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On one-local retract in modular metrics

Olivier Olela Otafudu* and Tlotlo Odacious Phawe

Dedicated to the 65^{th} birthday of Professor Themba Dube

Abstract. We continue the study of the concept of one-local retract in the settings of modular metrics. This concept has been studied in metric spaces and quasi-metric spaces by different authors with different motivations. In this article, we extend the well-known results on one-local retract in metric point of view to the framework of modular metrics. In particular, we show that any self-map $\psi : X_w \longrightarrow X_w$ satisfying the property $w(\lambda, \psi(x), \psi(y)) \leq w(\lambda, x, y)$ for all $x, y \in X$ and $\lambda > 0$, has at least one fixed point whenever the collection of all q_w-admissible subsets of X_w is both compact and normal.

1 Introduction

The concept of modular metric spaces was introduced by Chistyakov [2] in 2010. He developed the theory of modular metric on an arbitrary set and investigated the theory of metric spaces induced by a modular metric. He defined a modular metric in the following way. Let X be a nonempty set. Then the function $w : (0, \infty) \times X \times X \to [0, \infty]$ is called a modular metric

* Corresponding author
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if it satisfies (a) $w(\lambda, x, y) = 0$ if and only if $x = y$ whenever $\lambda > 0$, (b) $w(\lambda, x, y) = w(\lambda, y, x)$ whenever $x, y \in X$ and $\lambda > 0$ and (c) $w(\lambda + \mu) \leq w(\lambda, x, z) + w(\mu, z, y)$ whenever $x, y, z \in X$ and $\lambda, \mu > 0$.

For $a \in X$, the modular set $X_w(a)$ is defined by

$$X_w(a) = \{x \in X : \lim_{\lambda \to \infty} w(\lambda, x, a) = 0\}.$$

In the sequel, we are going to write X_w in place of $X_w(a)$. We point out that Chistyakov equipped the set X_w with the metric q_w, where

$$q_w(x, y) := \inf\{\lambda > 0 : w(\lambda, x, y) < \lambda\}$$

whenever $x, y \in X_w$.

We are aware that a similar concept was studied by Abdou in [1]. Our approach is different from what was done in [1], the author used the set $B_w(x, r) := \{y \in X_w : w(1, x, y) \leq r\}$, where $x \in X_w$ and $r \geq 0$ which she called modular ball to define w-boundeness and other concepts related to this modular ball. In this article, we use the concept of entourage $B_{\lambda,\mu}(x) = \{y \in X_w : w(\lambda, x, y) < \mu\}$, where $\lambda, \mu > 0$ and $x \in X_w$ (see below) introduced in [4] to defined the w-boundedness and the topology induced by a modular metric w on a modular set X_w. It turns out that the modular ball due to [1] is just the entourage $B_{\lambda,\mu}(x)$, where $\lambda = 1$.

Moreover, we continue the study of the concept of one-local retract on modular metric in more general settings and we attempt to make connections between this concept in metric and modular metric frameworks. Furthermore, we extend some well-known results from [7, 8] in metric settings to the structure of modular metrics. For instance, we show that if a subset A of X_w is q_w-bounded then A is w-bounded. In addition, we show that most results of [1] on fixed point theorem on a modular set still hold in our context.

2 Basic definitions

Let us consider the set w equipped with a modular metric W. For any $x \in X_w$ and $\lambda, \mu > 0$, the sets $B_{\lambda,\mu}(x)$ and $C_{\lambda,\mu}(x)$ are defined by

$$B_{\lambda,\mu}(x) := \big\{z \in X_w : w(\lambda, x, z) < \mu\big\}$$

and

$$C_{\lambda,\mu}(x) := \{z \in X_w : w(\lambda, x, z) \le \mu\}.$$

The set $B_{\lambda,\mu}(x)$ is called a w <-*entourage* about x relative to λ and μ, and the set $C_{\lambda,\mu}(x)$ is called a w $\le$-*entourage* about x relative to λ and μ.

Note that if $0 < \mu < \lambda$ and $x \in X_w$, then

$$C_{\mu,\mu}(x) \subseteq C_{\lambda,\lambda}(x) \quad \text{and} \quad B_{\mu,\mu}(x) \subseteq B_{\lambda,\lambda}(x).$$

Definition 2.1. [4] Let w be a modular metric on a set X. Given $x, y \in X$,

(i) the limit from the right of w at each point $\lambda > 0$ denoted by $w_{+0}(\lambda, x, y)$ is defined by $w_{+0}(\lambda, x, y) = \lim\limits_{\mu \to \lambda^+} w(\mu, x, y) = \sup\{w(\mu, x, y) : \mu > \lambda\}$.

(ii) the limit from the left of w at each point $\lambda > 0$ denoted by $w_{-0}(\lambda, x, y)$ is defined by $w_{-0}(\lambda, x, y) = \lim\limits_{\mu \to \lambda^-} w(\mu, x, y) = \inf\{w(\mu, x, y) : 0 < \mu < \lambda\}$. Furthermore,

(iii) w is said to be continuous from the right on $(0, \infty)$ if for any $\lambda > 0$ we have $w(\lambda, x, y) = w_{+0}(\lambda, x, y)$.

(iv) w is said to be continuous from the left on $(0, \infty)$ if for any $\lambda > 0$ we have $w(\lambda, x, y) = w_{-0}(\lambda, x, y)$.

(v) w is said to be continuous on $(0, \infty)$ if w is continuous from the right and continuous from the left on $(0, \infty)$.

Remark 2.2. If w is continuous from the right on $(0, \infty)$, then for any $x, y \in X_w$ and $\lambda > 0$ we have $q_w(x, y) \le \lambda$ if and only if $w(\lambda, x, y) \le \lambda$.

