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Bayer noise quasisymmetric functions and some combinatorial algebraic structures

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Abstract. Recently, quasisymmetric functions have been widely studied due to their big connection to enumerative combinatorics, combinatorial Hopf algebra and number theory. The Bayer filter mosaic, named due to Bryce Bayer (1929-2012), is a color filter array used to arrange RGB color filters on a square grid of photosensors. It is the most common pattern of filters, and almost all professional digital cameras are applications of this filter. We use this filter to introduce the Bayer Noise quasisymmetric functions, and we study some combinatorial algebraic and coalgebraic structures on Quasi-Bayer Noise modules and on Quasi-Bayer GB-Noise modules. We explicitly describe the primitive basis elements for each comultiplication defined on Quasi-Bayer Noise modules, and we calculate different kinds of comultiplications defined on Quasi-Bayer Noises module over a fixed commutative ring $\mathbf{k}$.

Keywords: Quasisymmetric functions, RGB, Bayer filter, algebra, coalgebra, noise, composition.
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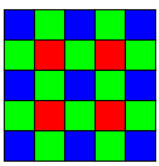
1 Introduction and Preliminaries

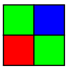
1.1 Introduction Quasisymmetric functions have been recently largely and interestingly studied due to their big relevant role to important theories, such as enumerative combinatorics, combinatorial Hopf algebra, number theory, representation theory, and graph theory. Their applications have been quickly extended to include Stanley's P-partition theory, Lyndon words, polynomial freeness, permutations, chains of posets, Schubert polynomials, Coxeter groups, Kazhdan–Lusztig polynomials, Shuffle algebra, and peak algebra. As a graded Hopf algebras with a single character, the algebra $Qsym_{\mathbf{k}}$ of quasisymmetric functions is the terminal object in category $\mathcal{H}_{\mathbf{k}}$ of graded Hopf algebras with a single character. Thus, for any object A in $\mathcal{H}_{\mathbf{k}}$, there exists exactly one morphism $A \to Qsym_{\mathbf{k}}$.
The Bayer filter pattern is the most common color filter array widely used in most digital "single chip" machine vision color camera. The Bayer filter mosaic, invented by Bryce Bayer (1929-2012) at Kodak, is a color filter array for arranging RGB color filters over each individual square grid of photosensors. This technique requires only one sensor and allows all the RGB color information to be recorded simultaneously. Consequently, with this technology, the cameras can be smaller, cheaper, and useful in performing high-quality part inspection. In this filter, 50% of the filter elements are green and the rest are comprised of blue and red (25% red and 25% blue). This gives an approximation for human photopic vision where the M and L cones amalgamate to produce a bias in the green spectral region [3, p. 124]. Basically, there are four patterns of this filter: GBRG, GRBG, BGGR and RGGB. A Bayer pattern array can be shown in the following figure.



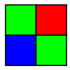
B	G	B	G	B
G	R	G	R	G
B	G	B	G	B
G	R	G	R	G
B	G	B	G	B

Bayer Filter Mosaic (in terms of colors) Bayer Filter Mosaic (in terms of letters).

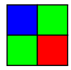
There are basically four patterns of this filter: GBRG, GRBG, BGGR and RGGB.



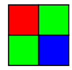
GBRG Pattern



GRBG Pattern



BGGR Pattern



RGGB Pattern

Every BGGR-Bayer Young diagram of shape α corresponds to a unique quasisymmetric monomial function whose degree equals to the number of its pixels. This monomial function (which we call the Quasi-Bayer Noise mial function) can be visualized as splitting an image into three parts GB-part, G-part and R-part. While the GB-part can be thought of as a full-size (free color (G, B)) image (the original image), the other parts can be seen as full-sizes (free color G) and (free color R) images respectively. We consider the set of all such monomial functions as a basis for (graded) modules called the Quasi-Bayer Noise module over a fixed commutative ring $\mathbf{k}$, and we study various kinds of combinatorial algebraic and coalgebraic structures on such modules. We also stress the imporatance of the (free color (G, B)) images to introduce and study a special kind of modules called Quasi-Bayer GB-Noise modules.

This paper has four sections. In section 1, the basic notions of quasisymmetric functions are recalled. Section 2 is concerned with introducing Noise Bayer Young composition diagrams and Quasi-Bayer Noise monomial functions. Section 3 and section 4 are devoted for studying some algebraic and coalgebraic structures on various kinds of Quasi-Bayer Noise modules.

Throughout this paper, $\mathbf{k}$ is a commutative ring, and $\otimes$ and $\oplus$ are the usual tensor products and direct sum respectively over $\mathbf{k}$.

1.2 Symmetric functions Following [5], we recall some basic concepts of symmetric functions. For the basic notions of symmetric functions, the reader is referred to [5], [6], [11], [8], [14], [13], [7] or [12]. Given an infinite variable set $\mathbf{x} = (x_1, x_2, \ldots)$, a monomial $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ is indexed by a sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ in $\mathbb{N}^\infty$ having finite support; such sequences α are called *weak compositions*. The nonzero entries of the sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ are called the *parts* of the weak composition α. The sum $\alpha_1 + \alpha_2 + \alpha_3 + \cdots$ of all entries of a weak composition $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ (or, equivalently, the sum of all parts of α) is called the *size* of α and denoted by

$|\alpha|$. A *composition* is a finite tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ of positive integers. In other words, it is a weak composition with no zero entries. We write $\varnothing$ or (0) for the empty composition $()$. Its *length* is defined to be m and denoted by $\ell(\alpha)$; its *size* is defined to be $\alpha_1 + \alpha_2 + \cdots + \alpha_m$ and denoted by $|\alpha|$; its *parts* are its entries $\alpha_1, \alpha_2, \ldots, \alpha_m$. The compositions of size n are called the *compositions of* n. Clearly, any partition of n is a composition of n. Let $Comp_n$ denote the set of all compositions of n, and let $Comp$ denote the set of all compositions. Let α in $Comp_n$. Define the *reverse* (composition) of a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ to be $\alpha^* = (\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$ [4]. We may define a partial order on the compositions of the integer m by defining the covering relation to be

$$(\alpha_1, \ldots \alpha_i + \alpha_{i+1}, \ldots, \alpha_m) \prec (\alpha_1, \ldots \alpha_i, \alpha_{i+1}, \ldots, \alpha_m).$$

This makes $Comp_n$ into a poset. In this order the composition $(1, 1, ..., 1)$ is the maximum element, and (n) the minimum element [4]. The *concatenation* $\alpha \odot \beta$ of two compositions $\alpha = (\alpha_1, \ldots, \alpha_l)$ and $\beta = (\beta_1, \ldots, \beta_m)$ is defined to be the composition $(\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_m)$. For α, β in $Comp_n$, say that α *refines* β or β *coarsens* α if, informally, one can obtain β from α by combining some of its adjacent parts. Alternatively, this can be defined as follows: One has a bijection $Comp_n \to 2^{[n-1]}$ where $[n-1] := \{1, 2, \ldots, n-1\}$ which sends $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ having length $\ell(\alpha) = \ell$ to its subset of partial sums

$$D(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{\ell-1}\},$$

and this sends the refinement ordering to the inclusion ordering on the Boolean algebra $2^{[n-1]}$ (to be more precise: a composition $\alpha \in Comp_n$ refines a composition $\beta \in Comp_n$ if and only if $D(\alpha) \supset D(\beta)$).

Consider the $\mathbf{k}$-algebra $\mathbf{k}[[\mathbf{x}]] := \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of all formal power series in the indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$; these series are infinite $\mathbf{k}$-linear combinations $\sum_\alpha c_\alpha \mathbf{x}^\alpha$ (with c_α in $\mathbf{k}$) of the monomials $\mathbf{x}^\alpha$ where α ranges over all weak compositions. The product of two such formal power series is well-defined by the usual multiplication rule. The *degree* of a monomial $\mathbf{x}^\alpha$ is defined to be the number $\deg(\mathbf{x}^\alpha) := \sum_i \alpha_i \in \mathbb{N}$. Given a number $d \in \mathbb{N}$, we say that a formal power series $f(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbf{k}[[\mathbf{x}]]$ (with c_α in $\mathbf{k}$) is *homogeneous of degree* d if every weak composition α satisfying $\deg(\mathbf{x}^\alpha) \neq d$ must satisfy $c_\alpha = 0$. In other words, a formal power series is homogeneous of degree d if it is an infinite $\mathbf{k}$-linear combination of monomials of degree

d. Every formal power series $f \in \mathbf{k}[[\mathbf{x}]]$ can be uniquely represented as an infinite sum $f_0 + f_1 + f_2 + \cdots$, where each f_d is homogeneous of degree d; in this case, we refer to each f_d as the *d-th homogeneous component of f*. Note that this does not make $\mathbf{k}[[\mathbf{x}]]$ into a graded $\mathbf{k}$-module, since these sums $f_0 + f_1 + f_2 + \cdots$ can have infinitely many nonzero addends. Nevertheless, if f and g are homogeneous power series of degrees d and e, then fg is homogeneous of degree $d+e$. A formal power series $f(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbf{k}[[\mathbf{x}]]$ (with c_α in $\mathbf{k}$) is said to be *of bounded degree* if there exists some bound $d = d(f) \in \mathbb{N}$ such that every weak composition $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ satisfying $\deg(\mathbf{x}^\alpha) > d$ must satisfy $c_\alpha = 0$. Equivalently, a formal power series $f \in \mathbf{k}[[\mathbf{x}]]$ is of bounded degree if all but finitely many of its homogeneous components are zero. (For example, $x_1^2 + x_2^2 + x_3^2 + \cdots$ and $1 + x_1 + x_2 + x_3 + \cdots$ are of bounded degree, while $x_1 + x_1x_2 + x_1x_2x_3 + \cdots$ and $1 + x_1 + x_1^2 + x_1^3 + \cdots$ are not.) It is easy to see that the sum and the product of two power series of bounded degree also have bounded degree. Thus, the formal power series of bounded degree form a $\mathbf{k}$-subalgebra of $\mathbf{k}[[\mathbf{x}]]$, which we call $R(\mathbf{x})$. This subalgebra $R(\mathbf{x})$ is graded (by degree). The symmetric group $\mathfrak{S}_n$ permuting the first n variables $x_1, \ldots, x_n$ acts as a group of automorphisms on $R(\mathbf{x})$, as does the union $\mathfrak{S}_{(\infty)} = \bigcup_{n \geq 0} \mathfrak{S}_n$ of the infinite ascending chain $\mathfrak{S}_0 \subset \mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \cdots$ of symmetric groups. This group $\mathfrak{S}_{(\infty)}$ can also be described as the group of all permutations of the set $\{1, 2, 3, \ldots\}$ which leave all but finitely many elements invariant. It is known as the *finitary symmetric group* on $\{1, 2, 3, \ldots\}$. The group $\mathfrak{S}_{(\infty)}$ also acts on the set of all weak compositions by permuting their entries:

$$\sigma(\alpha_1, \alpha_2, \alpha_3, \ldots) = \left(\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \alpha_{\sigma^{-1}(3)}, \ldots\right)$$

for any weak composition $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ and any $\sigma \in \mathfrak{S}_{(\infty)}$. These two actions are connected by the equality $\sigma(\mathbf{x}^\alpha) = \mathbf{x}^{\sigma\alpha}$ for any weak composition α and any $\sigma \in \mathfrak{S}_{(\infty)}$. The *ring of symmetric functions in* $\mathbf{x}$ *with coefficients in* $\mathbf{k}$, denoted $\alpha = \alpha(\mathbf{k}) = \alpha(\mathbf{x}) = \alpha(\mathbf{k})(\mathbf{x})$, is the $\mathfrak{S}_{(\infty)}$-invariant subalgebra $R(\mathbf{x})^{\mathfrak{S}_{(\infty)}}$ of $R(\mathbf{x})$:

$$\begin{aligned} \alpha &:= \left\{ f \in R(\mathbf{x}) : \sigma(f) = f \text{ for all } \sigma \in \mathfrak{S}_{(\infty)} \right\} \\ &= \left\{ f = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in R(\mathbf{x}) : c_\alpha = c_\beta \text{ if } \alpha, \beta \text{ lie in the same } \mathfrak{S}_{(\infty)}\text{-orbit} \right\}. \end{aligned}$$

We refer to the elements of α as *symmetric functions* (over $\mathbf{k}$); however, despite this terminology, they are not functions in the usual sense.

Note that Λ is a graded $\mathbf{k}$-algebra, since $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ where Λ_n are the symmetric functions $f = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ which are *homogeneous of degree* n, meaning $\deg(\mathbf{x}^\alpha) = n$ for all $c_\alpha \neq 0$. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell, 0, 0, \ldots)$ is a weak composition whose entries weakly decrease: $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$. The (uniquely defined) ℓ is said to be the *length* of the partition λ and denoted by $\ell(\lambda)$. Thus, $\ell(\lambda)$ is the number of parts of λ. One sometimes omits trailing zeroes from a partition: e.g., one can write the partition $(3, 1, 0, 0, 0, \ldots)$ as $(3, 1)$. We will often (but not always) write λ_i for the i-th entry of the partition λ (for instance, if $\lambda = (5, 3, 1, 1)$, then $\lambda_2 = 3$ and $\lambda_5 = 0$). If λ_i is nonzero, we will also call it the *i-th part* of λ. The sum $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = \lambda_1 + \lambda_2 + \cdots$ (where $\ell = \ell(\lambda)$) of all entries of λ (or, equivalently, of all parts of λ) is the size $|\lambda|$ of λ. For a given integer n, the partitions of size n are referred to as the *partitions of n*. The empty partition $() = (0, 0, 0, \ldots)$ is denoted by $\varnothing$. Every weak composition α lies in the $\mathfrak{S}_{(\infty)}$-orbit of a unique partition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell, 0, 0, \ldots)$ with $\alpha_1 \geq \cdots \geq \alpha_\ell > 0$. For any partition λ, define the *monomial symmetric function*

$$m_\lambda := \sum_{\alpha \in \mathfrak{S}_{(\infty)}\lambda} \mathbf{x}^\alpha. \tag{1.1}$$

Letting λ run through the set Par of all partitions, this gives the *monomial* $\mathbf{k}$-*basis* $\{m_\lambda\}$ of Λ. Letting λ run only through the set Par_n of partitions of n gives the monomial $\mathbf{k}$-basis for Λ_n.
It is straightforward to check that $(\Lambda, \hat{\eta}, \hat{u}, \hat{\Delta}, \hat{\epsilon})$ is a connected graded $\mathbf{k}$-bialgebra of finite type, and hence also a Hopf algebra, where

- The multiplication is the map

$$\Lambda \otimes \Lambda \xrightarrow{\hat{\eta}} \Lambda, \ m_\mu \otimes m_\nu \mapsto m_\mu m_\nu.$$

- The unit is the inclusion map

$$\mathbf{k} = \Lambda_0 \xrightarrow{\hat{u}} \Lambda.$$

- The comultiplication is the map

$$\Lambda \xrightarrow{\hat{\Delta}} \Lambda \otimes \Lambda,\ m_\lambda \mapsto \sum_{\substack{(\mu,\nu):\\ \mu \sqcup \nu = \lambda}} m_\mu \otimes m_\nu,$$

  in which $\mu \sqcup \nu$ is the partition obtained by taking the multiset union of the parts of μ and ν, and then reordering them to make them weakly decreasing.

