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Topological spaces versus frames in the topos of M-sets

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Dedicated to Themba Dube on the occasion of his 65^{th} birthday.

Abstract. In this paper we study topological spaces, frames, and their confrontation in the presheaf topos of M-sets for a monoid M. We introduce the internalization, of the frame of open subsets for topologies, and of topologies of points for frames, in our universe. Then we find functors between the categories of topological spaces and of frames in our universe. We show that, in contrast to the classical case, the obtained functors do not have an adjoint relation for a general monoid, but in some cases such as when M is a group, they form an adjunction. Furthermore, we define and study soberity and spatialness for our topological spaces and frames, respectively. It is shown that if M is a group then the restriction of the adjunction to sober spaces and spatial frames becomes into an isomorphism.

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1 Introduction and Preliminaries

The study of topological spaces via their open subsets, rather than their elements, formed a branch, called the "frame theory" or "pointfree topology". The set $O(X)$ of open subsets of a topological space X forms a frame, a complete lattice in which binary meets distributes over arbitrary joins. In this way, a functor O from the category **Top**, of topological spaces, to the dual of the category **Frm**, of frames, is obtained. But, not every frame L is of the form $O(X)$ for some topological space X, such frames are called "spatial". The functor O has a left adjoint which is given by taking the set ΣL, of "points" of a frame L, that is frame homomorphisms from L to the two element frame **2**. The idea of this definition of "points" obtains from the fact that each element x of X can be considered as a continuous map $\mathbf{1} \to X$, and it gives rise to a frame homomorphism $O(X) \to O(1) \cong \mathbf{2}$. The topology on ΣL is constructed by taking $\Sigma_a = \{f \in \Sigma L : f(a) = 1\}$, for $a \in L$, as open subsets. Again, it is shown that not every topological space X is of the form ΣL, such topological spaces are called "sober spaces". Sober spaces play important roles in the theory of topological spaces as well as in theoretical computer science. In [13], Paul Taylor used sober spaces to model meta observations in computation, and soberity as Leibniz Principle for spaces. Also, Jimmie Lawson in [9] applied soberification to get a completion for abstract bases (see [1]). For more detail about frames one can see [8].

On the other hand, topos theory is a branch of mathematics that studies categories behaving similar to the category of sets, however the mathematics that one develops in a topos is constructive. Also, it is proposed by Dana Scott that a topos is a suitable discourse to study models of computations ([6]).

In this paper, taking the presheaf topos $\mathbf{Set}^M$, for a monoid M, considered as a one object category, as our universe, we define the notion of a topology, and study the fundamentals of frame theory. Our definition of the notion of topology is inspired from Hyland ([6]), where he specifies an object $\mathbf{S}$ and regards the subobjects that classifies by $\mathbf{S}$, as "open" subobjects. In the classical case, the Sierpinki space plays the role of $\mathbf{S}$. We find the counterpart of Sierpinki for topological spaces in our universe. For the notion of a frame in a topos, we refer to [7]. The internalization of the (open subset) functor O and (point) functor Σ, between topological spaces and frames are then constructed. It is shown that the obtained functors do not generally form an adjoint pair. We also define and study spatial frames and sober spaces in our topos. Finally, we consider the results for the case where M is a group, and in particular, show that the adjunction holds in that case.

In the following, we recall preliminary notions needed in this paper. For more information about the category of M-sets one can see [3] and [5]; and about topos theory one may see [5] and [7].

M-sets. Let M be a monoid with e as its identity. An *M-set* is a set X with a function $\mu : M \times X \to X$, called the *action* of M on X, such that, denoting $\mu(s,x) = sx$, we have

$$ex = x, \qquad (st)x = s(tx).$$

A subset A of an M-set X is called a *sub M-set* of X if it is closed under the action of M on X. That is, for each $x \in A$ and $s \in M$ we have $sx \in A$.

A function $f : X \to Y$ between M-sets is called *equivariant* (or *action-preserving*) if for each $s \in M$ and $x \in X$ we have $f(sx) = sf(x)$.

By a *zero element* of an M-set X we mean an element x of X such that for each $s \in M$, $sx = x$. The sub M-set of all zero elements of X is denoted by $Z(X)$. In fact, $Z(X) \cong Hom_{M-\mathbf{Set}}(1, X)$, where M-**Set** is the category of all M-sets with equivariant maps between them.

The equational category of M-sets. One can consider an M-set X as a unary universal algebra with operations $\lambda_s : X \to X$ given by $\lambda_s(x) = sx$, for all $s \in M$. In this way, M-**Set** would be an equational category of algebras. This implies that it is a complete, and cocomplete category in which products are cartesian products with componentwise operations. In particular, the singleton M-set $\mathbf{1}$ is the terminal object, and the monomorphisms are one-one action-preserving maps, and therefore subobjects can be identified by sub M-sets.

The topos of M-sets. Considering a monoid M as a one object category with elements of M as morphisms and the binary operation of M as the composition, the category of M-sets is isomorphic to the functor (presheaf) category $\mathbf{Set}^M$. Therefore, M-**Set** is a topos.

The *subobject classifier* Ω in this topos is the set L_M of all *left ideal* of M, in other words the sub M-set of the M-set M (with its binary operation as the action), with the action of M on it defined by $s.I = \{t \in M : ts \in I\}$, for all $I \in L_M$ and $s \in S$. The truth arrow is $t : \mathbf{1} = \{*\} \to L_M$ given by $t(*) = M$. Also, for every sub M-set A of an M-set X, the classifying morphism χ_A which makes the diagram

$$\begin{array}{ccc} A & \hookrightarrow & X \\ ! \downarrow & & \downarrow \chi_A \\ \mathbf{1} & \xrightarrow{t} & \Omega \end{array}$$

into a pullback square, is defined by $\chi_A(a) = \{s \in M : sa \in A\}$.