Definition 2.3. ([4, Definition 4.3.1]) Let w be a modular metric on a set X and $\emptyset \ne O \subseteq X$. Then O is called $\tau(w)$-*open* (or *modular open*) if for any $x \in O$ and $\lambda > 0$, there exists $\mu > 0$ such that $B_{\lambda,\mu}(x) \subseteq O$.

Remark 2.4. Note that in Definition 2.3, one can use $C_{\lambda,\mu'}(x)$ in place of $B_{\lambda,\mu}(x)$ by taking $\mu' = \frac{\mu}{2}$.

Remark 2.5. Let w be a modular metric on a set X and $\varphi : (0, \infty) \to (0, \infty)$ be a function. For any $x \in X_w$, we have $\bigcup_{\lambda>0} B_{\lambda,\varphi(\lambda)}(x)$ is $\tau(w)$-open whenever the following two conditions are satisfied:

(1) φ is nondecreasing on $(0, \infty)$.

(2) w is convex and $\lambda \mapsto \lambda\varphi(\lambda)$ is nondecreasing on $(0,\infty)$.
(3) In view of (1) above and [4, Remark 4.3.3] note that

$$\left\{ \bigcup_{\lambda>0} B_{\lambda,\epsilon}(x) : \epsilon > 0 \right\}$$

may not form a neighborhood base for $\tau(w)$.
(4) For any $\lambda > 0$ and $n \in \mathbb{N}$, the set $B_{\lambda,1/n}(x)$ is $\tau(w)$-open for any $x \in X_w$.

It is very useful to note that if w is a modular metric on a set X, then for any $x, y \in X_w$ and $0 < \mu < \lambda$, we have

$$w(\lambda, x, y) = w(\lambda - \mu + \mu, x, y) \leq (w(\lambda - \mu, x, x) + w(\mu.x, y) = w(\mu, x, y). \tag{2.1}$$

3 w-boundedness

In this section we introduce and discuss concepts of w-boundedness and diameter function on a subset of a modular set.

Lemma 3.1. *Let w be a modular metric on a set X. Then for all $x, y \in X_w$ and $\lambda > 0$ we have*:

(a) $B_{q_w}(x, \lambda) \subseteq B_{\lambda,\lambda}(x)$,

(b) $C_{q_w}(x, \lambda) \subseteq C_{\lambda,\lambda}(x)$,

where the sets $B_{q_w}(x, \lambda)$ and $C_{q_w}(x, \lambda)$ are known as open ball and closed ball centred at x with radius λ, respectively.

Proof. We only prove (b) and (a) follows by similar arguments.
Let $y \in C_{q_w}(x, \lambda)$. Then $q_w(x, y) \leq \lambda$. It follows that

$$\mu' = \inf\{\mu > 0 : w(\mu, x, y) < \mu\} \leq \lambda.$$

Thus we have $w(\mu', x, y) < \mu' \leq \lambda$, it follows that

$$w(\lambda, x, y) \leq w(\mu', x, y) < \mu' \leq \lambda \text{ by the inquality (2.1).}$$

Hence $y \in C_{\lambda,\lambda}(x)$. $\square$

Example 3.2. Let $\mathbb{R}$ be equipped with its usual metric $q(x,y) = |x-y|$ for all $x, y \in \mathbb{R}$. Then for any $x, y \in \mathbb{R}$, the function $w(\lambda, x, y) = \dfrac{q(x,y)}{\lambda^2}$ is modular metric. For any $\lambda > 0$ and $x \in \mathbb{R}_w$, we have

$$B_{q_w}(x,\lambda) = \{y \in \mathbb{R}_w : q_w(x,y) = q(x,y)^{\frac{1}{3}} < \lambda\} = (x-\lambda^3, x+\lambda^3)$$

and

$$B_{\lambda,\lambda}(x) = \{y \in \mathbb{R}_w : w(\lambda,x,y) < \lambda\} = (x-\lambda^3, x+\lambda^3).$$

Clearly, $B_{q_w}(x,\lambda) = B_{\lambda,\lambda}(x)$ for any $\lambda > 0$.

Example 3.3. (compare [4, Example 4.2.2 (2)]) Let (X, q) be a metric space. Then for any $x, y \in X$, the function

$$w(\lambda, x, y) = \begin{cases} \infty & \text{if } 0 < \lambda < q(x,y) \\ 0 & \text{if } \lambda > q(x,y) \end{cases} \tag{3.1}$$

is modular metric on X. It is readily checked that for any $\lambda > 0$ we have

$$B_{q_w}(x,\lambda) = B_q(x,\lambda) \subset C_q(x,\lambda) = B_{\lambda,\lambda}(x)$$

whenever $x \in X_w = X$.

Definition 3.4. ([9, p.99]) Let w be a modular metric on X. A nonempty subset A of X_w is said to be w-bounded if there exists $x \in X_w$ such that $A \subseteq C_{\lambda,\lambda}(x)$ for some $\lambda > 0$.

Remark 3.5. Let w be a modular metric on a set X and $A \subseteq X_w$. If A is q_w-bounded, then A is w-bounded.

Proof. Suppose that A is q_w-bounded. Then there exist $x \in X_w$ and $\lambda > 0$ such that $A \subseteq C_{q_w}(x,\lambda)$. Since $C_{q_w}(x,\lambda) \subseteq C_{\lambda,\lambda}(x)$, it is follows that A is w-bounded. □

The following observation follows from Remarks 2.2 and 3.5.