- The counit is the $\mathbf{k}$-linear map

$$\mathbf{k} = \Lambda_0 \xrightarrow{\hat{\epsilon}} \Lambda$$

  with $\hat{\epsilon}|_{\Lambda_0 = \mathbf{k}} = id_{\mathbf{k}}$ and $\hat{\epsilon}|_{I = \bigoplus_{n>0} \Lambda_n} = 0$.

1.3 Quasisymmetric functions To define quasisymmetric functions, we need a totally ordered variable set. Following [5], we usually use a variable set denoted $\mathbf{x} = (x_1, x_2, \ldots)$ with the usual ordering $x_1 < x_2 < \cdots$. However, it is good to have some flexibility in changing the ordering, which is why we make the following definition. Given any totally ordered set I, create a totally ordered variable set $\{x_i\}_{i \in I}$, and then let $R(\{x_i\}_{i\in I})$ denote the power series of bounded degree in $\{x_i\}_{i\in I}$ having coefficients in $\mathbf{k}$. The ring of *quasisymmetric functions* $Qsym(\{x_i\}_{i\in I})$ *over the alphabet* $\{x_i\}_{i\in I}$ will be the $\mathbf{k}$-submodule consisting of the elements f in $R(\{x_i\}_{i\in I})$ that have the same coefficient on the monomials $x_{i_1}^{\alpha_1} \cdots x_{i_\ell}^{\alpha_\ell}$ and $x_{j_1}^{\alpha_1} \cdots x_{j_\ell}^{\alpha_\ell}$ whenever both $i_1 < \cdots < i_\ell$ and $j_1 < \cdots < j_\ell$ in the total order on I. We write $Qsym_{\mathbf{k}}(\{x_i\}_{i\in I})$ instead of $Qsym(\{x_i\}_{i\in I})$ to stress the choice of base ring $\mathbf{k}$. It immediately follows from this definition that $Qsym(\{x_i\}_{i\in I})$ is a free $\mathbf{k}$-submodule of $R(\{x_i\}_{i\in I})$, having as $\mathbf{k}$-basis elements the *monomial quasisymmetric functions*

$$M_\alpha(\{x_i\}_{i\in I}) := \sum_{i_1 < \cdots < i_\ell \text{ in } I} x_{i_1}^{\alpha_1} \cdots x_{i_\ell}^{\alpha_\ell}$$

for all compositions α satisfying $\ell(\alpha) \leq |I|$. Taking the variable set $\mathbf{x} = (x_1 < x_2 < \cdots)$, for example, to define $Qsym(\mathbf{x})$, for $n = 0, 1, 2, 3$, one has

$$\begin{aligned}
M_{()} = M_{\varnothing} &= 1 & \\
M_{(1)} &= x_1 + x_2 + x_3 + \cdots &= m_{(1)} \\
M_{(2)} &= x_1^2 + x_2^2 + x_3^2 + \cdots &= m_{(2)} \\
M_{(1,1)} &= x_1x_2 + x_1x_3 + x_2x_3 + \cdots &= m_{(1,1)} \\
M_{(3)} &= x_1^3 + x_2^3 + x_3^3 + \cdots &= m_{(3)} \\
M_{(2,1)} &= x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + \cdots & \\
M_{(1,2)} &= x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + \cdots & \\
M_{(1,1,1)} &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + \cdots &= m_{(1,1,1)}
\end{aligned}$$

When I is infinite, this means that the M_α for all compositions α form a basis of $Qsym(\{x_i\}_{i\in I})$. Note that $Qsym(\{x_i\}_{i\in I}) = \bigoplus_{n\geq 0} Qsym_n(\{x_i\}_{i\in I})$ is a graded $\mathbf{k}$-module of finite type, where $Qsym_n(\{x_i\}_{i\in I})$ is the $\mathbf{k}$-submodule of quasisymmetric functions which are homogeneous of degree n. Letting $Comp$ denote the set of all compositions α, and $Comp_n$ the compositions α of n (that is, compositions whose parts sum to n), the subset $\{M_\alpha\}_{\alpha\in Comp_n;\ \ell(\alpha)\leq|I|}$ gives a $\mathbf{k}$-basis for $Qsym_n(\{x_i\}_{i\in I})$.

Recall that α is a weak composition if it can include parts equal to zero. An expansion of a composition α is a weak composition $\bar{\alpha}$ such that removing the zeros from $\bar{\alpha}$ one obtains α. If $\alpha, \beta, \gamma \in Comp$, then we say γ is a shuffle sum of the other two compositions if there are expansions $\bar{\alpha}$ and $\bar{\beta}$ of α and β, respectively, which have length $\ell(\gamma)$ such that $\gamma = \bar{\alpha} + \bar{\beta}$. Here, addition is componentwise [10].

Proposition 1.1. [5], [10] *For any infinite totally ordered set I, the $\mathbf{k}$-module $Qsym$ is a $\mathbf{k}$-algebra with multiplication*

$$\underline{\eta} : Qsym \otimes Qsym \to Qsym, \ \ \underline{\eta}(M_\alpha \otimes M_\beta) = \sum_{\gamma} c^{\gamma}_{\alpha,\beta}\, M_\gamma \tag{1.2}$$

where $c^{\gamma}_{\alpha,\beta}$ is the number of ways of writing γ as a shuffle sum of α and β.

The multiplication rule (1.2) shows that the $\mathbf{k}$-algebra $Qsym(\{x_i\}_{i\in I})$ does not depend much on I, as long as I is infinite. More precisely, all such $\mathbf{k}$-algebras are mutually isomorphic. We can use this to define a $\mathbf{k}$-algebra of quasisymmetric functions without any reference to I:

Let $Qsym$ be the $\mathbf{k}$-algebra defined as having $\mathbf{k}$-basis $\{M_\alpha\}_{\alpha\in Comp}$ and with multiplication defined $\mathbf{k}$-linearly by (1.2). This is called the $\mathbf{k}$-algebra of quasisymmetric functions. We write $Qsym_{\mathbf{k}}$ instead of $Qsym$ to stress the choice of base ring $\mathbf{k}$.

The $\mathbf{k}$-algebra $Qsym$ is graded, and its n-th graded component $Qsym_n$ has $\mathbf{k}$-basis $\{M_\alpha\}_{\alpha\in Comp_n}$, and hence $dim_{\mathbf{k}}(Qsym_n) = 2^{n-1}$ [10].

For every infinite totally ordered set I, there is a $\mathbf{k}$-algebra isomorphism

$$\Theta : Qsym \to Qsym(\{x_i\}_{i\in I}),\ M_\alpha \longmapsto M_\alpha(\{x_i\}_{i\in I}).$$

In particular, we obtain the isomorphism $Qsym \cong Qsym\,(\mathbf{x})$ for $\mathbf{x}$ being the infinite chain

$$(x_1 < x_2 < x_3 < \cdots).$$

We will identify $Qsym$ with $Qsym\,(\mathbf{x})$ along this isomorphism. This allows us to regard quasisymmetric functions either as power series in a specific set of variables ("alphabet"), or as formal linear combinations of M_α's, whatever is more convenient. For any infinite alphabet $\{x_i\}_{i\in I}$ and any $f \in Qsym$, we denote by $f\left(\{x_i\}_{i\in I}\right)$ the image of f under the algebra isomorphism Θ. One has the following description of the comultiplication in the $\{M_\alpha\}$ basis.

Theorem 1.2. [5] *The quasisymmetric functions $Qsym$ form a connected graded Hopf algebra $(Qsym, \underline{\eta}, \underline{u}, \underline{\Delta}, \underline{\epsilon}, \underline{S})$ of finite type, which is commutative, and contains the symmetric functions Λ as a Hopf subalgebra, where*

- *The map $\underline{\eta}$ is the multiplication map defined in Proposition* (1.1)*; that is the map*

$$Qsym \otimes Qsym \xrightarrow{\eta} Qsym,\ M_\alpha \otimes M_\beta \mapsto \sum_\gamma c^\gamma_{\alpha,\beta}\, M_\gamma,$$

where $c^\gamma_{\alpha,\beta}$ is the number of ways of writing γ as a shuffle sum of α and β.

- *The the inclusion map*

$$\mathbf{k} = Qsym_0 \xrightarrow{u} Qsym$$

is the unit map.

- *The map*

$$Qsym \xrightarrow{\Delta} Qsym \otimes Qsym, \ M_\alpha \mapsto \sum_{\substack{(\mu,\nu): \\ \mu \odot \nu = \alpha}} M_\mu \otimes M_\nu$$

is the comultiplication map.

- *The* $\mathbf{k}$*-linear map*

$$\mathbf{k} = Qsym_0 \xrightarrow{\epsilon} Qsym$$

with $\underline{\epsilon}|_{Qsym_0 = \mathbf{k}} = id_{\mathbf{k}}$ *and* $\underline{\epsilon}|_{\bigoplus_{n>0} Qsym_n} = 0$ *is the counit map.*

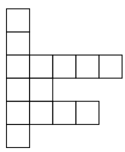
- *The* $\mathbf{k}$*-linear map*

$$\underline{S} : Qsym \to Qsym, \ M_\alpha \mapsto (-1)^{\ell(\alpha)} \sum_{\substack{\gamma \in Comp: \\ \gamma \ coarsens \ \alpha^*}} M_\gamma$$

is the antipode map.

2 Bayer composition Young diagrams

It is well-known that every partition corresponds to a unique Young diagram. Analogously, one might extend this to get a more general approach compatible to the compositions. For example, the composition $\alpha = (1, 1, 5, 2, 4, 1)$ corresponds to a unique diagram $T(\alpha)$ given by



$T(\alpha)$

We call the diagram $T(\alpha)$ as the composition Young diagram of shape α. This gives rise to a $1-1$ correspondence between compositions and such diagrams.

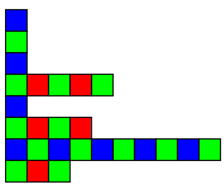
Definition 2.1. Let α be a composition.

1. A *colored Young composition diagram of shape* α is a composition Young diagram of shape α whose cells are colored with green, blue or red. In other words, colored Young composition diagram correspond can be seen (in the sense of [1]) as a particular kind of 3-color compositions whose colors are restricted to the set $\{green, blue, red\}$.
2. A Young composition diagram of shape α is called a *BGGR-Bayer Young composition diagram* of shape α if the corresponding Young composition diagram of α has a BGGR pattern. Similarly, *GBRG-Bayer Young composition diagram*, *GRBG-Bayer Young composition diagram* and *RGGB-Bayer Young composition diagram* can be defined.

Remark 2.2.

1. Clearly, Bayer Young composition diagrams are colored Young composition diagrams. The converse, however, needs not be true.
2. By Bayer Young composition Diagrams, we will simply mean the BGGR-Bayer Young composition Diagrams (since the other Bayer Young Diagrams can be characterized similarly).

Let $Comp$ be the set of all compositions, and let $\mathcal{YD}$ be the set of all Young composition diagrams. Let $T : Comp \to \mathcal{YD}$ be the bijective map that takes any composition α to its corresponding Young composition diagram $T(\alpha)$. Let $\mathcal{BYD}$ be the set of all Bayer Young composition diagrams. There is a bijective map $\mathcal{B} : Comp \to \mathcal{BYD}, \alpha \mapsto \mathcal{B}(\alpha)$. For example, if $\alpha = (1, 1, 1, 5, 1, 4, 10, 3)$, we have



$\mathcal{B}((1, 1, 1, 5, 1, 4, 10, 3))$

Let $\mathcal{B}(\alpha)$ be a Bayer Young composition diagram of shape α. Let $\mathcal{C}^{(1)}(\alpha, GBR)$ be the (colored) Young composition diagram obtained by rearranging the colored cells of $\mathcal{B}(\alpha)$ using the order $G < B < R$ as follows.

First, we rearrange the colored cells of $\mathcal{B}(\alpha)$ to be weakly increasing left-to-right in rows, and then we rearrange the colored cells of the resulting colored Young composition diagram to be weakly increasing top-to-bottom in columns. For any positive integer i, let $\mathcal{C}^{(i)}(\alpha, GBR)$ denote the (colored) Young composition diagram obtained by applying the above rearrangement process i-times. It will be convenient to write $\mathcal{C}^{(0)}(\alpha, GBR) = \mathcal{B}(\alpha)$. Analogously, one could define $\mathcal{C}^{(i)}(\alpha, RBG)$, $\mathcal{C}^{(i)}(\alpha, RGB)$, $\mathcal{C}^{(i)}(\alpha, BRG)$, $\mathcal{C}^{(i)}(\alpha, BGR)$ and $\mathcal{C}^{(i)}(\alpha, GRB)$ for any positive integer i.

Definition 2.3. Let α be a composition. Then $\mathcal{C}(\alpha, GBR) = \mathcal{C}^{(m)}(\alpha, GBR)$, where m is the least positive integer with $\mathcal{C}^{(m+1)}(\alpha, GBR) = \mathcal{C}^{(m)}(\alpha, GBR)$. The diagram $\mathcal{C}(\alpha, GBR)$ is called the (GBR) *Noise Bayer Young composition diagram* of α while such m is called the GBR-order of α. In this case, we write $m = |\alpha|_{GBR}$. Analogously, one could define GRB, BGR, BRG, RGB and RBG *Noise Bayer Young composition diagrams.*

Proposition 2.4. *Let $\alpha \in Comp$ and $m = |\alpha|_{GBR}$. Then we have the following:*

(i) $\mathcal{C}^{(m+t)}(\alpha, GBR) = \mathcal{C}^{(m)}(\alpha, GBR)$ *for any* $t \in \mathbb{N}$.