The *Exponentiation* Y^X of two M-sets X and Y in this topos is the set

$$\{(f_s)_{s\in M} : f_s : X \to Y \text{ is a function, } tf_s(x) = f_{ts}(tx)\ \forall t, s \in M,\ x \in X\}$$

equipped with the action $m(f_s)_{s\in M} = (f_{sm})_{s\in M}$ for all $(f_s)_{s\in M} \in Y^X$ and $m \in M$. The evaluation arrow $ev_Y^X : Y^X \times X \to Y$ is defined by $ev_Y^X((f_s)_{s\in M}, x) = f_e(x)$. Also, given an equivariant map $g : Z \times X \to Y$, the unique arrow $\hat{g}$, which makes the diagram

$$\begin{array}{ccc} Y^X \times X & \xrightarrow{ev_Y^X} & Y \\ {\scriptstyle \hat{g}\times id_X} \uparrow & \nearrow_{g} & \\ Z \times X & & \end{array}$$

commutative, is defined by $\hat{g}(z)_s(x) = g(sz, x)$, for $z \in Z,\ s \in M, x \in X$.

The *power object* Ω^X of an M-set X in this topos is the set

$$\{(X_s)_{s\in M} : X_s \subseteq X,\ tX_s \subseteq X_{ts},\ \forall t, s \in M\}$$

with the action given by $m(X_s)_{s\in M} = (X_{sm})_{s\in M}$, for all $(X_s)_{s\in M} \in \Omega^X$ and $m \in M$.

Finally, notice that for the case where M is a group, $Y^X \cong Hom_{\mathbf{Set}}(X, Y)$ under the assignment $(f_s)_{s\in M} \mapsto f_e$, also $\Omega = \{\emptyset, M\}$ and $\Omega^X \cong \mathcal{P}(X)$ under the assignment $(X_s)_{s\in M} \mapsto X_e$.

2 M-topological spaces and M-frames

In this section, we give the concepts of M-topological spaces and M-frames, for a monoid M. These are intended to play the role of topological spaces and frames in the topos of M-sets.

2.1 M-topological spaces We first recall the concepts of M-topological spaces and M-continuous maps, also some facts about the category they form, from [4].

Definition 2.1. (a) An *M-topological space* (*M-space*, for short) is an M-set X with a topology on it such that its open subsets are sub M-sets of X.

We call an open sub M-set of an M-topological space X, an *M-open* subset.

(b) We call an action-preserving continuous map between M-topological spaces, an *M-continuous map*.

We denote the category of M-topological spaces together with M-continuous maps by M-**Top**.

Also, recall that for any M-set X, $\{X, \varnothing\}$ and $Sub(X)$, the set of all sub M-sets, are the smallest and the largest M-topologies on X, respectively.

Further, by a *base* for an M-topology on an M-set X, we mean a base for its related topology whose elements are sub M-sets. That is, a collection $\mathcal{B}$ of sub M-sets of X such that (1) $\bigcup \mathcal{B} = X$; (2) for $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.

As in the classical case, we obtained in [4] that

If $\mathcal{B}$ is a base for an M-set X, then the smallest M-topology on X containing $\mathcal{B}$ exists.

Remark 2.2. [4] The category M-**Top** is a complete as well as a cocomplete category. In fact, the product of a family $\{X_i : i \in I\}$ of M-topological spaces, is their product M-set $\prod_{i\in I} X_i$ (with the pointwise actions) equipped with the M-topology generated by the base

$$\mathcal{B} = \{\prod_{i\in I} U_i : U_i \in O(X_i) \text{ and } U_i = X_i \text{ for all but a finite number}\},$$

and with the classical projection maps. Also, the classical equalizers work here with the subspace M-topology.

The coproduct of a given family $\{X_i : i \in I\}$ is their disjoint union $\bigsqcup_{i\in I} X_i$ with the action sx for $s \in M$ and $x \in X_i$ to be the same action in X_i, and with the M-topology generated by the union of $O(X_i)$, for all $i \in I$. Finally, the coequalizers are computed as in M-**Set** with the quotient topology, that is, if $q : Y \to Z$ is the coequalizer in M-**Set**, then one takes the sub M-sets U of Z for which $q^{-1}(U) \in O(Y)$ as M-open subsets of Z.

Finally, we recall from [4] that there exists an M-topological space which classifies M-open subsets. It is the counterpart of the Sierpinski space which we call it the "Sierpinski M-space".

Definition 2.3. We call the M-topological space $\mathbb{S} = (L_M; \{L_M, \{M\}, \emptyset\})$, with the same action as Ω, the *Sierpinski M-space*.

Lemma 2.4. [4] *$\mathbb{S}$ is the M-open subset classifier in M-**Top**, that is, for each M-topological space X, there is a natural frame isomorphism*

$$O(X) \cong Hom_{M-\mathbf{Top}}(X, \mathbb{S}).$$

Recall that for each M-open subset U of an M-topological space X, the classifying arrow $\chi_U : X \to \mathbb{S}$ is given by $x \mapsto \chi_U(x) = \{s \in M : sx \in U\}$.

2.2 M-frames Here we study the notion of frames in the topos of M-sets, namely M-frames. For more information about the notion of posets and frames in a topos, we refer to [7].

Notice that an *internal poset* in M-**Set**, or simply an *M-poset*, is a poset P which is also an M-set such that for each $x, y \in P$ and $s \in M$ we have $x \leq y$ implies $sx \leq sy$.