Remark 3.6. Let w be a modular metric on a set X which is continuous from the right on $(0,\infty)$. Then $C_{\lambda,\lambda}(x) = C_{q_w}(x,\lambda)$ whenever $\lambda > 0$ and $x \in X_w$.

The following result is a consequence of Remarks 3.5 and 3.6.

Lemma 3.7. *Let w be a modular metric on a set X which is continuous from the right on $(0, \infty)$. Then boundedness in (X_w, q_w) is equivalent to w-boundedness.*

We next introduce the diameter function on a subset of a modular set.

Definition 3.8. Let w be a modular pseudometric on X and $\emptyset \neq A \subseteq X_w$. Let a function $\Phi_A : (0, \infty) \longrightarrow [0, \infty]$ defined by

$$\Phi_A(\lambda) = \sup\{w(\lambda, x, y) : x, y \in A\}.$$

The *modular metric diameter* of A is defined by $\Phi_A(\lambda)$ for some $\lambda > 0$.

Lemma 3.9. *Let w be a modular metric on X and $\emptyset \neq A \subseteq X_w$. It is easy to see that the function Φ_A is well defined for any $A \subseteq X_w$. Then we have the following properties*:

(a) *if $0 < \lambda < \mu$, then $\Phi_A(\mu) \leq \Phi_A(\lambda)$,*

(b) *if $A \subseteq B$, then $\Phi_A(\lambda) \leq \Phi_B(\lambda)$ for any $\lambda > 0$,*

(c) *$\Phi_A(\lambda) = 0$ for some $\lambda > 0$ if and only if A is a singleton set.*

Proof. (a) Suppose that $0 < \lambda < \mu$. Let $x, y \in A$. Then

$$w(\mu, x, y) \leq w(\lambda, x, y).$$

It follows that

$$\sup\{w(\mu, x, y) : x, y \in A\} \leq \sup\{w(\lambda, x, y) : x, y \in A\}.$$

Thus

$$\Phi_A(\mu) \leq \Phi_A(\lambda).$$

(b) Suppose $A \subseteq B$ and $\lambda > 0$. Let $x, y \in A \subseteq B$. Then

$$w(\lambda, x, y) \leq \Phi_B(\lambda).$$

Moreover,

$$\sup\{w(\lambda, x, y) : x, y \in A\} \leq \Phi_B(\lambda).$$

So $\Phi_A(\lambda) \leq \Phi_B(\lambda)$.

(c) Suppose A is not a singleton set. There exist $x, y \in A$ with $x \neq y$. Then $w(\lambda, x, y) \neq 0$ for any $\lambda > 0$. Then

$$\sup\{w(\lambda, x, y) : x, y \in A\} \neq 0.$$

Thus $\Phi_A(\lambda) \neq 0$ for any $\lambda > 0$.
Conversely, suppose that $\Phi_A(\lambda) \neq 0$ for some $\lambda > 0$. It follows that for any $x, y \in A$ we have $w(\lambda, x, y) = 0$ for some $\lambda > 0$. Thus $x = y$. □

Lemma 3.10. *Let w be a modular pseudometric on X and $\emptyset \neq A \subseteq X_w$. Then we have $\Phi_A(\lambda) \leq diam_{q_w}(A)$ for some $\lambda > 0$.*

Proof. Let $x, y \in A$. By the definition of q_w we have

$$q_w(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \leq \lambda\}.$$

So it follows that $w(\lambda, x, y) \leq q_w(x, y)$ for some $\lambda > 0$ such that $w(\lambda, x, y) \leq \lambda$. Thus for some $\lambda > 0$

$$\begin{aligned} \Phi_A(\lambda) &= \sup\{w(\lambda, x, y) : x, y \in A\} \\ &\leq \sup\{q_w(x, y) : x, y \in A\} \\ &= \mathrm{diam}_{q_w}(A). \end{aligned}$$

□

Lemma 3.11. *Let w be a modular pseudometric on X. If A is a w-bounded subset of X_w, then $\Phi_A(\lambda) < \infty$.*

Proof. Suppose that A is w-bounded. Then for some $\lambda > 0$ we have $A \subseteq C^w_{\lambda,\lambda}(x)$ for some $x \in X_w$.

If $z, y \in A$, then $w(\lambda, x, z) \leq \lambda$. Thus

$$w(2\lambda, y, z) \leq (w(\lambda, y, x) + w(\lambda, x, z) \leq 2\lambda.$$

Moreover,

$$\sup\{w(\lambda', y, z) : z, y \in A\} \leq 2\lambda < \infty \quad \text{for some } \lambda' = 2\lambda > 0.$$

Therefore, $\Phi_A(\lambda') < \infty$ for some $\lambda' > 0$. □

Suppose that w is a modular pseudometric on a set X. For $\lambda > 0$, we set:

$$\begin{aligned}
r_A^x(\lambda) &:= \sup\{w(\lambda, x, y) : y \in A\} \\
r_A(\lambda) &:= \inf\{r_A^x(\lambda) : x \in X_w\} \\
R_A(\lambda) &:= \inf\{r_A^x(\lambda) : x \in A\} \\
C_A(\lambda) &:= \{x \in X_w : r_A^x(\lambda) = r_A(\lambda)\} \\
\mathrm{cov}_w(A) &:= \bigcap\{\mathcal{C} : \mathcal{C} \leq -entourage \quad \text{and} \quad A \subseteq \mathcal{C}\}.
\end{aligned}$$

Lemma 3.12. *Let w be a modular pseudometric on a set X and A be a w-bounded subset of X_w. Then*:

(1) *$cov_w(A) = \bigcap\{C_{r_A^x(\lambda), r_A^x(\lambda)}(x) : x \in X_w$ and $\lambda > 0\}$.*