(ii) $m \leq 2$.

(iii) $\mathcal{C}^{(m+2)}(\alpha, GBR) = \mathcal{C}^{(m)}(\alpha, GBR)$.

Proof. The proof of (i) is obvious, and (iii) is a direct consequence of part (i). To prove (ii), let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition. If $\ell = 0$ or 1, then it is clear that $m = 1$. Otherwise, we have two cases:

Case 1: There are positive numbers t and k such that t and α_t are even, and k is the largest odd number with $t < k$ and $\alpha_t < \alpha_k$. Write $T(\mathcal{C}^{(1)}(\alpha, GBR)) = (\beta_1, \cdots, \beta_\ell)$. It follows that β_k has a red pixel, and the color of the last pixel of β_k (in $\mathcal{C}^{(1)}(\alpha, GBR)$) must be blue. Consequently, the colored cells of each row of $\mathcal{C}^{(2)}(\alpha, GBR)$ are ordered using the order $G < B < R$. Thus, $m = 2$.

Case 2: There are no such positive numbers t and k. Then the colored cells of each row of $\mathcal{C}^{(1)}(\alpha, GBR)$ are ordered using the order $G < B < R$, and hence $m = 1$. □

Remark 2.5.

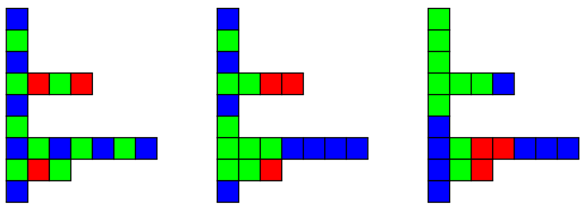
(i) Obviously, there is a 1-1 correspondence between compositions and their corresponding Noise Bayer Young composition diagrams

(ii) One might note that the green part of $\mathcal{C}(\alpha, GBR)$ forms a colored Young composition subdiagram, denoted by $\mathcal{C}(\alpha, GBR, G)$, of $\mathcal{C}(\alpha, GBR)$ (of shape α_G) while the region of both the green part and the blue part of $\mathcal{C}(\alpha, GBR)$ forms a colored Young composition subdiagram, denoted by $\mathcal{C}(\alpha, GBR, GB)$, of $\mathcal{C}(\alpha, GBR)$ (of shape α_{GB}). Here, α_G and α_{GB} are the shapes of the colored Young diagrams $\mathcal{C}(\alpha, GBR, G)$ and $\mathcal{C}(\alpha, GBR, GB)$ respectively.

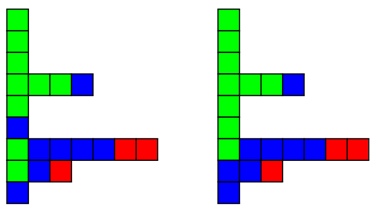
(iii) Clearly, if α is a partition, then $\mathcal{C}(\alpha, E) = \mathcal{C}^{(1)}(\alpha, E)$.

(iv) For any $\alpha \in Comp$, we have $\alpha_{GB} = \varnothing$ if and only if $\alpha = \varnothing$.

Example 2.6. Consider $\alpha = (1, 1, 1, 4, 1, 1, 7, 3, 1)$, we have



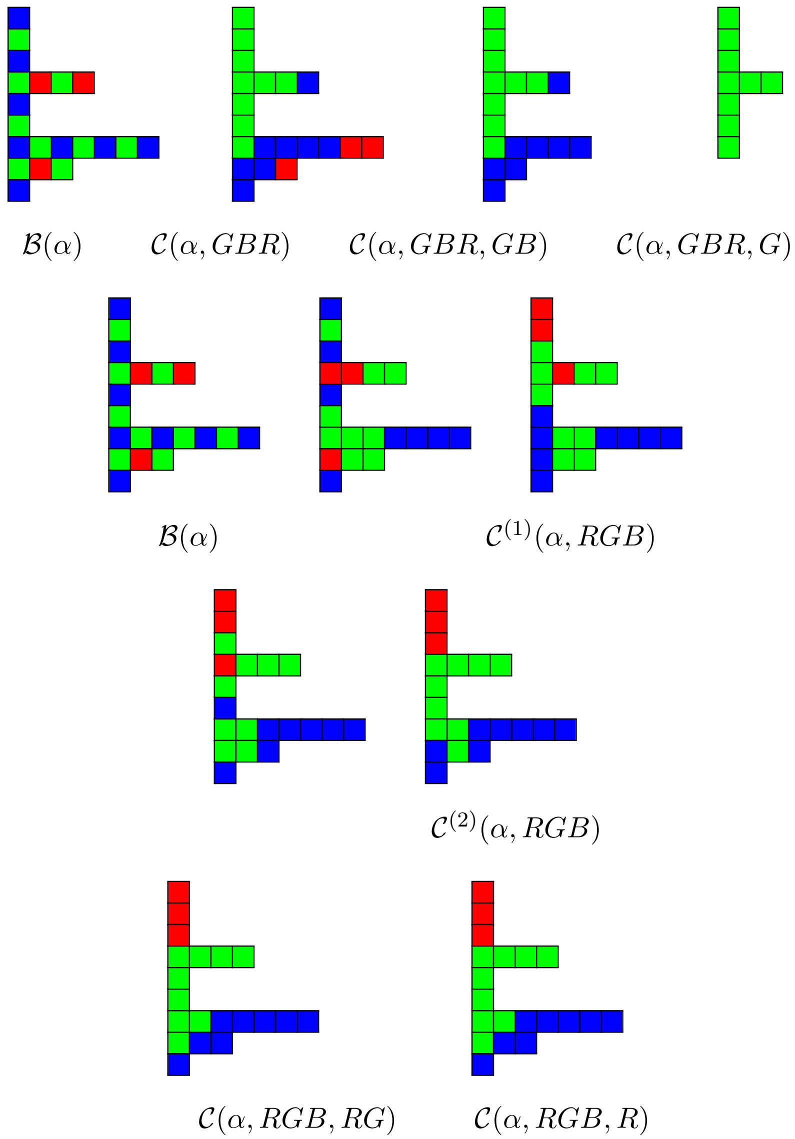
$\mathcal{B}(\alpha)$ $\qquad\qquad$ $\mathcal{C}^{(1)}(\alpha, GBR)$



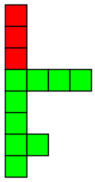
$\mathcal{C}^{(2)}(\alpha, GBR)$

Notably, $\mathcal{C}(\alpha, GBR) = \mathcal{C}^{(2)}(\alpha, GBR)$, and hence we have
On the other hand, we have
We observe that $\mathcal{C}(\alpha, RGB) = \mathcal{C}^{(3)}(\alpha, RGB)$. So, we have

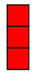
Remark 2.7. The color channels for a color image are represented by three distinct $2D$ arrays with dimension $m \times n$ for an image with m rows and n



columns, with one array for each color, red (color channel 1), green (color channel 2), blue (color channel 3). A pixel color is modeled as 1×3 array [9]. It is also well-known that the spatial domain of each RGB image can be represented as a 3D vector of 2D arrays. In MATLAB, the syntax



$\mathcal{C}(\alpha, RGB, RG)$



$\mathcal{C}(\alpha, RGB, R)$

$A = imread(filename)$ reads the image from the file specified by filename, inferring the format of the file from its contents. The Bayer Noise Young works as a machinery that provides us by an approach by which every Bayer Young diagram can be represented by three special types of colored (noise) diagrams RG, G and R diagrams. This can be visualized in the following example.

Example 2.8. We consider the following image.

Original RGB Image

Note that this image corresponding to the composition $\alpha = (1, 1, 1, 4, 1, 1, 7, 3)$. We have

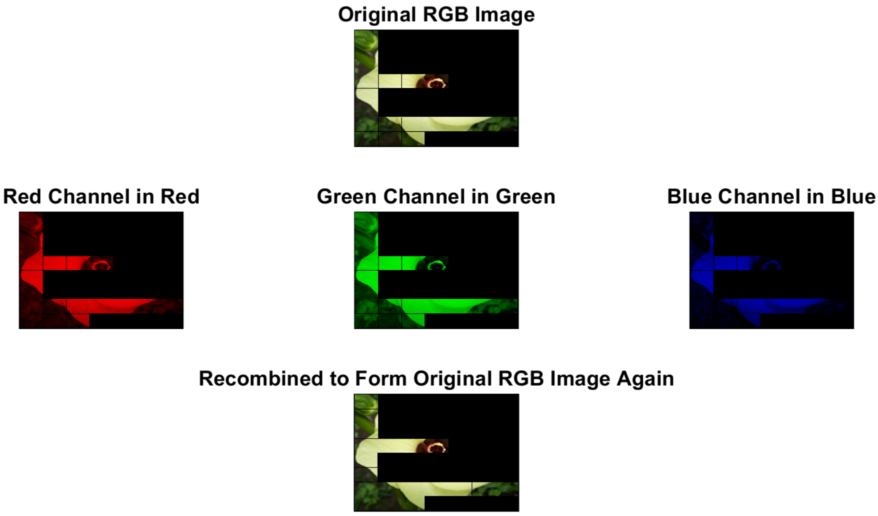
The following proposition is an obvious consequence.

Proposition 2.9.

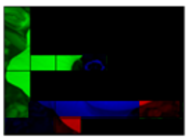
(i) *The colored Young composition diagrams* $\mathcal{C}(\alpha, GBR, G)$ *and* $\mathcal{C}(\alpha, GRB, G)$ *have the same shape* α_G*. Analogously,* $\mathcal{C}(\alpha, BGR, B)$ *and* $\mathcal{C}(\alpha, BRG, B)$ *have the same shape* α_B *while* $\mathcal{C}(\alpha, RGB, R)$ *and* $\mathcal{C}(\alpha, RBG, R)$ *have the same shape* α_R*.*

(ii) *We have* $\alpha_{GB} = \alpha_{BG}$*,* $\alpha_{GR} = \alpha_{RG}$ *and* $\alpha_{BR} = \alpha_{RB}$*.*

Definition 2.10.



3 RGB Channels



Bayer Noise Young Channels

1. For any composition $\alpha \in Comp$, the *Quasi-Bayer Noise monomial* is defined to be the monomial
$$\xi_\alpha(x, y, z) = M_{\alpha_{GB}}(x) \otimes M_{\alpha_G}(y) \otimes M_{\alpha_R}(z).$$

2. Let $\Omega_n(\mathbf{k})$ be the free $\mathbf{k}$-module with the basis $\{\xi_\alpha\}_{\alpha \in Comp_n}$, where $Comp_n$ is the set of compositions of n. We have $\Omega_n(\mathbf{k}) \cong Qsym_n$ (as vector spaces over $\mathbf{k}$) for any $n \in \mathbb{N}$, and hence $dim_{\mathbf{k}}(Qsym_n(\mathbf{k})) = 2^{n-1}$. Let $\Omega(\mathbf{k}) = \bigoplus_{n \geq 0} \Omega_n(\mathbf{k})$. Then the set $\{\xi_\alpha\}_{\alpha \in Comp}$ forms a basis for $\Omega(\mathbf{k})$ over $\mathbf{k}$, and hence $\Omega_n(\mathbf{k}) \cong Qsym_{\mathbf{k}}$ (as vector spaces). The $\mathbf{k}$-module $\Omega(\mathbf{k})$ is called the *Quasi-Bayer Noise module*.

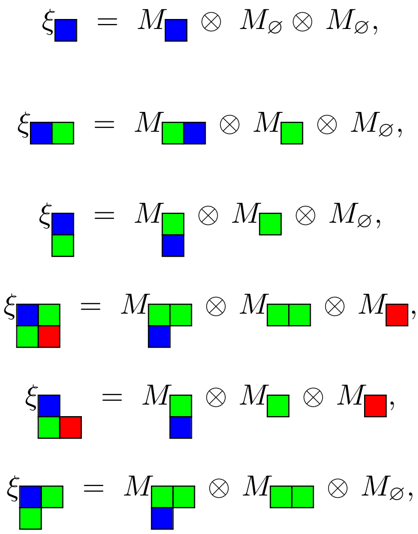
3. Let $Comp_n^{(1)} = Comp_n \bigcap Comp^{(1)}$, and let $Comp^e = \{\alpha \in Comp^{(1)} :$

all α- parts are even$\}$. Let $\Omega^{(e,n)}(\mathbf{k})$ be the free $\mathbf{k}$-module with the basis $\{\xi_\alpha\}_{\alpha\in Comp^{(e,n)}}$, where

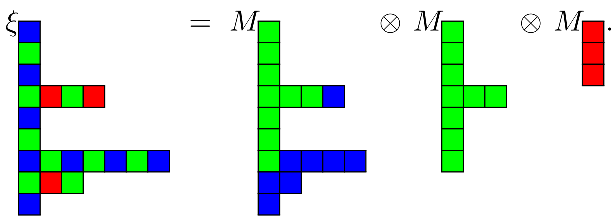
$$Comp^{(e,n)} = Comp^{(1)}_n \bigcap Comp^e.$$

Consider the $\mathbf{k}$-vector space $\Omega^e(\mathbf{k}) = \bigoplus_{n\geq 0}\Omega^{(e,n)}(\mathbf{k})$. Obviously, the set $\{\xi_\alpha\}_{\alpha\in Comp^e}$ forms a basis for $\Omega^e(\mathbf{k})$ over $\mathbf{k}$.

Example 2.11. We have



and

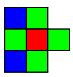
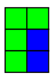
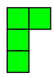


Remark 2.12.

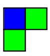
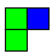
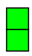
(i) When no confusion is possible, we will simply write Ω_n and Ω instead of $\Omega_n(\mathbf{k})$ and $\Omega(\mathbf{k})$ respectively.

