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Celebrating Professor Themba A. Dube (A TAD Celebration I)

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Commemorating Themba Dube on his 65th birthday

Abstract. This is the first in a series of survey papers featuring the mathematical contributions of Themba Dube to pointfree topology and ordered algebraic structures. We cover Dube's distinguished career and benefactions to the discipline with the early beginnings in nearness frames. We envelope the essential aspects of Dube's work in structured frames. The paper radars across the initial themes of nearness, metrization, and uniform structures that Dube conceives and presents in his independent and joint published papers. Pertinent subcategories of these structured frames are discussed. We also feature Dube's imprints on certain categorical aspects of his work on $\beta L, \lambda L, \upsilon L$ and ßL.

1 Introduction

It is a great honour and a privilege for me to narrate the academic story of Themba Dube, to author this first survey article and to write about the

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mathematical work of my colleague, Professor Themba Dube.

My first encounter with Dube was with his pronounced voice in 1996. He called me telephonically at the former University of Durban-Westville (UDW) (now known as the University of KwaZulu-Natal post the merger of some of the public Higher Education Institutions in South Africa) where I was a member of staff of the Department of Mathematics and Applied Mathematics studying towards a Master's degree under the lead supervision and mentorship of Professor Dharmanand (Dharms) Baboolal and co-supervision by Professor Ramesh Ori. I had applied for my first academic position with tenure at the University of Zululand (UniZul). The nature of Dube's call was in his capacity as Chair of Department and in his typical demeanour, that I would soon come to know of being rather firm and jovial, to enquire if I really wanted to join their department and faculty in somewhat remote rural KwaZulu-Natal being a Dubanite, within the segregated suburb of Chatsworth, hailing from one of the major urban cities of South Africa at the time. I, of course, certainly obliged in confirmation. He then propositioned me over up north to the KwaDlangezwa campus of UniZul and arranged an interview with their Department and the university Human Resources. There were also other candidates that were interviewed for the position. I then had the nervous pleasure of first physically meeting Dube at the scheduled interview for the lectureship at UniZul in the summer of 1996. I was eventually appointed and spent the next five years at UniZul intersecting with Dube every now and again whilst he occupied many seats of management at the university. Our careers overlapped and eventually converged to us being together at the University of South Africa (UNISA) in April 2009. I was tenured at the University of the Witwatersrand in Johannesburg and it was the second telephone call of a similar nature that I received from Dube in 2009. This time around it was an invitation to me to consider a position at UNISA and to join the Department of Mathematical Sciences in Pretoria at the behest of the Head of Department, Professor John Hartney, who was actioning on the decisions of the departmental recruitment and shortlisting committee. Since the nexus of 1996 it has been a treasured, memorable, exciting and an unforgettable journey in the academic company alongside Themba Dube. I am indebted to Themba for many things especially for the vibrant career in academia that he was instrumental in shaping for me unintentionally (and others like Martin Mugochi, Oghenetega Ighedo and Mack Matlabyana) as well as for being our pointfree talisman, academic big brother and mentor that we have had the privilege and fortune to engage with and enjoy at home in South Africa. We have forged and cultivated many ideas together that culminated in fruitful joint publications [36, 42, 47, 49, 53, 56–60, 66, 81, 106], ignited a lively and energetic working Topology Research Group at the university, been road, rail and air travel companions in collaboration across Africa, America and Europe, and by good fortune have been involved in many other research activities together at local and international conferences, workshops and writing retreats engaging with our Master's and Doctoral students. As luck would have it in 2015, the Vice-Chancellor, Professor Mandla Makhanya paired us together for the period 2016 - 2017 to serve as Chair and Deputy Chair of the highly challenging Tender Committee of the university. There were also occasions in which the tag-team of Dube and Naidoo were requested by the leadership of the college to deliberate and advocate on matters. It is therefore indeed befitting and a prestige for me to begin in this survey paper to commemorate and revere the illustrious scholarly work and mathematical dexterity of my dear friend, Professor Themba Dube. In this article we provide a comprehensive treatment of Dube's contributions in structured frames as the beginning theme in the launch of the series celebrating Dube's mathematical works.

We begin in the next Section §2 by providing a background and academic biography of Dube that captures his early formative years and his academic career. This is an additional supplement to the delightful interview that I engaged with Dube in [150](2023) in which he kindly shared some of his memoirs. Dube's scholarly work until his retirement spans three decades over the period 1992 - 2022. Briefly highlighting Dube's contributions does not do justice to his immense influence, participation and impactful involvement in the discipline, especially his living legacy within the African continent and the commonwealth of ordered algebraic structures. It is therefore the intention of the author, with this first survey paper, to begin celebrating the scholarship of Themba Dube with the platform created by the international conference TACT2022 in honour of his 65^{th} birthday. The article narrates Dube's early life as a mathematics virtuoso in Meadowlands, Soweto up until his retirement in 2022. We begin with focussing our attention on the various classical and localic thespians that are cast and scripted to take the stage in Dube's academic scholarship that is dramatised and featured in his published works. We give prominence to the main academic contributions of Dube that unfolded at his very beginning with the encounter with pointfree thinking and the theory of frames. We explain the forthcoming sections below. We fulfil the credits at the end of this paper in the References section by concluding with a comprehensive bibliography of Dube's research publications during this 30-year period.

In Section §3 we give the necessary preliminaries that are required for a self-contained exposition of the categories **Frm**, σ-**Frm**, **NFrm**, **UFrm** and **Near**. We discuss the Stone-Čech compactfication βL, the regular Lindelöf coreflection λL, and the Hewitt realcompactification υL, of a completely regular frame L. We highlight the dual equivalence between spatial frames and sober spaces and the adjoint relation between **Top** and **Frm** that was initially given by Dowker and Papert in [127](1966). We focus on the concept of a nucleus $j : L \longrightarrow L$ on a frame L and the resulting frame $\mathrm{Fix}(j)$ as well as the frame $\mathscr{D}L$ of all downsets of L. We also narrate on the strict extension of a frame and its role in the construction of the Cauchy completion of a nearness frame. Section §4 hovers over the foundations and findings in Dube's doctoral thesis and covers **NFrm**, the category of nearness frames, **Near** the category of nearness spaces and its full subcategory **BNear** of those nearness spaces that arise from B-spaces as introduced by Dube in [6](1992). The extended or structured version of the open set functor and the spectrum functor, $\mathfrak{O}$: **BNear** $\longrightarrow$ **NFrm** and Σ : **NFrm** $\longrightarrow$ **BNear** are shown to induce an equivalence between spatial nearness frames and sober B-spaces. We give a thorough account of the adjoint relation between **BNear** and **NFrm** that Dube ushers in [6]. A comprehensive treatment of this adjunction and bringing it into the limelight is warranted for the significant role that it plays in enhancing the discourse of nearness frames. Moreover, this adjunction is the genesis of Dube's work in nearness frames and it is historically the *second structured version* of the adjoint relation between **Top** and **Frm** that we succinctly present in Section §3. The next part of Section §4 deals with Dube's independent construction of the completion of a nearness frame via generators and defining relations. We give due recognition to the 1992 Dube - construction of the completion of a nearness frame *via a precongruence* in detail which has been silent and perhaps overlooked and gone unnoticed in the literature on nearness frames since its provisioning in 1992 by Dube in [6]. We bring this alternate construction to the fore to share the limelight with the popular version of the Banaschewski-Pultr-construction *via a prenucleus* that was published some time later in [118](1996). The third part of §4 catapults Dube's construction of the binary coproduct of nearness frames which Dube independently realised to scenes of autonomous jubilation at the University of Cambridge during his doctoral studies circa 1992. We taper Section §4 with a brief exchange on the various concepts, terminologies and subcategories of nearness frames that Dube illuminates in [6] that were required in his publications and which is needed in the remaining sections of this survey paper.

We then consider the period 1993 - 1997 post the award of Dube's doctorate from UDW. Section §5 embraces the notion of *uniform complete regularity* appearing in [11](1996) and the binary relation $\lhd\!\lhd$ on a frame that Dube developed and termed *completely regular* for a nearness frame in [6]. Featured in Dube's postdoctoral work, in Section §5, is the concept of *uniform normality* and *total boundedness* of a nearness, the uniform coreflection of a nearness frame, and strong nearness frames that Dube presented in 1994 at the University of Cape Town (UCT) during the SoCAT94 symposium that celebrated the 60^{th} birthday of Professor Guillaume Brümmer [17](1999). Apart from the contributions in nearness and uniform frames, Dube has also made significant strides in **MFrm**, the category of metric frames during the postdoctoral period in his career. We capture the notion of *separability* that Dube introduces in his very first research publication [7](1994). The localic version of the famous Urysohn's Metrization Theorem (UMT) is established independently by Dube in [7] that we highlight in Section §5. The marvel behind this localic result is Dube's injection of the notion of pointfree separability in **Loc** and the creativity of the concept in realising the UMT for locales. Pursuant to his invention of separable locales in [7], Dube continued with concomitant results in **MFrm** in [10](1996) which we recount and conclude with in Section §5. In Section §6 we orbit around Dube's further inputs into the categories **NFrm** and **UFrm**. We circumnavigate certain properties introduced by Dube on the objects in these two categories and track how these properties relate to their completions. Section §7 focusses on the notion of commutativity of functors that Dube investigates on the reflectors and coreflectors that permeate his life's work. The last section, Section §8 is an invaluable inclusion of congratulatory homages to Dube that aggregates Dube's mathematical influence in South Africa and on the African continent, the interview [150], his mathematical life, research contributions and the TACT2022 International Conference celebration, all of which this special issue of Categories and General Algebraic Structures with Applications is dedicated.

2 A TAD Journey

We commence by providing a brief account of the early life of Themba Dube and his trajectory in his academic career. The next two paragraphs is an introductory summary of the preamble that emanated from the interview in [150](2023). In the remaining part of this section we chronicle Dube's academic biography, some of which are briefly highlighted in the narratives that are provided in [150].

Themba Andrew Dube was born on 20 June 1957. He is the only child of Mrs Nolwandle Dube (a trained nurse) and Mr Masimini Dube. Andrew was the Catholic name given to him as an included middle. Themba (meaning Trust, Hope and Faith in the indigenous languages of South Africa) was raised and schooled in Meadowlands in Soweto which is a well known suburb within the City of Johannesburg in South Africa. Dube found himself growing up in a township that is associated with immense talent and brave heroes of human rights activism. Developing as a child in the *So*uth *We*stern *To*wnship, the youthful Dube fell in love with mathematics at a very young age. The affinity for the subject continued at secondary school level whilst he attended the Meadowlands High School in Soweto. The township was abuzz and would become familiar with Themba Dube as the mathematics whizz-kid of Meadowlands. Dube also learned the tactical game of chess, and its art of strategy and tussle, whilst at high school. Aside from the academe, he is very proud of his athletic prowess in one of the 100 metres sporting sprint events at Meadowlands High (he was the third fastest in that event). Dube performed exceptionally well in his final year at high school and his matriculation results immediately summed up his academic talent. He was interviewed by *The World* (a daily newspaper of Johannesburg "intended for an audience of black middle-class elite" - which was eventually banned by the apartheid government) on his secondary schooling accolades and for his achievements in matric. *The World* had acquired Dube's matriculation results prior to it being released in the public domain and the newspaper was particularly interested in an exposé of the beautiful mind of the mathematical prodigy from Meadowlands in Soweto.

Dube acknowledges Mrs Kona, his mathematics teacher during his early high schooling, up to grade 10, for her influence and dedication. The greatest mathematical influence during his high schooling is attributed to his teacher, Mr Lesole Gadinabokao, who had a Master's degree in Mathematics. Mr Gadinabokao continued with his studies and eclipsed secondary school teaching and transcended into tertiary academia to become Professor of Physics at the then University of the Bophuthatswana (now known as the University of the North-West, abbreviated NWU). Dube pursued higher degree studies in three major subjects at undergraduate level, namely, Mathematics, Applied Mathematics and Statistics and obtained a BSc from the University of Fort Hare. He further went on to obtain a BSc(Hons) (cum-laude) and an MSc (cum-laude) from his alma mater in the Eastern Cape province of South Africa. During the period 1980 - 1982 he lectured at the University of Fort Hare and thereafter continued as Senior Lecturer until the end of December 1985.

His next stint in tertiary tuition and research was at the Department of Mathematics and Applied Mathematics at the University of Venda (UniVen) (1986 - 1988). He thereafter joined the faculty in the Department of Mathematical Sciences at the main KwaDlangezwa campus of the University of Zululand as Senior Lecturer. Whilst at UniZul he registered for doctoral studies at the University of Durban-Westville in KwaZulu Natal. He was supervised, in lead, by Professor Dharmanand Baboolal (who introduced him to the art of pointfree thinking) and co-supervised by Professor Ramesh Ori. During his doctoral studies he was hosted during the Lent Term in 1992 by the prominent Dr Peter Johnston in the Department of Mathematics and Mathematical Statistics of the University of Cambridge in the United Kingdom. His thesis [6] entitled *Structures in Frames* presented a general foundation for the study of the concept of nearness in pointfree topology and his doctorate was awarded by UDW in December 1992. At UniZul he was promoted to Associate Professor in 1995 and Full Professor in 1996. A career in management at UniZul then succeeded him where he served as Vice-Rector (Deputy Vice-Chancellor) for Academic Affairs and Research (1997 - 2003). He also acted as Vice Chancellor during his tenure at UniZul. Dube thereafter joined the Department of Mathematical Sciences at the University of South Africa in 2003 as Full Professor tenured with a focus primarily on his research and the training of master's and doctoral students.

He served as Vice-President of the South African Mathematical Society (SAMS) (2008 - 2009) and Associate Editor of Quaestiones Mathematicae (QM), the journal of the SAMS, during the period 2010 - 2015. Dube was the Editor-in-Chief of QM for the period 2016 - 2022 and was the first person of colour to be such in the history of the SAMS. He is also an editorial board member of the journal Categories and General Algebraic Structures with Applications (CGASA) and an Associate Editor for the journal Afrika Mathematika.

Numerous research accolades have been awarded to Dube. Early in his studies, in 1982, he was awarded the University of Fort Hare Council Research Award for a Master's dissertation. In 2010 he received the second highest internal UNISA prize that of the Principal's Prize for Research Excellence. In 2013, Dube was awarded the most prestigious prize of the SAMS that for Research Distinction. The Chancellor's Prize for Excellence in Research is *the* most prestigious prize for research at UNISA. Dube received the latter in 2013 and repeated the fête in 2017. Also in 2013, Dube was inaugurated as a member of the Academy of Science of South Africa (ASSAf).

Dube has widely published over a 100 research articles in peer-reviewed journals spanning pointfree topology, rings of continuous functions and ordered algebraic structures. He continues to be active in research post his formal retirement. He has supervised to graduation seven doctoral students: Martin Mandirevesa Mugochi (graduated in 2010), Mack Zackaria Matlabyana (graduated in 2012), Oghenetega Ighedo (graduated in 2014), Jissy Nsonde-Nsayi (graduated in 2016), Mohammad Zarghani (Hakim Sabzevari University, Iran, co-supervised with Dr A.A. Estaji and Dr A.K. Feizabadi, graduated in 2017), Batsile Tlharesakgosi (co-supervised with Dr O. Ighedo) and Dorca Nyamusi Stephen both graduated in 2021. He has also supervised to completion seven masters students: Mr Fanyana Ncongwane (graduated in 2016); Mr Batsile Tlharesakgosi co-supervised with Dr O. Ighedo, graduated in 2017; Mr Shegu Mayila and Ms Elizabeth Mrema (both students at the University of Dar es Salaam, Tanzania, co-supervised with Dr K. Mpimbo) that graduated in 2018. Ms Lindiwe Maria Sithole, Ms Annette Flavie Ngo Babem and Mr Siphamandla Blose graduated with their MSc in 2019.

Dube truly epitomises the meaning in his indigenous namesake - *Themba*. From my own many experiences with Dube, his trust and belief in you is solid and grounded, he has given hope to many a graduate master's and doctoral student and colleague. He certainly is a loyal and steadfast parent to his only daughter Linda. He is a loyal friend and colleague and is categorically likewise to the pointfree topology fraternity. Professor Dube formally retired at the end of December 2022 as a Research Professor at UNISA, as a B-3 National Research Foundation (NRF) rated researcher, an astute pure mathematician of the African diaspora of distinguished note, a dreamer, an avid chess-player and an engaged community scholar with profound interest in reggae music. He continues to supervise Master's and Doctoral students in the Department of Mathematical Sciences at UNISA that is supported and kindly extenuated by the university.

3 Preliminaries

In this section we provide the pertinent basic background required for the purpose of this article and for it to be a self-contained exposition of Dube's mathematical works. We begin with a brief discussion on frames, σ-frames, locales and the entities required in the sequel. Thereafter we look at structures on frames, particularly nearness and uniformity. For a detailed background into frames, locales and structures on frames we recommend the text by Picado and Pultr [154](2012). For aspects concerning the countable analogue of frames (σ-frames) the papers by Banaschewski [110](1993), and Banaschewski and Gilmour [113](1996) are suggested readings. For the categorical abstractions that are alluded to, we refer the reader to the text by Borceux [125](1990).

3.1 Frames We recall that a *frame* is a bounded lattice $(L, \leq)$ with bottom element 0_L and top element 1_L, in which every subset S of L has a join (supremum) $\bigvee S$ (that is to say that L is complete) such that the distributive law

$$x \wedge \bigvee S = \bigvee_{s \in S} (x \wedge s) \qquad (\dagger)$$

holds for each $x \in L$ and any $S \subseteq L$. For frames L and M, a *frame homomorphism* is a map $h : L \to M$ which preserves finite meets and all joins. The resulting category of frames and their homomorphisms will be denoted by **Frm**. For a topological space X, the collection of open sets of X is denoted by $\mathfrak{O}X$ which forms a frame (the order provided by $\subseteq$). A frame L is said to be *spatial* if there is some topological space X for which $L \simeq \mathfrak{O}X$. **Loc** is the opposite category of **Frm**.