Also, it is known that for each M-set X, Ω^X is an internal poset with the componentwise order:

$$(X_s)_{s\in M} \leq (Y_s)_{s\in M} \Leftrightarrow X_s \subseteq Y_s, \ \forall s \in M.$$

Definition 2.5. An *M-frame* is an M-set L which is also a frame such that for all $s \in M$ and $\{a_i : i \in I\} \subseteq L$,

$$s \bigvee_{i\in I} a_i = \bigvee_{i\in I} sa_i, \;\; s(a \wedge b) = (sa) \wedge (sb), \;\; s0 = 0, \;\; s1 = 1.$$

An *M-frame homomorphism* between M-frames L_1 and L_2 is an action-preserving map which is also a frame map.

We denote the category of M-frames and their homomorphisms by M-**Frm**.

Remark 2.6. For each M-set A, the M-poset Ω^A is an M-frame, in which

$$\bigvee_{i\in I} U_i = (\bigcup_{i\in I}(U_i)_s)_{s\in M}, \quad (U_s)_{s\in M} \wedge (V_s)_{s\in M} = (U_s \cap V_s)_{s\in M},$$

Also, $0 = (\emptyset)_{s\in M},\ 1 = (X)_{s\in M}$.

Theorem 2.7. *The category M-**Frm** is small complete.*

Proof. It can be easily see that for a family $\{L_i : i \in I\}$ of M-frames, the cartesian product $\prod_{i\in I} L_i$ with the componentwise operations and actions, is the product of this family in M-**Frm**. Also, for M-frame homomorphisms $\phi, \psi : L_1 \to L_2$, their equalizer in M-**Set** is an M-frame which becomes also the equalizer of ϕ, ψ in M-**Frm**. □

3 M-frames versus M-topological spaces

In this section, we find the functors from M-topological spaces to M-frames, and vice versa.

3.1 M-frames related to M-topological spaces As we recalled in the introduction, an important functorial relation from topological spaces to frames, is the one which relates to a topological space X, the frame $O(X)$ of its open subsets. In the classical topology, it is also well-known that $Hom_{\mathbf{Top}}(X, \mathbf{S}) \cong O(X)$, where $\mathbf{S}$ is the Sierpinski space. Although we found the counterpart of this isomorphism in Lemma 2.4, but unfortunately no natural actions were found to make the sides of the isomorphism into M-frames. We solved this problem, by internalizing both of $O(X)$ and $Hom_{\mathbf{M-Top}}(X, \mathbb{S})$ in the topos of M-sets.

To internalize $Hom_{\mathbf{M-Top}}(X, \mathbb{S})$, we applied a similar notion to internal homomorphisms between algebras in a topos (from [2]) for topological spaces. More precisely, considering the notion of the internal homomorphism object $[A, B]$ from an algebra A to an algebra B in the topos M-**Set** (see [12]):

$$[A, B] = \{(f_s)_{s\in M} \in B^A : \forall s \in M,\ f_s \text{ is an algebra homomorphism}\},$$

we define the *internal continuous map object* in the topos of M-sets from an M-topological space X to an M-topological space Y as:

$$[X,Y] = \{(f_s)_{s\in M} \in Y^X : \forall s \in M,\ f_s \text{ is a continuous map}\}.$$

Theorem 3.1. *For M-topological spaces X, Y, $[X,Y]$ is the largest sub M-set of Y^X such that for every $(f_s)_{s\in M} \in Y^X$, $ev_Y^X((f_s)_{s\in M}, -) : X \to Y$ is a continuous map.*

Proof. First notice that $[X,Y]$ satisfies the stated condition, because for each $(f_s)_{s\in M} \in [X,Y]$, $ev_Y^X((f_s)_{s\in M}, -) = f_e$ is a continuous map.

Let $\mathcal{A}$ be a sub M-set of Y^X satisfying the stated property. Then taking $(f_s)_{s\in M} \in \mathcal{A}$, we have $ev_Y^X((f_s)_{s\in M}, -) = f_e$ is a continuous map. Also, for each $t \in M$, $t(f_s)_{s\in M} \in \mathcal{A}$, and so $f_t = (t(f_s))_e$ is continuous. Therefore, $\mathcal{A} \subseteq [X,Y]$. □

In the following, we compute $[X,\mathbb{S}]$, and borrowing the above mentioned idea from the classical setting, we define "internal open subobjects".

Lemma 3.2. *For each M-topological space X, $[X,\mathbb{S}]$ is isomorphic (as M-sets) to $\{(U_s)_{s\in M} \in \Omega^X : \forall s \in M \quad U_s$ is M-open in $X\}$, which is a sub M-set of Ω^X.*

Proof. Taking $f = (f_s)_{s\in M} \in [X,\mathbb{S}]$, with a direct calculation we get that $(f_s^{-1}\{M\})_{s\in M}$ belongs to Ω^X and each of its components are M-open subset of X. Conversely, given a family $(U_s)_{s\in M} \in \Omega^X$ such that each U_s is an M-open subset of X, it is straightforward to see that $(g_s)_{s\in M}$, where $g_s : X \to \mathbb{S}$ is defined as $g_s(x) = \{t \in M : tx \in U_{ts}\}$, is an internal continuous map. This correspondence is clearly an M-set isomorphism. □

Definition 3.3. Let X be an M-topological space. We define *the object of internal open subobjects* as

$$\mathcal{O}(X) \doteq \{(U_s)_{s\in M} \in \Omega^X : \forall s \in M,\ U_s \text{ is } M\text{-open in } X\}.$$

We may call each element $U = (U_s)_{s\in M}$ of $\mathcal{O}(X)$ an "internal open subobject".