(ii) Let $\alpha, \alpha' \in Comp$. Then, in general, $(\alpha_{GB} + \alpha'_{GB}, \alpha_G + \alpha'_G, \alpha_R + \alpha'_R)$ need not be in $\mathfrak{D}_{GBR}(Comp)$, and hence $\xi_\alpha \xi'_\alpha$ need not be in Ω. For example, if $\alpha = (1,1) = \alpha'$, then $\alpha_{GB} = \alpha'_{GB} = (1,1)$, $\alpha_G = \alpha'_G = (1)$ and $\alpha_R = \alpha'_R = (0)$ (the empty composition). However, $(\alpha_{GB} + \alpha'_{GB}, \alpha_G + \alpha'_G, \alpha_R + \alpha'_R) = ((2,2),(2),(0))$ which is clearly not in $\mathfrak{D}_{GBR}(Comp)$. It turns out that the operation $(\xi_\alpha, \xi'_\alpha) \mapsto \xi_\alpha \xi'_\alpha$ does not define an algebra structure on Ω.

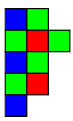
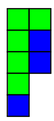
(iii) One might notice that in general if $(\alpha, \alpha') \in Comp \times Comp$, then $((\alpha \odot \alpha')_{GB}, ((\alpha \odot \alpha')_G, (\alpha \odot \alpha')_R) \neq (\alpha_{GB} \odot \alpha'_{GB}, \alpha_G \odot \alpha'_G, \alpha_R \odot \alpha'_R)$ and $(\alpha_{GB} \odot \alpha'_{GB}, \alpha_G \odot \alpha'_G, \alpha_R \odot \alpha'_R)$ need not be in $\mathfrak{D}_{GBR}(Comp)$. For example, if $\alpha = (2,3,2)$, $\alpha' = (2,1)$, then $((\alpha \odot \alpha')_{GB}, (\alpha \odot \alpha')_R) \neq (\alpha_{GB} \odot \alpha'_{GB}, \alpha_R \odot \alpha'_R)$ and $(\alpha_{GB} \odot \alpha'_{GB}, \alpha_R \odot \alpha'_R) \notin \mathfrak{D}_{GBR}(Comp)$. Explicitly, we have the following:

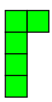
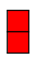
$\mathcal{B}(\alpha)$ $\mathcal{C}(\alpha, GBR, GB)$ $\mathcal{C}(\alpha, GBR, G)$ $\mathcal{C}(\alpha, GBR, R)$

   $\varnothing$

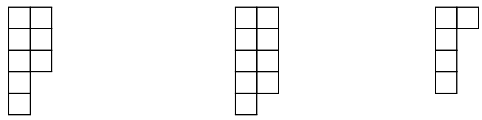
$\mathcal{B}(\alpha')$ $\mathcal{C}(\alpha', GBR, GB)$ $\mathcal{C}(\alpha', GBR, G)$ $\mathcal{C}(\alpha', GBR, R)$

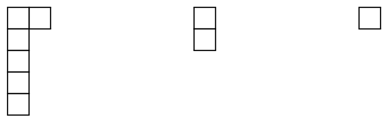
$\mathcal{B}((\alpha \odot \alpha'))$ $\mathcal{C}((\alpha \odot \alpha'), GBR, GB)$

$\mathcal{C}((\alpha \odot \alpha'), GBR, G)$ $\mathcal{C}((\alpha \odot \alpha'), GBR, R)$



$T((\alpha \odot \alpha')_{GB})$ $\quad$ $T(\alpha_{GB} \odot \alpha'_{GB})$ $\quad$ $T((\alpha \odot \alpha')_{G})$



$T(\alpha_{G} \odot \alpha'_{G})$ $\quad$ $T((\alpha \odot \alpha')_{R})$ $\quad$ $T(\alpha_{R} \odot \alpha'_{R})$

3 Algebraic structures

Recall that for any $n, m_1, \cdots, m_t \in \mathbb{N}$ with $\sum_{i=1}^{t} m_i = n$ and $t \geq 2$, the multinomial coefficient, denoted by $\binom{n}{m_1, \cdots, m_t}$, is defined by

$$\binom{n}{m_1, \cdots, m_t} = \frac{(\sum_{i=1}^{t} m_i)!}{(m_1)! \cdots (m_t)!}$$

One can easily verify the following.

Proposition 3.1. *Let* $\mathbf{k} = \Omega_0 \xrightarrow{u} \Omega$ *be the inclusion map. We have the following:*

1. *The triple* (Ω, η, u) *is a* $\mathbf{k}$*-algebra with a multiplication*

$$\eta : \Omega \otimes \Omega \to \Omega, \;\; \eta(\xi_\alpha \otimes \xi_\beta) = \sum_{\gamma} c_{\alpha,\beta}^{\gamma} \, \xi_\gamma,$$

   where $c_{\alpha,\beta}^{\gamma}$ *is the number of ways of writing* γ *as a shuffle sum of* α *and* β*.*

2. $\binom{|\alpha|+|\alpha'|}{|\alpha|,\, |\alpha'|} \binom{|\alpha|+|\alpha'|+|\alpha''|}{|\alpha\odot\alpha'|,\, |\alpha''|} = \binom{|\alpha|+|\alpha'|+|\alpha''|}{|\alpha|,\, |\alpha'|,\, |\alpha''|}$ *for every* $\alpha, \alpha' \in Comp$*.*

3. *The triple* $(\Omega, \widehat{\eta}, u)$ *is a* $\mathbf{k}$*-algebra, where* η *is the map*

$$\widehat{\eta} : \Omega \otimes \Omega \to \Omega, \;\; \xi_\alpha \otimes \xi_{\alpha'} \;\mapsto \binom{|\alpha| + |\alpha'|}{|\alpha|, \;|\alpha'|} \, \xi_{\alpha+\alpha'}.$$

The proof of the following lemma is clear and left to the reader.

Lemma 3.2. *We have* $(\alpha+\alpha')_{GB} = \alpha_{GB} + \alpha'_{GB}$ *and* $(\alpha+\alpha')_{R} = \alpha_{R} + \alpha'_{R}$ *for every* $\alpha, \alpha' \in Comp^e$.

The proof of the following theorem follows directly from Lemma 3.2.

Theorem 3.3. *We have the following:*

(i) $\binom{|(\alpha+\alpha')_{GB}|}{|\alpha_{GB}|,\, |\alpha'_{GB}|} = \binom{|\alpha_{GB}|+|\alpha'_{GB}|}{|\alpha_{GB}|,\, |\alpha'_{GB}|}$ *for any* $\alpha, \alpha' \in Comp^e$.

(ii) $\binom{|(2(\alpha+\alpha'))_{GB}|}{|2\alpha_{GB}|,\, |2\alpha'_{GB}|} = \binom{2(|\alpha_{GB}|+|\alpha'_{GB}|)}{2|\alpha_{GB}|,\, 2|\alpha'_{GB}|}$ *for any* $\alpha, \alpha' \in Comp^{(1)}$.

(iii) $\binom{|(\alpha+\alpha')_{GB}|}{|\alpha_{GB}|,\, |\alpha'_{GB}|}\binom{|((\alpha+\alpha'+\alpha'')_{GB}|}{|(\alpha+\alpha')_{GB}|,\, |\alpha''_{GB}|} = \binom{|\alpha_{GB}|+|\alpha'_{GB}|+|\alpha''_{GB}|}{|\alpha_{GB}|,\, |\alpha'_{GB}|,\, |\alpha''_{GB}|}$ *for any* $\alpha, \alpha', \alpha'' \in Comp^e$.

(iv) *More generally, we have*

$$\binom{|(\alpha^{(1)}+\alpha^{(2)}+\ldots+\alpha^{(t)})_{GB}|}{|\alpha^{(1)}_{GB}|,\, |\alpha^{(2)}_{GB}|,\, \ldots,\, |\alpha^{(t)}_{GB}|} = \binom{|\alpha^{(1)}_{GB}|+|\alpha^{(2)}_{GB}|+\ldots+|\alpha^{(t)}_{GB}|}{|\alpha^{(1)}_{GB}|,\, |\alpha^{(2)}_{GB}|,\, \ldots,\, |\alpha^{(t)}_{GB}|}$$

for every $(\alpha^{(1)}, \alpha^{(2)}, ..., \alpha^{(t)}) \in Comp^e$, *and*

$$\binom{|(2(\alpha^{(1)}+\alpha^{(2)}+\ldots+\alpha^{(t)}))_{GB}|}{|2\alpha^{(1)}_{GB}|,\, |2\alpha^{(2)}_{GB}|,\, \ldots,\, |2\alpha^{(t)}_{GB}|} = \binom{2|\alpha^{(1)}_{GB}|+2|\alpha^{(2)}_{GB}|+\ldots+2|\alpha^{(t)}_{GB}|}{2|\alpha^{(1)}_{GB}|,\, 2|\alpha^{(2)}_{GB}|,\, \ldots,\, 2|\alpha^{(t)}_{GB}|}$$

for every $(\alpha^{(1)}, \alpha^{(2)}, ..., \alpha^{(t)}) \in Comp^{(1)}$.

(v) *The triple* $(\Omega^e(\mathbf{k}), \eta_e, u_e)$ *is a* $\mathbf{k}$*-algebra, where* η_e *is the map*

$$\eta_e : \Omega^e(\mathbf{k}) \otimes \Omega^e(\mathbf{k}) \to \Omega^e(\mathbf{k}), \;\; \xi_\alpha \otimes \xi_{\alpha'} \mapsto \binom{|(2(\alpha+\alpha'))_{GB}|}{|2\alpha_{GB}|,\; |2\alpha'_{GB}|}\xi_{\alpha\odot\alpha'}$$

and

$$\mathbf{k} = \Omega^{(e,0)}(\mathbf{k}) \xrightarrow{u_e} \Omega^e(\mathbf{k})$$

is the inclusion map.

Definition 3.4.

(i) Define the sets

$$Comp^n_{GB} = \{\alpha \in Comp_n \,:\, \alpha = \alpha_{GB}\}$$

and

$$Comp_{GB} = \{\alpha \in Comp \,:\, \alpha = \alpha_{GB}\}.$$

Since $\alpha = \alpha_{GB}$ if and only if $\alpha_R = \varnothing$, the sets $Comp^n_{GB}$ and $Comp_{GB}$ can be respectively given by the following equivalent forms: $Comp^n_{GB} = \{\alpha \in Comp_n \,:\, \alpha_R = \varnothing\}$ and $Comp_{GB} = \{\alpha \in Comp \,:\, \alpha_R = \varnothing\}$. Clearly, the elements of $Comp_{GB}$ are precisely the empty composition $\emptyset$ and the compositions of the form $\alpha = (\alpha_1, \cdots, \alpha_t)$ with $\alpha_i = 1$ if i is even.

(ii) Let $\Omega^n_{GB}(\mathbf{k})$ be the free $\mathbf{k}$-module with the basis $\{\xi_\alpha\}_{\alpha\in Comp^n_{GB}}$, and let $\Omega_{GB}(\mathbf{k}) = \bigoplus_{n\geq 0} \Omega^n_{GB}(\mathbf{k})$. Then the set $\{\xi_\alpha\}_{\alpha\in Comp_{GB}}$ forms a basis for $\Omega_{GB}(\mathbf{k})$ over $\mathbf{k}$ and is called the *Quasi-Bayer GB-Noise module.*

(iii) Let $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)})$, $\beta = (\beta_1, \ldots, \beta_{\ell(\beta)}) \in Comp_{GB}$. The *GB-concatenation* of α and β is the composition $\alpha \boxdot \beta$ defined by

$$\alpha \boxdot \beta = \begin{cases} \alpha \odot \beta & \text{if } \ell(\alpha) \text{ is even or } \beta = \emptyset \\ (\alpha_1, \ldots, \alpha_{\ell(\alpha)}, 1, \beta_1, \ldots, \beta_{\ell(\beta)}) & \text{if } \ell(\alpha) \text{ is odd and } \beta \neq \emptyset \end{cases}$$

Consequently, we have the following proposition.

Proposition 3.5. *Let* $u_{GB} : \mathbf{k} = \Omega^0_{GB} \hookrightarrow \Omega_{GB}$ *be the obvious inclusion map. Then*

(i) *We have* $\alpha \boxdot \beta \in Comp_{GB}$ *for every* $\alpha, \beta \in Comp_{GB}$.

(ii) *The triple* $(\Omega_{GB}, \eta_{GB}, u_{GB})$ *is a* $\mathbf{k}$*-algebra with a multiplication*

$$\eta_{GB} : \Omega_{GB} \otimes \Omega_{GB} \to \Omega_{GB}, \;\; \eta_{GB}(\xi_\alpha \otimes \xi_\beta) = \binom{|(\alpha+\alpha')_{GB}|}{|\alpha_{GB}|,\; |\alpha'_{GB}|} \xi_{\alpha\boxdot\beta}.$$

Proof. The proof of (i) follows immediately from Definition (3.4). We prove part (ii). Let $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)})$, $\alpha' = (\alpha'_1, \ldots, \alpha'_{\ell(\alpha')})$, $\alpha'' =$

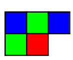
$(\alpha''_1, \ldots, \alpha''_{\ell(\alpha'')}) \in Comp_{GB}$. To prove the associativity axiom, by using part (2) of proposition, (3.1), it suffices to show that $(\alpha \boxdot \alpha') \boxdot \alpha'' = \alpha \boxdot (\alpha' \boxdot \alpha'')$ for any $\alpha, \alpha', \alpha'' \in Comp_{GB}$. If α, α' or α'' is $\emptyset$ (the empty composition), then we obviously have $(\alpha \boxdot \alpha') \boxdot \alpha'' = \alpha \boxdot (\alpha' \boxdot \alpha'')$ for any $\alpha, \alpha', \alpha'' \in Comp_{GB}$. If none of them is $\emptyset$, then we have

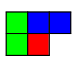
$\ell(\alpha)$	$\ell(\alpha')$	$\ell(\alpha'')$	$\ell(\alpha \boxdot \alpha')$	$\ell(\alpha' \boxdot \alpha'')$
even	even	even	even	even
odd	odd	odd	odd	odd
even	even	odd	even	odd
even	odd	even	odd	even
odd	even	even	even	even
odd	odd	even	odd	even
odd	even	odd	even	odd
even	odd	odd	odd	odd