3.1.1 The adjunction between Top and Frm

A topological space X is represented in the category **Frm** by its frame of open sets $\mathfrak{O}X$. For any continuous function $h : X \longrightarrow Y$ in **Top** we have the frame homomorphism $\mathfrak{O}h : \mathfrak{O}Y \longrightarrow \mathfrak{O}X$ given by $\mathfrak{O}h(U) = h^{-1}(U)$ for $U \in \mathfrak{O}Y$. This correspondence of objects and morphisms between topological spaces and frames results in the contravariant (*open set*) functor $\mathfrak{O}$: **Top** $\longrightarrow$ **Frm**. On the flip side, the passage from frames to topological spaces is routed via the *spectrum* of a frame. Associated with any frame L, we have its *spectrum* ΣL whose elements are the frame homomorphisms $\xi : L \longrightarrow \mathbb{2}$ of L into the two-element frame $\mathbb{2} = \{0, 1\}$. If for each $x \in L$, we let

$$\Sigma_x = \{\xi : L \longrightarrow \mathbb{2} \ : \ \xi(x) = 1\},$$

we then have (see, for instance, in Pultr and Sichler [158](2014))

$$\Sigma_{0_L} = \emptyset, \Sigma_{1_L} = \Sigma L, \Sigma_{x \wedge y} = \Sigma_x \cap \Sigma_y \text{ and } \Sigma_{\bigvee x_i} = \bigcup \Sigma_{x_i},$$

so that $\{\Sigma_x : x \in L\}$ is a topology (called the *spectral topology*) on ΣL. For any frame homomorphism $f : L \longrightarrow M$ we have the continuous function $\Sigma f : \Sigma M \longrightarrow \Sigma L$ given by $\Sigma f(\xi) = \xi \circ f$ for each $\xi \in \Sigma M$. The resulting correspondence is the contravariant (*spectrum*) functor given by Σ : **Frm** $\longrightarrow$ **Top**. The actions of $\mathfrak{O}$ and Σ on objects and morphisms are depicted below:

$$\begin{array}{lccc|ccc}
 & \mathfrak{O} : \mathbf{Top} \longrightarrow \mathbf{Frm} & & & \Sigma : \mathbf{Frm} \longrightarrow \mathbf{Top} & & \\
\text{Objects} & X \dashrightarrow \mathfrak{O}X & & & L \dashrightarrow \Sigma L & & \\
\text{Morphisms} & h : X \to Y & \dashrightarrow & \mathfrak{O}h : \mathfrak{O}Y \to \mathfrak{O}X, \quad U \mapsto h^{-1}(U) & f : L \to M & \dashrightarrow & \Sigma f : \Sigma M \to \Sigma L, \quad \xi \mapsto \xi \circ f
\end{array}$$

The functors $\mathfrak{O}$ and Σ are adjoint on the right with natural transformations η and ε defined as follows. $\varepsilon : 1_{\mathbf{Top}} \longrightarrow \Sigma\mathfrak{O}$ where for any topological space X, $\varepsilon_X : X \longrightarrow \Sigma\mathfrak{O}X$ is a continuous map taking any $x \in X$ to $\tilde{x} = \varepsilon_X(x) : \mathfrak{O}X \longrightarrow \mathbb{2}$ where $\tilde{x}(U) = 1$ iff $x \in U$. On the other hand, $\eta : 1_{\mathbf{Frm}} \longrightarrow \mathfrak{O}\Sigma$ where for any frame L, $\eta_L : L \longrightarrow \mathfrak{O}\Sigma L$ is a frame homomorphism defined by $\eta_L(x) = \Sigma_x$. The naturality of η and ε are expressed in the commutative squares below.

$$\begin{array}{ccccc}
L & \xrightarrow{\eta_L} & \mathfrak{O}\Sigma L & & \Sigma_x \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathfrak{O}\Sigma f} & & \downarrow \\
M & \xrightarrow[\eta_M]{} & \mathfrak{O}\Sigma M & & (\Sigma f)^{-1}(\Sigma_x) = \Sigma_{f(x)}
\end{array}
\qquad
\begin{array}{ccccc}
X & \xrightarrow{\varepsilon_X} & \Sigma\mathfrak{O}X & & \tilde{x} \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Sigma\mathfrak{O}h} & & \downarrow \\
Y & \xrightarrow[\varepsilon_Y]{} & \Sigma\mathfrak{O}Y & & \tilde{x} \circ \mathfrak{O}h = \widetilde{h(x)}
\end{array}$$

The adjointness of $\mathfrak{O}$ and Σ is given by the following commutative triangles.

$$\begin{array}{ccc}
\mathfrak{O}X & & \\
{\scriptstyle \eta_{\mathfrak{O}X}}\downarrow & \searrow^{\mathrm{id}_{\mathfrak{O}X}} & \\
\mathfrak{O}\Sigma\mathfrak{O}X & \xrightarrow[\mathfrak{O}\varepsilon_X]{} & \mathfrak{O}X
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Sigma L & \xrightarrow{\varepsilon_{\Sigma L}} & \Sigma\mathfrak{O}\Sigma L \\
 & \searrow_{\mathrm{id}_{\Sigma L}} & \downarrow{\scriptstyle \Sigma\eta_L} \\
 & & \Sigma L
\end{array}$$

L is a spatial frame provided that η_L is an isomorphism. The topological space X is *sober* if ε_X is a homeomorphism. Furthermore, Σ and $\mathfrak{O}$ induce a dual equivalence between spatial frames and sober spaces with Σ being full and faithful. The adjoint relation between **Top** and **Frm** is originally given in the Appendix of the paper by Dowker and Papert [127](1966) and is required in Section §4.2.

3.1.2 The right adjoint of a frame homomorphism

Each frame homomorphism $h : L \longrightarrow M$ has a *right* (*Galois*) *adjoint*, which is a $\wedge$-semilattice homomorphism $h_* : M \longrightarrow L$ that is explicitly given by the formula $h_*(y) = \bigvee\{x \in L : h(x) \leq y\}$. We call h a *dense* frame homomorphism if $h(x) = 0_M$ implies that $x = 0_L$ (equivalently, $h_*(0_M) = 0_L$) and h is *codense* if it maps only the top to the top. Any dense homomorphism between regular frames is monic, and any codense homomorphism between regular frames is one-one. Regular frames are described in the next section §3.1.3. An onto frame homomorphism will be called a *quotient map*. We also recall that h is a quotient map iff $hh_* = \mathrm{id}_M$ iff h_* is one-to-one, and dually, h is one-to-one iff $h_*h = \mathrm{id}_L$ iff h_* is onto.

3.1.3 Separation properties

We will use the notation $\subseteq_{<\omega}$ to denote a *finite* subset and $\subseteq_\omega$ for a *countable* subset. A subset $B \subseteq L$ is called a *base* (or *join-base*) for the frame L if for each $x \in L$, we have that $x = \bigvee\{t \in T : t \leq x\}$ for some $T \subseteq B$. Each element x in a frame L has a *pseudocomplement* denoted by x^* which is the largest element in L which misses x. The pseudocomplement, its properties and machinery play a major rôle in pointfree topology. It features in many definitions that are required and given in the remarks below.

Remark 3.1. Let x, y, c, d and p be elements of a frame L.

(1) We write $x \prec y$ (and say, *x is rather below y*) to mean that there is $s \in L$ such that $x \wedge s = 0$ and $s \vee y = 1$ (equivalently, $x^* \vee y = 1$). L is a *regular* frame if each element of L is the join of the elements rather below it. **RegFrm** is the subcategory of regular frames and frame homomorphisms.

(2) We also have the *completely below* relation $\prec\!\!\prec$ where $x \prec\!\!\prec y$ means that there is a *scale* $\{c_\alpha : \alpha \in \mathbb{Q} \cap [0,1]\}$ in L with $c_0 = x, c_1 = y$ and $c_\alpha \prec c_\beta$ whenever $\alpha < \beta$. L is called a *completely regular* frame if for each $x \in L$, $x = \bigvee\{y \in L : y \prec\!\!\prec x\}$. **CRegFrm** is the subcategory of completely regular frames.

(3) L is a *normal* frame if whenever $x \vee y = 1$ there is $s, t \in L$ such that $s \prec x$, $t \prec y$ and $s \wedge t = 0$.

(4) c is called a *compact* (respectively, *Lindelöf*) element if whenever $c \leq \bigvee X$ for $X \subseteq L$ we have that $c \leq \bigvee Y$ for some $Y \subseteq_{<\omega} X$ (respectively, $Y \subseteq_\omega X$). The collection of compact elements of a frame L is denoted by $\mathfrak{k}L$ and the Lindelöf ones is given by σL. The frame L is a *compact* frame if $1_L \in \mathfrak{k}L$ and it is a *Lindelöf* frame if $1_L \in \sigma L$.

(5) x is a *complemented* element if $x \vee x^* = 1$. The set of all complemented elements of the frame L is denoted by $\mathfrak{c}L$. L is a *zero-dimensional frame* if $\mathfrak{c}L$ is a base for L.

(6) If $x = x^{**}$, then x is called a *regular element* of L. The collection of all regular elements of L is a Boolean frame denoted by ßL which is regular. Meets in ßL are calculated as in L whilst for any $S \subseteq$ ßL, the join of S in ßL is given by $\bigsqcup S = (\bigvee S)^{**}$. ßL is called the *Booleanization* of L. The map $ß_L : L \longrightarrow ßL$ given by $ß_L(x) = x^{**}$ is a dense onto frame homomorphism. **ßFrm** is the category of Boolean frames. Various aspects of **ßFrm** may be found in Banaschewski [112](1996).

(7) p is called a *point* of L if $p \neq 1$ and whenever $x \wedge y \leq p$ we have either $x \leq p$ or $y \leq p$. The points of a frame are the *prime* or *meet-irreducible* elements. $\mathrm{Pt}(L)$ is the collection of all points of L.

(8) x is a *small* element if x is *continuously below* or *well below* the top, that is $x \ll 1_L$. The continuously below relation $\ll$ on L is defined by $c \ll d$ iff $S \subseteq L$ and $d \leq \bigvee S$ implies $c \leq \bigvee T$ for some $T \subseteq_{<\omega} S$. We will denote the collection of small elements of L by $L_\ll$. L is a *continuous frame* if for each $x \in L$, $x = \bigvee\{y \in L : y \ll x\}$.

3.1.4 The frames $\mathrm{Fix}(j)$ and $\mathscr{D}L$

The concept of a *nucleus* on a frame L is defined by Simmons in [161](1977) and again (for an *idiom*[1], which a frame is) in [162, Definition 1.2.](1989) as a map $j : L \longrightarrow L$ that is inflationary ($x \leq j(x)$), monotone ($x \leq y \Rightarrow j(x) \leq j(y)$) and idempotent ($j^2(x) = j(j(x)) = j(x)$) (so that j is a closure operator) on L which preserves finitary meet ($j(x \wedge y) = j(x) \wedge j(y)$) for each $x, y \in L$. For a quotient $f : L \longrightarrow M$ between frames L and M defining $j : L \longrightarrow L$ by $j(x) = (f_* \circ f)(x)$ for each $x \in L$ we obtain a 1-1 correspondence between the quotients and nuclei of the frame L. For a nucleus j on a frame L we define $\mathrm{Fix}(j) = \{x \in L : j(x) = x\}$.

Lemma 3.2. $\mathrm{Fix}(j)$ *is a frame and* $j : L \longrightarrow \mathrm{Fix}(j)$ *where* $x \longmapsto j(x)$ *for each* $x \in L$ *is a frame homomorphism whose right adjoint is the inclusion* $\mathrm{Fix}(j) \longrightarrow L$.

For any $S \subseteq \mathrm{Fix}(j)$, $\bigvee_{\mathrm{Fix}(j)} S = j(\bigvee_{L} S)$ whilst finite meets are the same as those in L. Banaschewski [112](1988) introduced the idea of a *prenucleus* on a frame. A map $k_0 : L \longrightarrow L$ on a frame L is called a *prenucleus* if k_0 is inflationary and monotone such that for each $x, y \in L$ we have $k_0(x) \wedge y \leq k_0(x \wedge y)$. Then $\mathrm{Fix}(k_0) = \{x \in L : k_0(x) = x\}$ is a closure system with associated closure operator given by $k : L \longrightarrow L$ where $k(x) = \bigwedge\{y \in \mathrm{Fix}(k_0) : x \leq y\}$.

Lemma 3.3 (Lemma 1 [112])**.** *The closure operator* k *is a nucleus such that the frame homomorphism* $k : L \longrightarrow L$ *is universal among frames* $h : L \longrightarrow M$ *for which* $h(x) = h(k_0(x))$ *for all* $x \in L$.

Simmons [162] also defines a prenucleus on a frame as an inflationary, monotone map that preserves finite meets. Banaschewski's notion is much more general and we will retain the definition of a prenucleus as given by Banaschewski [112] in the remaining parts of the paper.

For any partially ordered set L, $U \subseteq L$ is called a *downset* if $x \in U$ implies that $\downarrow x = \{y \in L : y \leq x\} \subseteq U$. We denote the set of all downsets

[1] An *idiom* is a complete lattice L which is both modular (i.e. for each $x, y, z \in L$ we have $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee (x \wedge z))$) and upper continuous (i.e. for each $a \in L$ we have the distributive law $a \wedge \bigvee X = \bigvee\{a \wedge x : x \in X\}$).

of L by $\mathscr{D}L$. If L is a bounded meet semi-lattice then $\mathscr{D}L$ is a frame partially ordered by set inclusion with $U \wedge V = U \cap V$ and $\bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i$ for any $\{U_i\}_{i \in I} \subseteq \mathscr{D}L$. The top element is $1_{\mathscr{D}L} = \downarrow 1_L = L$ and the bottom is $0_{\mathscr{D}L} = \downarrow 0_L = \{0\}$. If L is a frame then the join map $\bigvee : \mathscr{D}L \longrightarrow L$ is a frame homomorphism with right adjoint given by $\downarrow : L \longrightarrow \mathscr{D}L$ where $x \longmapsto \downarrow x$ for each $x \in L$. If $U \in \mathscr{D}L$, then $U = \bigvee\{\downarrow x : x \in U\}$. Consequently, $B = \{\downarrow x : x \in L\}$ is a base for the frame $\mathscr{D}L$. For each $x, y \in L$ we have that $\downarrow x \cap \downarrow y = \downarrow (x \wedge y)$ so that B is a base for $\mathscr{D}L$ which is closed under finite meets.

3.1.5 Covers of a frame

A *cover* of the frame L is any subset whose join is the top element. The collection of all covers of L is denoted by $\operatorname{Cov} L$. Compact (resp. Lindelöf) frames are those frames in which each cover has a finite (resp. countable) subcover. A *quasi-cover* of L is any subset $B \subseteq L$ whose join is dense in L, that is $\bigvee B \in \mathfrak{d}L = \{x \in L : x^* = 0\}$. The collection of all quasi-covers of L will be denoted by $\operatorname{Cov}_q L$. Madden and Vermeer in [143](1986) define a frame L to be *weakly Lindelöf* if for each $A \in \operatorname{Cov} L$ there is $B \subseteq_\omega A$ such that $B \in \operatorname{Cov}_q L$. We next provide a calculus for covers on a frame L.

Let $A, B \in \operatorname{Cov} L$ and $x \in L$. We say that A *refines* B (or A *is a refinement of* B) and write $A \leq B$ if for each $a \in A$ there is $b \in B$ such that $a \leq b$. The meet of the two covers A and B is the cover defined as $A \wedge B = \{a \wedge b : a \in A \,\text{and}\, b \in B\}$. The *star of* x *with respect to the cover* A is the set $Ax = \bigvee\{a \in A : a \wedge x \neq 0\}$. We also have the covers $AB = \{Ab : b \in B\}$ and $A^* = AA$. We say that A *star refines* B if $A^* \leq B$ and we also write this as $A \leq^* B$. We will write $A \leq_{<\omega} B$ (respectively, $A \leq_\omega B$) to mean that A is finite (respectively, countable) and refines B. The cover A is a *normal cover* if there is sequence of covers $\{A_n\}_{n \in \mathbb{N}}$ such that $A = A_1$ and $A_{n+1} \leq^* A_n$ for each $n \in \mathbb{N}$. We will denote the collection of all normal covers of a frame L by $\operatorname{Cov}_n L$. A subset $S \subseteq L$ is *locallly finite* if there is $T \in \operatorname{Cov} L$ such that for each $s \in S$, $T_s = \{t \in T : t \wedge s \neq 0\} \subseteq_{<\omega} T$. L is a *paracompact frame* if each cover of L has a locally finite refinement. **ParFrm** is the subcategory of paracompact frames and frame homomorphisms that we will encounter throughout the paper.

3.2 σ-Frames If we have the requirement that $(L, \leq)$ is a countably complete bounded lattice and satisfies the distributive law (†) (given in §3.1.) for only countable subsets S, then L is called a *σ-frame*. *σ-Frame homomorphisms* are bounded lattice homomorphisms that preserves all countable joins. We then have the corresponding category $\sigma\mathbf{Frm}$ of σ-frames and their homomorphisms. The concepts of regularity, complete regularity and normality that are described above in §3.1.2. also carry over to σ-frames with the appropriate modifications incorporating countable joins. Regular σ-frames are always normal. In general, elements of σ-frames do not possess pseudocomplements. For any frame L the set of all its *cozero* elements, namely,

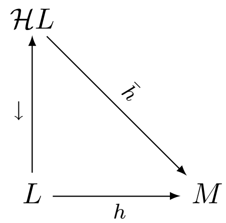
$$\operatorname{Coz} L = \{a \in L \,:\, a = h(\mathbb{R} - \{0\}) \text{ for some } h : \mathfrak{O}\mathbb{R} \longrightarrow L \text{ in } \mathbf{Frm}\}$$

is a σ-frame called the *cozero part of L*. $\operatorname{Coz} L$ features prominently in Dube's work. Banaschewski and Gilmour in [113, Proposition 1] provide the following descriptions of cozero elements that are frequently used:

$$\begin{aligned} a \in \operatorname{Coz} L \quad \text{iff} \quad & a = \bigvee x_n \text{ where } x_n \prec\!\prec a, \text{ for all } n = 1, 2, \ldots \\ \text{iff} \quad & a = \bigvee a_n \text{ where } a_n \prec\!\prec a_{n+1}, \text{ for all } n = 1, 2, \ldots \end{aligned}$$

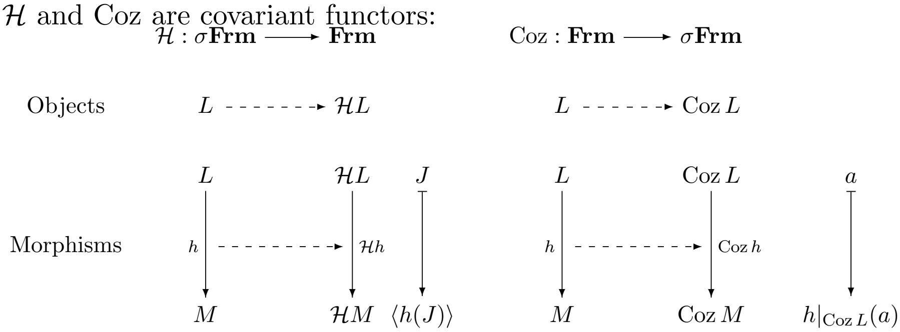
$\operatorname{Coz} L$ is the largest regular sub-σ-frame of L (as a σ-frame) (see for instance, [113, Corollary 2]). For completely regular frames L, $\operatorname{Coz} L$ generates L as a frame (it is a base for L). Frame homomorphisms preserve cozero elements. For any frame homomorphism $h : L \longrightarrow M$, the restriction $\operatorname{Coz} h = h|_{\operatorname{Coz} L}$ is a σ-frame homomorphism. For any σ-frame L, an ideal J is a *σ-ideal* in case J is closed under countable joins. The collection of all σ-ideals of the σ-frame L, denoted $\mathcal{H}L$, is a frame and is called the *free frame* over L or the *frame envelope* of L by Banaschewski in [110](1993). For any $a \in L$, the principal ideals (principal down sets) $\downarrow a = \{x \in L \,:\, x \leq a\}$ are σ-ideals of L. This realises a σ-frame homomorphism $\downarrow : L \longrightarrow \mathcal{H}L$ which is an embedding that is the universal σ-frame homomorphism to frames. That is, given any σ-frame homomorphism $h : L \longrightarrow M$ from L to a frame M there is a unique frame homomorphism $\bar{h} : \mathcal{H}L \longrightarrow M$ such that $\bar{h}\circ \downarrow = h$

i.e. the following diagram commutes

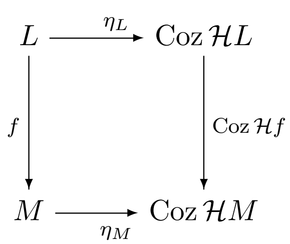


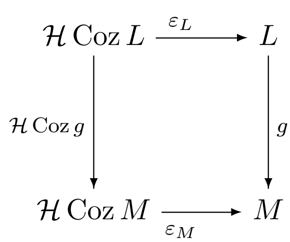
$\bar{h} : \mathcal{H}L \longrightarrow M$ is explicitly given by $\bar{h}(J) = \bigvee\{h(x) : x \in J\}$ in Banaschewski [110, Proposition 1]. Given any σ-frame homomorphism $h : L \longrightarrow M$, $\mathcal{H}h : \mathcal{H}L \longrightarrow \mathcal{H}M$ defines a frame homomorphism where $\mathcal{H}h(J) = \langle h(J)\rangle$ is the σ-ideal generated by $h(J)$.