Theorem 3.4. *For an M-topological space X, $O(X) \cong Z\mathcal{O}(X)$.*

Proof. Let $(U_s)_{s\in M}$ be a member of $Z\mathcal{O}(X)$. Then for each $t \in M$, we have $(U_{st})_{s\in M} = t \cdot (U_s)_{s\in M} = (U_s)_{s\in M}$. Thus, for each $t \in M$, we have $U_t = U_e$. Now, the assignment $(U_s)_{s\in M} \mapsto U_e$, gives the desired one-one correspondence between $O(X)$ and $Z\mathcal{O}(X)$. □

Lemma 3.5. *For any M-topological space X, $\mathcal{O}(X)$ is a sub M-frame of Ω^X.*

Finally, we have the following functorial relation.

Theorem 3.6. *The assignment $X \mapsto \mathcal{O}(X)$ from M-**Top** to $(M\text{-}\mathbf{Frm})^{op}$ is functorial.*

Proof. For an M-continuous map $\phi : X \to Y$, we define $\mathcal{O}(\phi) : \mathcal{O}(Y) \to \mathcal{O}(X)$ as $\mathcal{O}(\phi)((V_s)_{s\in M}) = (\phi^{-1}(V_s))_{s\in M}$, for each $(V_s)_{s\in M} \in \mathcal{O}(Y)$. First notice that, since ϕ is continuous and action-preserving, $\phi^{-1}(V_s)$ is and M-open subset of X, for each $s \in M$. Also, for all $t, s \in M$, $t\phi^{-1}(V_s) \subseteq \phi^{-1}(V_{ts})$, since ϕ is action-preserving. Therefore, $\mathcal{O}(\phi)$ is well-defined. The fact that $\mathcal{O}(\phi)$ is an M-frame map follows from the definition of the inverse image. □

In the following we see one of the useful properties of the functor $\mathcal{O}$.

Theorem 3.7. *The functor $\mathcal{O} : M\text{-}\mathbf{Top} \to M\text{-}\mathbf{Frm}^{op}$ preserves finite colimits.*

Proof. It is obvious that $\mathcal{O}(\varnothing)$ is the one element M-frame, and so $\mathcal{O}$ preserves initial objects. Next, we see that

$$\mathcal{O}(X \sqcup Y) \cong \mathcal{O}(X) \times \mathcal{O}(Y).$$

In fact, it is straightforward to check that the map $\psi : \mathcal{O}(X \sqcup Y) \to \mathcal{O}(X) \times \mathcal{O}(Y)$ given by $\psi((U_s)_{s\in M}) = ((U_s \cap X)_{s\in M}, (U_s \cap Y)_{s\in M})$ is an M-frame isomorphism. □

3.2 M-topological spaces related to M-frames As we mentioned in the introduction, given a frame L, the set ΣL of "points" of L, that is, frame homomorphisms from L to $\mathbf{2}$, can be made into a topological space whose open subsets are the sets $\Sigma_a = \{f \in \Sigma L : f(a) = 1\}$, for $a \in L$.

In this subsection, we introduce the internalization of ΣL in the topos of M-sets. Similar to the former subsection, we do this by considering the internal frame maps in M-**Set** instead of classical frame maps.

The following definition is the counterpart of the definition of internal homomorphism object between algebras in a Grothendieck topos mentioned in [2].

Definition 3.8. Let L_1 and L_2 be two M-frames. Then, we define the *internal frame maps object* in the topos of M-sets from L_1 to L_2 as:

$$[L_1, L_2] = \{(f_s)_{s\in M} \in L_2^{L_1} : \forall s \in M,\ f_s \text{ is a frame homomorphism}\}.$$

In [2], it is shown that the internal homomorphism object between algebras in a Grothendieck topos is the largest subobject of the exponential which is in some sense compatible with the algebra operations (is locally homomorphism).

In the following, we show that $[L_1, L_2]$ is the largest sub M-set of $L_2^{L_1}$ all whose elements are families of frame homomorphisms.

Theorem 3.9. *For M-frames L_1, L_2, $[L_1, L_2]$ is the largest sub M-set of $L_1^{L_2}$ such that for every $(f_s)_{s\in M} \in [L_1, L_2]$, $ev_{L_2}^{L_1}((f_s)_{s\in M}, -) : L_1 \to L_2$ is a frame homomorphism.*

Proof. First notice that $[L_1, L_2]$ satisfies the stated property, because for each $(f_s)_{s\in M} \in [L_1, L_2]$, $ev_{L_2}^{L_1}((f_s)_{s\in M}, -) = f_e$ is a frame homomorphism.

Let $\mathcal{A}$ be a sub M-set of $L_2^{L_1}$ such that $ev_{L_2}^{L_1}((f_s)_{s\in M}, -) : L_1 \to L_2$, for every $(f_s)_{s\in M} \in \mathcal{A}$, is a frame homomorphism. Then taking $(f_s)_{s\in M} \in \mathcal{A}$, for each $t \in M$, we have $(f_{st})_{s\in M} = t(f_s)_{s\in M} \in \mathcal{A}$, and so, by the hypothesis, $ev_{L_2}^{L_1}((f_{st})_{s\in M}, -)$ is a frame homomorphism. But, $ev_{L_2}^{L_1}((f_{st})_{s\in M}, -) = f_t$, and so each f_t is a frame homomorphism. Therefore, $\mathcal{A} \subseteq [L_1, L_2]$, as required. □

Definition 3.10. Let L be an M-frame. We call $[L, \Omega]$, the *object of internal points* of L, and denote it by $\mathbf{\Sigma} L$.

Also, we call each member of it an *internal point* of L.

Remark 3.11. (1) For any M-frame L, $Z\mathbf{\Sigma}L \cong Hom_{M-Frm}(L, \Omega)$. This is because,

$$(f_s)_{s\in M} \in Z\mathbf{\Sigma}L \Leftrightarrow t(f_s)_{s\in M} = (f_s)_{s\in M},\ \forall t \in M \Leftrightarrow f_{st} = f_s, \forall s, t \in M,$$

which means that, for any $(f_s)_{s\in M} \in Z\mathbf{\Sigma}L$, we have $f_s = f_e$ for all $s \in M$. Then, the compatibility property of the constant family $(f_s)_{s\in M}$ shows that f_e is equivariant.