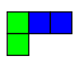
$(\alpha \boxdot \alpha') \boxdot \alpha''$	$\alpha \boxdot (\alpha' \boxdot \alpha'')$
$(\alpha \odot \alpha') \odot \alpha''$	$\alpha \odot (\alpha' \odot \alpha'')$
$\alpha \odot (1) \odot \alpha' \odot (1) \odot \alpha''$	$\alpha \odot (1) \odot \alpha' \odot (1) \odot \alpha''$
$(\alpha \odot \alpha') \odot \alpha''$	$\alpha \odot (\alpha' \odot \alpha'')$
$\alpha \odot \alpha' \odot (1) \odot \alpha''$	$\alpha \odot \alpha' \odot (1) \odot \alpha''$
$\alpha \odot (1) \odot \alpha' \odot \alpha''$	$\alpha \odot (1) \odot \alpha' \odot \alpha''$
$\alpha \odot (1) \odot \alpha' \odot (1) \odot \alpha''$	$\alpha \odot (1) \odot \alpha' \odot (1) \odot \alpha''$
$\alpha \odot (1) \odot \alpha' \odot \alpha''$	$\alpha \odot (1) \odot \alpha' \odot \alpha''$
$\alpha \odot \alpha' \odot (1) \odot \alpha''$	$\alpha \odot \alpha' \odot (1) \odot \alpha''$

Thus, $(\alpha \boxdot \alpha') \boxdot \alpha'' = \alpha \boxdot (\alpha' \boxdot \alpha'')$ for any $\alpha, \alpha', \alpha'' \in Comp_{GB}$. One can easily show that the unity axiom holds. $\square$

Example 3.6. Let $\alpha = (3, 2)$, $\beta = (4, 5, 3)$, $\mu = (2, 1, 4, 1, 5)$ and $\nu = (3, 1)$. We have





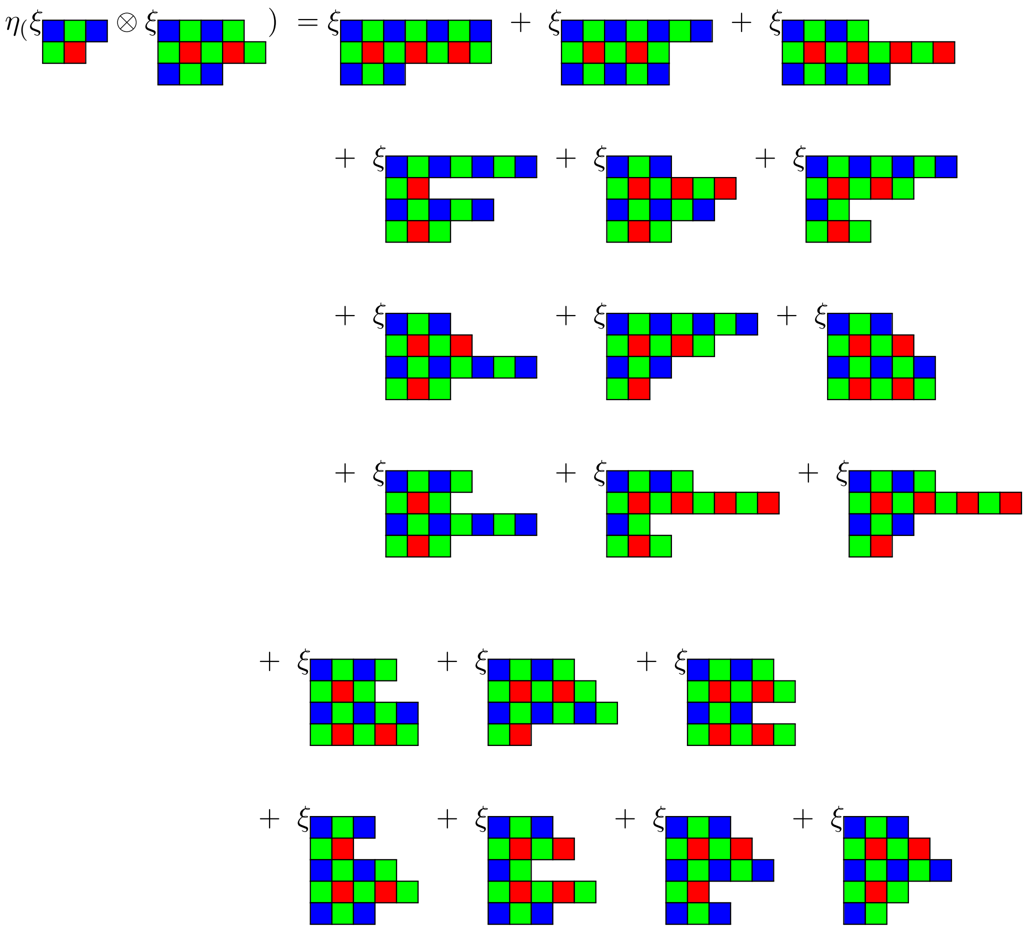


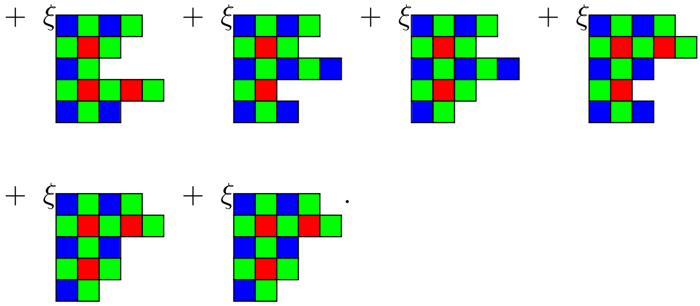
$\mathcal{B}(\alpha)$ $\mathcal{C}(\alpha, GBR)$ $\mathcal{C}(\alpha, GBR, GB)$
$\mathcal{C}(\alpha, GBR, R)$

One could similarly show that the corresponding diagrams for β are

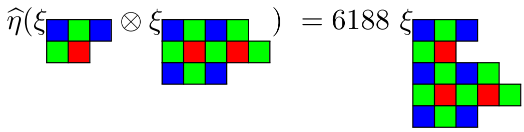
$\mathcal{B}(\beta)$ $\mathcal{C}(\beta, GBR)$ $\mathcal{C}(\beta, GBR, GB)$ $\mathcal{C}(\beta, GBR, R)$

One can calculate $\eta(\xi_\alpha \otimes \xi_\beta)$, $\widehat{\eta}(\xi_\alpha \otimes \xi_\beta)$ and $\eta_{GB}(\xi_\mu \otimes \xi_\nu)$ as follows:

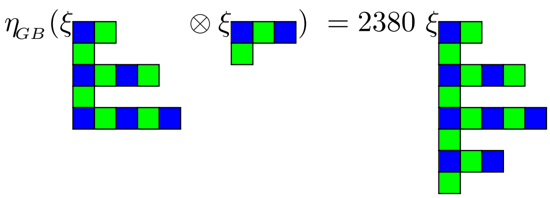




One might calculate $\widehat{\eta}(\xi_\alpha \otimes \xi_\beta)$ as follows:



Similarly, one could easily calculate $\eta_{GB}(\xi_\mu \otimes \xi_\nu)$ as follows:



4 Coalgebraic structures

Naturally, one might dually think of breaking a basis element ξ_α into tensors of basis elements in such a compatible way. Consider the map

$$\Delta\xi_\alpha = \sum_{\substack{(\mu,\nu)\in Comp\times Comp:\\ \mu\odot\nu=\alpha}} \xi_\mu \otimes \xi_\nu, \tag{4.1}$$

Interestingly, one might define the map $\tilde{\Delta} : \Omega \to \Omega \otimes \Omega$ defined **k**-linearly by

$$\tilde{\Delta}\xi_\alpha = \sum_{\substack{(\mu,\nu)\in Comp\times Comp:\\ \mu_U\odot\nu_U=\alpha_U,\forall U\in\{GB,R\}}} \binom{|\mu+\nu|}{|\mu|,\ |\nu|} \xi_\mu \otimes \xi_\nu, \tag{4.2}$$

We have the following theorem.

Theorem 4.1. *Let* $\Omega \xrightarrow{\epsilon} \mathbf{k}$ *be the map defined* $\mathbf{k}$*-linearly by*

$$\epsilon|_{\Omega_0=\mathbf{k}} = id_{\mathbf{k}} \text{ and } \epsilon|_{I=\bigoplus_{n>0}\Omega_n} = 0.$$

Then

(i) *The triple* $(\Omega, \Delta, \epsilon)$ *is a* $\mathbf{k}$*-coalgebra.*

(ii) *The triple* $(\Omega, \tilde{\Delta}, \epsilon)$ *is a* $\mathbf{k}$*-coalgebra.*

Proof. The proof of (i) is obvious. To prove part (ii), we have to show that the following diagrams are commutative.

$$\begin{tikzcd}
 & \Omega\otimes\Omega\otimes\Omega & \\
\Omega\otimes\Omega \arrow[ur, "\tilde{\Delta}\otimes id"] & & \Omega\otimes\Omega \arrow[ul, "id\otimes\tilde{\Delta}"'] \\
 & \Omega \arrow[ul, "\tilde{\Delta}"] \arrow[ur, "\tilde{\Delta}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega\otimes\mathbf{k} \arrow[r, "\Psi"] & \Omega & \mathbf{k}\otimes\Omega \arrow[l, "\Phi"'] \\
\Omega\otimes\Omega \arrow[u, "id\otimes\epsilon"] & \Omega \arrow[u, "id"] \arrow[l, "\tilde{\Delta}"] \arrow[r, "\tilde{\Delta}"'] & \Omega\otimes\Omega \arrow[u, "\epsilon\otimes id"']
\end{tikzcd} \tag{4.3}$$

Here Ψ and Φ are the isomorphisms $\Psi : \Omega\otimes\mathbf{k} \to \Omega,\ \xi_\alpha \otimes 1 \mapsto \xi_\alpha$ and $\Phi : \mathbf{k}\otimes\Omega \to \Omega,\ 1\otimes\xi_\alpha \mapsto \xi_\alpha$. For any $\alpha \in Par$, we have

$$\begin{aligned}
(\tilde{\Delta}\otimes id)\tilde{\Delta}\xi_\alpha &= (\tilde{\Delta}\otimes id)\Big(\sum_{\substack{(\mu,\mu')\in Comp\times Comp:\\ \mu_U\odot\mu'_U=\alpha_U,\forall U\in\{GB,R\}}} \binom{|\mu+\mu'|}{|\mu|,\ |\mu'|}\ \xi_\mu\otimes\xi_{\mu'}\Big) \\
&= \sum_{\substack{(\mu,\mu')\in Comp\times Comp:\\ \mu_U\odot\mu'_U=\alpha_U,\forall U\in\{GB,R\}}} \binom{|\mu+\mu'|}{|\mu|,\ |\mu'|}\ \tilde{\Delta}\xi_\mu\otimes\xi_{\mu'} \\
&= \Big(\sum_{\substack{(\mu,\mu')\in Comp\times Comp:\\ \mu_U\odot\mu'_U=\alpha_U,\forall U\in\{GB,R\}}} \binom{|\mu+\mu'|}{|\mu|,\ |\mu'|} \\
&\qquad \sum_{\substack{(\nu,\nu')\in Comp\times Comp:\\ \nu_W\odot\nu'_W=\mu_W,\forall W\in\{GB,R\}}} \binom{|\nu+\nu'|}{|\nu|,\ |\nu'|}\ (\xi_\nu\otimes\xi_{\nu'})\Big)\otimes\xi_{\mu'}
\end{aligned}$$

$$= \sum_{\substack{(\nu,\nu',\mu')\in Comp\times Comp:\\ \nu_U\odot\nu'_U\odot\mu'_U=\alpha_U,\forall U\in\{GB,R\}}} \binom{|\mu+\mu'|}{|\mu|,\ |\mu'|}\binom{|\nu+\nu'|}{|\nu|,\ |\nu'|}\xi_\nu\otimes\xi_{\nu'}\otimes\xi_{\mu'}$$

One can easily check the following:

$$|\alpha| = |\alpha_{GB}| + |\alpha_R| = |\mu_{GB}| + |\mu'_{GB}| + |\mu_R| + |\mu'_R| = |\mu| + |\mu'|$$

and

$$|\mu| = |\mu_{GB}| + |\mu_R| = |\nu_{GB}| + |\nu'_{GB}| + |\nu_R| + |\nu'_R| = |\nu| + |\nu'|.$$

As a consequence, we have

$$\binom{|\mu|+|\mu'|}{|\mu|,\ |\mu'|}\binom{|\nu|+|\nu'|}{|\nu|,\ |\nu'|} = \binom{|\nu|+|\nu'|+|\mu'|}{|\nu|+|\nu'|,\ |\mu'|}\binom{|\nu|+|\nu'|}{|\nu|,\ |\nu'|}$$

$$= \binom{|\nu|+|\nu'|+|\mu'|}{|\nu|,\ |\nu'|,\ |\mu'|} \text{ (by part (2) of proposition (3.1)).}$$