$\mathcal{H}$ and Coz are covariant functors:



$\mathcal{H}$ is left adjoint to Coz with unit $\eta_L : L \longrightarrow \mathrm{Coz}\,\mathcal{H}L$ defined by $\eta_L(x) = \downarrow x$ and counit $\varepsilon_L : \mathcal{H}\,\mathrm{Coz}\, L \longrightarrow L$ given by the join map $\varepsilon_L(J) = \bigvee J$ *i.e* η and ε are *natural* (the following diagrams commute)





and (the adjointness) $\varepsilon_{\mathcal{H}L} \circ \mathcal{H}\eta_L = id_{\mathcal{H}L}$ and $\mathrm{Coz}\,\varepsilon_M \circ \eta_{\mathrm{Coz}\, M} = id_{\mathrm{Coz}\, M}$.

Reynolds in [159] first showed that the functor $\mathcal{H}$ is left adjoint to the functor Coz. In effect, $\mathcal{H}$ and Coz induce an equivalence between the categories of regular σ-frames and regular Lindelöf frames. The regular Lindelöf frame $\mathcal{H}\,\mathrm{Coz}\,L$ in the above description plays a formidable role in many of Dube's results pertaining to Lindelöfications. It is precisely the frame λL that has a nuanced meaning in Section §3.3.2.

3.3 The trinity of coreflections Dube's fascination with particularly the trio βL, λL and υL, is articulated in many of his papers. These coreflections are intimately related to the cozero part $\mathrm{Coz}\,L$ of the completely regular frame L that is considered. We give the necessary details concerning these coreflections animated by Dube in his mathematical works. The bedrocks of these coreflections are certain distinguished subsets of a frame, namely the ideals. Recall that in any frame L, an *ideal* is any nonempty subset $I \subseteq L$ such that $\bigvee F \in I$ for any $F \subseteq_{<\omega} I$, and $a \in I$ whenever $a \leq b$ and $b \in I$. $\mathbf{Id}\,L$ denotes the lattice of (nonempty) ideals of L where for a collection of ideals $\{J_\alpha\}_{\alpha\in\Lambda}$, the supremum in $\mathbf{Id}\,L$ is given by $\bigvee\limits_{\alpha\in\Lambda} J_\alpha = \{\bigvee T : T \subseteq_{<\omega} \bigcup\limits_{\alpha\in\Lambda} J_\alpha\}$. $\mathbf{Id}\,L$ ordered by inclusion is then a compact frame.

3.3.1 The Stone-Čech compactification

Recall that a *compactification* of a frame L is a pair (M, h) were $h : M \longrightarrow L$ is a dense onto homomorphism and M is a compact regular frame. We either loosely refer to the frame M or the map h as the compactification of L. A frame which has compactifications is called *compactifiable*. Completely regular frames are such types of frames that are compactifiable. The pointfree (localic) analogue of the Stone-Čech compactification of Tychonoff spaces was first contrived by Banaschewski and Mulvey in [115](1980). An ideal $J \in \mathbf{Id}\,L$ is *completely regular* (or *creg* for brevity) if for each $a \in J$, $a \prec\!\prec b$ for some $b \in J$. We denote $\mathbf{Id}_{\prec\!\prec}\,L = \{J \in \mathbf{Id}\,L : J \text{ is creg}\}$. Then $\mathbf{Id}_{\prec\!\prec}\,L$ is a completely regular subframe of $\mathbf{Id}\,L$ that is also compact, and is the compact completely regular coreflection of the completely regular frame L. $\mathbf{Id}_{\prec\!\prec}\,L$ is referred to as the Stone-Čech compactification of L and is commonly denoted by βL. The coreflection map is given by join $\beta_L : \beta L \longrightarrow L$ $(J \rightsquigarrow \bigvee J)$ with right adjoint given by $(\beta_L)_* : L \longrightarrow \beta L$

where $(\beta_L)_*(a) = \{c \in \operatorname{Coz} L : c \prec\!\prec a\}$ for each $a \in L$. Instead of this original construction, βL may also be characterised using the regular ideals of the σ-frame $\operatorname{Coz} L$ as given by Banaschewski and Gilmour [113](1996).

3.3.2 The regular Lindelöf coreflection

In [143](1986), Madden and Vermeer, working in the category **Loc**, gave a localic construction of λL, the regular Lindelöf reflection for any completely regular locale L. In the category **Frm**, the regular Lindelöf coreflection λL of a completely regular frame L is given by the frame of σ-ideals of $\operatorname{Coz} L$. The σ-ideals are the ones that are closed under countable join. The coreflection map, $\lambda_L : \lambda L \longrightarrow L$, is the dense onto frame homomorphism given by join. The right adjoint of λ_L is $(\lambda_L)_* : L \longrightarrow \lambda L$ given by $(\lambda_L)_*(a) = \{c \in \operatorname{Coz} L : c \leq a\}$ for each $a \in L$. This turns out to be precisely the principal ideal in $\operatorname{Coz} L$ generated by a, that is, $(\lambda_L)_*(a) = \downarrow_{\operatorname{Coz} L} a$, for each $a \in \operatorname{Coz} L$.

3.3.3 The realcompact coreflection

Reynolds in [159](1979) first introduced a notion of *realcompactness* for frames. Marcus carried out a detailed study into realcompact frames in his Master's thesis [144](1993) and presented the salient aspects of realcompactifications of frames in [145](1995). We recall, from Marcus [145](1995), that an ideal J of a frame L is *σ-proper* if $\bigvee T \neq 1$ for any $T \subseteq_\omega J$. J is *completely proper* if $\bigvee J \neq 1$. A frame L is called *realcompact* if any σ-proper maximal ideal in $\operatorname{Coz} L$ is completely proper. The realcompact completely regular frames are shown to form a coreflective subcategory of completely regular frames by Banaschewski and Gilmour in [114](2001). The realcompact coreflection (or *Hewitt realcompactification*) is the frame denoted by υL which is constructed from $\operatorname{Pt}(\lambda L)$, the points of the regular Lindelöf coreflection λL, via the map $\ell : \lambda L \longrightarrow \lambda L$ defined on λL by

$$\ell(P) = \lambda_*\Big(\bigvee P\Big) \wedge \bigwedge\Big\{Q \in \operatorname{Pt}(\lambda L) : P \leq Q\Big\}.$$

ℓ is a nucleus on λL so that, by Lemma 3.2, $\upsilon L = \operatorname{Fix}(\ell)$ is a frame. υL is realcompact and the resulting dense onto frame homomorphism $\upsilon_L : \upsilon L \longrightarrow L$ given by join is then the coreflection map. The right adjoint of the latter coreflection is given by $(\upsilon_L)_* : L \longrightarrow \upsilon L$, where $(\upsilon_L)_*(a) = \{c \in \operatorname{Coz} L : c \leq a\}$ for each $a \in L$.

We remark that for any completely regular frame L,

$$\mathrm{Pt}(\lambda L) = \mathrm{Pt}(\upsilon L)$$

and for any $a \in L$,

$$(\lambda_L)_*(a) = (\upsilon_L)_*(a) = \{c \in \operatorname{Coz} L : c \leq a\}.$$

We will also focus our attention on a fourth coreflection, namely the *paracompact coreflection* πL, that we discuss in Section §7, Remark 7.4. πL forms the umbrella that constitutes the tetrad of indispensable coreflections that feature in Dube's work.

3.4 Structured frames For a frame L, let $x, y \in L$ and $\mathfrak{N}L \subseteq \operatorname{Cov} L$. We will say that x is *$\mathfrak{N}L$-below* (or *uniformly below*) y and write $x \lhd_{\mathfrak{N}L} y$ (or for brevity $x \lhd y$) if there is $A \in \mathfrak{N}L$ such that $Ax \leq y$. $\mathfrak{N}L$ is called an *admissible* system of covers if for each $a \in L$, $a = \bigvee\{b \in L : b \lhd_{\mathfrak{N}L} a\}$. $\mathfrak{N}L$ is called a *nearness* on L provided that $\mathfrak{N}L$ is an admissible system of covers that is a filter with respect to refinement $\leq$ on covers. The pair $(L, \mathfrak{N}L)$ is then called a *nearness frame* (or *N-frame* as Dube calls it in [6]) and the members of $\mathfrak{N}L$ are called *uniform covers*. In any N-frame $(L, \mathfrak{N}L)$ we have that $x \lhd y$ iff $\{x^*, y\} \in \mathfrak{N}L$. A frame homomorphism $h : (L, \mathfrak{N}L) \longrightarrow (M, \mathfrak{N}M)$ between nearness frames is a *uniform frame homomorphism* if h preserves uniform covers. It is well-known that a frame L has a nearness iff $L \in$ **RegFrm**. $\operatorname{Cov} L$ is a nearness on L (where $\lhd_{\operatorname{Cov} L} = \prec$) called the *fine nearness*. A nearness $\mathfrak{N}L$ is called *fine* if $\mathfrak{N}L = \operatorname{Cov} L$. A *nearness base* or an *N-base* on a frame L is any $\nu \subseteq \operatorname{Cov} L$ such that ν is an admissible system of covers and if $A, B \in \nu$ then $C \leq A \wedge B$ for some $C \in \nu$. If ν is a N-base on a frame L then $\mathfrak{N}_\nu L = \{A \in \operatorname{Cov} L : B \leq A \text{ for some } B \in \nu\}$ is the nearness on L generated by ν.

For a nearness frame $(L, \mathfrak{N}L)$ and $A \in \mathfrak{N}L$ we have the associated covers $\check{A} = \{x \in L : x \lhd_{\mathfrak{N}L} a \text{ for some } a \in A\}$ and A^*. $\mathfrak{N}L$ is called a *strong* *nearness* if it is the case that $\check{A} \in \mathfrak{N}L$ whenever $A \in \mathfrak{N}L$. $\mathfrak{N}L$ is *almost uniform* if $\mathfrak{N}L$ is strong and if for each $x, z \in L$ with $x \lhd_{\mathfrak{N}L} z$ there is $y \in L$ such that $x \lhd_{\mathfrak{N}L} y \lhd_{\mathfrak{N}L} z$ (that is, $\lhd_{\mathfrak{N}L}$ *interpolates*). $\mathfrak{N}L$ is *uniform* or a *uniformity* if each uniform cover has a uniform star refinement. Furthermore, a *preuniformity* on a frame L is a filter $\mathfrak{U}L$ in $\operatorname{Cov} L$ such that for each $A \in \mathfrak{U}L$ there is $B \in \mathfrak{U}L$ such that $B \leq^* A$. A uniformity is then just a preuniformity that is an admissible filter of covers. We thus have the resulting category **NFrm** of nearness frames and subcategories **FNFrm** of fine nearness frames, **StrNFrm** of strong nearness frames, **AuNFrm** of almost uniform nearness frames, and **UFrm** of uniform frames. It is also well-known that a frame L is *uniformizable* provide that it is completely regular with Countable Dependent Choice (CDC). The fine nearness $\operatorname{Cov} L$ is a uniformity provided that L is a paracompact frame. If $L \in$ **CRegFrm** then the collection of all uniformities on L is a uniformity $\mathfrak{U}_F L$ called the *fine uniformity* on L which consists of all the normal covers of L ($\mathfrak{U}_F L = \operatorname{Cov}_n L$).

Hong in [136](1995) introduces the concept of a strict extension of a frame. Given a frame L, let X be the set of all filters (dual ideals) in L and $\wp(X)$ be the frame of the power set lattice of X. Furthermore, let

$$s_X L = \{(x, \Sigma) \in L \times \wp(X) \,:\, \text{for any } F \in \Sigma, x \in F\}$$

and let $s : s_X L \longrightarrow L$ be the restriction of the first projection to $s_X L$ so that $s((x, \Sigma)) = x$. Then s is a frame homomorphism being merely the identity on the first projection. Moreover, $s_X L$ is a subframe of the product frame $L \times \wp(X)$ and s is an open, dense and onto frame homomorphism which is called the *simple extension* of L with respect to X. Now let $s_* : L \to s_X L$ be the right adjoint of s which is explicitly given by

$$s_*(x) = \bigvee\{(y, \Sigma) \in s_X L \,:\, s((y, \Sigma)) = y \leq x\} = (x, \Sigma_x),$$

where $\Sigma_x = \{F \in X \,:\, x \in F\}$. Then $s_*(L) = \{s_*(x) \,:\, x \in L\}$ is closed under finite meets (since the right adjoint preserves finite meets). We now let $t_X L$ be the subframe of $s_X L$ generated by $s_*(L)$. Then

$$t_X L = \left\{\bigvee\{(x, \Sigma_x) \,:\, x \in A\} \,:\, A \subseteq L\right\}.$$

Now let $t = s\,|_{t_X L}$, the restriction of s to $t_X L$. Then $t : t_X L \longrightarrow L$ is a dense and onto frame homomorphism (since $t(s_*(x)) = x$) and is called the *strict extension* of L with respect to X.

Recall that a filter F in a frame L is *completely prime* iff $\forall\ S \subseteq L$, $\bigvee S \in F$ implies $S \cap F \neq \emptyset$. Furthermore, F *converges* iff $F \cap A \neq \emptyset$ $\forall\ A \in \operatorname{Cov} L$ (see Hong [136, Definition 1.1]). Given a nearness frame $(L, \mathfrak{N}L)$, a filter F in L is a *Cauchy filter* iff $\forall\ A \in \mathfrak{N}L$, $A \cap F \neq \emptyset$. F is called a *regular Cauchy filter* if F is a Cauchy filter and for any $x \in F$, there is $y \in F$ such that $y \lhd x$. The nearness frame $(L, \mathfrak{N}L)$ is said to be *complete* if every dense surjection $h : (M, \mathfrak{N}M) \longrightarrow (L, \mathfrak{N}L)$ is an isomorphism and it is *Cauchy complete* provided that every regular Cauchy filter G in L is a completely prime filter iff G is convergent in the sense of Hong [136]. The *completion* of a nearness frame and its construction is discussed in Section §4.3. In [137](1995), Hong and Kim construct the *Cauchy completion* of a nearness frame via a strict extension. They consider the set X of *regular Cauchy filters* in L and the strict extension $t_X L$ of L associated with X (as described above). We denote $t_X L$ by cL and $t : cL \longrightarrow L$ by c_L or c. A nearness $\mathfrak{N}cL$ on cL is introduced generated by $\{c_*(A) : A \in \mathfrak{N}L\}$ where $c_* : L \longrightarrow cL$ is the right adjoint of c i.e. $c_*(x) = (x, \Sigma_x)$ for each $x \in L$. They then go on to show that $c : (cL, \mathfrak{N}cL) \longrightarrow (L, \mathfrak{N}L)$ is a dense surjection between nearness frames and that the nearness frame $(cL, \mathfrak{N}cL)$ is Cauchy complete. $(cL, \mathfrak{N}cL)$ is called the *Cauchy completion* of the nearness frame $(L, \mathfrak{N}L)$. We will require the strict extension of an N-frame and its Cauchy completion in Section §6.

4 Dube's Doctoral Studies

The theory of frames (locales) is an abstraction of topological spaces in which the primitive notion of the *open* sets is the chief protagonist. The frame of open subsets $\mathfrak{O}X$ of a topological space is the archetype and the point of reference in viewing topology without points through a lattice theoretic lens. A concept or property P is *conservative* provided that P is possesed by a topological space X iff the frame of its open subsets $\mathfrak{O}X$ also has the said property P. The typical example of regularity is a conservative property (X is a regular topological space iff $\mathfrak{O}X$ is a regular frame).

Dube's doctoral study focusses on the *concept of nearness* introduced by Herrlich [133](1974). The comprehensive study is through the looking glass of frames and with concepts, discoveries and findings that are possibly conservative. Our exposition of Dube's mathematical contributions is certainly not exhaustive. We will not mention all the classical topological concepts (and references) that Dube presents in (conservative) pointfree form. We focus on highlighting certain of Dube's contributions including the (conservative) pointfree definitions that he conceives and the formidable results that are realised. The proofs of the latter are omitted since they are directly accessible from his published works given in the bibliography.

4.1 Nearness frames and Nearness Spaces The structure of a *uniformity* on a space traces originally to Weil [168](1937), thereafter to Tukey [166, Chapter VI. Structs](1940) and then to the widely referenced book by Isbell [139](1964). The introduction of the structure of a *uniformity* into locales dates back to the Paris *Séminaire Ehresmann* (1957 - 1958) with the first talk in the seminar presented by the Papert's [153]. Therein is included, for the first time, a definition of a *uniformizable locale* [153, Definitions. vi. p.1-05]. A further sample of theorems was presented where they enunciated that uniformizability implies regularity and that normality together with regularity gives uniformizability [153, Théorèmes. v. a., b.]. Isbell [140, Section 3](1972) further looked at generalizing some of his own work on uniform spaces [139](1964) into the pointfree context. A more comprehensive treatment of uniformity via a covering approach is given by Pultr [155](1984) and Frith [129, 130](1986, 1990). Pultr and Frith both show by different means that a frame is uniformizable provided that it is completely regular.

Herrlich [133](1974) introduced the concept of a *nearness* and the category **Near** of nearness spaces and uniformly continuous maps. For any set X, $\wp X = \{A : A \subseteq X\}$ denotes the power set of X. A *cover* of X is any collection $\mathscr{C} \subseteq \wp(X)$ with $\bigcup \mathscr{C} = X$. If $\mathscr{C}$ and $\mathscr{D}$ are covers of X we have the meet cover of $\mathscr{C}$ and $\mathscr{D}$ defined by $\mathscr{C} \wedge \mathscr{D} = \{C \cap D : C \in \mathscr{C}, D \in \mathscr{D}\}$. The cover $\mathscr{C}$ *refines* the cover $\mathscr{D}$ (written $\mathscr{C} \leq \mathscr{D}$) if for each $C \in \mathscr{C}$, $C \subseteq D$ for some $D \in \mathscr{D}$. A *nearness space* or *N-space* is a pair (X, μ) where X is a set and μ is a nonempty collection of covers of X (called *uniform covers*) that satisfy:

(N1) If $\mathscr{A} \in \mu$ and $\mathscr{A} \leq \mathscr{B}$ then $\mathscr{B} \in \mu$.

(N2) If $\mathscr{A}, \mathscr{B} \in \mu$ then $\mathscr{A} \wedge \mathscr{B} \in \mu$.

(N3) If $\mathscr{A} \in \mu$ then $\operatorname{int}_\mu \mathscr{A} = \{\operatorname{int}_\mu A : A \in \mathscr{A}\} \in \mu$ where

$$\operatorname{int}_\mu A = \{x \in X : \{A, \{X - \{x\}\} \in \mu\}.$$

A morphism $f : (X, \mu) \longrightarrow (Y, \nu)$ between nearness spaces is *uniformly continuous* if f is a function on the underlying sets for which the preimage $f^{-1}(\mathscr{A}) = \{f^{-1}(A) : A \in \mathscr{A}\} \in \mu$ whenever $\mathscr{A} \in \nu$. The resulting category **Near** contains the category of all uniform spaces and uniformly continuous maps and the theory of nearness spaces mainly aimed to unify the various types of topological structures.

Frith in [129, Notes on Chapter 7(5)](1986) posed the question of whether a theory of nearness frames is possible. In his doctoral thesis [6](1992), Dube independently presented a rich foundation for a general theory of *nearness* on frames which responded to the open question of Frith. A suitable definition of a nearness frame was suggested by Dube's doctoral advisors, Baboolal and Ori (in 1990), as that of a uniform frame *without* the star-refinement property. In the same year, Banaschewski [109](1990) presented a second series of lectures, this occasion on *Cauchy points of nearness frames* at the University of Cape Town which Dube attended. The first series of lectures by Banaschewski at UCT was in 1988 on the category of frames. Banaschewski's definition of a *nearness* in [109] incidentally coincided with the one suggested to Dube by Baboolal and Ori. The lectures of Banaschewski [109] eventually culminated in the paper [118](1996).