(2) For any M-frame L, if $Z\mathbf{\Sigma}L = \mathbf{\Sigma}L$ then $f_s = f_e$, for all $(f_s)_{s\in M} \in \mathbf{\Sigma}L$. This is implied by the proof of part (1) of this remark.

But, the converse is also true, if M is commutative. To see this, let $f = (f_s)_{s\in M} \in \mathbf{\Sigma}L$, $a \in L$, and $t \in M$. Then applying the hypothesis for $t(f_s)_{s\in M} = (f_{st})_{s\in M}$, we have

$$\begin{aligned} t \in f_s(a) \Leftrightarrow tf_s(a) = M \Leftrightarrow f_{st}(ta) = f_{ts}(ta) = M \Leftrightarrow (tf)_s(ta) = M \\ \Leftrightarrow (tf)_e(ta) = M \Leftrightarrow f_t(ta) = M \Leftrightarrow tf_e(a) = M \Leftrightarrow t \in f_e(a). \end{aligned}$$

Theorem 3.12. *For any M-frame L, $\mathbf{\Sigma}L$ can be made into an M-topological space.*

Proof. Let L be an M-frame, and for $a \in L$ denote the set

$$\{(f_s)_{s\in M} \in \mathbf{\Sigma}L : \forall s \in M \quad f_s(a) = M\}$$

by $\mathbf{\Sigma}_a$. Then we show that each $\mathbf{\Sigma}_a$ is a sub M-set of $\mathbf{\Sigma}L$, and the set $\{\mathbf{\Sigma}_a : a \in L\}$ is an M-topology on $\mathbf{\Sigma}L$. First notice that given $(f_s)_{s\in M}$ in $\mathbf{\Sigma}_a$ and $t \in M$, since we have $t(f_s)_{s\in M} = (f_{st})_{s\in M}$, we conclude that if $f_s(a) = M$ for all $s \in M$ then, in particular, $f_{st}(a) = M$ for all $s, t \in M$. Therefore, $\mathbf{\Sigma}_a$ is a sub M-set of $\mathbf{\Sigma}L$. Also, since for all $s \in M$, f_s is a frame homomorphism, we have $f_s(1) = M$, and so $\mathbf{\Sigma}_1 = \mathbf{\Sigma}L$. One can also see that for all $a, b \in L$ we have $\mathbf{\Sigma}_{a\wedge b} = \mathbf{\Sigma}_a \cap \mathbf{\Sigma}_b$. Therefore, $\{\mathbf{\Sigma}_a : a \in L\}$ is an M-topology on $\mathbf{\Sigma}L$. □

Remark 3.13. For each M-frame L and $a \in L$, $\mathbf{\Sigma}_a$ is the largest sub M-set of $\mathbf{\Sigma}L$ such that for all $(f_s)_{s\in M} \in \mathbf{\Sigma}_a$, $ev^L_\Omega((f_s)_{s\in M}, a) = M$.

This is because, if A is a sub M-set of $\mathbf{\Sigma}L$ with the mentioned property, then for each $(f_s)_{s\in M} \in A$ and $t \in M$, we have $tf \in A$, and so

$$ev^L_\Omega(t(f_s)_{s\in M}, a) = M.$$

This means that $f_t(a) = (tf)_e(a) = M$, and hence $A \subseteq \mathbf{\Sigma}L$.

Theorem 3.14. *The assignment $L \mapsto \mathbf{\Sigma}L$ from $(M$-$\mathbf{Frm})^{op}$ to M-$\mathbf{Top}$ is functorial.*

Proof. Let $h : L_1 \to L_2$ be an M-frame map. Define $\mathbf{\Sigma}h : \mathbf{\Sigma}L_2 \to \mathbf{\Sigma}L_1$ by

$$(\mathbf{\Sigma}h)(f_s)_{s\in M} = (f_s h)_{s\in M},$$

for every $(f_s)_{s\in M} \in \mathbf{\Sigma}L_2$. First note that for every $(f_s)_{s\in M} \in \mathbf{\Sigma}L_2$, $(f_s h)_{s\in M} \in \mathbf{\Sigma}(L_1)$.

We see that $\mathbf{\Sigma}(h)$ is an M-continuous map, since it is clearly equivariant, also it is continuous, since for every $a \in L_2$, $(\mathbf{\Sigma}h)^{-1}(\mathbf{\Sigma}_a) = \mathbf{\Sigma}_{h(a)}$. $\square$

4 The relation between the functors $\mathcal{O}$ and Σ

In this section, we study the relations between $\mathcal{O}$ and $\mathbf{\Sigma}$. To see whether there exists an adjunction between them or not, we introduce and study the intend to be "unit" and "counit" of the desired adjunction.

Definition 4.1. For any M-topological space X, define $\epsilon_X : X \to \mathbf{\Sigma}\mathcal{O}(X)$ by

$$\begin{aligned} \epsilon_X(x)_s : & \mathcal{O}(X) \to \Omega \\ & (U_m)_{m\in M} \mapsto \{t \in M : t(sx) \in U_t\}, \end{aligned}$$

for $x \in X$ and $s \in M$.

In other words, $\epsilon_X(x) \in \mathbb{S}^{\mathbb{S}^X}$ is determined by $ev_{\mathbb{S}} \circ (id_M \times \mathbb{S}^{\rho_x})$, where $\rho_x : M \to X$ is given by $\rho_x(s) = sx$.