Thus, we have

$$\begin{aligned}(\tilde\Delta\otimes id)\tilde\Delta\xi_\alpha &= \sum_{\substack{(\nu,\nu',\mu')\in Comp\times Comp:\\ \nu_U\odot\nu'_U\odot\mu'_U=\alpha_U,\forall U\in\{GB,R\}}} \binom{|\nu|+|\nu'|+|\mu'|}{|\nu|,\ |\nu'|,\ |\mu'|}\xi_\nu\otimes\xi_{\nu'}\otimes\xi_{\mu'}\\
&= \sum_{\substack{(\mu,\mu')\in Comp\times Comp:\\ \mu_U\odot\mu'_U=\alpha_U,\forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,\ |\mu'|}\\
&\quad \sum_{\substack{(\nu,\nu')\in Comp\times Comp:\\ \nu_W\odot\nu'_W=\mu'_W,\forall W\in\{GB,R\}}} \binom{|\nu|+|\nu'|}{|\nu|,\ |\nu'|}\xi_\mu\otimes(\xi_\nu\otimes\xi_{\nu'})\\
&= \sum_{\substack{(\mu,\mu')\in Comp\times Comp:\\ \mu_U\odot\mu'_U=\alpha_U,\forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,\ |\mu'|}\end{aligned}$$

$$\xi_\mu \otimes \sum_{\substack{(\nu,\nu')\in Comp\times Comp:\\ \nu_W \odot \nu'_W = \mu'_W, \forall W\in\{GB,R\}}} \binom{|\nu|+|\nu'|}{|\nu|,\ |\nu'|} (\xi_\nu \otimes \xi_{\nu'})$$

$$= \sum_{\substack{(\mu,\mu')\in Comp\times Comp:\\ \mu_U \odot \mu'_U = \alpha_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,\ |\mu'|}\ \xi_\mu \otimes \tilde{\Delta}\xi_{\mu'}$$

$$= (id \otimes \tilde{\Delta})(\sum_{\substack{(\mu,\mu')\in Comp\times Comp:\\ \mu_U \odot \mu'_U = \alpha_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,\ |\mu'|}\ \xi_\mu \otimes \xi_{\mu'})$$

$$= (id \otimes \tilde{\Delta})\tilde{\Delta}\xi_\alpha.$$

Therefore, the commutativity of the associativity diagram follows. Checking the commutativity of the unity diagram can be done as follows:

$$\Phi(\epsilon \otimes id)\tilde{\Delta}\xi_\alpha = \Phi(\epsilon \otimes id)(\sum_{\substack{(\mu,\mu')\in Comp\times Comp:\\ \mu_U \odot \mu'_U = \alpha_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,\ |\mu'|}\ \xi_\mu \otimes \xi_{\mu'})$$

$$= \Phi(\sum_{\substack{(\mu,\mu')\in Comp\times Comp:\\ \mu_U \odot \mu'_U = \alpha_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,\ |\mu'|}\ \epsilon(\xi_\mu) \otimes \xi_{\mu'})$$

$$= \sum_{\substack{(\mu,\mu')\in Comp\times Comp:\\ \mu_U \odot \mu'_U = \alpha_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,\ |\mu'|}\ \epsilon(\xi_\mu)\, \xi_{\mu'}$$

$$= \xi_\alpha \text{ (since } \epsilon|_{\mathbf{k}} = id_{\mathbf{k}} \text{ and } \epsilon|_{I=\bigoplus_{n>0}\Omega_n} = 0).$$

$$= id(\xi_\alpha)$$

$$= \sum_{\substack{(\mu,\mu')\in Comp\times Comp:\\ \mu_U \odot \mu'_U = \alpha_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,\ |\mu'|}\ \xi_\mu\, \epsilon(\xi_{\mu'})$$

$$= \Psi(\sum_{\substack{(\mu,\mu')\in Comp\times Comp:\\ \mu_U \odot \mu'_U = \alpha_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,\ |\mu'|}\ \xi_\mu \otimes \epsilon(\xi_{\mu'}))$$

$$= \Psi(id \otimes \epsilon)(\sum_{\substack{(\mu,\mu')\in Comp\times Comp: \\ \mu_U \odot \mu'_U = \alpha_U, \forall U \in \{GB,R\}}} \binom{|\mu| + |\mu'|}{|\mu|,\ |\mu'|} \xi_\mu \otimes \xi_{\mu'})$$
$$= \Psi(id \otimes \epsilon)\tilde{\Delta}\xi_\alpha.$$

It follows that $(\Omega, \tilde{\Delta}, \epsilon)$ is a $\mathbf{k}$-coalgebra. □

We call the $\mathbf{k}$-coalgebra $(\Omega, \tilde{\Delta}, \epsilon)$ as the *Bayer coalgebra* over $\mathbf{k}$. The following proposition gives an explicit description for primitive with respect to the comultiplication $\tilde{\Delta}$.

Proposition 4.2. *The primitive basis elements for Ω (with respect to the comultiplication $\tilde{\Delta}$) are precisely of the form ξ_α, where $\alpha = (m)$ for some non-negative integer m.*

Proof. It is clear that $\tilde{\Delta}\xi_\lambda = \xi_\lambda \otimes 1 + 1 \otimes \xi_\lambda$ if and only if $\lambda = (m)$ for some non-negative integer m. Thus, ξ_α is primitive if and only if $\lambda = (m)$ for some non-negative integer m. □

Let $\widehat{\Delta} : \Omega^e(\mathbf{k}) \to \Omega^e(\mathbf{k}) \otimes \Omega^e(\mathbf{k})$ be the map defined $\mathbf{k}$-linearly by

$$\widehat{\Delta}\xi_\alpha = \sum_{\substack{(\mu,\nu)\in Comp^e\times Comp^e: \\ \mu_U \odot \nu_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\mu + \nu|}{|\mu|,\ |\nu|} \xi_\mu \otimes \xi_\nu, \tag{4.4}$$

Using part (3) of Theorem (3.3), the following theorem can be proved similarly to the proof of Theorem (4.1).

Theorem 4.3. *The triple $(\Omega^e(\mathbf{k}), \widehat{\Delta}, \widehat{\epsilon})$ is a $\mathbf{k}$-coalgebra, where $\Omega^e(\mathbf{k}) \xrightarrow{\widehat{\epsilon}} \mathbf{k}$ is the map defined $\mathbf{k}$-linearly by*

$$\widehat{\epsilon}|_{\Omega^{(e,0)}=\mathbf{k}} = id_{\mathbf{k}} \text{ and } \widehat{\epsilon}|_{I=\bigoplus_{n>0}\Omega^{(e,n)}} = 0.$$

The primitive elements in $\Omega^e(\mathbf{k})$ (with respect to the comultiplication $\widehat{\Delta}$) can be explicitly described as follows:

Proposition 4.4. *The primitive basis elements for $\Omega^e(\mathbf{k})$ (with respect to the comultiplication $\widehat{\Delta}$) are precisely of the form ξ_α, where $\alpha = (m)$ for some $m \in 2\mathbb{N} = \{0, 2, 4, \ldots\}$.*

Proof. The proof is very similar to the proof of Proposition 4.2. □

Definition 4.5.

1. Let $\Omega_n^{(1)}(\mathbf{k})$ be the free $\mathbf{k}$-module with the basis $\{\xi_\alpha\}_{\alpha\in Comp_n^{(1)}}$. Consider the $\mathbf{k}$-module
$$\Omega^{(1)}(\mathbf{k}) = \bigoplus_{n\geq 0} \Omega_n^{(1)}(\mathbf{k}).$$
Clearly, the set $\{\xi_\alpha\}_{\alpha\in Comp^{(1)}}$ forms a basis for the free $\mathbf{k}$-module $\Omega^{(1)}(\mathbf{k})$.
2. Consider the map $\Delta^{(e)} : \Omega^{(1)}(\mathbf{k}) \to \Omega^{(1)}(\mathbf{k})\otimes\Omega^{(1)}(\mathbf{k})$ defined $\mathbf{k}$-linearly by
$$\Delta^{(e)}\xi_\alpha = \sum_{\substack{(\mu,\nu)\in Comp^{(1)}\times Comp^{(1)}:\\ \mu_U\odot\nu_U=\alpha_U,\forall U\in\{GB,R\}}} \binom{2|\mu+\nu|}{2|\mu|,\ 2|\nu|} \xi_\mu\otimes\xi_\nu. \tag{4.5}$$

The following are analogous consequences to those of Theorem 4.3 and Proposition 4.4, respectively.

Theorem 4.6. *The triple* $(\Omega^{(1)}(\mathbf{k}), \Delta^{(e)}, \epsilon^{(e)})$ *is a* $\mathbf{k}$*-coalgebra, where* $\Omega^{(1)}(\mathbf{k}) \xrightarrow{\epsilon^{(e)}} \mathbf{k}$ *is the map defined* $\mathbf{k}$*-linearly by*
$$\epsilon^{(e)}|_{\Omega_0^{(1)}(\mathbf{k})=\mathbf{k}} = id_{\mathbf{k}} \text{ and } \epsilon^{(e)}|_{I=\bigoplus_{n>0}\Omega_n^{(1)}(\mathbf{k})} = 0.$$

Proposition 4.7. *The primitive basis elements for* $\Omega^{(1)}$ *(with respect to the comultiplication* $\Delta^{(e)}$*) are precisely of the form* ξ_α*, where* $\alpha = (m)$ *for some* $m \in 2\mathbb{N} = \{0, 2, 4, \ldots\}$*.*

Define the $\mathbf{k}$-linear map
$$\Delta_{GB} : \Omega_{GB} \to \Omega_{GB}\otimes\Omega_{GB},\ \ \Delta_{GB}\xi_\alpha = \sum_{\substack{(\mu,\nu)\in Comp_{GB}\times Comp_{GB}:\\ \mu\boxdot\nu=\alpha}} \binom{|\mu+\nu|}{|\mu|,\ |\nu|} \xi_\mu\otimes\xi_\nu.$$

The proof of the following proposition is obvious and left to the reader.

Proposition 4.8. *Let $\Omega_{GB} \xrightarrow{\epsilon_{GB}} \mathbf{k}$ be the $\mathbf{k}$-linear map defined by*

$$\epsilon_{GB}|_{\Omega^0_{GB}=\mathbf{k}} = id_{\mathbf{k}} \text{ and } \epsilon_{GB}|_{I=\bigoplus_{n>0}\Omega^n_{GB}} = 0.$$

Then $(\Omega_{GB}, \Delta_{GB}, \epsilon_{GB})$ is a $\mathbf{k}$-coalgebra.

The following proposition gives an explicit description of the primitive basis elements of $\{\xi_\alpha\}_{\alpha\in Comp_{GB}}$ (with respect to the comultiplication Δ_{GB}).

Proposition 4.9. *The primitive basis elements of $\{\xi_\alpha\}_{\alpha\in Comp_{GB}}$ (with respect to the comultiplication Δ_{GB}) are precisely of the form ξ_θ, where $\theta \in Comp_{GB}$ with $\ell(\theta) \leq 2$.*

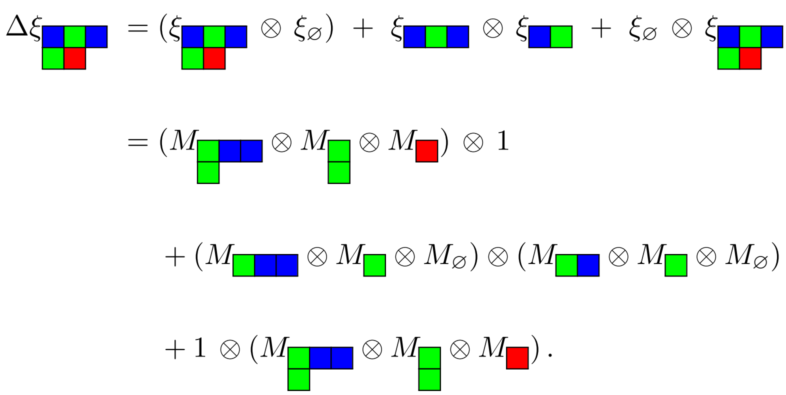
Proof. For any $\theta \in Comp_{GB}$, we have $\Delta_{GB}\xi_\theta = \xi_\theta \otimes 1 + 1 \otimes \xi_\theta$ if and only if $\ell(\theta) \leq 2$ which completes the proof. $\square$

Example 4.10.

1. Let $\alpha = (3, 2)$. One can easily verify that

$$\begin{aligned}
\Delta\xi_\alpha &= \Delta\xi_{(3,2)} \\
&= \xi_{(3,2)} \otimes \xi_\varnothing + \xi_{(3)} \otimes \xi_{(2)} + \xi_\varnothing \otimes \xi_{(3,2)} \\
&= (M_{(3,1)} \otimes M_{(1,1)} \otimes M_{(1)}) \otimes 1 + (M_{(3)} \otimes M_{(1)} \otimes M_\varnothing) \otimes (M_{(2)} \otimes M_{(1)} \otimes M_\varnothing) \\
&\quad + 1 \otimes (M_{(3,1)} \otimes M_{(1,1)} \otimes M_{(1)}).
\end{aligned}$$

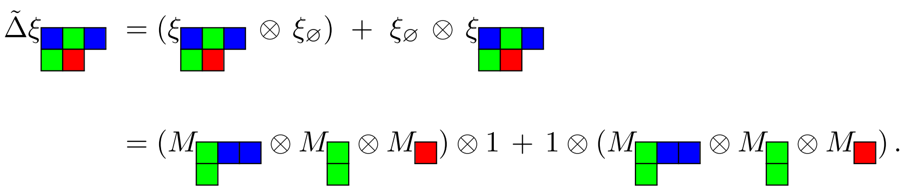
This can be pictured as



On the other hand, we have

$$\begin{aligned}\tilde{\Delta}\xi_\alpha &= \tilde{\Delta}\xi_{(3,2)}\\ &= \xi_{(3,2)} \otimes \xi_\varnothing + \xi_\varnothing \otimes \xi_{(3,2)}\\ &= (M_{(3,1)} \otimes M_{(1,1)} \otimes M_{(1)}) \otimes 1 + 1 \otimes (M_{(3,1)} \otimes M_{(1,1)} \otimes M_{(1)}).\end{aligned}$$

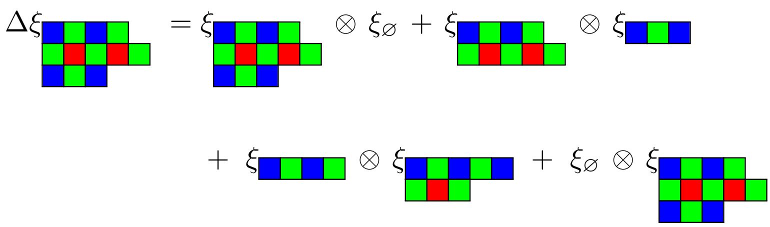
This can be visualized as

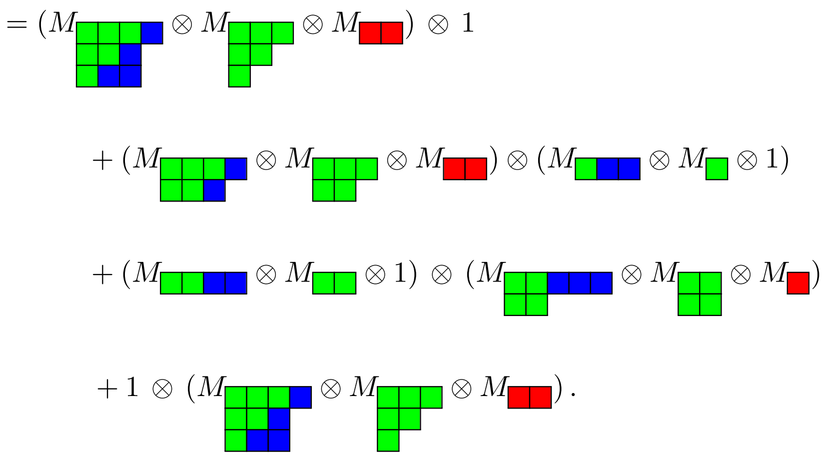


2. To see the difference between Δ and $\tilde{\Delta}$ more clearly, let $\alpha = (4,5,3)$. Clearly, we have

$$\begin{aligned}\Delta\xi_\alpha &= \Delta\xi_{(4,5,3)}\\ &= \xi_{(4,5,3)} \otimes \xi_\varnothing + \xi_{(4,5)} \otimes \xi_{(3)} + \xi_{(4)} \otimes \xi_{(5,3)} + \xi_\varnothing \otimes \xi_{(4,5,3)}\\ &= (M_{(4,3,3)} \otimes M_{(3,2,1)} \otimes M_{(2)}) \otimes 1\\ &\quad + (M_{(4,3)} \otimes M_{(3,2)} \otimes M_{(2)}) \otimes (M_{(3)} \otimes M_{(1)} \otimes 1)\\ &\quad + (M_{(4)} \otimes M_{(2)} \otimes 1) \otimes (M_{(5,2)} \otimes M_{(2,2)} \otimes M_{(1)})\\ &\quad + 1 \otimes (M_{(4,3,3)} \otimes M_{(3,2,1)} \otimes M_{(2)}).\end{aligned}$$