(2) *$r^x_{cov_w(A)}(\lambda) = r_A^x(\lambda)$ for any $x \in X_w$ and some $\lambda > 0$.*

(3) *$r_{cov_w(A)}(\lambda) = r_A(\lambda)$ for some $\lambda > 0$.*

Proof. (1) Let $x \in X_w$ and $y \in A$. Then

$$w(r_A^x(\lambda), x, y) \leq \sup\{w(r_A^x(\lambda), x, y) : y \in A\} = r_A^x(\lambda).$$

Then $y \in C_{r_A^x(\lambda), r_A^x(\lambda)}(x)$ for some $\lambda > 0$. It follows that

$$A \subseteq C_{r_A^x(\lambda), r_A^x(\lambda)}(x) \text{ for some } \lambda > 0.$$

Thus

$$\mathrm{cov}_w(A) \subseteq \bigcap\{C_{r_A^x(\lambda), r_A^x(\lambda)}(x) : x \in X_w \text{ and } \lambda > 0\}. \tag{3.2}$$

Suppose that A is a w-bounded. Then for some $x \in X_w$ and $\lambda > 0$, $A \subseteq C_{\lambda,\lambda}(x)$. For any $y \in A$, we have $w(\lambda, x, y) \leq \lambda$ for some $\lambda > 0$.
Then

$$r_A^x(\lambda) = \sup\{w(\lambda, x, y) : y \in A\} \leq \lambda \text{ for some } \lambda > 0.$$

It follows that

$$C_{r_A^x(\lambda), r_A^x(\lambda)}(x) \subseteq C_{\lambda,\lambda}(x) \text{ for some } \lambda > 0.$$

Hence

$$\bigcap\{C_{r_A^x(\lambda),r_A^x(\lambda)}(x) : x \in X_w \text{ and } \lambda > 0\} \subseteq C_{\lambda,\lambda}(x) \ \text{ for some } \lambda > 0.$$

Thus

$$\bigcap\{C_{r_A^x(\lambda),r_A^x(\lambda)}(x) : x \in X_w \text{ and } \lambda > 0\} \subseteq \text{cov}_w(A). \tag{3.3}$$

Therefore, we have $\text{cov}_w(A) = \bigcap\{C_{r_A^x(\lambda),r_A^x(\lambda)}(x) : x \in X_w \text{ and } \lambda > 0\}$ from (3.2) and (3.3).

(2) Let $x \in X_w$, we have

$$r^x_{\text{cov}_w(A)}(\lambda) = \sup\{w(r_A^x(\lambda), x, y) : y \in \text{cov}_w(A)\}.$$

By (1), we have $y \in \bigcap\{C_{r_A^x(\lambda),r_A^x(\lambda)}(x) : x \in X_w \text{ and } \lambda > 0\}$. Thus

$$y \in C_{r_A^x(\lambda),r_A^x(\lambda)}(x) \text{ for some } \lambda > 0.$$

Hence $w(r_A^x(\lambda), x, y) \le r_A^x(\lambda)$ for some $\lambda > 0$. Furthermore,

$$r^x_{\text{cov}_w(A)}(\lambda) = \sup\{w(r_A^x(\lambda), x, y) : y \in \text{cov}_w(A)\} \le r_A^x(\lambda) \text{ for some } \lambda > 0.$$

Thus

$$r^x_{\text{cov}_w(A)}(\lambda) \le r_A^x(\lambda) \quad \text{for some } \lambda > 0. \tag{3.4}$$

Since $A \subseteq r^x_{\text{cov}_w}(A)$ by definition, it follows

$$r^x_{\text{cov}_w(A)}(\lambda) \ge r_A^x(\lambda) \quad \text{for some } \lambda > 0. \tag{3.5}$$

From (3.4) and (3.5) we have

$$r^x_{\text{cov}_w(A)}(\lambda) = r_A^x(\lambda)$$

for any $x \in X_w$ and some $\lambda > 0$.

(3) Let $x \in X_w$. From the axiom (2) above we have

$$r^x_{\text{cov}_w(A)}(\lambda) = r_A^x(\lambda)$$

for some $\lambda > 0$. Therefore,

$$r_{\text{cov}_w(A)}(\lambda) = \inf\{r^x_{\text{cov}_w(A)}(\lambda) : x \in X_w\} = \inf\{r_A^x(\lambda) : x \in X_w\} = r_A(\lambda)$$

for some $\lambda > 0$. $\square$

Remark 3.13. Note that a w-admissible subset of X_w can be written as the intersection of a family of the form $C_{\lambda,\lambda}(x)$, where $x \in X_w$ and $\lambda > 0$.

Definition 3.14. [9] Let w be a modular quasi-pseudometric on a nonempty set X. We say that X_w is w-Isbell-convex if for any family of points $(x_i)_{i\in I}$ in X_w and family of point $(\lambda_i)_{i\in I}$ in $(0,\infty)$ such that

$$w(\lambda_i + \lambda_j, x_i, x_j) \leq \lambda_i + \lambda_j,$$

for all $i, j \in I$, then

$$\bigcap_{i\in I}\Big[C_{\lambda_i,\lambda_i}(x_i)\Big] \neq \emptyset.$$

Lemma 3.15. *Let w be a modular pseudometric on X. If X_w is w-Isbell-convex and $A \subseteq X_w$. Then:*

(1) $r_A(\lambda) = \dfrac{\Phi_A(\lambda)}{2}$ *for some* $\lambda > 0$.

(2) $\Phi_A(\lambda) = \Phi_{cov_w(A)}(\lambda)$ *for some* $\lambda > 0$.