Notice that $(\epsilon_X(x)_s)_{s\in M}$ satisfies the compatibility property of the elements of $\mathbf{\Sigma}\mathcal{O}(X)$. This is because, given $(U_m)_{m\in M} \in \mathcal{O}(X)$ and $t \in M$, we have $(t\epsilon_X(x)_s)(U_m)_{m\in M} = \epsilon_X(x)_{ts}(U_{mt})_{m\in M}$, since

$$\begin{aligned} n \in (t\epsilon_X(x)_s)(U_m)_{m\in M} & \Leftrightarrow (nt)(sx) \in U_{nt} \\ & \Leftrightarrow n(tsx) \in U_{nt} \\ & \Leftrightarrow n \in \epsilon_X(x)_{ts}(U_{mt})_{m\in M}. \end{aligned}$$

Lemma 4.2. *For every $x \in X$ and $s \in M$, $(\epsilon_X(x))_s$ is a frame homomorphism.*

Proof. Let $x \in X$ and $s \in M$. Then $(\epsilon_X(x))_s$ is well-defined, because for each $(U_m)_{m\in M} \in \mathcal{O}(X)$, the set $\{t \in M : t \in \chi_{U_t}(sx)\}$ is a left ideal of M. Take $t \in M$ with $t \in \chi_{U_t}(sx)$ and $r \in M$. Then $(ts)x = t(sx) \in U_t$, and so $(rt)(sx) = r((ts)x) \in U_{rt}$ which means $rt \in \chi_{U_{rt}}(sx)$. Also, applying the definition of the classifying arrow, we get that $(\epsilon_X(x))_s$ preserves zero and unit, also since $\chi_{U\cap V}(x) = \chi_U(x) \cap \chi_V(x)$, we obtain that $(\epsilon_X(x))_s$ preserves binary meets and arbitrary joins, and therefore it is a frame homomorphism. Finally, we show that $(\epsilon_X(x))_s$ is action-preserving. Let $r \in M$ and $(U_m)_{m\in M} \in \mathcal{O}(X)$. Then

$$\begin{aligned} t \in r(\epsilon_X(x)_s)((U_m)_{m\in M}) &\Leftrightarrow tr \in \epsilon_X(x)_s((U_m)_{m\in M}) \\ &\Leftrightarrow tr \in \chi_{U_{tr}}(sx) \\ &\Leftrightarrow trsx \in U_{tr} \\ &\Leftrightarrow t \in \chi_{U_{tr}}(tsx) \\ &\Leftrightarrow t \in (\epsilon_X(x)_{rs})((U_{mr})_{m\in M}), \end{aligned}$$

and so

$$r(\epsilon_X(x)_s)((U_m)_{m\in M}) = (\epsilon_X(x)_{rs})((U_{mr})_{m\in M}).$$

□

Theorem 4.3. *For any M-topological space X, ϵ_X is an M-continuous map.*

Proof. First notice that ϵ_X is action-preserving. Let $s \in M$ and $x \in X$. Then, for every $m \in M$, we have

$$(s\epsilon_X(x))_m = (\epsilon_X(x))_{ms}.$$

This is because, for every $t \in M$ and $(U_r)_{r\in M} \in \mathcal{O}(X)$, we get

$$t \in (\epsilon_X(x)_{ms})((U_r)_{r\in M}) \Leftrightarrow t \in \chi_{U_t}(msx) \Leftrightarrow t \in (\epsilon_X(sx))_m((U_r)_{r\in M}),$$

and so $(\epsilon_X(x))_{ms}((U_r)_{r\in M}) = (\epsilon_X(sx))((U_r)_{r\in M})$.

Also, ϵ_X is continuous, since for $(U_m)_{m\in M} \in \mathcal{O}(X)$, we have

$$\begin{aligned} \epsilon_X^{-1}(\mathbf{\Sigma}_{(U_m)_{m\in M}}) &= \{x \in X : \epsilon_X(x) \in \mathbf{\Sigma}_{(U_m)_{m\in M}}\} \\ &= \{x \in X : \forall s \in M \quad (\epsilon_X(x))_s((U_m)_{m\in M}) = M\} \end{aligned}$$

$$
\begin{aligned}
&= \{x \in X : \forall s \in M \quad \{t \in M : t \in \chi_{U_t}(sx) = M\}\} \\
&= \{x \in X : \forall s \in M \quad \chi_{U_e}(sx) = M\} \\
&= \{x \in X : \forall s \in M \quad s \in \chi_{U_e}(X)\} \\
&= \{x \in X : \chi_{U_e}(x) = M\} \\
&= \{x \in X : x \in U_e\} \\
&= U_e.
\end{aligned}
$$

□

Proposition 4.4. $\epsilon = (\epsilon_X)_{X \in M-\mathbf{Top}} : id_{M-\mathbf{Top}} \to \mathbf{\Sigma}\mathcal{O}$ *is a natural transformation.*

Proof. We have to prove that for an M-continuous map $f : X \to Y$, the following square is commutative:

$$
\begin{array}{ccc}
X & \xrightarrow{\epsilon_X} & \mathbf{\Sigma}\mathcal{O}(X) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathbf{\Sigma}\mathcal{O}(f)} \\
Y & \xrightarrow{\epsilon_Y} & \mathbf{\Sigma}\mathcal{O}(Y)
\end{array}
$$

By definitions, for every $x \in X$, we have

$$(\mathbf{\Sigma}\mathcal{O}(f))(\epsilon_X(x)_s)s \in M) = (\epsilon_X(x)_s\mathcal{O}(f))_{s \in M} \in [\mathcal{O}(Y), \Omega].$$

Now, for every $(V_m)_{m \in M} \in \mathcal{O}(Y)$, we have

$$
\begin{aligned}
(\epsilon_X(x)_s\mathcal{O}(f))_{s \in M}((V_m)_{m \in M}) &= (\epsilon_X(x)_s(f^{-1}(V_m)_{m \in M}) \\
&= (\{t \in M : tsx \in f^{-1}(V_t)\})_{s \in M} \\
&= (\{t \in M : f(tsx) \in V_t\})_{s \in M} \\
&= (\{t \in M : ts \cdot f(x) \in V_t\})_{s \in M} \\
&= (\epsilon_Y f(x)_s)_{s \in M}((V_m)_{m \in M}).
\end{aligned}
$$

□

In the classical case, as we mentioned before, Σ and O form an adjunction between the categories of frames and topological spaces, and $\varepsilon = (\varepsilon_X)_{X \in \mathbf{Top}} : id_{\mathbf{Top}} \to \Sigma O$, where $\varepsilon_X : x \mapsto \Sigma_x$, is the unit of the adjunction.