One might visualize it as

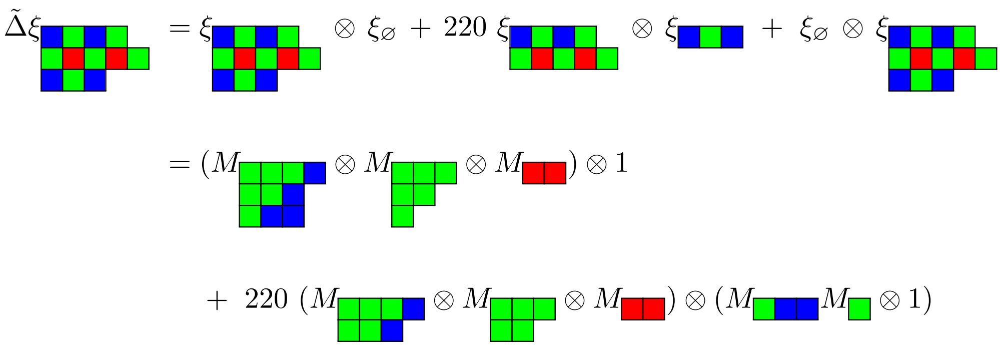


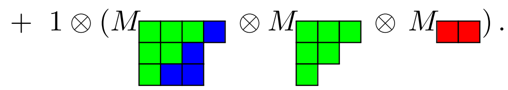


It is easy to check that $\tilde{\Delta}\xi_\alpha$ is given by

$$\begin{aligned}
\tilde{\Delta}\xi_\alpha &= \tilde{\Delta}\xi_{(4,5,3)} \\
&= \xi_{(4,5,3)} \otimes \xi_\varnothing + 220\ \xi_{(4,5)} \otimes \xi_{(3)} + \xi_\varnothing \otimes \xi_{(4,5,3)} \\
&= (M_{(4,3,3)} \otimes (M_{(3,2,1)} \otimes M_{(2)}) \otimes 1 \\
&\quad + (M_{(4,3)} \otimes (M_{(3,2)} \otimes M_{(2)}) \otimes (M_{(3)} \otimes (M_{(1)} \otimes 1) \\
&\quad + 1 \otimes (M_{(4,3,3)} \otimes (M_{(3,2,1)} \otimes M_{(2)}),
\end{aligned}$$

which can be visualized as the following.

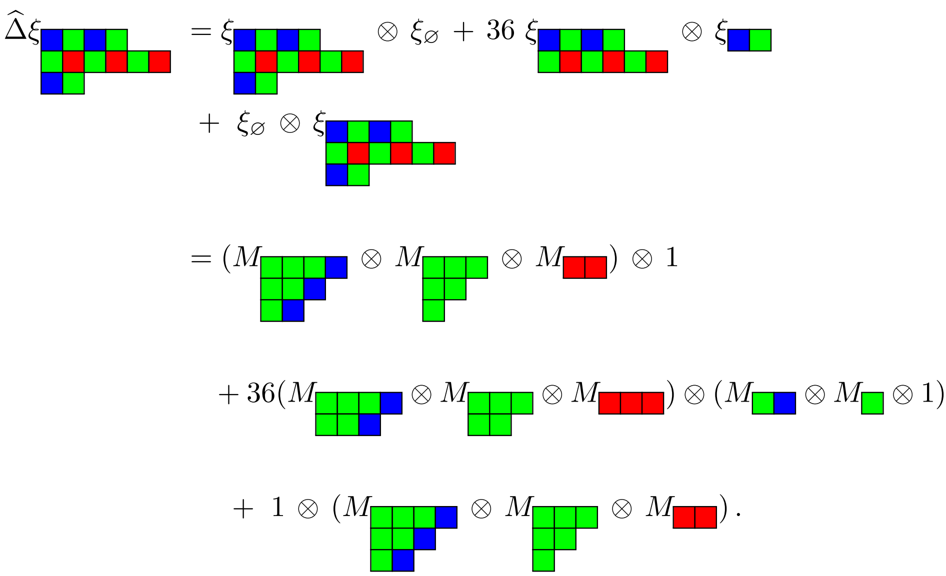




3. Let $\alpha = (4,6,2)$ and $\beta = (3,6,2)$. A direct calculation shows that

$$\begin{aligned}
\widehat{\Delta}\xi_\alpha &= \widehat{\Delta}\xi_{(4,6,2)} \\
&= \xi_{(4,6,2)} \otimes \xi_\varnothing + 36\ \xi_{(4,6)} \otimes \xi_{(2)} + \xi_\varnothing \otimes \xi_{(4,6,2)} \\
&= (M_{(4,3,2)} \otimes M_{(3,2,1)} \otimes M_{(2)}) \otimes 1 \\
&\quad + 36\ (M_{(4,3)} \otimes M_{(3,2)} \otimes M_{(3)}) \otimes (M_{(2)} \otimes M_{(1)} \otimes 1) \\
&\quad + 1 \otimes (M_{(4,3,2)} \otimes M_{(3,2,1)} \otimes M_{(2)}),
\end{aligned}$$

One can be picture this as follows:

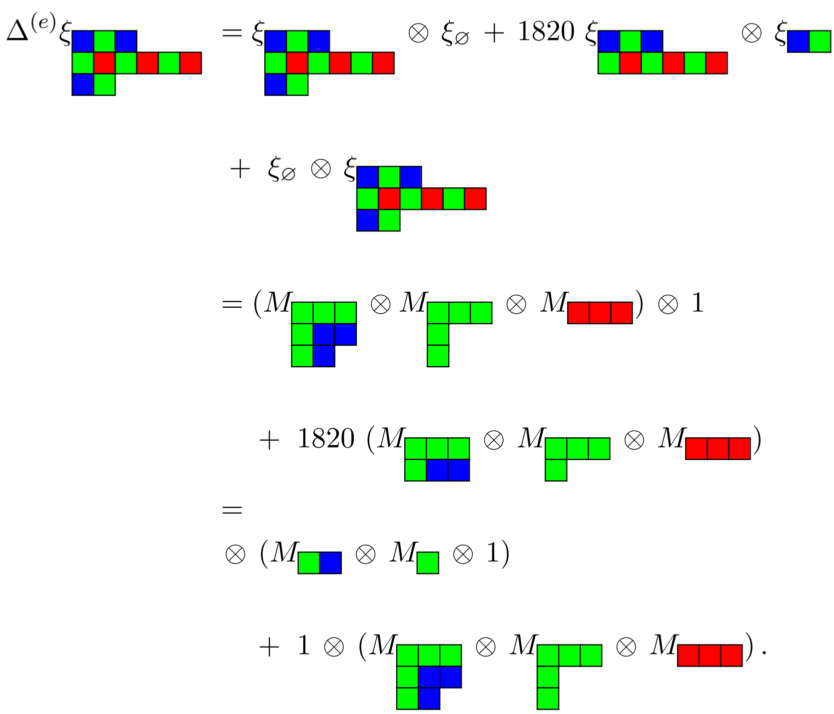


A similar calculation shows that

$$\begin{aligned}
\Delta^{(e)}\xi_\beta &= \Delta^{(e)}\xi_{(3,6,2)} \\
&= \xi_{(3,6,2)} \otimes \xi_\varnothing + 1820\ \xi_{(3,6)} \otimes \xi_{(2)} + \xi_\varnothing \otimes \xi_{(3,6,2)}
\end{aligned}$$

$$
\begin{aligned}
&= (M_{(3,3,2)} \otimes M_{(3,1,1)} \otimes M_{(3)}) \otimes 1 \\
&+ 1820\ (M_{(3,3)} \otimes M_{(3,1)} \otimes M_{(3)}) \otimes (M_{(2)} \otimes M_{(1)} \otimes 1) \\
&\quad + 1 \otimes (M_{(3,3,2)} \otimes M_{(3,1,1)} \otimes M_{(3)}).
\end{aligned}
$$

One can visualize this calculation as the following:

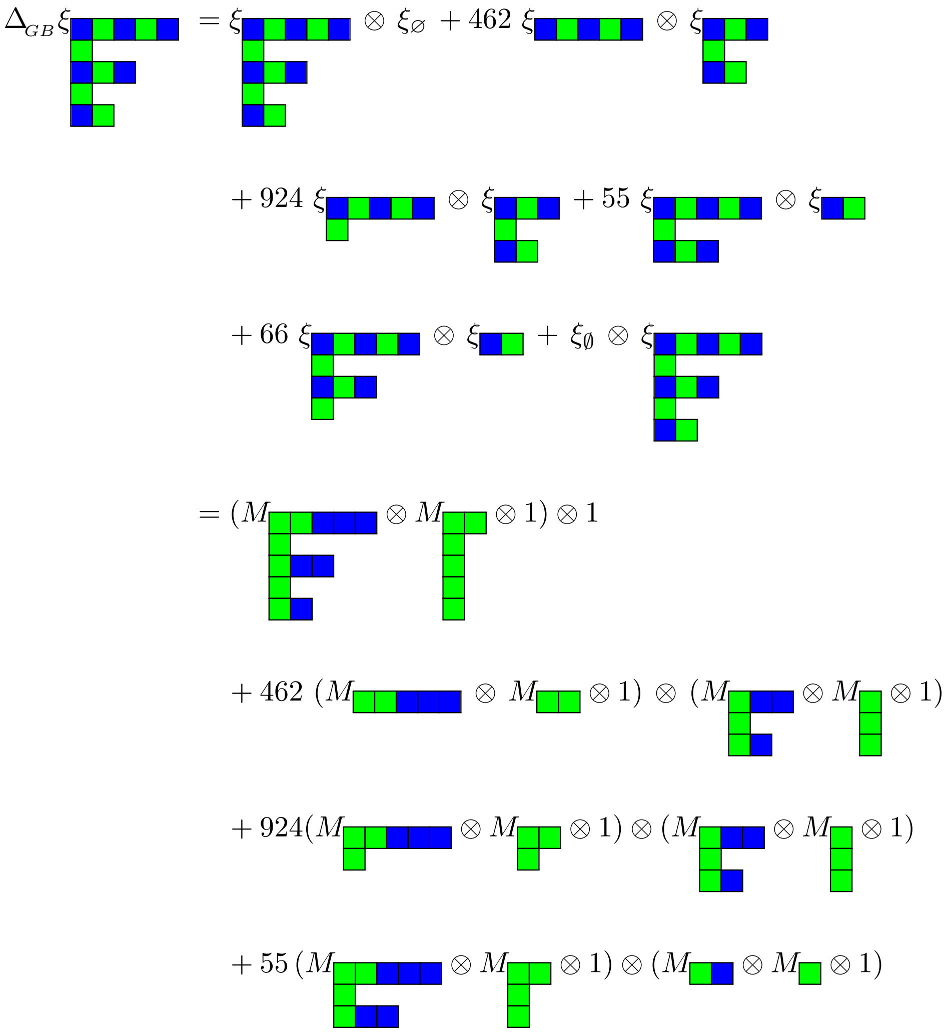


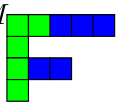
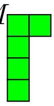
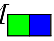
4. Let $\sigma = (5,1,3,1,2)$. Clearly, we have

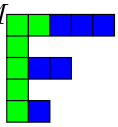
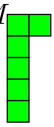
$$
\begin{aligned}
\Delta_{GB}\xi_\sigma &= \Delta_{GB}\xi_{(5,1,3,1,2)} \\
&= \xi_{(5,1,3,1,2)} \otimes \xi_\varnothing + 462\ \xi_{(5)} \otimes \xi_{(3,1,2)} + 924\ \xi_{(5,1)} \otimes \xi_{(3,1,2)} \\
&\quad + 55\ \xi_{(5,1,3)} \otimes \xi_{(2)} + 66\ \xi_{(5,1,3,1)} \otimes \xi_{(2)} + \xi_\varnothing \otimes \xi_{(5,1,3,1,2)} \\
&= (M_{(5,1,3,1,2)} \otimes M_{(2,1,1,1,1)} \otimes 1) \otimes 1 \\
&\quad + 462\ (M_{(5)} \otimes M_{(2)} \otimes 1) \otimes (M_{(3,1,2)} \otimes M_{(1,1,1)} \otimes 1) \\
&\quad + 924\ (M_{(5,1)} \otimes M_{(2,1)} \otimes 1) \otimes (M_{(3,1,2)} \otimes M_{(1,1,1)} \otimes 1)
\end{aligned}
$$

$$+\,55\ (M_{(5,1,3)} \otimes M_{(2,1,1)} \otimes 1) \otimes (M_{(2)} \otimes M_{(1)} \otimes 1)$$
$$+\,66\ (M_{(5,1,3,1)} \otimes M_{(2,1,1,1)} \otimes 1) \otimes (M_{(2)} \otimes M_{(1)} \otimes 1)$$
$$+\,1 \otimes (M_{(5,1,3,1,2)} \otimes M_{(2,1,1,1,1)} \otimes 1).$$

One might visualize it as



$+\, 66(M$  $\otimes M$  $\otimes 1) \otimes (M$  $\otimes M$  $\otimes 1)$

$+\, 1 \otimes (M$  $\otimes M$  $\otimes 1).$

Definition 4.11. Let $\alpha \in Comp$.