(3) *If* $A = cov_w(A)$, *then* $r_A(\lambda) = R_A(\lambda)$ *and* $R_A(\lambda) = 1/2\Phi_A(\lambda)$ *for some* $\lambda > 0$.

Proof. (1) Let us consider the set $\{C_{\Phi_A(t)/2,\Phi_A(\lambda)/2}(a) : a \in A\}$ for some $\lambda > 0$.
If $a, b \in A$, then

$$w(\Phi_A(\lambda), a, b) \leq \Phi_A(\lambda) = \Phi_A(\lambda)/2 + \Phi_a(\lambda)/2.$$

Then we have by the w-Isbell-convexity,

$$\bigcap_{a\in A}\Big[C_{\Phi_A(t)/2,\Phi_A(\lambda)/2}(a)\Big] \neq \emptyset.$$

Let

$$x \in \bigcap_{a\in A}\Big[C_{\Phi_A(t)/2,\Phi_A(\lambda)/2}(a)\Big],$$

thus

$$w(\Phi_A(\lambda)/2, a, x) \leq \Phi_A(\lambda)/2 \text{ for some } \lambda > 0.$$

So $r_A^x(\lambda) \leq \Phi_A(\lambda)/2$.

Let $x \in X_w$ and $a, b \in A$. We have

$$w(\Phi_A(\lambda), a, b) \leq w(\Phi_A(\lambda)/2, a, x) + w(\Phi_A(\lambda)/2, x, b).$$

Then

$$\begin{aligned} \Phi_A(\lambda) &= \sup\{w(\Phi_A(\lambda), a, b) : a, b \in A\} \\ &\leq \inf\{w(\Phi_A(\lambda)/2, a, x) : x \in X_w\} + \inf\{w(\Phi_A(\lambda)/2, x, b) : x \in X_w\} \\ &= r_A(\lambda) + r_A(\lambda). \end{aligned}$$

Thus $\Phi_A(\lambda) \leq 2r_A(\lambda)$. Therefore, we have

$$\Phi_A(\lambda) \leq 2r_A(\lambda) \leq 2r_A^x(\lambda) \leq \Phi_A(\lambda).$$

Hence $r_A(\lambda) = \dfrac{\Phi_A(\lambda)}{2}$ for any $\lambda > 0$.

(2) The result follows from (1) above and Lemma 3.12(3).

(3) Indeed for some $\lambda > 0$ we have

$$\frac{\Phi_A(\lambda)}{2} \leq r_A(\lambda) \leq R_A(\lambda). \tag{3.6}$$

Since $A = \bigcap_{i \in I} \mathcal{C}_i$, where $\mathcal{C}_i$ is $\leq$-entourages with $A \subseteq \mathcal{C}_i$ for any $i \in I$. Since

$$\bigcap_{a \in A} C_{\frac{\Phi_A(\lambda)}{2}, \frac{\Phi_A(\lambda)}{2}}(a) \neq \emptyset,$$

it follows that the collection of sets

$$\{\mathcal{C}_i : i \in I\} \cup \{C_{\frac{\Phi_A(\lambda)}{2}, \frac{\Phi_A(\lambda)}{2}}(a) : a \in A\}$$

has the mixed binary intersection property. By the w-Isbell-convexity of X_w, we have

$$\mathcal{C} = A \cap \{C_{\frac{\Phi_A(\lambda)}{2}, \frac{\Phi_A(\lambda)}{2}}(a) : a \in A\} = \bigcap_{i \in I} \mathcal{C}_i \cap \{C_{\frac{\Phi_A(\lambda)}{2}, \frac{\Phi_A(\lambda)}{2}}(a) : a \in A\} \neq \emptyset.$$

Let $x \in \mathcal{C}$. Then

$$r_A^x(\lambda) \le \frac{\Phi_A(\lambda)}{2} \text{ since } w\left(\frac{\Phi_A(\lambda)}{2}, a, x\right) \le \frac{\Phi_A(\lambda)}{2}. \tag{3.7}$$

Combining inequalities (3.6), (3.7) and the definition of $r_A^x(\lambda)$, we have

$$r_A^x(\lambda) \le \frac{\Phi_A(\lambda)}{2} \le r_A(\lambda) \le R_A(\lambda) \le r_A^x(\lambda).$$

Therefore, for some $\lambda > 0$.

$$r_A(\lambda) = R_A(\lambda) = 1/2\Phi_A(\lambda).$$

□

Definition 3.16. Let w be a modular pseudometric on X. Given a subset A of X_w, for $\lambda > 0$, the λ-parallel set of A is defined as

$$P_\lambda(A) = \bigcup_{a \in A} \left[C^w_{\lambda,\lambda}(a)\right].$$

Proposition 3.17. *Let w be a modular pseudometric on X. If X_w is w-Isbell-convex and A is a w-admissible subset of X_w, that is, $A = \bigcap_{i \in I} C_{\lambda_i,\lambda_i}(x_i)$ where $x_i \in X_w$ and $\lambda_i > 0$ for each $i \in I \neq \emptyset$, then*

$$P_\lambda(A) = \bigcap_{i \in I} \left[C_{\lambda_i+\lambda,\lambda_i+\lambda}(x_i)\right] \tag{3.8}$$

whenever $\lambda > 0$.

Proof. Let $y \in P_\lambda(A)$. Then we have $w(\lambda, a, y) \le \lambda$ for some $a \in A$. Moreover, for each $i \in I$,

$$w(\lambda_i + \lambda, x, y) \le w(\lambda_i, x_i, a) + w(\lambda, a, y) \le \lambda_i + \lambda.$$

It follows that $y \in C_{\lambda_i+\lambda,\lambda_i+\lambda}(x_i)$ whenever $i \in I$. Hence,

$$P_\lambda(A) \subseteq \bigcap_{i \in I} \left[C_{\lambda_i+\lambda,\lambda_i+\lambda}(x_i)\right].$$

Suppose that $y \in \bigcap_{i\in I} \left[C_{\lambda_i+\lambda,\lambda_i+\lambda}(x_i)\right]$.