Dube's thesis [6] is a comprehensive disquistion into the concept of nearness in the pointfree context. His subsequent papers on nearness frames, namely [7–11], emanated from the original work that he investigated in his thesis. There are three specific areas covered by Dube in [6] that we highlight below that is seldom made reference to that is profound in its very nature and fundamental within the developments in the category **NFrm**. Firstly, the adjoint relation between nearness spaces and nearness frames, secondly the construction of the completion of a nearness frame, and lastly the construction of the coproduct of nearness frames.

4.2 Adjoint relations In [6] Dube introduces *B-spaces*. The concept was initially materialised by Banaschewski [109] as the spatial counterparts of nearness frames that were called nearness spaces. However, they were not nearness spaces in the original sense of Herrlich. The category **BNear** of those nearness spaces that arise from B-spaces (see Theorem 4.2 below) is a full subcategory of **Near**. Dube investigates how a B-space can be associated categorically in a natural way with a nearness frame and vice versa producing an adjoint relation. A *B-space* is a pair (X, μ) where X is a topological space and μ is a filter (with respect to $\wedge$ and $\leq$) in $\mathfrak{O}X$ such that for each $U \in \mathfrak{O}X$ and each $x \in U$ there is $V \in \mathfrak{O}X$ and $\mathscr{C} \in \mu$ with $x \in V$ and $\mathrm{St}(V, \mathscr{C}) = \mathscr{C}V = \bigcup\{C \in \mathscr{C} : C \cap V \neq \emptyset\} \subseteq U$. Dube provides a structured version of the adjoint relation between **Top** and **Frm** (given in §3.1.) for B-spaces and nearness frames. This is the second of the type following the first structured version of $\mathfrak{O}$ and Σ shown by Frith [129, 130] for uniform spaces and uniform frames.

For any B-space (X, μ), $(\mathfrak{O}X, \mathfrak{O}\mu)$ is a nearness frame where $\mathfrak{O}\mu$ is the filter that makes (X, μ) a B-space. The passage from N-frames to B-spaces is given by the following Proposition.

Proposition 4.1. [6, Proposition 1.3.2 and 1.3.3]. *Let $(L, \mathfrak{N}L)$ be a N-frame. For each $A \in \mathfrak{N}L$ and $a \in A$, $\Sigma_a = \{\xi : L \longrightarrow \mathbb{2} : \xi(a) = 1\}$.*

(*a*) *If we let $\Sigma_A = \{\Sigma_a : a \in A\}$ and*

$$\mu_{\mathfrak{N}L} = \{\mathscr{C} : \mathscr{C} \in \mathrm{Cov}\,\mathfrak{O}\Sigma L \text{ and } \Sigma_A \leq \mathscr{C} \text{ for some } A \in \mathfrak{N}L\},$$

then $(\Sigma L, \mu_{\mathfrak{N}L})$ is a B-space.

(*b*) *If $(L, \mathfrak{N}L)$ is a spatial N-frame, then $\mu_{\mathfrak{N}L} = \{\Sigma_A : A \in \mathfrak{N}L\}$.*

A N-space is realised from a B-space by the following result.

Theorem 4.2. [6, Theorem 1.3.4]. *If $(X, \mathfrak{O}\beta)$ is a B-space, then (X, β) is an N-space where $\beta = \{\mathscr{A} \subseteq \wp(X) : \mathscr{B} \leq \mathscr{A} \text{ for some } \mathscr{B} \in \mathfrak{O}\beta\}$.*

In the above result, if the canonical topology on the B-space $(X, \mathfrak{O}\beta)$ is τ_X and the topology on the induced N-space (X, β) is τ_β, then $\tau_X = \tau_\beta$. Furthermore, uniformly continuous functions between B-spaces are continuous functions between their associated topological spaces. As a consequence of the above theorem, the path from an N-frame to an N-space is captured in the following result.

Corollary 4.3. [6, Corollary 1.3.5].
For an N-frame $(L, \mathfrak{N}L)$, $(\Sigma L, \Sigma\mathfrak{N}L)$ is an N-space where

$$\Sigma\mathfrak{N}L = \{\mathscr{A} \subseteq \wp(\Sigma L) : \Sigma_A \leq \mathscr{A} \text{ for some } A \in \mathfrak{N}L\}.$$

The functors $\mathfrak{O}$: **BNear** $\longrightarrow$ **NFrm** and Σ : **NFrm** $\longrightarrow$ **BNear** are contravariant functors.

	$\mathfrak{O}$: **BNear** $\longrightarrow$ **NFrm**		Σ : **NFrm** $\longrightarrow$ **BNear**	
Objects	$(X,\beta) \dashrightarrow (\mathfrak{O}X, \mathfrak{O}\beta)$		$(L,\mathfrak{N}L) \dashrightarrow (\Sigma L, \Sigma\mathfrak{N}L)$	
Morphisms	$h: (X,\beta) \to (Y,\gamma) \dashrightarrow \mathfrak{O}h: (\mathfrak{O}Y,\mathfrak{O}\gamma) \to (\mathfrak{O}X,\mathfrak{O}\beta)$	$U \mapsto h^{-1}(U)$	$f: (L,\mathfrak{N}L) \to (M,\mathfrak{N}M) \dashrightarrow \Sigma f: (\Sigma M,\Sigma\mathfrak{N}M) \to (\Sigma L,\Sigma\mathfrak{N}L)$	$\xi \mapsto \xi \circ f$

For all B-spaces (X, β) and N-frames $(L, \mathfrak{N}L)$, along the similar lines of uniform frames given by Frith [129, Theorem 2.18], Dube [6, Theorem 1.3.11] shows that the pair of functors $\mathfrak{O}$ and Σ is an adjunction as an isomorphism of the hom-sets

$$\hom_{\mathbf{NFrm}}((L, \mathfrak{N}L), (\mathfrak{O}X, \mathfrak{O}\beta)) \simeq \hom_{\mathbf{BNear}}((X, \beta), (\Sigma L, \Sigma\mathfrak{N}L)) \qquad (\ddagger)$$

by exhibiting a bijection between the respective morphism sets which is natural in $(L, \mathfrak{N}L)$ and (X, β). For any uniform frame homomorphism $h \in \hom_{\mathbf{NFrm}}((L, \mathfrak{N}L), (\mathfrak{O}X, \mathfrak{O}\beta))$, a uniformly continuous function $\bar{h} \in \hom_{\mathbf{BNear}}((X, \beta), (\Sigma L, \Sigma\mathfrak{N}L))$ where $\bar{h}(x) : L \longrightarrow \mathbb{2}$ ($\bar{h}(x) \in \Sigma L$) for each $x \in X$ is defined by $\bar{h}(x)(a) = 1$ iff $x \in h(a)$. For any uniformly continuous function $f \in \hom_{\mathbf{BNear}}((X, \beta), (\Sigma L, \Sigma\mathfrak{N}L))$, a uniform frame homomorphism is defined by

$$\hat{f} \in \hom_{\mathbf{NFrm}}((L, \mathfrak{N}L), (\mathfrak{O}X, \mathfrak{O}\beta)) \quad \text{where} \quad \hat{f}(a) = f^{-1}(\Sigma_a)$$

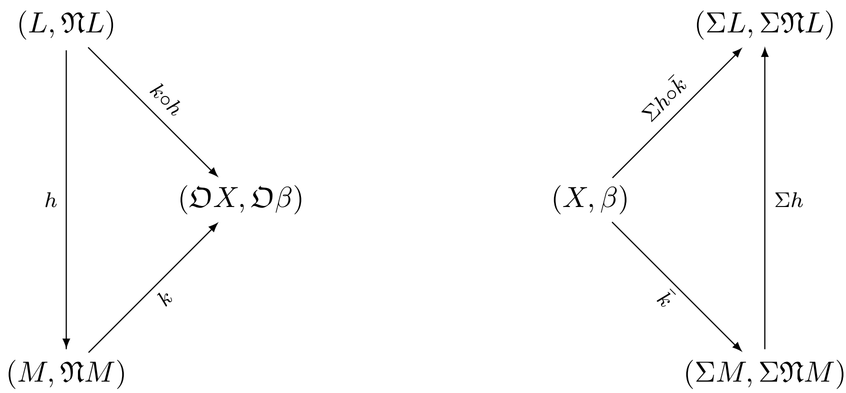
for each $a \in L$. We then define $\varphi = \varphi(\mathbf{NFrm}, \mathbf{BNear})$ by $\varphi(h) = \bar{h}$ and $\psi = \psi(\mathbf{BNear}, \mathbf{NFrm})$ by $\psi(f) = \hat{f}$:

$$\begin{array}{ccc} & h \longmapsto \bar{h} & \\ \hom_{\mathbf{NFrm}}((L, \mathfrak{N}L), (\mathfrak{O}X, \mathfrak{O}\beta)) & \underset{\psi}{\overset{\varphi}{\rightleftarrows}} & \hom_{\mathbf{BNear}}((X, \beta), (\Sigma L, \Sigma\mathfrak{N}L)) \\ & \hat{f} \longleftarrow\!\mid f & \end{array}$$

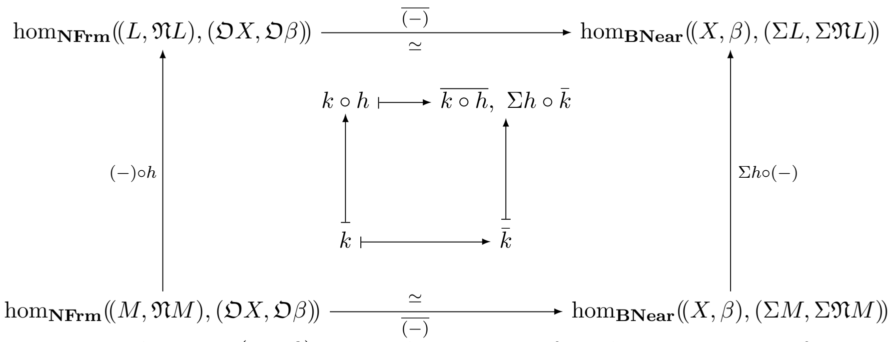
Since $\hat{\bar{h}} = h$ and $\bar{\hat{f}} = f$ we have that $\psi(\varphi(h)) = \psi(\bar{h}) = \hat{\bar{h}} = h$ and $\varphi(\psi(f)) = \varphi(\hat{f})) = \bar{\hat{f}} = f$. Consequently $\varphi^{-1} = \psi$. Thus φ (and ψ) is a bijection between the hom-sets which realises (‡). Thus for each B-space (X, β) and each N-frame $(L, \mathfrak{N}L)$, we have a bijection

$$\hom_{\mathbf{NFrm}}((L, \mathfrak{N}L), (\mathfrak{O}X, \mathfrak{O}\beta)) \xrightarrow[\simeq]{\varphi((L,\mathfrak{N}L),(\mathfrak{O}X,\mathfrak{O}\beta))} \hom_{\mathbf{BNear}}((X, \beta), (\Sigma L, \Sigma\mathfrak{N}L)).$$

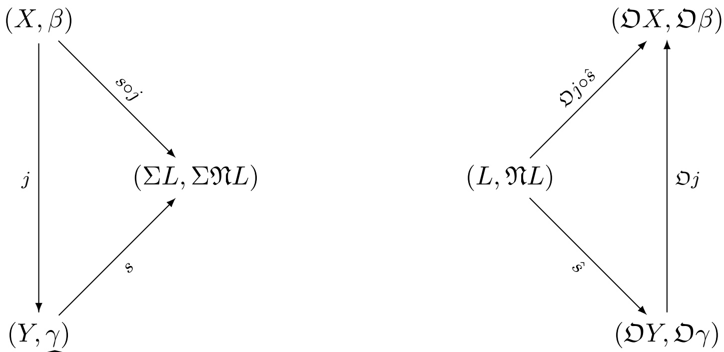
Naturality follows exactly as in [129, Theorem 2.18] which we highlight below, particularly for the categories **NFrm** and **BNear**. For naturality in $(L, \mathfrak{N}L)$ we take any N-map $(L, \mathfrak{N}L) \xrightarrow{h} (M, \mathfrak{N}M)$. Let $k \in \hom_{\mathbf{NFrm}}((M, \mathfrak{N}M), (\mathfrak{O}X, \mathfrak{O}\beta))$, then $(\Sigma M, \Sigma\mathfrak{N}M) \xrightarrow{\Sigma h} (\Sigma L, \Sigma\mathfrak{N}L)$, $k \circ h \in \hom_{\mathbf{NFrm}}((L, \mathfrak{N}L), (\mathfrak{O}X, \mathfrak{O}\beta))$, $\bar{k} \in \hom_{\mathbf{BNear}}((X, \beta), (\Sigma L, \Sigma\mathfrak{N}L))$ and $\overline{k \circ h}, \Sigma h \circ \bar{k} \in \hom_{\mathbf{BNear}}((X, \beta), (\Sigma L, \Sigma\mathfrak{N}L))$ as illustrated in the triangles below.



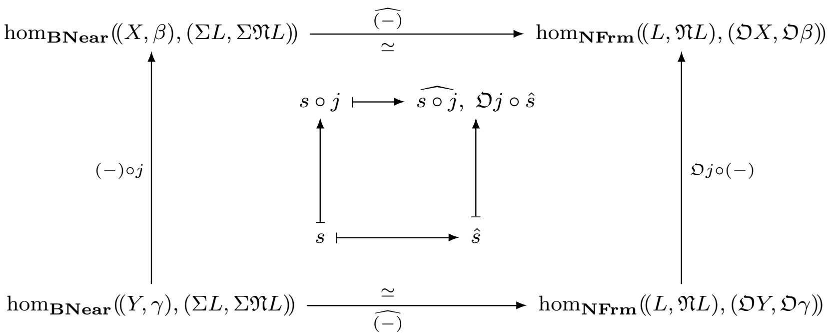
Since $\overline{k \circ h} = \Sigma h \circ \bar{k}$, the following diagram is commutative as required.



For naturality in (X,β) we take any uniformly continuous function $(X,\beta)\xrightarrow{j}(Y,\gamma)$. Let $s\in\hom_{\mathbf{BNear}}((Y,\gamma),(\Sigma L,\Sigma\mathfrak{N}L))$, we then have $(\mathfrak{O}Y,\mathfrak{O}\gamma)\xrightarrow{\mathfrak{O}j}(\mathfrak{O}X,\mathfrak{O}\beta)$, $s\circ j\in\hom_{\mathbf{BNear}}((X,\beta),(\Sigma L,\Sigma\mathfrak{N}L))$ and $\hat{s}\in\hom_{\mathbf{NFrm}}((L,\mathfrak{N}L),(\mathfrak{O}Y,\mathfrak{O}\gamma))$. Furthermore, the morphisms $\widehat{s\circ j},\mathfrak{O}j\circ\hat{s}\in\hom_{\mathbf{NFrm}}((L,\mathfrak{N}L),(\mathfrak{O}X,\mathfrak{O}\beta))$. These morphisms are represented in the triangles below.



Since $\widehat{s\circ j}=\mathfrak{O}j\circ\hat{s}$, the following diagram is commutative as required.



We obtain the unit and counit of the adjunction using the specific identity morphisms (see for instance, Borceux [125, Theorem 3.1.5] or Simmons [163, §5.4.]) as followed in [129]. We fix $(L, \mathfrak{N}L) = (\mathfrak{O}X, \mathfrak{O}\beta)$. We then have the isomorphism

$$\hom_{\mathbf{NFrm}}((\mathfrak{O}X, \mathfrak{O}\beta), (\mathfrak{O}X, \mathfrak{O}\beta)) \xrightarrow[\simeq]{\varphi((\mathfrak{O}X,\mathfrak{O}\beta),(\mathfrak{O}X,\mathfrak{O}\beta))} \hom_{\mathbf{BNear}}((X, \beta), (\Sigma\mathfrak{O}X, \Sigma\mathfrak{O}\beta)).$$

In particular, the identity morphism $\mathrm{id}_{(\mathfrak{O}X,\mathfrak{O}\beta)}$ in **NFrm** corresponds via the adjunction isomorphism φ to a morphism ε_X in **BNear** as follows:

$$\varphi(\mathrm{id}_{(\mathfrak{O}X,\mathfrak{O}\beta)}) = \overline{\mathrm{id}}_{(\mathfrak{O}X,\mathfrak{O}\beta)} := \varepsilon_X.$$

Thus for each $x \in X$, $\varepsilon_X(x) = \overline{\mathrm{id}}_{(\mathfrak{O}X,\mathfrak{O}\beta)}(x) : \mathfrak{O}X \longrightarrow 2$ where $\varepsilon_X(x)(U) = 1$ iff $x \in U$ so that $\varepsilon_X(x) = \tilde{x}$ which defines a natural transformation $\varepsilon : 1_{\mathbf{BNear}} \longrightarrow \Sigma\mathfrak{O}$ similar as in §3.1.1.

Similarly, if we fix $(X, \beta) = (\Sigma L, \Sigma\mathfrak{N}L)$, we have the isomorphism

$$\hom_{\mathbf{BNear}}((\Sigma L, \Sigma\mathfrak{N}L), (\Sigma L, \Sigma\mathfrak{N}L)) \xrightarrow[\simeq]{\varphi^{-1}((\Sigma L,\Sigma\mathfrak{N}L,(\Sigma L,\Sigma\mathfrak{N}L))} \hom_{\mathbf{NFrm}}((L, \mathfrak{N}L), (\mathfrak{O}\Sigma L, \mathfrak{O}\Sigma\mathfrak{N}L)).$$

We then obtain for the identity morphism $\mathrm{id}_{(\Sigma L,\Sigma\mu)}$ in **BNear**, corresponding via the adjunction isomorphism φ^{-1}, a morphism η_L in **NFrm** as follows:

$$\varphi^{-1}(\mathrm{id}_{(\Sigma L,\Sigma\mathfrak{N}L)}) = \widehat{\mathrm{id}}_{(\Sigma L,\Sigma\mathfrak{N}L)} := \eta_L.$$

Thus for each $a \in L$, we have that

$$\eta_L(a) = \widehat{\mathrm{id}}_{(\Sigma L,\Sigma\mathfrak{N}L)}(a) = \mathrm{id}^{-1}_{(\Sigma L,\Sigma\mathfrak{N}L)}(\Sigma_a) = \Sigma_a,$$

which defines a natural transformation $\eta : 1_{\mathbf{NFrm}} \longrightarrow \mathfrak{O}\Sigma$ that is similar as in §3.1.1.

In summary, Dube [6, Theorem 1.3.11] shows a structured version of the adjunction between **Top** and **Frm** extended to **BNear** and **NFrm** as was done by Frith [129, Theorem 2.18] for the category **Unif** of uniform spaces and uniformly continuous maps, and **UFrm**. Furthermore, the functors $\mathfrak{O}$ and Σ induce a dual equivalence between spatial N-frames and sober B-spaces. For a spatial nearness frame $(L, \mathfrak{N}L)$, $\eta_L : (L, \mathfrak{N}L) \longrightarrow (\mathfrak{O}\Sigma L, \mathfrak{O}\Sigma\mathfrak{N}L)$ is an isomorphism whilst for any sober B-space (X, β),

$$\varepsilon_X : (X, \beta) \longrightarrow (\Sigma\mathfrak{O}X, \Sigma\mathfrak{O}\beta)$$

is an isomorphism.