1. Write $\alpha^{(GB,0)} = \alpha$, $\alpha^{(GB,1)} = \alpha_{GB}$, and $\alpha^{(GB,2)} = (\alpha_{GB})_{GB} = (\alpha^{(GB,1)})_{GB}$. Inductively, we define

$$\alpha^{(GB,t)} = (\alpha^{(GB,t-1)})_{GB}$$

for any $t \in \mathbb{N}$ with $t \geq 1$. Similarly, one could define $\alpha^{(R,t)}$.

2. The GB-order of α, denoted by $|\alpha|^{GB}$, is the least positive integer t with $\alpha^{(GB,t)} = (\alpha^{(GB,t-1)})_{GB}$. Note that $|\alpha|^{GB} \geq 1$.

3. Let $t = |\alpha|^{GB}$. Define the sets

$$Comp^{(GB)^t} = \{\alpha \in Comp \,:\, |\alpha|^{GB} \leq t\}$$

and

$$Comp^{(GB,n)^t} = \{\alpha \in Comp_n \,:\, |\alpha|^{GB} \leq t\}.$$

Example 4.12.

1. To find $(6,6,6,6,6,6)^{(GB}$, we calculate

Thus, $|(6,6,6,6,6,6)|^{GB} = 8$.

2. One could check that

Therefore, $|(1,4,2,2)|^{GB} = 4$. Similarly, one might verify that $|(2,4,2,4)|^{GB} = 4$.

Remark 4.13. Let $\alpha \in Comp$ and $t \in \mathbb{N}$ with $t \geq 2$.

$(6,6,6,6,6,6)^{(GB,0)}$	$(6,6,6,6,6,6)$
$(6,6,6,6,6,6)^{(GB,1)}$	$(6,6,6,3,3,3)$
$(6,6,6,6,6,6)^{(GB,2)}$	$(6,6,3,3,2,2)$
$(6,6,6,6,6,6)^{(GB,3)}$	$(6,3,3,2,2,1)$
$(6,6,6,6,6,6)^{(GB,4)}$	$(6,3,2,2,1,1)$
$(6,6,6,6,6,6)^{(GB,5)}$	$(6,2,2,1,1,1)$
$(6,6,6,6,6,6)^{(GB,6)}$	$(6,2,1,1,1,1)$
$(6,6,6,6,6,6)^{(GB,7)}$	$(6,1,1,1,1,1)$

$(1,4,2,2)^{(GB,0)}$	$(1,4,2,2)$
$(1,4,2,2)^{(GB,1)}$	$(1,2,2,1)$
$(1,4,2,2)^{(GB,2)}$	$(1,2,1,1)$
$(1,4,2,2)^{(GB,3)}$	$(1,1,1,1)$

1. Clearly, $|\alpha|^{GB} \leq |\alpha|$ for $\alpha \in Comp$ with $|\alpha| \geq 1$.
2. If $\alpha \in Comp^{(GB)^t}$, then $\alpha_{GB} \in Comp^{(GB)^{(t-1)}}$.
3. If $\alpha \in Comp^{(GB)^t}$, then $(\alpha^{(GB,t-1)})_R = \emptyset$. In particular, if $\alpha \in Comp^{(GB)^2}$, then $(\alpha^{(GB,2)})_R = \emptyset$, $\alpha^{(R,2)} = \emptyset$ and $(\alpha^{(GB,1)})_R = \emptyset$.

Consider the map

$$\Omega(\mathbf{k}) \xrightarrow{\Delta^{GB}} \Omega(\mathbf{k}) \otimes \Omega(\mathbf{k}) \tag{4.6}$$

defined $\mathbf{k}$-linearly by

$$\Delta^{GB}\xi_\alpha = \begin{cases} 1 \otimes 1 & \text{if } \alpha = \emptyset \\ \xi_{\alpha^{(GB,|\alpha|^{GB}-1)}} \otimes 1 + 1 \otimes \xi_{\alpha^{(GB,|\alpha|^{GB}-1)}} & \text{if } \alpha \neq \emptyset \end{cases}$$

We have the following proposition.

Proposition 4.14. $(\Omega(\mathbf{k}), \Delta^{GB})$ *is a nonunital* $\mathbf{k}$*-coalgebra.*

Proof. We have to show that the following diagram is satisfied.

$$\begin{tikzcd} & (\Omega(\mathbf{k}))^{\otimes 3} & \\ (\Omega(\mathbf{k}))^{\otimes 2} \arrow[ur, "\Delta^{GB}\otimes id"] & & (\Omega(\mathbf{k}))^{\otimes 2} \arrow[ul, "id\otimes\Delta^{GB}"'] \\ & \Omega(\mathbf{k}) \arrow[ul, "\Delta^{GB}"] \arrow[ur, "\Delta^{GB}"'] & \end{tikzcd} \tag{4.7}$$

$$\begin{aligned}(\Delta^{GB}\otimes id)\Delta^{GB}\xi_\alpha &= (\Delta^{GB}\otimes id)(\xi_{\alpha^{(GB,|\alpha|^{GB}-1)}}\otimes 1 + 1\otimes \xi_{\alpha^{(GB,|\alpha|^{GB}-1)}}) \\ &= \Delta^{GB}(\xi_{\alpha^{(GB,|\alpha|^{GB}-1)}})\otimes 1 + \Delta^{GB}(1)\otimes \xi_{\alpha^{(GB,|\alpha|^{GB}-1)}} \\ &= (\xi_{\alpha^{(GB,|\alpha|^{GB})}}\otimes 1 + 1\otimes \xi_{\alpha^{(GB,|\alpha|^{GB})}})\otimes 1 + 1\otimes 1\otimes \xi_{\alpha^{(GB,|\alpha|^{GB}-1)}} \\ &= \xi_{\alpha^{(GB,|\alpha|^{GB})}}\otimes 1\otimes 1 + 1\otimes \xi_{\alpha^{(GB,|\alpha|^{GB})}}\otimes 1 + 1\otimes 1\otimes \xi_{\alpha^{(GB,|\alpha|^{GB}-1)}}\end{aligned}$$

$$\begin{aligned}(id\otimes\Delta^{GB})\Delta^{GB}\xi_\alpha &= (id\otimes\Delta^{GB})(\xi_{\alpha^{(GB,|\alpha|^{GB}-1)}}\otimes 1 + 1\otimes \xi_{\alpha^{(GB,|\alpha|^{GB}-1)}}) \\ &= \xi_{\alpha^{(GB,|\alpha|^{GB}-1)}}\otimes \Delta^{GB}(1) + 1\otimes \Delta^{GB}(\xi_{\alpha^{(GB,|\alpha|^{GB}-1)}}) \\ &= \xi_{\alpha^{(GB,|\alpha|^{GB}-1)}}\otimes (1\otimes 1) + 1\otimes (\xi_{\alpha^{(GB,|\alpha|^{GB})}}\otimes 1\otimes 1 + 1\otimes \xi_{\alpha^{(GB,|\alpha|^{GB})}}) \\ &= \xi_{\alpha^{(GB,|\alpha|^{GB})}}\otimes 1\otimes 1 + 1\otimes \xi_{\alpha^{(GB,|\alpha|^{GB})}}\otimes 1 + 1\otimes 1\otimes \xi_{\alpha^{(GB,|\alpha|^{GB})}}\end{aligned}$$

For any $\alpha \in Comp$, $\alpha^{(GB,|\alpha|^{GB})} = \alpha^{(GB,|\alpha|^{GB}-1)}$ (by the definition of $|\alpha|^{GB}$). It follows that $(\Delta^{GB}\otimes id)\Delta^{GB} = (id\otimes\Delta^{GB})\Delta^{GB}$. Thus, the diagram (4.7) is commutative, and hence $(\Omega(\mathbf{k}), \Delta^{GB})$ is a nonunital $\mathbf{k}$-coalgebra. □

Definition 4.15. Fix $t \in \mathbb{N}$ with $t \geq 1$. Let $\Omega^{(GB,n)^t}(\mathbf{k})$ be the free $\mathbf{k}$-module with the basis $\{\xi_\alpha\}_{\alpha\in Comp^{(GB,n)^t}}$. Let $\Omega^{(GB)^t}(\mathbf{k}) = \bigoplus_{n\geq 0}\Omega^{(GB,n)^t}(\mathbf{k})$. Then the set $\{\xi_\alpha\}_{\alpha\in Comp^{(GB)^t}}$ forms a basis for $\Omega^{(GB)^t}(\mathbf{k})$ over $\mathbf{k}$, and $\Omega^{(GB)^t}(\mathbf{k})$ is called the (GB,t)-*Bayer Noise module* over $\mathbf{k}$.

Now consider the map

$$\Omega^{(GB)^t}(\mathbf{k}) \xrightarrow{\Delta^{(GB)^t}} \Omega^{(GB)^t}(\mathbf{k}) \otimes \Omega^{(GB)^t}(\mathbf{k}) \tag{4.8}$$

defined $\mathbf{k}$-linearly by

$$\Delta^{(GB)^t}\xi_\alpha = \begin{cases} 1 \otimes 1 & \text{if } \alpha = \emptyset \\ \xi_{\alpha^{(GB,t-1)}} \otimes 1 + 1 \otimes \xi_{\alpha^{(GB,t-1)}} & \text{if } \alpha \neq \emptyset. \end{cases}$$

We have the following proposition.

Proposition 4.16. $(\Omega^{(GB)^t}(\mathbf{k}), \Delta^{(GB)^t})$ *is a nonunital* $\mathbf{k}$*-coalgebra.*

Proof. The proof is very similar to the proof of Proposition 4.14. □

It is well known that the nonunital $\mathbf{k}$-coalgebras $(\Omega(\mathbf{k}), \Delta^{GB})$ can be extended for a unital $\mathbf{k}$-coalgebra $(\overline{\Omega(\mathbf{k})}, \overline{\Delta^{GB}}, \overline{\epsilon^{GB}})$, where $\overline{\Omega(\mathbf{k})} = \Omega(\mathbf{k}) \oplus \mathbf{k}$, and $\overline{\epsilon^{GB}} : \overline{\Omega(\mathbf{k})} = \Omega(\mathbf{k}) \oplus \mathbf{k} \to \mathbf{k}$ is the projection map, and $\overline{\Delta^{GB}}$ is the map

$$\overline{\Omega(\mathbf{k})} \xrightarrow{\overline{\Delta^{GB}}} \overline{\Omega(\mathbf{k})} \otimes \overline{\Omega(\mathbf{k})} \tag{4.9}$$

defined by

$$\overline{\Delta^{GB}}(f + a) = \Delta^{GB}(f) + f \otimes 1 + 1 \otimes f + a(1 \otimes 1)$$

for any $f \in \overline{\Omega(\mathbf{k})}$ and $a \in \mathbf{k}$. Similarly, the nonunital $(\Omega^{(GB)^t}(\mathbf{k}), \Delta^{(GB)^t})$ can be extended for a unital $\mathbf{k}$-coalgebra $(\overline{\Omega^{(GB)^t}(\mathbf{k})}, \overline{\Delta^{(GB)^t}}, \overline{\epsilon^{(GB)^t}})$, where $\overline{\Omega^{(GB)^t}(\mathbf{k})} = \Omega^{(GB)^t}(\mathbf{k}) \oplus \mathbf{k}$, and $\overline{\epsilon^{(GB)^t}} : \overline{\Omega^{(GB)^t}(\mathbf{k})} = \Omega^{(GB)^t}(\mathbf{k}) \oplus \mathbf{k} \to \mathbf{k}$ is the projection map, and $\overline{\Delta^{(GB)^t}}$ is the map

$$\overline{\Omega^{(GB)^t}(\mathbf{k})} \xrightarrow{\overline{\Delta^{(GB)^t}}} \overline{\Omega^{(GB)^t}(\mathbf{k})} \otimes \overline{\Omega^{(GB)^t}(\mathbf{k})} \tag{4.10}$$

defined by

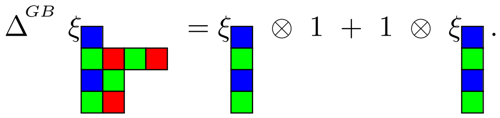
$$\overline{\Delta^{(GB)^t}}(f + a) = \Delta^{(GB)^t}(f) + f \otimes 1 + 1 \otimes f + a(1 \otimes 1)$$

for any $f \in \overline{\Omega^{(GB)^t}(\mathbf{k})}$ and $a \in \mathbf{k}$. Consequently, we have the following.

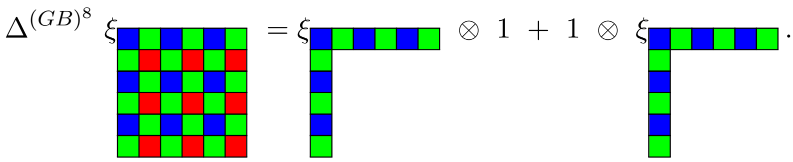
Proposition 4.17. $(\overline{\Omega(\mathbf{k})}, \overline{\Delta^{GB}}, \overline{\epsilon^{GB}})$ *and* $(\overline{\Omega^{(GB)^t}(\mathbf{k})}, \overline{\Delta^{(GB)^t}}, \overline{\epsilon^{(GB)^t}})$ *are (unital)* $\mathbf{k}$*-coalgebras.*

Example 4.18.

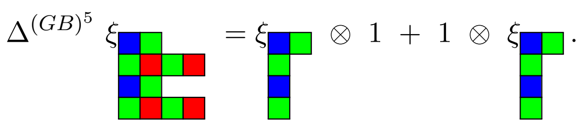
1. A direct Calculation for $\Delta^{GB}\, \xi_{(1,4,2,2)}$ gives the following:



2. Calculating $\Delta^{(GB)^8}\, \xi_{(6,6,6,6,6,6)}$ gives the following:



3. One could calculate $\Delta^{(GB)^5}\, \xi_{(2,4,2,4)}$ as follows:



Fix a commutative ring $\mathbf{k}$. We end the paper with a few things as suggestions to the reader. The interested reader could think of the following:

- Establishing other bases for the Quasi-Bayer Noise module over $\mathbf{k}$.
- Defining noise quasisymmetric functions using other filters.
- Defining quasisymmetric functions based on the denoising concept.
- Studying Stanley's P-partition theory

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