Then

$$w(\lambda_i+\lambda, x_i, y) \leq \lambda_i+\lambda$$

for any $i \in I$. For any $a \in A$ and $i, j \in I$ we have

$$w(\lambda_i+\lambda_j, x_i, x_j) \leq w(\lambda_i, x_i, a) + w(\lambda_j, a, x_j) \leq \lambda_i+\lambda_j$$

by the definition of A and the triangle inequality.

Thus, the families of w $\leq$-entourages

$$\left[(C_{\lambda_i,\lambda_i}(x_i))_{i\in I}; (C_{\lambda,\lambda}(y))\right]$$

satisfy the hypothesis of w-Isbell-convexity of X_w. Then

$$\begin{aligned} \emptyset &\neq \left(\bigcap_{i\in I} C_{\lambda_i,\lambda_i}(x_i)\right) \cap \left(C_{\lambda,\lambda}(y)\right) \\ &= A \cap C_{\lambda,\lambda}(y). \end{aligned}$$

It then follows that $w(\lambda, y, a) \leq \lambda$ for some $a \in A$. Therefore, $y \in P_\lambda(A)$. $\square$

Definition 3.18. (compare [10, Definition 2.6]) Let w be a modular pseudometric on X. A nonempty and w-bounded subset A of X_w is called *w-admissible* if $A = \mathrm{cov}_w(A)$.

Remark 3.19. Note that a w-admissible subset of X_w can be written as the intersection of a family of the form $C^w_{\lambda,\lambda}(x)$, where $x \in X_w$ and $\lambda > 0$.

It should be observed that the collection of all w-admissible subsets of X_w will be denoted by $\mathcal{A}_w(X_w)$.

Definition 3.20. Let w be a modular metric on X. We say that:

(i) The collection $\mathcal{A}_w(X_w)$ is *compact* if every descending chain of nonempty subsets of $\mathcal{A}_w(X_w)$ has a nonempty intersection.

(ii) The collection $\mathcal{A}_w(X_w)$ is *w-normal* (or has a *w-normal structure*) if for any $A \in \mathcal{A}_w(X_w)$ with A having more than one point, there exists $\lambda > 0$ such that $\lambda < \Phi_A(\lambda)$ and for $a \in A$ with $A \subseteq C^w_{\lambda,\lambda}(a)$.

Remark 3.21. In line of Remark 3.5 it is easy to see that $\mathcal{A}_{q_w}(X_w) \subseteq \mathcal{A}_w(X_w)$. Then the compactness of $\mathcal{A}_{q_w}(X_w)$ implies the compactness of $\mathcal{A}_w(X_w)$.

Theorem 3.22. *Let w be a modular metric on X. If X_w is q_w-bounded and $\psi : X_w \to X_w$ is a map such that $w(\lambda, \psi(x), \psi(y)) \leq w(\lambda, x, y)$ for all $x, y \in X_w$ and $\lambda > 0$, then ψ has at least one fixed point whenever $\mathcal{A}_w(X_w)$ is compact and normal.*

Proof. Suppose that X_w is q_w-bounded and $\mathcal{A}_{q_w}(X_w)$ is compact and normal from the compactness. Since the map $\psi : X_w \to X_w$ satisfies the property

$$w(\lambda, \psi(x), \psi(y)) \leq w(\lambda, x, y)$$

for all $x, y \in X_w$ and $\lambda > 0$, it follows from the corollary of [3, Theorem 5.2] with $k = 1$ that

$$q_w(\psi(x), \psi(y)) \leq q_w(x, y)$$

for all $x, y \in X_w$. Thus $\psi : (X_w, q_w) \to (X_w, q_w)$ is a nonexpansive map and $\mathcal{A}_{q_w}(X_w)$ is compact and normal by the hypothesis. By [8, Theorem 5.1], the map $\psi : (X_w, q_w) \to (X_w, q_w)$ has at least one fixed point. $\square$

4 One-local retract

In this section we study the concept of one-local retract and we also investigate some fixed point theorems. We recommend to the reader [5, 6] for more details about one-local retract on metric spaces.

Definition 4.1. Let w be a modular metric on X. A subset A of X_w is said to be a 1-*local retract* of X_w if for any family $\{\mathcal{A}_i\}_{i\in I}$ of $\leq$-entourages on A for which

$$\bigcap_{i\in I} \mathcal{A}_i \neq \emptyset$$

it follows that $A \cap (\bigcap_{i\in I} \mathcal{A}_i) \neq \emptyset$.

The following lemma is obvious therefore we leave the proof to the reader.

Proposition 4.2. *Let w be a modular metric on X and $A \subseteq X_w$. If A is a* 1*-local retract of (X_w, q_w), then A is a* 1*-local retract of X_w in the sense of Definition* 4.1*.*

Let us recall that the fixed point set $\mathrm{Fix}(\psi)$ of a map $\psi : X_w \longrightarrow X_w$ is defined by $\mathrm{Fix}(\psi) = \{x \in X_w : \psi(x) = x\}$.