We will see that this does not happen in the topos of M-sets for a general monoid M. Let first assume that this happens! Then we have the following remark.

Remark 4.5. Assume that the functor $\mathbf{\Sigma}$ is the right adjoint to the functor $\mathcal{O}$, and ϵ is the unit of this adjunction. Then for each M-continuous map $f : X \to \mathbf{\Sigma}L$ there exists a unique M-frame homomorphism $\hat{f} : L \to \mathcal{O}(X)$ such that

$$\mathbf{\Sigma}(\hat{f})\epsilon_X(x) = f(x), \ \forall x \in X.$$

This means that for each $x \in X$, $s \in M$, and $a \in L$, we have

$$\epsilon_X(x)_s\hat{f}(a) = f(x)_s(a).$$

In particular, for each $x \in X$ and $a \in L$, taking $s \doteq e$, we should have

$$\epsilon_X(x)_e\hat{f}(a) = f(x)_e(a).$$

This implies that

$$x \in \hat{f}(a)_e \Leftrightarrow f(x)_e(a) = M.$$

This is because, if $x \in \hat{f}(a)_e$, then for each $t \in M$ we have $tx \in \hat{f}(a)_t$, and hence $\epsilon_X(x)_e\hat{f}(a) = M$, which means $f(x)_e(a) = M$. The converse is clear. Now, since $\hat{f}$ is action-preserving, we get

$$x \in \hat{f}(a)_m \Leftrightarrow x \in \hat{f}(ma)_e \Leftrightarrow f(x)_e(ma) = M.$$

for every $m \in M$.

Therefore, assuming that $\mathbf{\Sigma}$ and $\mathcal{O}$ form an adjunction with the unit ϵ, then $\hat{f} : L \to \mathcal{O}(X)$, the adjoint transpose of an M-continuous map $f : X \to \mathbf{\Sigma}L$, is defined as

$$\hat{f}(a)_m = \{x \in X : f(x)_e(ma) = M\}.$$

In the following, we will see that the above defined $\hat{f}$ is not generally well-defined, although it preserves the action and is a frame map (if it is well-defined).

Lemma 4.6. *If $f : X \to \mathbf{\Sigma}L$ is an M-continuous map, then for each $m \in M$, $(\hat{f}(a))_m$ is an M-open subset of X if and only if $\hat{f}(a)_m = f^{-1}(\mathbf{\Sigma}_{ma})$.*

Proof. If $\hat{f}(a)_m$ is an M-open subset, then being a sub M-set, it is closed under the action, and so if $x \in \hat{f}(a)_m$ then also $sx \in \hat{f}(a)_m$, for all $s \in M$, that is, $f(x)_s(ma) = (sf(x))_e(ma) = f(sx)_e(ma) = M$. Therefore,

$$\begin{aligned}(\hat{f}(a))_m &= \{x \in X : \forall s \in M,\ f(x)_s(ma) = M\}\\ &= \{x \in X : f(x) \in \mathbf{\Sigma}_{ma}\}\\ &= f^{-1}(\mathbf{\Sigma}_{ma}).\end{aligned}$$

The converse is true because f is M-continuous and $\mathbf{\Sigma}_{ma}$ is an M-open subset of $\mathbf{\Sigma}L$. □

As an immediate consequence of the above lemma, we have

Corollary 4.7. *If $f : X \to \mathbf{\Sigma}L$ is an M-continuous map, then $\hat{f}$ is well-defined if and only if $\hat{f}(a)_m = f^{-1}(\mathbf{\Sigma}_{ma})$, for all $a \in L$ and $m \in M$.*

Lemma 4.8. *If $f : X \to \mathbf{\Sigma}L$ is an M-continuous map and $\hat{f}$ is well-defined, then it is an M-frame homomorphism.*

Proof. First we see that $\hat{f}$ preserves zero and unit, since for each $x \in X$, $f(x)_e$ is a frame homomorphism. From the same reason and the fact that for any $I, J \in \Omega$, $I \cap J = M$ if and only if $I = J = M$, we get that $\hat{f}$ preserves binary meets. Finally, since for any family $\{I_\gamma : \gamma \in \Gamma\}$ of elements of Ω, we have $\bigcup_{\gamma\in\Gamma} I_\gamma = M$ if and only if $I_\gamma = M$, for some $\gamma \in \Gamma$, and $f(x)_e$ is a frame homomorphism, we conclude that $\hat{f}$ preserves arbitrary joins. □

As a corollary of the above lemma and Remark 4.5, we have

Theorem 4.9. *The functors $\mathcal{O}$ and $\mathbf{\Sigma}$ form an adjoint pair with ϵ as the unit if and only if for each M-continuous map $f : X \to \mathbf{\Sigma}L$, $a \in L$, and $m \in M$, $\hat{f}(a)_m = f^{-1}(\mathbf{\Sigma}_{ma})$.*

The following example, shows that $\hat{f}$ is not generally well-defined, and so we do not generally have adjoint relation between $\mathbf{\Sigma}$ and $\mathcal{O}$.

