 6.5. Calculating L_d for the space $X = \{(s, s^2) : s \in [\frac{1}{2}, 1]\}$ with the usual metric:

As in Example 6.3,

$$
\begin{aligned}
E = L_d &= \sup\left\{ \tfrac{w(x_0,x_1)}{d(x_0,x_1)} : x_0, x_1 \in X, x_0 \neq x_1 \right\} \\
&= \sup\left\{ \frac{\sqrt{1+4s_1^2}}{\sqrt{1+(s_0+s_1)^2}} : s_0, s_1 \in [\tfrac{1}{2}, 1], s_0 < s_1 \right\} \\
&= \sup\left\{ \frac{\sqrt{1+4s_1^2}}{\sqrt{1+(\frac{1}{2}+s_1)^2}} : s_1 \in [\tfrac{1}{2}, 1] \right\} \\
&= \sup\left\{ \frac{2\sqrt{1+4s_1^2}}{\sqrt{4+(1+2s_1)^2}} : s_1 \in [\tfrac{1}{2}, 1] \right\} \\
&= \tfrac{2\sqrt{5}}{\sqrt{13}} \\
&\approx 1,240347346.
\end{aligned}
$$

This can be seen by considering the function $f(s) = \frac{2\sqrt{1+4s^2}}{\sqrt{4+(1+2s)^2}}$ to get $f'(s) = \frac{4(4s^2+8s-1)}{[4+(1+2s)^2]\sqrt{1+4s^2}\sqrt{4+(1+2s)^2}}$. The critical points are $z_1 = -1 - \frac{1}{2}\sqrt{5}$ and $z_2 = -1 + \frac{1}{2}\sqrt{5}$. So in particular, f is increasing for $s > z_2$, and since $z_2 < \frac{1}{2}$ we can conclude that f is increasing on $[\frac{1}{2}, 1]$. Thus the maximum of f on $[\frac{1}{2}, 1]$ occurs at 1, giving us the value of L_d. $\square$

Remark 6.6. (1) All spaces in the above examples are uniformly Lipschitz-connected since all the L_d values are finite (see [1]).

(2) The values of L_d obtained in all these examples concurs with our conjecture that L_d is a measure of convexity. We would expect E to be the closest to 1 in all the examples as it is closest to a straight line, the parabola is more convex than the circle, and we have $B = C > A > D > E > 1$ which confirms what we expected.

We note that all spaces considered in the examples above are exclusively curves in the plane with the usual metric. It would be interesting to study the relationship between L_d and the curvature of a smooth curve. This is something we can consider for future study.

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