Theorem 4.3. *Let w be a modular metric on X. If X_w is q_w-bounded for which $\mathcal{A}_{q_w}(X_w)$ is compact and normal and $\psi : X_w \to X_w$ is a map such that $w(\lambda, \psi(x), \psi(y)) \leq w(\lambda, x, y)$ for all $x, y \in X_w$ and $\lambda > 0$, then $Fix(\psi)$ of ψ is nonempty* 1*-local retract of X_w. Furthermore, $Fix(\psi)$ is compact and w-normal in the sense of Definitions* 3.20 *and* 4.1*, respectively.*

Proof. Indeed the fixed point set $\mathrm{Fix}(\psi) \neq \emptyset$ by Theorem 3.22. In order to show that $\mathrm{Fix}(\psi)$ is a 1-local retract of X_w, we consider a family of $\leq$-entourages

$$\{C_{\lambda_\alpha,\lambda_\alpha}(x_\alpha)\}_{\alpha\in\Gamma},$$

where $x_\alpha \in \mathrm{Fix}(\psi)$ and $\lambda_\alpha > 0$ for all $\alpha \in \Gamma$ such that

$$A = \bigcap_{\alpha\in\Gamma} C_{\lambda_\alpha,\lambda_\alpha}(x_\alpha) \neq \emptyset.$$

It follows that A is w-admissible and w-normal. Then the map $\psi : A \longrightarrow A$ satisfies the same property with ψ.

Therefore, ψ has a fixed point by Theorem 3.22 and then

$$\emptyset \neq \mathrm{Fix}(\psi).$$

Thus the fixed point set $\mathrm{Fix}(\psi)$ is a 1-local retract of S. Furthermore, the definition of 1-local retract assures that $\mathcal{A}_w(\mathrm{Fix}(\psi))$ is compact.

To finish, we need to show that $\mathcal{A}_w(\mathrm{Fix}(\psi))$ is w-normal. Let $C \in \mathcal{A}_w(\mathrm{Fix}(\psi))$. From Lemmas 3.12 and 3.15 we have

$$\Phi_{\mathrm{cov}_w(C)}(\lambda) = \Phi_C(\lambda)$$

and

$$r_{\mathrm{cov}_w(C)}(\lambda) = r_C(\lambda) \quad \text{for some} \quad \lambda > 0.$$

Moreover, the w-normality of $\mathcal{C}_w(X_w)$ implies that

$$\lambda < \Phi_{\mathrm{cov}_w(C)}(\lambda) \quad \text{for some} \quad \lambda > 0.$$

Then it follows that

$$\lambda < \Phi_C(\lambda) \quad \text{for some} \quad \lambda > 0.$$

Thus $\mathcal{A}_w(\mathrm{Fix}(\psi))$ is w-normal. $\square$

Theorem 4.4. *Let w be a modular metric on X. If X_w is nonempty q_w-bounded for which $\mathcal{A}_w(X_w)$ is compact and normal, then any commuting family of maps $\{\psi_\alpha\}_{\alpha\in\{1,\cdots,n\}}$, (with for all α, $\psi_\alpha : X_w \longrightarrow X_w$ satisfies the property of the map ψ in Theorem* 3.22*) has a nonempty common fixed point set. Moreover, the common fixed point set $\bigcap_{\alpha=1}^{n} Fix(\psi_\alpha)$ is a* 1*-local retract of X_w in the sense of Definition* 4.1.

Proof. We note first that $\mathrm{Fix}(\psi_\alpha) \neq \emptyset$ by Theorem 3.22 for any $\alpha \in \{1,\cdots,n\}$. Thus there exists $x \in X_w$ such that $\psi_\alpha(x) = x$ for all $\alpha \in \{1,\cdots,n\}$.

Since ψ_1 and ψ_2 commute, let us show that $\psi_2(\mathrm{Fix}(\psi_1)) \subseteq \mathrm{Fix}(\psi_1)$. If for some $x \in X_w$, then we have $x = \psi_1(x)$ and $\psi_2(x) = \psi_2(\psi_1(x)) = \psi_1(\psi_2(x))$. Thus $\psi_2(x) \in \mathrm{Fix}(\psi_1)$.

We conclude that $\psi_2 : \mathrm{Fix}(\psi_1) \longrightarrow \mathrm{Fix}(\psi_1)$ has a fixed point $z \in \mathrm{Fix}(\psi_1)$, which is a fixed point of ψ_2 and ψ_1. By mathematical induction for each finite family $\{\psi_\alpha\}_{\alpha\in\{1,\cdots,n\}}$ of self-maps on X_w satisfying the same property of the map ψ in Theorem 3.22, the set of common fixed point $\bigcap_{\alpha=1}^{n} \mathrm{Fix}(\psi_\alpha) \neq \emptyset$.

To complete the proof, let us show that $\bigcap_{\alpha=1}^{n} \mathrm{Fix}(\psi_\alpha)$ is 1-local retract. Consider a family of $\leq$-entourages $\{C_{\lambda_\alpha,\lambda_\alpha}(x_\alpha)\}_{\alpha\in\Gamma}$, where $x_\alpha \in \bigcap_{\alpha=1}^{n} \mathrm{Fix}(\psi_\alpha)$

and $\lambda_\alpha > 0$ for all $\alpha \in \Gamma$ such that

$$A = \bigcap_{\alpha=1}^{n} \mathrm{Fix}(\psi_\alpha) \neq \emptyset.$$

For any $\alpha \in \{1, \cdots, n\}$, we have $\psi_\alpha : A \longrightarrow A$ is such that for all $x, y \in A$ and $\lambda > 0$: $w(\lambda, \psi_\alpha(x), \psi_\alpha(y)) \leq w(\lambda, x, y)$.