4.3 The completion of a nearness frame Following the method of Kříž [141](1986), Dube provides a direct description of the completion of a nearness frame in terms of generators and defining relations. We begin with the notion of a *precongruence* R on a frame L, defined by Kříž [141], which is a relation $R \subseteq L \times L$ where for all $x, y \in L$, $(x, y) \in R$ implies that $\{a \in L : (a \wedge x, a \wedge y) \in R\}$ is a base for L. Let $S \subseteq L \times L$ and B be a base for L that is closed under finite meets. Then

$$R(S, B) = \{(s \wedge b, t \wedge b) : (s, t) \in S, b \in B\} \qquad (\diamond)$$

is a precongruence on L. If $R \subseteq L \times L$ and $x \in L$, then x is *R-coherent* if for any pair $a, b \in L$ with $(a, b) \in R$ we have $a \leq x$ iff $b \leq x$. The set of all R-coherent elements of L is denoted by $\mathrm{Coh}(R)$ and is closed under all meets. We then have the following theorem concerning a precongruence R and $\mathrm{Coh}(R)$.

Theorem 4.4 (Theorem 2.2.3 [141])**.** *Let R be a precongruence on a frame L. Then $\mathrm{Coh}(R)$ together with the induced ordering is a frame and there exists a nucleus $j : L \longrightarrow L$ such that $\mathrm{Fix}(j) = \mathrm{Coh}(R)$. Moreover, $j : L \longrightarrow \mathrm{Coh}(R)$ is universal among the join-preserving mappings f from L to complete lattices B satisfying $(a, b) \in R$ implies $f(a) = f(b)$. (More exactly, for any such f there exists a unique join-preserving $f_j : \mathrm{Coh}(R) \longrightarrow B$ such that $f = f_j \circ j$).*

The nucleus $j : L \longrightarrow L$ in the above theorem is given by

$$j(x) = \wedge\{u \in \mathrm{Coh}(R) : x \leq u\},$$

for each $x \in L$. Now let $(L, \mathfrak{N}L)$ be a nearness frame and consider the frame of downsets $\mathscr{D}L$ of L with base $B = \{\downarrow x : x \in L\}$ which is closed under finite meets as described in §3.1.3. We now let $S \subseteq \mathscr{D}L \times \mathscr{D}L$ be the system of all pairs

$$(\downarrow a, k(a)), \quad (L, c(U)), \quad (\downarrow 0_L, \emptyset)$$

with $a \in L$ and $U \in \mathfrak{N}L$ where

$$k(a) = \bigcup\{\downarrow b : b \lhd_{\mathfrak{N}L} a\} \text{ and } c(U) = \bigcup\{\downarrow x : x \in U\}.$$

Let $R = R(S, B)$ as defined in $(\diamond)$. We also let $\bar{L} = \mathrm{Coh}(R)$ and $j : \mathscr{D}L \longrightarrow \mathscr{D}L$ be the nucleus in Theorem 4.4 with $\bar{L} = \mathrm{Fix}(j)$. Dube then shows that

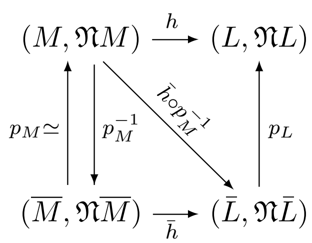
(1) the base $B \subseteq \bar{L}$,

(2) for each $A \in \mathfrak{N}L$, $\downarrow A = \{\downarrow a \,:\, a \in A\} \in \operatorname{Cov} \bar{L}$ and

(3) $\mathfrak{N}_0 \bar{L} = \{\downarrow A \,:\, A \in \mathfrak{N}L\}$ is a N-base for the frame $\bar{L}$.

We then let $\mathfrak{N}\bar{L}$ be the nearness on $\bar{L}$ with N-base $\mathfrak{N}_0\bar{L}$ and define $p : (\bar{L}, \mathfrak{N}\bar{L}) \longrightarrow (L, \mathfrak{N}L)$ by join, $p(D) = \bigvee D$ for each $D \in \bar{L}$. Then p is a uniform frame homomorphism (an N-map) and $((\bar{L}, \mathfrak{N}\bar{L}), p)$ is the completion of the nearness frame $(L, \mathfrak{N}L)$.

Dube [6] also goes a step further and shows that the completion of a nearness frame may be realised out of a nucleus. For a nearness frame $(L, \mathfrak{N}L)$ and for its completion that he describes as $((\bar{L}, \mathfrak{N}\bar{L}), p)$ the map $k : \bar{L} \longrightarrow \bar{L}$ defined by $D \longmapsto \downarrow(\bigvee D)$ is a nucleus on $\bar{L}$. With $\tilde{L} = \{\downarrow a \,:\, a \in L\}$, $\tilde{L} = \operatorname{Fix}(k)$. Then $\tilde{L}$ is a frame with $\wedge$ defined as $\downarrow a \wedge \downarrow b = \downarrow (a \wedge b)$ for $a, b \in \tilde{L}$ and for any $\{\downarrow x_i\}_{i \in I}$ in $\tilde{L}$, $\bigvee_{\tilde{L}}(\downarrow x_i) = \downarrow(\bigvee_{i \in I} x_i)$. Furthermore, the map $\downarrow : L \longrightarrow \tilde{L}$ given by $a \longmapsto \downarrow a$ for each $a \in L$ is a frame homomorphism. As a result defining $\widetilde{\mathfrak{N}L} = \{\downarrow A \,:\, A \in \mathfrak{N}L\}$ realises a nearness on the frame $\tilde{L}$. Dube [6, Theorem 4.2.5] then concludes that the nearness frame $(L, \mathfrak{N}L)$ is complete iff $(\bar{L}, \mathfrak{N}\bar{L}) = (\tilde{L}, \widetilde{\mathfrak{N}L})$. Furthermore, if $(M, \mathfrak{N}M)$ is a complete nearness frame and $h : (M, \mathfrak{N}M) \longrightarrow (L, \mathfrak{N}L)$ is any N-map, then h factors uniquely via $\bar{h}$ through the completion map $p_L : (\bar{L}, \mathfrak{N}\bar{L}) \longrightarrow (L, \mathfrak{N}L)$ where $\bar{h} : \overline{M} \longrightarrow \bar{L}$ is given by $\downarrow x \overset{\bar{h}}{\longmapsto} \downarrow h(x)$ for each $x \in M$. Furthermore, $h = p_L \circ (\bar{h} \circ p_M^{-1})$ (the diagram below is commutative):



Pursuant to the above independent construction by Dube in 1992, Banaschewski's lecture series presented at the 1994 Symposium on Categorical Topology (SoCat94) in Cape Town, where a description of the completion of a nearness frame (without the use of generators and relations) is given, was published by the Department of Mathematics and Applied Mathematics at UCT as lectures notes (Banaschewski [111](1996)). The latter construction also appears in the paper by Banaschewski and Pultr [118](1996) which is the more widely referenced and popularised version of the completion of a nearness frame described as a frame of specific downsets. For a nearness frame $(L, \mathfrak{N}L)$, $x \in L$ and $A \in \mathfrak{N}L$, let $k(x) = \{y \in L : y \lhd_{\mathfrak{N}L} x\}$ and $x \wedge A = \{x \wedge a : a \in A\}$. Let γL be the closure system of all $U \in (\mathscr{D}L, \subseteq)$ determined by

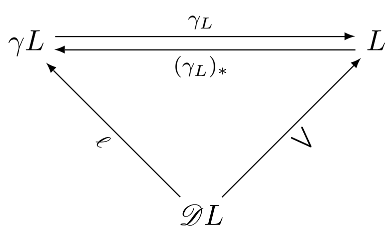
(1) if $k(x) \subseteq U$, then $x \in U$ and

(2) if $x \wedge A \subseteq U$ for some $A \in \mathfrak{N}L$, then $x \in U$.

We define $\ell_0 : \mathscr{D}L \longrightarrow \mathscr{D}L$ such that for each $U \in \mathscr{D}L$,

$$\ell_0(U) = \{x \in L : k(x) \subseteq U\} \cup \{x \in L : x \wedge A \subseteq U \text{ for some } A \in \mathfrak{N}L\}.$$

Then ℓ_0 is a prenucleus and $U \in \gamma L$ iff $\ell_0(U) = U$. By Lemma 3.3 and Lemma 3.2, the closure operator ℓ on $\mathscr{D}L$ determined by γL is a nucleus so that γL is a frame. Furthermore, as described in §3.1.3., on the underlying frames the join map $\bigvee : \mathscr{D}L \longrightarrow L$ is a frame homomorphism with right adjoint $\downarrow : L \longrightarrow \mathscr{D}L$. The frame homomorphism $\bigvee : \mathscr{D}L \longrightarrow L$ factors through the frame homomorphism (the nucleus) $\ell : \mathscr{D}L \longrightarrow \gamma L$ via the frame homomorphism $\gamma_L : \gamma L \longrightarrow L$ given by the join map restricted to γL



The right adjoint $(\gamma_L)_* : L \longrightarrow \gamma L$ is given by $(\gamma_L)_*(x) = \downarrow x$ for each $x \in L$. For each $A \in \mathfrak{N}L$, $(\gamma_L)_*(A) \in \operatorname{Cov} \gamma L$. The system of covers $\nu = \{(\gamma_L)_*(A) : A \in \mathfrak{N}L\} = \{\downarrow A : A \in \mathfrak{N}L\}$ is an N-base on γL generating a nearness $\mathfrak{N}_\nu \gamma L = \gamma \mathfrak{N}L$. We then have the N-map $\gamma_{(L,\mathfrak{N}L)} : (\gamma L, \gamma\mathfrak{N}L) \longrightarrow (L, \mathfrak{N}L)$ given by join is a dense surjection rendering $((\gamma L, \gamma\mathfrak{N}L), \gamma_{(L,\mathfrak{N}L)})$ as the completion of the nearness frame $(L, \mathfrak{N}L)$.

In the above two independent constructions of the completion of a nearness frame, the Dube description involves a precongruence whilst the Banaschewski-Pultr version involves a prenucleus. Furthermore, the result of Kříž (Theorem 4.4) on precongruences has a striking resemblance and an identical form to Banaschewski's Lemma 3.3 on prenuclei. It is therefore quite natural to then consider the relationship between prenuclei and precongruences. They are seemingly distinct concepts, nevertheless they are related through the concept of a nucleus. Townsend [165, Lemma 2.3.1](1996) shows a passage from a prenucleus $\ell_0 : L \longrightarrow L$ on a frame L to a precongruence in the following manner. Define $R_{\ell_0} : L \longrightarrow L$ by

$$(a, b) \in R_{\ell_0} \quad \text{iff} \quad \forall\, u \in L, \ell_0(u) = u \Rightarrow (a \leq u \text{ iff } b \leq u).$$

Then $(\ell_0(u), u) \in R_{\ell_0}$ for each $u \in L$ and furthermore, R_{ℓ_0} is a precongruence on L such that $\mathrm{Coh}(R_{\ell_0}) = \mathrm{Fix}(\ell_0)$. Townsend also shows that for any subset T of $L \times L$ one realises a prenucleus by defining $p_0 : L \times L$ by $p_0(u) = u \vee \bigvee\{a \wedge b \,:\, \exists\, c, (c, a) \in T, c \wedge b \leq u\}$. T does not necessarily have to be a precongruence.

4.4 Coproduct of nearnesss frames Dube in [150] relates his excitement of independently constructing the coproduct of a nearness frame whilst he was on a British grant visiting at the University of Cambridge during his doctoral studies in 1992. Conveying the results to his supervisor (Baboolal) from the UK to South Africa was one of his most memorable moments. We highlight the need for the coproduct and its construction below.

Bentley [123](1991) gave an analogue of the classical theorems of Tamano [123, Theorem 21] and Dowker [123, Theorem 22] for nearness spaces. The classical theorems involved the topological product $X \times Y$ of certain topological spaces X and Y. Dube [6, Theorem 6.2.3] provides a pointfree version of Bentley's result [123, Theorem 23] on the Tamano-Dowker type theorems for nearness frames. In so doing, the binary coproduct of two nearness frames needed to be defined and constructed. We require the notion and applications of a precongruence introduced by Kříž [141] which we have described in §4.2. Kříž and Pultr [142](1989) construct the coproduct of two frames using a precongruence. For frames L and M let $S \subseteq \mathscr{D}(L \times M)$ be the precongruence relation consisting of all pairs

$$(\downarrow (\bigvee_{i\in I} a_i, b), \bigcup_{i\in I}(\downarrow (a_i, b)), (\downarrow (a, \bigvee_{i\in I} b_j), \bigcup_{i\in I} \downarrow (a, b_i)), (\downarrow (0, b), \emptyset), (\downarrow (a, 0), \emptyset)$$

for all $a, a_i \in L$ and $b, b_i \in M$ with $i \in I$. Then $X \in \mathscr{D}(L \times M)$ is S-coherent iff

(1) $(x, 0), (0, x) \in X$ for all $x, y \in X$ and

(2) $(\bigvee_{i\in I} x_i, y), (x, \bigvee_{i\in I} y_i) \in X$ whenever $(x_i, y), (x, y_i) \in X \ \forall i \in I$.

The *coproduct* of the frames L and M is given by $L \oplus M = \mathrm{Coh}(S)$. Then $L \oplus M = \mathrm{Fix}(j)$ for the nucleus j emanating from Theorem 4.4 and we write $x \oplus y$ for an element of $L \oplus M$, i.e. $x \oplus y = j(\downarrow (x, y))$ for $x \in L, y \in M$. The set $B_\oplus = \{x \oplus y : x \in L, y \in M\}$ is a base for $L \oplus M$. The top element of the frame $L \oplus M$ is $1_{L\oplus M} = \downarrow (1_L, 1_M)$ and the bottom is $0_{L\oplus M} = \downarrow (1_L, 0_M) \cup \downarrow (0_L, 1_M)$. The operations of $\oplus$ with $\wedge, \bigvee$ and $\leq$ are fairly standard where for $x, x_i, c \in L$ and $y, y_i, d \in M$ for $i \in I$ we have

(1) $x \oplus y = 0_{L\oplus M}$ iff $a = 0_L$ or $b = 0_M$,

(2) $x \oplus \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \oplus y_i)$ and $(\bigvee_{i\in I} x_i) \oplus y = \bigvee_{i\in I}(x_i \oplus y)$,

(3) $(x \oplus y) \wedge (c \oplus d) = (a \wedge c) \oplus (y \oplus d)$,

(4) $x \oplus y \leq c \oplus d$ whenever $x \leq c$ and $y \leq d$ and

(5) $0_{L\oplus M} \neq x \oplus y \leq c \oplus d$ implies that $x \leq c$ and $y \leq d$.

For nearness frames $(L, \mathfrak{N}L)$ and $(M, \mathfrak{N}M)$, Dube uses the coproduct $L \oplus M$ of the underlying frames given by Kříž and Pultr that is described above and constructs a nearness on this coproduct frame. Noting that for $A \in \mathfrak{N}L$ and $B \in \mathfrak{N}M$,

$$A \oplus B = \{a \oplus b : a \in A, b \in B\} \in \mathrm{Cov}(L \oplus M),$$

Dube proceeds to define

$\mathfrak{N}(L \oplus M)$

$$= \{C \in \mathrm{Cov}(L \oplus M) \,:\, A \oplus B \leq C \text{ for some } A \in \mathfrak{N}L, B \in \mathfrak{N}M\},$$

and using the operations of $\oplus$ shows that $\mathfrak{N}(L \oplus M)$ is a nearness on the coproduct $L \oplus M$. Hence, the coproduct of the nearness frames $(L, \mathfrak{N}L)$ and $(M, \mathfrak{N}M)$ is the nearness frame $(L \oplus M, \mathfrak{N}(L \oplus M))$. If $\mathfrak{N}L$ and $\mathfrak{N}M$ are uniformities, then $(L \oplus M, \mathfrak{N}(L \oplus M))$ is a uniform frame which is the coproduct of the uniform frames $(L, \mathfrak{N}L)$ and $(M, \mathfrak{N}M)$. With this formalisation of the coproduct of a nearness frame a pointfree version of the Tamano-Dowker type theorems for nearness frames is also presented by Dube in [8](1995). The structure of binary coproducts of frames is further studied by Chen [126](1992). Further categorical properties of **NFrm** are investigated by Seo and Lee [160](1998) and they particularly show that **NFrm** is complete (has equalizers and products) and also cocomplete. The arbitrary product of a family of nearness frames $((L_i, \mathfrak{N}L_i))_{i \in I}$ is constructed and Seo and Lee show that the categories **UFrm**, **StrNFrm** and **AUNFrm** are closed under the formation of products. The arbitrary coproduct of nearness frames is given by Picado and Pultr [154, §3.5. pp. 155].

Dube's doctoral thesis, *Structures in frames* [6], covers various other subcategories of nearness frames, including *contigual*, *locally fine* (**LfNFrm**) and *paracompact* (**ParNFrm**) nearness frames. The contigual nearness frames in a sense generalize compactness. In Dube [9, Proposition 4](1995) **ParNFrm** is shown to be a coreflective subcategory of **NFrm** whilst **LfNFrm** is a reflective subcategory of **NFrm** [9, Proposition 10]. We recall that a nearness frame $(L, \mathfrak{N}L)$ is *locally fine* iff whenever $A \in \mathfrak{N}L$ and $(B_a)_A = \{B_a : a \in A\} \subseteq \mathfrak{N}L$ then the collection $A \wedge (B_a)_A = \{a \wedge b : a \in A \text{ and } b \in B_a\} \in \mathfrak{N}L$. If $\mathfrak{N}L$ and $\mathfrak{M}L$ are nearnesses on L, Dube defines the following nearness on L:

$$\mathfrak{N}L/\mathfrak{M}L = \{A \in \mathrm{Cov}\, L : B \wedge (C_b)_B \leq A \text{ for some } B \in \mathfrak{M}L, (C_b)_B \subseteq \mathfrak{N}L\}.$$

We will call the above nearness the *Ginsberg-Isbell nearness* $\mathfrak{N}L/\mathfrak{M}L$. The locally fine reflection (which we will denote by ϑL) of a nearness frame $(L, \mathfrak{N}L)$ is constructed by the Ginsberg-Isbell derivatives which are defined by transfinite induction:

$$\begin{aligned} \mathfrak{N}^{(0)}L &= \mathfrak{N}L, \\ \mathfrak{N}^{(\alpha+1)}L &= \mathfrak{N}^{(\alpha)}L/\mathfrak{N}^{(\alpha)}L, \end{aligned}$$

$$\mathfrak{N}^{(\beta)}L \;=\; \bigcup_{\alpha<\beta} \mathfrak{N}^{(\alpha)}L \text{ for limit ordinal } \beta.$$

$\mathfrak{N}^{(\alpha)}L$ are nearnesses on L for each α. If $\mathfrak{N}_f L$ is the first $\mathfrak{N}^{(\alpha)}L$ such that $\mathfrak{N}^{(\alpha)}L = \mathfrak{N}^{(\alpha+1)}L$ then $\mathfrak{N}L \subseteq \mathfrak{N}_f L$ and $\vartheta L = (L, \mathfrak{N}_f L)$ is the locally fine reflection of L with reflection N-map mapping L identically given by $\mathrm{id}_L : (L, \mathfrak{N}L) \longrightarrow (L, \mathfrak{N}_f L)$ $(x \mapsto x \ \forall\ x \in L)$. Dube also introduces the concepts of *regular, uniformly completely regular, separated* and *uniformly connected* for nearness frames and investigates some of their properties in his thesis.

Remark 4.5. We note that, in the terminology of Dube, a nearness frame $(L, \mathfrak{N}L)$ (or the nearness $\mathfrak{N}L$) is called

(1) *contigual* iff $\forall\ A \in \mathfrak{N}L\ \exists\ B \subseteq_{<\omega} A$, $B \in \mathfrak{N}L$. This is essentially the notion of total boundedness that is explained in the forthcoming Section §5.1.

(2) *paracompact* iff $\forall\ A \in \mathfrak{N}L\ \exists$ ulf $B \in \mathfrak{N}L$, $B \leq A$. A subset $B \subseteq L$ is *uniformly locally finite* (or ulf for brevity) if there is $C \in \mathfrak{N}L$ such that for each $b \in B$, $C_b = \{c \in C\ :\ b \wedge c \neq 0\} \subseteq_{<\omega} C$. **ParNFrm** is the subcategory of paracompact nearness frames.