Since A is w-admissible, $\mathcal{A}_{q_w}(A)$ is compact and normal. Then by Theorem 3.22, the map ψ_α has a fixed point in A, that is

$$\bigcap_{\alpha=1}^{n} \mathrm{Fix}(\psi_\alpha) \cap A \neq \emptyset.$$

This proves that $\bigcap_{\alpha=1}^{n} \mathrm{Fix}(\psi_\alpha)$ is a 1-local retract of X_w. □

Theorem 4.5. *Let w be a modular metric on X. Also, let X_w be nonempty q_w-bounded for which $\mathcal{A}_w(X_w)$ is compact and w-normal. Suppose that $(H_\alpha)_{\alpha\in\Gamma}$ be a descending family of 1-local retracts of X_w, where we assume that Γ is totally ordered such that $\alpha_1, \alpha_2 \in \Gamma$ and $\alpha_1 \leq \alpha_2$ holds if and only if $H_{\alpha_1} \subseteq H_{\alpha_2}$. Then $\bigcap_{\alpha\in\Gamma} H_\alpha$ is nonempty and is a 1-local retract of X_w.*

Proof. Indeed, the descending family $(H_\alpha)_{\alpha\in\Gamma}$ is 1-local retract of (X_w, q_w) since the descending family $(H_\alpha)_{\alpha\in\Gamma}$ is a 1-local retracts of X_w by Proposition 4.2. From the well-known result of Khamsi [7, Theorem 6] we have $\bigcap_{\alpha\in\Gamma} H_\alpha \neq \emptyset$.

We now show that $H := \bigcap_{\alpha\in\Gamma} H_\alpha$ is 1-local retract of X_w. Let us consider a family of $\leq$-entourages $\{C_{\lambda_\beta,\lambda_\beta}(x_\beta)\}_{\beta\in\Gamma'}$, where $\lambda_\beta > 0$ and $x_\beta \in H$ for all $\beta \in \Gamma'$ for which

$$\bigcap_{\beta\in\Gamma'} C_{\lambda_\beta,\lambda_\beta}(x_\beta) \neq \emptyset.$$

By fixing $\alpha \in \Gamma$, since H_α is 1-local retract of X_w and since $x_\beta \in H_\alpha$ whenever $\beta \in \Gamma'$, thus $\mathcal{A}_\alpha = \bigcap_{\beta\in\Gamma'} C_{\lambda_\beta,\lambda_\beta}(x_\beta) \cap H_\alpha \neq \emptyset$.

$$\emptyset \;\neq\; \bigcap_{\alpha\in\Gamma} \mathcal{A}_\alpha = \bigcap_{\alpha\in\Gamma} \left[\bigcap_{\beta\in\Gamma'} C_{\lambda_\beta,\lambda_\beta}(x_\beta) \cap H_\alpha \right]$$

$$
\begin{aligned}
&= \bigcap_{\beta\in\Gamma'} C_{\lambda_\beta,\lambda_\beta}(x_\beta) \cap \bigcap_{\alpha\in\Gamma} H_\alpha \\
&= \bigcap_{\beta\in\Gamma'} C_{\lambda_\beta,\lambda_\beta}(x_\beta) \cap H,
\end{aligned}
$$

since the family $\{\mathcal{A}_\alpha\}_{\alpha\in\Gamma}$ is descending. Therefore $H = \bigcap_{\alpha\in\Gamma} H_\alpha$ is 1-local retract of X_w. □

The next result is a consequence of Theorem 4.5 and an application of Zorn's lemma.

Corollary 4.6. *Let w be a modular metric on a set X and X_w be a nonempty q_w-bounded. If $\{H_\alpha\}_{\alpha\in\Gamma}$ is a family of 1-local retract of subsets of X_w such that $\bigcap_{\alpha\in\Psi} H_\alpha$ is 1-local retract of X_w whenever $\Psi \subseteq \Gamma$ is finite, then $\bigcap_{\alpha\in\Gamma} H_\alpha$ is nonempty and 1-local retract of X_w.*

Theorem 4.7. *Let w be a modular metric on X. If X_w is nonempty q_w-bounded for which $\mathcal{A}_{q_w}(X_w)$ is compact and normal, then any commuting family of maps $\{\psi_\alpha\}_{\alpha\in\Gamma}$ satisfying the property of the map ψ in Theorem 3.22, has a common fixed point. Furthermore, the common fixed point set $\bigcap_{\alpha\in\Gamma} Fix(\psi_\alpha)$ is 1-local retract of X_w.*

Proof. For any $\alpha \in \Gamma$ and $x, y \in X_w$ and $\lambda > 0$, we have $w(\lambda, \psi(x), \psi(y)) \leq w(\lambda, x, y)$. It follows the corollary of [3, Theorem 5.2] with $k = 1$ that $\psi_\alpha : (X_w, q_w) \longrightarrow (X_w, q_w)$ is a nonexpansive map for all $\alpha \in \Gamma$ and since $\mathcal{A}_w(X_w)$ is compact and normal on (X_w, q_w). We have the family of maps $\{\psi_\alpha\}_{\alpha\in\Gamma}$ has a common fixed point by Theorem [7, Theorem 8]. Moreover, the set $\bigcap_{\alpha\in\Gamma} \mathrm{Fix}(\psi_\alpha)$ is 1-local retract of X_w by Theorems 3.22, 4.3 and Corollary 4.6. □

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Olivier Olela Otafudu *School of Mathematical and Statistical Sciences North-West University, Potchefstroom Campus, Potchefstroom 2520, South Africa.*

Email: olivier.olelaotafudu@nwu.ac.za

Tlotlo Odacious Phawe *School of Mathematical and Statistical Sciences North-West University, Potchefstroom Campus, Potchefstroom 2520, South Africa.*

Email: tlotlo.odacious@gmail.com