(3) *regular* iff $\forall\ A \in \mathfrak{N}L\ \exists\ B \in \mathfrak{N}L$, $B \lhd A$ (B *strongly refines* A) iff $\forall$ $A \in \mathfrak{N}L$, $\check{A} = \{b \in L : \exists\ a \in A, b \lhd a\} \in \mathfrak{N}L$. This is precisely the notion of a strong nearness.

(4) *uniformly completely regular* (ucr) iff $\forall\ A \in \mathfrak{N}L\ \exists\ B \in \mathfrak{N}L$, $B \lhd\!\lhd A$ where $B \lhd\!\lhd A$ means that $\forall\ b \in B\ \exists\ a \in A$, $b \lhd\!\lhd a$. Here, $b \lhd\!\lhd a$ iff $\exists$ an N-map $h : (\mathfrak{O}[0,1], \mathfrak{O}\nu) \longrightarrow (L, \mathfrak{N}L)$ such that $a \wedge h((0,1]) = 0_L$ and $h([0,1)) \leq b$ where $[0,1]$ is the nearness space with usual nearness ν.

(5) *separated* iff for each near and Cauchy $A \subseteq L$, $\eth A$ is near where $\eth A = \{y \in L : A \cup \{y\}$ is near$\}$. We recall from [6, Chapter 3, p.38] that for $(L, \mathfrak{N}L) \in$ **NFrm**, a subset $A \subseteq L$ is *near* iff for each $B \in \mathfrak{N}L$, $B \cap \sec A \neq \emptyset$ where $\sec A = \{x \in L : x \wedge a \neq 0\ \forall\, a \in A\}$. Equivalently, A is near iff $A^{\neg} \notin \mathfrak{N}L$ where $A^{\neg} = \{a^* : a \in A\}$. Furthermore, A is *Cauchy* iff $\sec A$ is near.

(6) *uniformly connected* iff each N-map $(\mathbb{4}, \mathrm{Cov}\,\mathbb{4}) \xrightarrow{\varphi} (L, \mathfrak{N}L)$ factors through the N-map $(\mathbb{2}, \mathrm{Cov}\,\mathbb{2}) \xrightarrow{\psi} (L, \mathfrak{N}L)$ iff there is $x \in \mathbb{4}\backslash\mathbb{2}$, $\varphi(x) = 0_L$.

5 Postdoctoral research (1993 - 1997)

Dube never ventured into or took up any postdoctoral fellowship locally in South Africa nor abroad but pursued his academic career at the University of Zululand after the award of his doctorate. He continued his research focussing on nearness frames and pointfree topology at the unstructured level publishing the work from his doctoral thesis with further new results and original postdoctoral work. The paper Dube [8](1995) focuses on presenting [6, Chapter 6.2] on the pointfree version of the Tamano-Dowker type theorems given by Bentley for nearness spaces in [123](1991). In [9](1995), Dube articulates the work in [6, Chapter 3] on paracompactness and locally fine nearness frames. The main aim of [9] is to show that the category **ParNFrm** forms a bicoreflective subcategory of **NFrm** whilst the category **LfNFrm** forms a bifreflective subcategory of **NFrm**. In the next two subsections we showcase the independent postdoctoral study of Dube [7](1994), Dube [11, 12] (1996) and the 1994 investigation of Dube on the category **StrNFrm** of strong nearness frames that appeared in Dube [12](1999).

5.1 Uniform complete regularity and uniform normality Ori (Dube's co-supervisor) together with Herrlich introduced the concept of *completely within* and *completely regular* for nearness spaces in their paper [134](1988). If (X, μ) is a nearness space and $A, B \subseteq X$, then A is *completely within* B (which we express as $A \blacktriangleleft\!\blacktriangleleft B$) if there is a uniformly continuous map $f : X \longrightarrow [0, 1]$ such that $f(A) \subseteq \{0\}$ and $f(X - B) \subseteq \{1\}$. The nearness space (X, μ) is then called *completely regular* if for each $\mathscr{U} \in \mu$, we have that $\mathscr{U}_{\blacktriangleleft\blacktriangleleft} \in \mu$ where $\mathscr{U}_{\blacktriangleleft\blacktriangleleft} = \{A \in \wp(X) : A \blacktriangleleft\!\blacktriangleleft U \text{ for some } U \in \mathscr{U}\}$.

In [11](1996), emanating from his thesis, Dube provides a pointfree analogue for the relation $\blacktriangleleft\!\blacktriangleleft$ which he represents as $\lhd\!\lhd$ and introduces *uniform complete regularity* (ucr) for nearness frames which we have described in Remark 4.5(4). It then turns out that an N-space $(X, \beta) \in$ **BNear** is completely regular iff the nearness frame $(\mathfrak{O}X, \mathfrak{O}\beta)$ is ucr [11, Proposition 2.3] and that the category **BNear** contains all *regular* N-spaces [11, Lemma 2.8]. A nearness space (X, μ) is *regular* if for each $\mathscr{A} \in \mu$ there is $\mathscr{B}$ such that $\mathscr{B} \leq_s \mathscr{A}$ (*$\mathscr{B}$ strongly refines $\mathscr{A}$*) which means that for each $A \in \mathscr{A}$ there is $B \in \mathscr{B}$ and $\mathscr{C} \in \mu$ such that $\mathrm{St}(A, \mathscr{C}) \subseteq B$. Consequently, $(X, \beta) \in$ **BNear** is regular iff $(\mathfrak{O}X, \mathfrak{O}\beta) \in$ **StrNFrm**. Baboolal [11, Proposition 2.5] communicated that a frame is completely regular iff it has a ucr nearness. As a consequence, Dube provides a new proof of the well-known result that a topological space is uniformizable iff it is completely regular [11, Corollary 2.10].

Dube also introduces *totally bounded* nearnesses as those that have a base consisting of finite uniform covers. **TbNFrm** denotes the subcategory of totally bounded nearness frames. Given any nearness $\mathfrak{N}L$ on a frame L, $\mathfrak{N}_{\mathfrak{t}}L = \{A \in \mathfrak{N}L \,:\, B \leq_{<\omega} A \,\text{for some}\, B \in \mathfrak{N}L\}$ is a totally bounded nearness on L. The pair $(L, \mathfrak{N}_{\mathfrak{t}}L)$ is then the *totally bounded coreflection* of the nearness frame $(L, \mathfrak{N}L)$ with coreflection N-map given by the identity $\mathrm{id}_L : (L, \mathfrak{N}_{\mathfrak{t}}L) \longrightarrow (L, \mathfrak{N}L)$. Thus **TbNFrm** is a coreflective subcategory of **NFrm**. By [11, Lemma 3] the totally bounded nearness frames that are strong are precisely the uniform frames. Banaschewski and Pultr in [120](2012) provide a more in-depth study into totally bounded nearness frames. We discuss more coreflective subcategories of **NFrm** developed by Dube in Section §6.1.

Dube then defines an N-frame $(L, \mathfrak{N}L)$ to be *uniformly normal* (we abbreviate this to *u-normal*) if both $\mathfrak{N}L$ and $\mathfrak{N}_{\mathfrak{t}}L$ are strong nearnesses on L ($\mathfrak{N}_{\mathfrak{t}}L$ is thus a uniformity). Accordingly, normality for regular frames may be characterised by u-normality of the fine nearness. We have that a regular frame is normal provided that the fine nearness is u-normal. With Countable Dependent Choice, u-normality implies ucr so that the underlying frame of a u-normal nearness frame is completely regular [11, Proposition 3.6]. We remark here that for a u-normal nearness frame $(L, \mathfrak{N}L)$, $(L, \mathfrak{N}_{\mathfrak{t}}L) \in$ **UFrm** which makes L completely regular (with CDC, since it is uniformizable).

The uniform coreflection of an N-frame is discussed for the very first time by Dube [11](1996), the construction of which was communicated to

Dube by Banaschewski. The research in [11] was carried out during the period 1993 -1994 and submitted for publication in September of 1994. For an N-frame $(L, \mathfrak{N}L)$, the normal uniform covers (recall from §3.1.4.), which we denote by $\mathfrak{N}_{\mathfrak{n}}L = \{A \in \mathrm{Cov}_n L : A \in \mathfrak{N}L\}$, is a preuniformity on L. Consequently, $\mathfrak{N}_{\mathfrak{n}}L$ determines an interior operator $k : L \longrightarrow L$ on L defined by $k(a) = \bigvee\{x \in L : x \triangleleft_{\mathfrak{N}_{\mathfrak{n}}L} a\}$ such that $\mu L = \mathrm{Fix}(k) = \{a \in L : a = k(a)\}$ is a subframe of L and $\mathfrak{U}\mu L = \{k(A) : A \in \mathfrak{N}_{\mathfrak{n}}L\}$ is a uniformity on μL which generates $\mathfrak{N}_{\mathfrak{n}}L$ (see Banaschewski and Pultr [117, Lemma 2](1993)). Then $\mu L = (\mu L, \mathfrak{U}\mu L)$ is the uniform coreflection of $(L, \mathfrak{N}L)$ with coreflection map given by the inclusion $i : (\mu L, \mathfrak{U}\mu L) \longrightarrow (L, \mathfrak{N}L)$. The construction of the uniform coreflection μL of an N-frame $(L, \mathfrak{N}L)$ is also given by Baboolal and Ori [107, Theorem 2.1](1994) which they use in their construction of the Samuel compactification of a nearness frame. Consequently, with the use of the uniform coreflection of a nearness frame, Dube establishes the pointfree analogue of the result in spaces given by Bentley [122, Proposition 2.6](1977) that a u-normal nearness frame has the same underlying frame as its uniform coreflection.

In 1994, Dube presented a focussed study on the category **StrNFrm** at the SoCat94 symposium at UCT that highlighted the relation between regular nearness spaces and strong nearness frames (which was published as [12](1999)). Characterizations of pointfree paracompactness was the main theme with investigations into local fineness, subfiness and paracompactness for nearness frames. Dube provides a neat consequence relating to the latter and strongness in showing the pointfree version of the classical result of Michael [147](1957) in the characterization of paracompact spaces that a regular Lindelöf frame is paracompact [12, Corollary 6]. The result here involves the use of entirely pointfree nearness concepts. The origins of this result traces back as a consequence of the work of Madden and Vermeer [143](1986) who show that localic regular Lindelöfness and realcompactness are equivalent. Alternate proofs of this novel result may be found either in Sun [164, Corollary 1](1989) who uses σ-locally finite refinements or in Walters-Wayland [167, Lemma, p.273](1999) who uses the cozero part of a frame. Mugochi (Dube's first doctoral student) pursued with studies into totally boundedness, strongness and ucr for nearness frames in his doctoral studies [148](2009). Dube's descendant in Mogochi's master's student

Haimene [132](2018) also looked at the properties of totally strong and ucr nearness frames and studied the relationship between them.

5.2 Separability and metrizability It is well-known that a topological space is called *separable* provided that it has a countable dense subset. The terminology was introduced in [128](1906) by Fréchet wherein the formal notion of a metric space was first established. *Metrizable* locales were first introduced by Isbell [140](1972) as those uniform locales for which the uniformity has a countable base. Pultr in [155, 156](1984) formally introduced metric diameters and provides an in-depth study of metric frames. Sun in [164](1989) continued with the study of metric frames with an emphasis on investigating paracompactness and solving an open question posed by Pultr [157](1984) on whether metrizable locales are paracompact. Sun provides an affirmative answer to this open question in [164, Corollary 2]. Dube [7](1994) introduces a pointfree translation of separability for locales. A locale L is separable if there is $S \subseteq_\omega L - \{1\}$ such that for each $0 \neq x \in L$, $x \vee s = 1$ for some $s \in S$. The concept is conservative for T_1 spaces in the sense that a T_1 space X is separable iff the locale of its open sets $\mathfrak{O}X$ is separable. An observation is that a regular locale with a countable base is separable. In so doing, Dube establishes a pointfree version of the classical Metrization Theorem of Urysohn [7, Proposition 3.3] given below.

Proposition 5.1 (Urysohn's Metrization Theorem for Locales)**.**
A locale is separable and metrizable iff it is regular and has a countable base.

Four years later, Banaschewski and Pultr in [119](1998) presented a unified treatment of pointfree metrization theorems based on an analysis of special properties of bases. Particularly, they considered a frame L with a basis B of the types σ-discrete (*Bing base*), σ-locally finite (*Nagata-Smirnov base*), regular (*Archangelskij base*), σ-admissible or σ-stratified. For regular frames these various bases types coincide [119, Theorem 3.6]. The authors relate that "any result which asserts the existence of a countable basis of uniformity for a certain class of frames is rightly considered a *metrization theorem*". Various metrization theorems are identified according to the basis type and the authors realise what we may refer to as the Moore Metrization Theorem , the Nagata-Smirnov Metrization Theorem, the Bing Metrization Theorem (BMT) and the Archangelskij Metrization Theorem.

Consequently, the BMT generalizes the original Urysohn's Metrization Theorem in pointfree form in that any regular frame with a countable basis is metrizable. Thus in [119], the UMT is a consequence of the BMT which also provides the compact variant of the UMT, namely that *a compact regular frame is metrizable iff it has a countable basis.* The Dube notion of separability and the consequential proof of the UMT may thus be included in the unified treatment of pointfree metrization theorems. The compact variant of the UMT can be realised from Dube's results in [10](1996) which we briefly consider next.

Building on the results on Lindelöfness, paracompactness and coproducts, Dube shows that a paracompact separable locale is Lindelöf [7, Proposition 3.6] and that, as in spaces, separability is countably productive. Consequently, since metrizable locales are paracompact (Sun [164, Corollary 2]), separable metrizable locales are Lindelöf. The Stone-Čech compactification βL features predominantly in the work of Dube post the award of his doctorate. It is in [7] that he first makes reference to βL and proves that if a normal regular locale L is separable, then so is βL [7, Proposition 3.9]. In [10](1996), Dube continues with the study on separability that he defined in [7] and shows, as in the classical case for spaces, that separability and Lindelöfness are equivalent for metrizable frames [10, Proposition 2]. Furthermore, quotients of separable metrizable frames are separable and regular subframes of compact metrizable frames are metrizable.

6 Structured frames and completion

Dube's direct description of the completion $(\bar{L}, \mathfrak{N}\bar{L})$ of a nearness frame $(L, \mathfrak{N}L)$ emanating from his doctoral thesis is described in Section §4.3. If $\mathfrak{N}L$ is a uniformity, then $(\bar{L}, \mathfrak{N}\bar{L})$ is also its uniform completion. Further discussions on the category **UFrm** and constructions of the completion of a uniform frame are given by Isbell [140](1972), Kříž [141](1986), in a series of papers by Banaschewski and Pultr [116](1990), [118](1996) and [120](2012), and individually in [111](1996) by Banaschewski. Apart from the foundations on nearness frames presented in [6] and the work on metrizable locales that we have encountered in Section §5.2, Dube has made further noteworthy contributions in the category **NFrm** and **UFrm**. Particularly, in

NFrm, various subcategories are discovered over and above the ones that he initially studied in [6]. Dube (together with Mugochi) investigate certain subcategories of **NFrm** and determine whether they are coreflective or reflective in the subcategories of nearness frames. We highlight these in Section §6.1. Thereafter we consider preservation of properties by the completion in Section 6.2 that Dube establishes for nearness frames and some of its subcategories. Lastly, in Section §6.3, we feature the notion of *uniform paracompactness* and its applications in **UFrm** that Dube invents and delves into.

6.1 Reflections and coreflections in NFrm In Section §4 we recalled the subcategory **LfNFrm** of locally fine nearness frames and the category **ParNFrm** of paracompact nearness frames and their homomorphisms. In this section we also refer to the category of strong nearness frames **StrNFrm** and almost uniform nearness frames **AuNFrm** that we have describe in Section §3.4 and the subcategory **TbNFrm** shared in Section §5. We consolidate the various subcategories emerging from Dube's workings. We recall the following classifications of nearness structures or a nearness frame.

Remark 6.1. A nearness frame $(L, \mathfrak{N}L)$ or the nearness $\mathfrak{N}L$ is called

(1) *fine* if $\mathfrak{N}L = \operatorname{Cov} L$. **FNFrm** is the subcategory of fine nearness frames (see Section §3.4). **FNFrm** is a reflective subcategory of **NFrm**.

(2) *quotient-fine* if there is an onto frame homomorphism $h : M \longrightarrow L$ such that $\mathfrak{N}L = \{h(C) : C \in \operatorname{Cov} M\}$. **QfNFrm** is the resulting subcategory of quotient-fine nearness frames. An alternate characterisation of quotient-fine nearness frames is given by its completion. $(L, \mathfrak{N}L)$ is quotient-fine iff its completion $(\bar{L}, \mathfrak{N}\bar{L})$ is fine [35, lemma 3.2]. Furthermore, **FNFrm** $\subset$ **QfNFrm** $\subseteq$ **StrNFrm**.

(3) *interpolative* if $x \lhd_{\mathfrak{N}L} z$ in L then $x \lhd_{\mathfrak{N}L} y \lhd_{\mathfrak{N}L} z$ for some $y \in L$. The subcategory of interpolative nearness frames is denoted by **IntNFrm**.

(4) *uniform prenormal* if $\mathfrak{N}L$ is strong and its totally bounded coreflection $\mathfrak{N}_{\mathfrak{t}}L$ is also strong. The nearness $\mathfrak{N}_{\mathfrak{t}}L$ is described in Section §5.1. **UpnNFrm** denotes the subcategory of uniformly prenormal nearness frames.

(5) *uniformly normal* if $\mathfrak{N}L$ is strong and the totally bounded coreflection of its completion $\mathfrak{N}_t\bar{L}$ is also strong. **UnNFrm** is the subcategory of uniformly normal nearness frames.

We then have the following strict containment of subcategories of **AuNFrm**.

Proposition 6.2 ([40], Proposition 4.3)**.**

$$\mathbf{UFrm} \subset \mathbf{UnNFrm} \subset \mathbf{UpnNFrn} \subset \mathbf{AuNFrn}.$$

We now summarize the various reflective and coreflective subcategories of nearness frames established by Dube in [6, 9, 40, 41, 56].

Proposition 6.3. *The following are reflective subcategories.*

(1) **LfNFrm** *is reflective in* **NFrm** [9, Proposition 10]*.*

(2) **FNFrm** *is reflective in* **NFrm** [40]*.*

(3) **QfNFrm** *is reflective in* **StrNFrm** [40, Proposition 3.4] *.*

Proposition 6.4. *We have the following coreflective subcategories.*

(1) **ParNFrm** *is coreflective in* **NFrm** [9, Proposition 4]*.*

(2) **TbNFrm** *is coreflective in* **NFrm** [11, 41]*.*

(3) **AuNFrm** *is coreflective in* **IntNFrm** [41, Proposition 3.3]*.*

Proposition 6.4(3) was proved by means of a direct construction of the coreflection in [41](2011). An open question arose in [41] as to whether strong nearness frames are coreflective in nearness frames. The result, however, could not be achieved by a direct construction and remained open. Frith and Schauerte in [131](2014) pursued with a study on coreflections of nearness frames and conceived a general method for constructing coreflections in the category of nearness frames. With their general method, they show that **StrNFrm** is coreflective in **NFrm** thereby answering the open question in [41] in the affirmative. The category of complete nearness frames is **CNFrm** and we note that completion is a coreflection for uniform frames (Banaschewski and Pultr [116](1990)), and also for strong nearness frames

(Banaschewski, Hong and Pultr [121, Proposition 2.4].(1998)). Further subcategories of nearness frames are created in Dube and Mugochi [35](2010), and Dube, Mugochi and Naidoo [56](2014). These are **ZdNFrm** the zero-dimensional nearness frames , **HzdNFrm** the H-zero-dimensional nearness frames, **ConNFrm** the constrained nearness frames, and **CntrNFrm** the controlled nearness frames. These subcategories are discussed in the next section.

6.2 Permanence of properties under completion If a nearness (or uniform) frame $(L, \mathfrak{N}L)$ has a property P, the necessary and sufficient conditions under which the completion $(\bar{L}, \mathfrak{N}\bar{L})$ has property P is a natural inquiry. Dube in his doctoral thesis [6](1992) investigates such and shows that a nearness frame $(L, \mathfrak{N}L)$ has property P iff its completion $(\bar{L}, \mathfrak{N}\bar{L})$ also has property P for all of the properties P given in Remark 4.5.

Proposition 6.5. *Let $(L, \mathfrak{N}L)$ be a nearness frame with completion $(\bar{L}, \mathfrak{N}\bar{L})$. Then $(L, \mathfrak{N}L)$ is*

(1) *locally fine iff $(\bar{L}, \mathfrak{N}\bar{L})$ is locally fine.*

(2) *paracompact iff $(\bar{L}, \mathfrak{N}\bar{L})$ is paracompact.*

(3) *contigual iff $(\bar{L}, \mathfrak{N}\bar{L})$ is contigual.*

(4) *uniform iff $(\bar{L}, \mathfrak{N}\bar{L})$ is uniform.*

(5) *regular iff $(\bar{L}, \mathfrak{N}\bar{L})$ is regular.*

(6) *ucr iff $(\bar{L}, \mathfrak{N}\bar{L})$ is ucr.*

(7) *separated iff $(\bar{L}, \mathfrak{N}\bar{L})$ is separated.*

(8) *uniformly connected iff $(\bar{L}, \mathfrak{N}\bar{L})$ is uniformly connected.*

The above properties are thus permanent under completion. This investigative theme continues in Dube [13](1999), Dube and Mugochi [35](2010), and Dube, Mugochi and Naidoo [56](2014) albeit not the sole intention.

6.2.1 σ-compactness

One of the essential purposes in [13](1999) is to establish permanence for the property of *σ-compactness* under completion that Dube introduces in the categories **Near** and **NFrm** akin to the same-named classical topological notion. We recall that a topological space is said to be σ-compact in case it is expressible as a union of compact subsets. Dube defines σ-compactness for a nearness frame $(L, \mathfrak{N}L)$ via totally bounded elements of the underlying frame and a base $\mathcal{B}$ for the nearness $\mathfrak{N}L$. An element $t \in L$ is *totally bounded* iff $\forall\ B \in \mathcal{B}\ \exists\ F \subseteq_{<\omega} B$ such that $t \leq \bigvee F$ (every basic uniform cover has a finite subcollection whose join is above t). We will denote the collection of all totally bounded elements of a nearness frame $(L, \mathfrak{N}L)$ by $\mathfrak{tb}L$. For a fine nearness frame, totally bounded elements are precisely the compact elements of the frame that are described in Remark 3.1(4) so that $\mathfrak{tb}L = \mathfrak{k}L$ for fine nearness frames. A nearness frame $(L, \mathfrak{N}L)$ is *σ-compact* if $\exists\ T \in \mathfrak{N}L$ such that $T \subseteq_{\omega} \mathfrak{tb}L$ (L has a *countable* uniform cover consisting of totally bounded elements). In the category **Near**, in a similar fashion, Dube defines a nearness space to be σ-compact if the space has a countable uniform cover consisting of totally bounded subspaces. Dube then goes on to show the permanence of σ-compactness under completion for regular nearness spaces [13, Proposition 6]. However, in **NFrm**, σ-compactness is permanent under Cauchy completion.

Theorem 6.6 ([13], Proposition 5)**.**
$(L, \mathfrak{N}L)$ is σ-compact iff $(cL, \mathfrak{N}cL)$ is σ-compact.

Permanence under Cauchy completion also features in Dube [18](2004). Dube provides and alternate proof that locally fine is permanent under completion and also includes the following upshot in the same result.

Theorem 6.7 ([18], Corollary 4)**.**
$(L, \mathfrak{N}L)$ is locally fine iff $(cL, \mathfrak{N}cL)$ is locally fine.

Hence, the locally fine property on a nearness frame is permanent under Cauchy completion.

6.2.2 Zero-dimensionality

We recall from Remark 3.1(5) that a frame L is zero-dimensional if for each $x \in L$, $x = \bigvee C$ for some $C \subseteq \mathfrak{c}L$ (every element of L is a join of complemented elements). **ZdFrm** is the category of zero-dimensional frames. In [35](2010), Dube together with Mugochi extend the notion of zero-dimensionality to nearness frames by calling an N-frame *uniformly zero-dimensional* if for each $A \in \mathfrak{N}L$ there is some $B \in \mathfrak{N}L$ in which $b \triangleleft_{\mathfrak{N}L} b$ for each $b \in B$ such that $B \leq A$. The resulting subcategory of uniformly zero-dimensional nearness frames is denoted **ZdNFrm**. The motivation behind formalizing the category **ZdNFrm** was to materialise the results of McKee [146, Theorem 3](1994) on zero-dimensional nearness spaces in pointfree form. For $(L, \mathfrak{N}L) \in$ **ZdNFrm**, the underlying frame $L \in$ **ZdFrm**. Furthermore, **ZdNFrm** $\subseteq$ **StrNFrm**. Mckee's result in pointfree form is realised in [35, Proposition 3.8]. Herrlich [133](1974) initially presented a notion of zero-dimensionality for nearness spaces by uniform partitions. Herrlich's notion is translated into the language of frames and a nearness frame is called H-zero-dimensional (H for Herrlich) if every uniform cover is refined by a uniform partition. A partition of a frame L is any cover in which its distinct elements miss each other. The resulting category of H-zero-dimensional nearness frames is denoted **HzdNFrm**. In comparison, it turns out that **HzdNFrm** $\subseteq$ **UFrm** and **HzdNFrm** $\subset$ **ZdNFrm**. Furthermore, both **ZdNFrm** and **HzdNFrm** are closed under coproducts [35, Proposition 3.10 & 3.20]. The following result is also solidified in both **ZdNFrm** and **HzdNFrm**.

Proposition 6.8 ([35], Corollary 3.6 & Lemma 3.17)**.** *$(L, \mathfrak{N}L)$ is uniformly zero-dimensional (resp. H-zero-dimensional) iff its completion $(\bar{L}, \mathfrak{N}\bar{L})$ is uniformly zero-dimensional (resp. H-zero-dimensional).*

For nearness frames, the properties of uniform zero-dimensionality and H-zero-dimensionality are thus permanent under completion.

6.2.3 Čech completeness

In [56](2014), the notion of *Čech-complete* and *strong Čech complete* is defined for any frame via the clustering and convergence of certain designated filters in a frame, the cover-dependent $\mathcal{C}-$Cauchy ones, as an adaptation of Hong's notion of convergence [136](1995). The intention of these concepts is to crystallize a pointfree version of Bentley and Hunsaker's notion of *Čech-comptete nearness spaces* [124](1992). For a frame L and $\mathcal{C} \subseteq \operatorname{Cov} L$, a filter F in L is *$\mathcal{C}$-Cauchy* if $F \cap C \neq \emptyset \ \forall \ C \in \mathcal{C}$. A frame L is then Čech-complete (resp. strongly Čech-complete) if there is a countable collection $\mathcal{C}$ of covers of L such that every $\mathcal{C}$-Cauchy filter F in L clusters (resp. converges). The notion of *constrained* (resp. *controlled*) in [56] is the adaptation of Čech-completeness (resp. strong Čech-completeness) to nearness frames. A nearness frame $(L, \mathfrak{N}L)$ is constrained if there is a countable collection $\mathcal{C}$ of uniform covers such that each every $\mathcal{C}$-Cauchy filter F in L is weakly Cauchy (that is, $F^{\neg} = \{x^* : x \in F\} \notin \mathfrak{N}L$), the latter notion defined by the author in [151](2005). We then have the subcategory of constrained nearness frames **ConNFrm** which is countably coproductive in **NFrm** [56, Proposition 4.11]. On the other hand, $(L, \mathfrak{N}L) \in$ **NFrm** is controlled if there is $\mathcal{C} \subseteq_\omega \mathfrak{N}L$ such that every $\mathcal{C}$-Cauchy filter F in L is Cauchy (F meets every uniform cover). The resulting subcategory of controlled nearness frames is denoted by **CntrNFrm** and **CntrNFrm** $\subseteq$ **ConNFrm**. The following result is then materialised in the categories **ConNFrm** and **CntrNFrm**.

Proposition 6.9 ([56], Corollary 4.6 & Corollary 4.17)**.** $(L, \mathfrak{N}L)$ *is constrained (resp. controlled) iff its completion* $(\bar{L}, \mathfrak{N}\bar{L})$ *is constrained (resp. controlled).*

Thus the property of a nearness frame being constrained or controlled is permanent under completion and we may add the above result to the list in Proposition 6.5.

6.3 Uniform paracompactness in UFrm Dube's study on uniform paracompactness independently in [18](2004) transpired over his busy term as Vice-Rector for Academic Affairs and Research at the University of Zululand. He was looking at realising a pointfree version of uniform paracompactness for uniform frames given in the study of Hohti [135](1981) in spaces on the same named classical concept. Particularly, Dube wanted to characterise uniform paracompactness in terms of C-normality in attempting to show that *a uniform frame* L *is uniformly paracompact iff* $L \oplus \beta L$ *is C-normal.* Coincidently, I was also an academic staff at the University of Zululand at that time working on my doctoral studies and labouring on the work of Howes [138](1992) on paracompactifications and preparacompactness in pointfree form. Little did I (or Dube) know that our independent work would coincide with uniform paracompactness. I recall an appointment with Dube at his deluxe executive management office at the

University of Zululand. He spent some time, over his busy schedule, relating to me his proof of [18, Proposition 8] which I really appreciated coming from him and in his capacity as Vice-Rector. We would then present our independent results at the South African Mathematical Society Congress hosted at our destiny UNISA in 2000. Professor Peter Witbooi, from the University of the Western Cape, attended our talks and commented that we should consider joint work in the future being at the same institution after all. Peter Witbooi's prophetic words eventually materialised some 11 years later. Moreover, a follow up joint publication on uniform paracompactness [66](2015) combined the initial work we had done independently in 1999.

In [18], Dube continues with his initial study on paracompactness and locally fine nearness frames that he presented in [9](1995) which ensued from [6]. Dube shows that **TbNFrm** $\subseteq$ **LfNFrm** [18, Proposition 1] and then considers the locally fine reflection ϑL of a nearness frame $(L, \mathfrak{N}L)$ that we have illustrated towards the end of Section §4. Each Ginsberg-Isbell derivative $(L, \mathfrak{N}^{(\alpha)}L) \in$ **StrNFrm** whenever $(L, \mathfrak{N}L) \in$ **StrNFrm**. Thus for every strong nearness frame $(L, \mathfrak{N}L)$ its locally fine reflection ϑL is also a strong nearness frame [18, Proposition 2]. Dube then considers the transactions of the locally fine reflector ϑ with the completion coreflector γ which we relate in the next Section §7. Dube turns the attention to the category **ParNFrm** but focuses on the subcategory **UFrm**. It is the first occasion that Dube departs from nearness frames and presents a focussed study just in the category **UFrm**. For a uniform frame $(L, \mathfrak{U}L)$, Dube extends the notion of a locally finite subset as done for nearness frames, retains the nomenclature and calls $A \subseteq L$ *uniformly locally finite* (ulf for brevity) if there is $U \in \mathfrak{U}M$ such that for each $u \in U$, $A_u = \{a \in A : a \wedge u \neq 0\} \subseteq_{<\omega} A$. $(L, \mathfrak{U}L)$ is *uniformly paracompact* if for each $A \in \operatorname{Cov} L$ there is ulf $B \in \operatorname{Cov} L$ such that $B \leq A$. The precompact coreflection (totally bounded terminology for nearness frames) $(L, \mathfrak{U}_{\mathfrak{t}}L)$ of the uniform frame $(L, \mathfrak{U}L)$ is then considered with the Ginsberg-Isbell nearness $\mathfrak{U}_{\mathfrak{t}}L/\mathfrak{U}L$. The following characterisations of uniform paracompactness is then proved by Dube.

Proposition 6.10. *The following statements are equivalent for any uniform frame* $(L, \mathfrak{U}L)$.

(1) *For each* $A \in \operatorname{Cov} L$, $A^{<\omega} = \{\bigvee B : B \subseteq_{<\omega} A\} \in \mathfrak{U}L$.

(2) L *is a paracompact frame and* $\operatorname{Cov} L = \operatorname{Cov}_{\mathfrak{t}} L/\mathfrak{U}L$.

(3) $(L, \mathfrak{U}L)$ *is uniformly paracompact.*

For (2) in the above proposition, L paracompact renders $\operatorname{Cov} L$ a uniformity and $\operatorname{Cov}_{\mathfrak{t}} L$ is the associated precompact uniformity. Dube then considers small elements of a frame (see Remark 3.1(8)) and defines a uniform frame $(L, \mathfrak{U}L)$ to be *uniformly locally compact* if there is $U \in \mathfrak{U}L$ such that $U \subseteq L_{\ll}$. Uniform paracompactness for continuous frames are then characterised as follows in [18, Proposition 13].

Proposition 6.11. *A uniform frame with a continuous underlying frame is uniformly paracompact iff it is uniformly locally compact.*

Other pertinent results that Dube achieves pertaining to uniform paracompactness are given next.

Theorem 6.12. *If a uniform frame* $(L, \mathfrak{U}L)$ *is uniformly paracompact, then* $(L, \mathfrak{U}L)$ *is complete, Cauchy complete and* C*-normal.*

Let $(L, \mathfrak{U}L)$ be a uniform (or nearness) frame, $A \subseteq L$ and $b \in L$. Dube [6] defines $b \wedge A = \{b \wedge a : a \in A\}$ and

$$b \wedge \mathfrak{U}L = \{C \in \operatorname{Cov}(\downarrow b) : b \wedge D \leq C \text{ for some } D \in \mathfrak{U}L\}.$$

A is *uniformly locally uniform* (ulu for brevity) if there is $B \in \mathfrak{U}L$ such that for each $b \in B$, $b \wedge A \in b \wedge \mathfrak{U}L$. The uniform frame $(L, \mathfrak{U}L)$ is *C-normal* if each of its two-covers (those $\{a, b\} \in \mathfrak{U}L, a, b \in L$) is ulu. Equivalently, a uniform frame is C-normal iff each of its finite covers is ulu [18, Corollary 17]. The result of Hohti that Dube was aiming to achieve still remains elusive and it is an open question whether a uniform frame L is uniformly paracompact iff $L \oplus \beta L$ is C-normal.

In [152](2007) an independent study of uniform paracompactness in uniform frames was ventured into with a different point of view focusing on convergence in frames. In [151](2005) *weakly Cauchy filters* and the notion of *strong Cauchy complete* for uniform frames were introduced. For $(L, \mathfrak{U}L) \in$ **UFrm** a filter F in L is weakly Cauchy if $\sec F \cap A \neq \emptyset$ for each

$A \in \mathfrak{U}L$ where $\sec F = \{y \in L : y \wedge x \neq 0 \, \forall x \in F\}$. The uniform frame $(L, \mathfrak{U})$ is strongly Cauchy complete if every weakly Cauchy filter in L clusters in the sense of Hong [136]. It is then shown in [152, Theorem 3.5] that if the underlying frame of a uniform frame is Boolean then strong Cauchy completeness and uniform paracompactness are equivalent. It was left an open question whether Booleaness could be dropped. This open question formed the basis for continuing with the study on uniform paracompactness in Dube and Naidoo [66](2015). The initial discussions surrounding providing a solution to the open question arose at the airport in Lisbon, Portugal in 2012. Both Dube and I were returning home to South Africa after participating at the Workshop on Category Theory hosted by the University of Coimbra in Coimbra. The conference was in honour of George Janelidze, on the occasion of his 60^{th} birthday. We had a long layover at the airport and decided to use the time profitably in addressing the solution. The result [66, Proposition 3.1] answers the open question in the affirmative. It is shown that uniform paracompactness and strong Cauchy completeness are equivalent for any uniform frame. Consequently, the three variants of completeness for uniform frames have the strict implication: strong Cauchy complete $\Rightarrow$ complete $\Rightarrow$ Cauchy complete. The article also brings to the fore the role of the Stone-Čech compactication βL in the completeness criteria for uniform frames under the guise of uniform paracompactness. Uniform paracompactness for uniform frames is then characterised in terms of compactifications as follows in [66, Proposition 3.3].

Proposition 6.13. *The following statements are equivalent for a uniform frame* $(L, \mathfrak{U}L)$.

(1) *L is uniformly paracompact.*

(2) *For any compactification $h : M \longrightarrow L$ of L, if $c \in M$, $h(c) = 1_L$, then $\exists\, U \in \mathfrak{U}L$, $h_*(u) \prec\!\prec c \;\forall\, u \in U$.*

(3) *For any $c \in \beta L$, $\beta_L(c) = 1_L \;\exists\, U \in \mathfrak{U}L$, $(\beta_L)_*(u) \prec\!\prec c \;\forall\, u \in U$.*

The above concepts are then adapted to rope in the regular Lindelöf coreflection λL and its associated coreflection map $\lambda L \longrightarrow L$. Furthermore, a countable version of uniform paracompactness is formulated and, as might be expected, the cozero part of a frame is enlisted. For a uniform frame

$(L, \mathfrak{U}L)$ and $A \subseteq L$, A is called *locally countable* (for brevity loctble) if $\exists$ $B \in \mathfrak{U}L$ such that $\forall\, b \in B$, $B_a = \{a \in A : b \wedge a \neq 0\} \subseteq_\omega A$. $(L, \mathfrak{U}L)$ is called *uniformly para-Lindelöf* if $\forall\ A \in \mathrm{Cov}\, L\ \exists$ loctble $B \in \mathfrak{U}L$, $B \leq A$ (every cover of the underlying frame has a uniformly locally countable refinement). The concept is a pointfree version of the uniformly para-Lindelöf spaces given in Hohti [135]. Analogous to Proposition 6.10, we have the following result for para-Lindelöf uniform frames.

Proposition 6.14. *The following statements are equivalent for a uniform frame* $(L, \mathfrak{U}L)$.

(1) *L is uniformly para-Lindelöf.*

(2) *For each* $A \in \mathrm{Cov}\, L$, $A^{<\omega_1} = \{\bigvee B : B \subseteq_\omega A\} \in \mathfrak{U}L$.

(3) *L is a paracompact frame and* $\mathrm{Cov}\, L = \mathrm{Cov}_\mathfrak{c}\, L/\mathfrak{U}L$.

In the above result, the Ginsberg-Isbell nearness $\mathrm{Cov}_\mathfrak{c}\, L/\mathfrak{U}L$ (now a uniformity) coincides with the fine uniformity, and the separable coreflection

$$\mathrm{Cov}_\mathfrak{c}\, L = \{A \in \mathrm{Cov}\, L : B \leq A \text{ for some countable } B \in \mathfrak{U}L\}$$

is used. Analogous to Proposition 6.13, $\lambda_L : \lambda L \longrightarrow L$ is now used together with its right adjoint in place of the Stone-Čech compactiication.

Proposition 6.15. *A uniform frame $(L, \mathfrak{U}L)$ is uniformly para-Lindelöf iff* $\forall\ a \in \lambda L$ *with* $\lambda_L(c) = 1_L\ \exists\ U \in \mathfrak{U}L$, $(\lambda_L)_*(u) \prec\!\prec a\ \forall\ u \in U$.

It is known that the properties of uniform paracompactness and uniform para-Lindelöfness coincide for fine uniform frames. The equivalence of the two properties for all uniform frames remains an open question, noting that every uniformly paracompact uniform frame is uniformly para-Lindelöf. Next, for the countable version: A uniform frame is *uniformly countably paracompact* (or uctpara for brevity) if every countable uniform cover of the underlying frame has a ulf refinement. For Lindelöf uniform frames uctpara and uniformly paracompact coincide. Recall that a completely regular frame is *countably paracompact* (ctpara for short) if every countable cover has a locally finite refinement. We then have the corresponding analogous results for uctpara uniform frames with the invocation of the cozero part of the underlying frame and weakly Cauchy filters.

Proposition 6.16. *The following statements are equivalent for a uniform frame* $(L, \mathfrak{U}L)$.

(1) *L is uctpara.*

(2) *L is ctpara, and for any compactiication* $h : M \longrightarrow L$ *of L, if* $c \in \operatorname{Coz} M$, $h(c) = 1_L$, *then* $\exists\, U \in \mathfrak{U}L$, $h_*(u) \prec\!\prec c \;\forall\, u \in U$.

(3) *L is ctpara and for any* $c \in \operatorname{Coz} \beta L$, $\beta_L(c) = 1_L \;\exists\, U \in \mathfrak{U}L$, $(\beta_L)_*(u) \prec\!\prec c \;\forall\, u \in U$.

(4) *L is ctpara and every countably-based weakly Cauchy filter in L clusters.*

7 Commutation of functors

If we have any functors $\varrho, \zeta : \mathscr{C} \longrightarrow \mathscr{C}$ in a category $\mathscr{C}$, we say that ϱ and ζ *commute* if for an $\mathscr{C}$-object X we have that $\varrho(\zeta X)$ is isomorphic to $\zeta(\varrho X)$. Dube's first encounter with the commutativity of functors appears in [18]. He considers the completion functor γ and the locally fine reflection functor ϑ in **NFrm**. Dube then shows that the locally fine reflection of the completion of a strong nearness frame is isomorphic to the completion of the locally fine reflection of the given strong nearness frame. In this sense the functors ϑ and γ commute.

Proposition 7.1 ([18], Proposition 7)**.** *For* $(L, \mathfrak{N}L) \in$ **StrNFrm**, $\vartheta(\gamma L) = \gamma(\vartheta L)$.

The next such occasion appears with Mugochi in [40]. The almost uniform coreflection functor $\alpha :$ **IntNFrm** $\longrightarrow$ **IntNFrm** is considered where for any interpolative nearness frame $(L, \mathfrak{N}L)$, αL denotes its almost uniform coreflection and for any $h : (L, \mathfrak{N}L) \longrightarrow (M, \mathfrak{N}M)$ between interpolative nearness frames, $\alpha h : \alpha L \longrightarrow \alpha M$ is the uniform homomorphism mapping as h. Next, for a nearness frame $(L, \mathfrak{N}L)$ the totally bounded coreflection $\tau L = (L, \mathfrak{N}_{\mathfrak{t}} L)$ is considered. We have illuminated τL in Section §5.1. Dube and Mugochi then show that L is interpolative iff τL is interpolative so that we have the functor $\tau :$ **IntNFrm** $\longrightarrow$ **IntNFrm** where $\tau(L) = \tau L$ and for any uniform homomorphism $h : (L, \mathfrak{N}L) \longrightarrow (M, \mathfrak{N}M)$ between interpolative nearness frames $\tau h : \tau L \longrightarrow \tau M$ is a uniform homomorphism mapping as h. In general, $\alpha(\tau L) \neq \tau(\alpha L)$. However, α and τ do commute on the category of those nearness frames that have a strong totally bounded coreflection.

Proposition 7.2 ([40], Proposition 3.10)**.** *Let* $(L, \mathfrak{N}L) \in$ **NFrm** *such that* $\tau L \in$ **StrNFrm**. *Then* $(L, \mathfrak{N}L) \in$ **IntNFrm** *and* $\alpha(\tau L) = \tau(\alpha L)$.

The uniform coreflection μL of a nearness frame $(L, \mathfrak{N}L)$ is depicted in Section §5.1. Let μ be the uniform coreflection functor. It is then shown in [40] that α and μ commute.

Proposition 7.3 ([40], Proposition 3.13)**.** *If* $(L, \mathfrak{N}L) \in$ **IntNFrm**, *then* $\alpha(\mu L) = \mu(\alpha L) = \mu L$.

Remark 7.4 (The paracompact coreflection). Isbell in [140][Corollary 3.10](1972) first showed that the paracompact completely regular locales form a full reflective subcategory of completely regular locales. A more palatable offering of the paracompact coreflection πL is given by Banaschewski and Pultr [117, Proposition 2](1993) for a completely regular frame L. The latter description may also be found in Picado and Pultr [154, Chapter IX] which we outline below. Let $L \in$ **CRegFrm**. Equip L with its fine uniformity $\mathfrak{U}_F L$ (described in Section §3.4) and consider the completion $\gamma(L, \mathfrak{U}_F L)$. Let $\pi L = \gamma L$ be the underlying frame of the completion and $\pi_L : \pi L \longrightarrow L$ be the underlying frame homomorphism induced by the completion (acts as $\gamma_L : \gamma L \longrightarrow L$). πL is paracompact and is the paracompact coreflection of L with coreflection map given by the dense and onto homomorphism π_L. We denote the coreflection functor (the paracompact coreflector) by $\pi :$ **CRegFrm** $\longrightarrow$ **ParFrm**.

Dube in [85](2017) considers necessary and sufficient conditions for the commutativity of the Boolean functor ß (see Remark 3.1(6) and Banaschewski [112]) with the regular Lindelöf coreflector λ, the realcompact (Hewitt) coreflector υ, the paracompact coreflector π and the Stone-Čech coreflector β. To this end, Dube adapts the notion of weakly Lindelöf frames (see Section §3.1.5 and Madden and Vermeer [143]) to quasi-covers. Dube defines a frame to be *q-Lindelöf* if for each $A \in \mathrm{Cov}_q\, L$ there is $B \subseteq_\omega A$ such that $B \in \mathrm{Cov}_q\, L$ (every quasi-cover has a countable subset which is itself a quasi-cover). Furthermore, extending the notion to realcompactness, a frame L is *q-realcompact* if each maximal ideal of $\operatorname{Coz} L$ with a dense join has a countable subset with a dense join. Equivalently, L is q-realcompact iff every prime ideal of $\operatorname{Coz} L$ which is a quasi-cover has a countable quasi-subcover. The regular Lindelöf coreflector λ is akin to q-Lindelöfness as the Hewitt reflector υ is to q-realcompactness. For $L \in \mathbf{CRegFrm}$, the commutativity of ß with λ is materialised under q-Lindelöfness whilst ß commutes with υ under q-realcompactness.

Proposition 7.5 ([85] Proposition 3.6 & Proposition 5.4)**.** *The following are equivalent for $L \in \mathbf{CRegFrm}$.*

(1) ß$(\lambda L) \simeq \lambda($ß$L)$ *(resp.* ß$(\upsilon L) \simeq \upsilon($ß$L))$*.*

(2) *L is q-Lindelöf (resp. q-realcompact).*

(3) *ßL is Lindelöf (resp. realcompact).*

(4) *ßL is q-Lindelöf (resp. q-realcompact).*

Concerning the commutativity of ß with π, since $\pi_L : \pi L \longrightarrow L$ is dense onto and Boolean frames are paracompact we immediately have that for any $L \in \mathbf{CRegFrm}$ ß commutes with π vacuously.

Proposition 7.6. ß$(\pi L) \simeq \pi($ß$L) \simeq$ ßL*.*

Since Boolean frames are compact iff they are finite we concisely have the following commutation.

Proposition 7.7. ß$(\beta L) \simeq \beta($ß$L)$ *iff L is finite.*

8 Celebratory tributes

The following personal tributes were homaged to Themba Dube at the festive Gala Dinner of the TACT2022 International Conference. The authors have kindly agreed to have their transcripts included. Parts 8.1 and 8.4 were delivered by Naidoo at the gala dinner on behalf of the authors. The recording of Part 8.2 was audio streamed at the conference Gala Dinner venue, The Kloofzicht Lodge and Spa at the Oevermeer Bistro.

8.1 Prof Loyiso Nongxa *Loyiso Nongxa is a former Vice Chancellor of the University of the Witwatersrand.*

Good evening to everyone who has graced this special occasion by their presence, from near and afar. A warm and BIG HELLO to Professor Themba Dube who is being deservedly honoured this week. I sometimes regard or look at him as a precocious younger brother, but of course cannot say that loudly, lest I get into trouble.

I wish to express deep thanks and gratitude to the organisers of this event. I am cognizant of the fact that this is not an honour bestowed on all-and-sundry. Great idea and I hope it will serve as an inspiration to the next generation of mathematicians to aim for a similar recognition down the road.

Themba, I deeply regret that I have not been able to attend the TACT2022 in your honour. I had previously informed the organisers that I would attend and present a paper: the subject of the presentation would have been "**Evolution of SA Algebra research**" and highlight your contributions to the topic. Originally I had been invited to the Heidelberg Laureate Forum taking place this week in Germany, but I am currently self-isolating at home after contracting the dreaded COVID (I suspect in the Republic of the Western Cape two weeks ago).

I remember vividly when you and I first met just over 45 years ago at the University of Fort Hare. You were a bright-eyed self-confident incoming first year student with a reputation for being an outstanding mathematics student. I was a beginning graduate student and I think silently we each 'sized" each other up wondering who had a better aptitude or talent for mathematics. Well the passage of time has long settled that question and clearly and categorically demonstrated who is. **Proof**: this week's event – QED.

I cringe when I recall that I taught you undergraduate mathematics with only a 4-year degree in mathematics – what did I know about mathematics? I was just an enthusiast for the subject, with a little bit of arrogance and self-belief? I suspect I may have blithely and naively proclaimed that all finite groups would be known (by the beginning of the 21^{st} century?) and that we simply needed to (a) classify all finite simple groups and (b) solve the "extension problem". I am relieved that my teaching did not lead to permanent damage to your mathematical potential.

I followed your academic progress from afar, having chosen to spend almost 8 years outside South Africa. I often wondered by how much you smashed my mathematics records at Fort Hare. I believed then, and still do, that one needed international exposure in order to blossom and realise their full potential. When you stayed put within the South African university system, I feared that your talent would be wasted. I think you found as a supervisor one of the finest human beings and most understated and underrated mathematicians in our system whose contributions have gone unnoticed and unrecognized. I may be biased because he is a very good friend of mine. You and he have made significant contributions in nurturing and training the next generation of young "pointless" topologists some of whom I have come to admire. I wish they emulate your example.

Few people are aware or realise that you rose to the highest echelons of university administration - Deputy Vice-Chancellor and Acting Vice-Chancellor. From experience, I know that this is the graveyard of many research careers in academia. I still recall the few occasions when I would visit the University of Zululand as a member of one or the other Advisory Committee of the Foundation for Research Development. I was always told you were busy – now I am not sure whether you would be in meetings or working behind closed doors proving theorems! I would make it a point to remind our UZ hosts that I was your mathematics lecturer, that I taught you Abstract Algebra; I hoped that then they would afford me more undeserved respect. I remember also that years later I made a vain attempt to 'drag' you back into university administration during my time at the University of the Witwatersrand. But you wisely and firmly tuned us down – I forgave you for the rebuff, almost immediately.

Recently we have been trying to map out the mathematical sciences research landscape in South Africa, from antiquity until the present. One simply comes up with boxes labelled: Graph Theory; Analysis; Topology; Fluid

Dynamics; Group Theory; Cosmology and Relativity, Symmetry Analysis and place each name in a box; and research how the contents of each box have evolved over time. We found your name popping up in different boxes: Topology; Category Theory; Algebraic Structures and Commutative Algebra. I am somewhat 'relieved' that there is a 'point' to your mathematics and you are a fellow Algebraist.

Mathematical discovery and the sheer joy, satisfaction and fulfilment that one derives from it is, I believe, the main reason why many mathematicians persecute research. It is not simply about the number of publications or citations. I have always been struck and inspired by the energy you bring to sharing with others, during your seminars, the joy of mathematical discovery. Your seminars are taking the audience along on a journey that you have traversed and reaching the mountain top (a proof of a good theorem). Your enthusiasm is always infectious and of course your smile that brightens the seminar room. I suspect this has been an inspiration to your graduate students.

I am reminded of those papers that from time to time you would share with me, sometimes in the early hours of a Saturday morning with the message "Loyiso, here's a nice piece of work that I am sure you will enjoy" . I may have forgotten to inform you that I have been retired for almost FIVE years! Did I read and enjoy all these gems? I choose to invoke the 5th Amendment.

We are all proud of you and bask in the glory of your achievements. You deserve all the accolades, past present and surely many more to come.

8.2 Prof Hlengani James Siweya *Hlengani Siweya is the Executive Dean of the Faculty of Science and Agriculture at the University of Limpopo.*

(Siweya wrote to the author indicating: "Indeed the occasion has arrived. Please receive a clip which is about 4 minutes and 30 seconds on my thoughts about Themba. Unfortunately, because of our own Faculty Postgraduate Day and demanding responsibilities, I am unable to join the celebration. However, there will be a few colleagues from here who will be in the audience. I wish you and all a successful celebration of one of our celebrated Mathematicians in our country. Stay safe, I remain: Prof Dr. HJ Siweya, Pr. Sci. Nat. Executive Dean: Science & Agriculture")

Good day Ladies and Gentlemen. I would like to express my sincerest and warmest appreciation to colleagues, the organisers of this celebration, Professor Inderasan Naidoo and Professor Ighedo for inviting me to say a few words at this occasion. It is my pleasure and honour and privilege for me on an august occasion such as this to say a few words about a man I respect and love, Professor Themba Dube. I got to know Themba some 35 years ago through our late Head of Department, Professor Sentsho Mashike (may his soul rest in peace). From then onwards, Themba became a role model to me and many of our students. Indeed, I followed him into pointfree topology because when Themba completed his PhD in pointfree topology under the supervision of Professor Dharms Baboolal, I trekked down to the then University of Durban-Westville to also do my PhD in pointfree topology under Professor Baboolal. Upon completing my Phd, I spent a week on a visit to the University of Zululand to continue my research with Professor Themba Dube. We did not write any joint papers together. However, our collaboration continued into the training of masters students that I had recruited from our honours group at the University of the North-West. Amongst these students were Silwana, Siwana and Matlabyana. Interestingly, of these students, Mack Matlabyana went on to complete a PhD under the mentorship of Professor Dube. In a way, Professor Dube has been an inspiration to the Mathematics Department at the University of Limpopo. His association with us, at Limpopo University, has lived until the present day. As we are gathered here today to celebrate Themba's birthday, for the days, the years, and the hours well spent in his resolve for the advancement and development of mathematics in our country and the globalised world, it is my wish and prayer - may my God hear me - that he Themba continues his selfless contributions in mathematics and pointfree topology in particular, so that the many young men and women in mathematics draw courage from one of our finest black mathematicians in our country - Themba Dube.

To Themba himself, I have this to say. You have been a brother and a mentor to me and so you remain. May this day be a reminder that you came and made lifelong contributions to the development of many and that's why we are celebrating you today. Happy birthday brother.

8.3 Prof Thekiso Seretlo *Thekiso Seretlo is a former Director of the School of Mathematical Sciences at the University of Limpopo* (UL).

Ladies and Gentlemen and my fellow Mathematicians. Thank you ever so much for inviting me to this great occasion to honour Professor Dube. May I also express my heartfelt gratitude for being granted the opportunity and privilege to say a few words about him as the guest of honour. Allow me therefore to begin by saying that I have known Prof Dube for almost 45 years. When I was a first year student, following the era of the Professor Nongxa's as Fort Harians, there was this Themba Dube everyone used to talk about all over that campus. This person was said to be an extremely talented mathematician. He was also good in mathematical courses. But it was whispered that he did not have any liking for the life sciences.

In 1980 when I was a third year student, Prof Dube had just finished his honours and had stepped in to lecture us in the Real Analysis course. That step was a blessing to us, because the professor who previously lectured the same course, had not been very popular. This was perhaps because there had been a remarkably high failure rate among students taking the course. Things changed for the better after the young Prof Dube took over lecturing Real Analysis. I seem to recall that many students, myself included, started to indulge in Mathematics with more dedication and vigour. This was a result of Prof Dube's youthful inspiration and example at that time. Indeed, it is interesting to note that at the end of this year (2022), Prof Dube will have finished 42 years as an academic. In particular, he will have lectured Mathematics for nearly half a century, after he first started in the early 80s, at Fort Hare, with people like me, having been his student. On a lighter note, my father also lectured at Fort Hare at the time that the young Prof Dube arrived to work at that institution. In those days, Prof Dube would occasionally visit our home. Since he was my lecturer, whenever he visited us, I would call my father to attend to him. One day, my father called me aside and gently pointed out that Prof Dube was more my age, than him, that is, my father. That was the beginning of my friendship with

Prof Dube. Prof Dube left Fort Hare at the beginning of 1985 and went to join UniZul. I lost contact with him as he shunted between UniZul and UniVen. I then met him again in 1994 at the SAMS conference in University of Natal. He was then at UniZul where he rose to the rank of DVC, including being the acting VC, at one stage. To me, he always seemed to prefer to do Mathematics than being in an administrative position.

From there he has not only been a friend; but he has intervened on my behalf in very crucial situations. He has advised me on very awkward and difficult decisions I had to take. He has also established very good relations with NWU (Mafikeng Campus) while I was there. He had established solid relations with UL (Turfloop) even before I joined this institution. The relations, I believe, got even stronger when I got here to be Director in the School for Mathematical Sciences. Prof Dube has contributed a great deal to the academic advancement of other students of Mathematics. For example, he supervised Dr Matlabyana on his PhD studies. The Executive Dean of the Faculty of Science and Agriculture at UL, Prof Siweya, told me that he was encouraged by Prof Dube to register for a PhD in Mathematics under Prof Baboolal who was also Prof Dube's supervisor. I am sure NWU is deeply grateful to him for the research contributions he has made to the University. I know that UL feels greatly indebted to Prof Dube. And, on a more personal level, I am certain that I am not exaggerating when I say, I owe him at least, half of my academic life. We all hope that his retirement will mean being relieved of administrative duties to allow him to focus on his beloved Mathematics. We all wish him very well in all his future endeavours. Let me leave you with the famous saying by Erdos. In it, Erdos probably captures what is Prof Dube's perspective on Mathematics. The saying goes like this: "Without Mathematics, there's nothing you can do." Everything around you is Mathematics. Mathematics is the most beautiful and most powerful creation of the human spirit.

Thank you for paying attention to this short delivery. Thank you all and I join you all in wishing Prof Dube and his loved ones the very best on his retirement. God Speed and Good Luck going forward.

8.4 Prof Mandirevesa Martin Mugochi *Martin Mugochi was the* *first doctoral student of Dube. He is currently the Head of Department (Mathematics) and the University of Namibia.*

I didn't know a Mathematician with such a loud commanding voice until I met and listened to Themba at a conference in Cape Town, speaking about nearness frames. That caught my attention and a few years later I was his PhD student. I would like to think that I am his first offspring in the PhD genealogy (unless there are some illegitimate ones before me). For a while, Tega and I were the only siblings feeding on his wisdom, until others came along. Themba transcended from being an advisor to a mentor and friend. In fact I'm still wondering why we are talking about his retirement, since he has always displayed this youthful demeanour in our interactions. I still continue to learn a lot from him, and continue to be inspired by his tremendous impact, especially championing the cause and relevance of Pure Mathematics in our world today. This is one occasion I would have loved to attend in person, but circumstances prevented me from that privilege. My best wishes to Themba on this new chapter in his life, and looking forward to even better things in the new era. Cheers, Martin.

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