Example 4.10. Consider the monoid $M \doteq M_2 \doteq \{e, a\}$ with $a.a = a$. Take $L \doteq \Omega$, and notice that $\Omega = \{\emptyset, \{a\}, M_2\}$. Put $f \doteq id_{\mathbf{\Sigma}L}$. Then f is an M-continuous map for which $\hat{f}$ is not well-defined. To see this, applying Corollary 4.7, we show that $\hat{f}(\{a\})_e \neq f^{-1}(\mathbf{\Sigma}_{e\{a\}})$. Define $g \in \mathbf{\Sigma}L$

as $g = (g_e, g_a)$, where $g_e, g_a : L \to \Omega$ are given as $g_e(\emptyset) = g_a(\emptyset) = \emptyset$, $g_e(\{a\}) = g_e(M) = M$, and $g_a = id_\Omega$. Then

$$g \in \hat{f}(\{a\})_e = \{h \in \mathbf{\Sigma}L : f(h)_e(e\{a\}) = M\} = \{h \in \mathbf{\Sigma}L : h_e(\{a\}) = M\}$$

while $g \notin f^{-1}(\mathbf{\Sigma}_{\{a\}})$, since

$$g = f(g) \notin \mathbf{\Sigma}_{\{a\}} = \{h \in \mathbf{\Sigma}L : h_s(\{a\}) = M,\ \forall s \in M\}.$$

Theorem 4.11. *If $Z\mathbf{\Sigma}L = \mathbf{\Sigma}L$, then the functors $\mathcal{O}$ and $\mathbf{\Sigma}$ form an adjoint pair with ϵ as the unit.*

Proof. Applying Corollary 4.7, we show that for each M-continuous map $f : X \to \mathbf{\Sigma}L$, $\hat{f} : L \to \mathcal{O}(X)$ given by

$$\hat{f}(a) = (\{x \in X : f(x)_e(sa) = M\})_{s\in M},$$

is well-defined. Then it will be the unique M-frame homomorphism such that $\mathbf{\Sigma}(\hat{f})\epsilon_X = f$. Since $Z\mathbf{\Sigma}L = \mathbf{\Sigma}L$, by the proof of Remark 3.11(1), for each $x \in X$, $f(x)$ is completely determined by the equivariant map $f(x)_e : L \to \Omega$, and so

$$x \in \hat{f}(a)_s = f(x)_e(sa) = M \Leftrightarrow f(x)_m(sa) = M,\ \forall m \in M \Leftrightarrow x \in f^{-1}(\mathbf{\Sigma}_{sa}).$$

This means that for all $s \in M$, $\hat{f}(a)_s = f^{-1}(\mathbf{\Sigma}_{sa})$, as required. □

Although we do not have adjoint relation between $\mathbf{\Sigma}$ and $\mathcal{O}$, we have a retraction between $\mathbf{\Sigma}$ and $\mathbf{\Sigma}\mathcal{O}\mathbf{\Sigma}$ in some cases. The cases we consider are when $X = \mathbb{S}$, the Sierpinski M-space, and when $X = \mathbf{\Sigma}L$, where L has the trivial action.

Theorem 4.12. *The Sierpinski M-space $\mathbb{S}$ is a retract of $\mathbf{\Sigma}\mathcal{O}(\mathbb{S})$ in M-* **Top**.

Proof. Define $\eta : \mathbf{\Sigma}\mathcal{O}(\mathbb{S}) \to \mathbb{S}$ by $\eta((f_s)_{s\in M}) = f_e((\{M\})_{m\in M})$. It is clear that η is an action-preserving map. It is also continuous, becasue

$$\begin{aligned}\eta^{-1}(\{M\}) &= \{(f_s)_{s\in M} \in \mathbf{\Sigma}\mathcal{O}(\mathbb{S}) : f_e((\{M\})_{m\in M}) = M\}\\ &= \{(f_s)_{s\in M} \in \mathbf{\Sigma}\mathcal{O}(\mathbb{S}) : f_s(\{M\})_{m\in M}) = M,\ \forall s \in M\}\\ &= \mathbf{\Sigma}_{(\{M\})_{m\in M}}.\end{aligned}$$

Also, we have $\eta\epsilon_{\mathbb{S}} = id_{\mathbb{S}}$. □

Remark 4.13. If L is an M-frame with the trivial action, then

$$\mathbf{\Sigma}L \cong Hom_{\mathbf{Frm}}(L,\Omega).$$

This is because, in this case, any $(f_s)_{s\in M} \in [L,\Omega]$ is completely determined by f_e. In fact, for each $s \in M$, $f_s = sf_e$, where sf_e is defined as $(sf_e)(a) = sf_e(a)$.

Notice that under the above isomorphism, the M-topology on $\mathbf{\Sigma}L$ corresponds to an M-topology on $Hom_{\mathbf{Frm}}(L,\Omega)$ whose M-open subsets are of the form $U_a = \{g \in Hom_{\mathbf{Frm}}(L,\Omega) : g(a) = M\}$, for $a \in L$.

Lemma 4.14. *If L is an M-frame with the trivial action then $\epsilon_{\mathbf{\Sigma}L}$ is one-one.*

Proof. Recalling the definition of $\epsilon_{\mathbf{\Sigma}L}$, and applying the above lemma, we consider

$$\epsilon_{\mathbf{\Sigma}L} : Hom_{\mathbf{Frm}}(L,\Omega) \to \mathbf{\Sigma}\mathcal{O}(Hom_{\mathbf{Frm}}(L,\Omega))$$

which takes $g \in Hom_{\mathbf{Frm}}(L,\Omega)$ to $(\epsilon_X(g)_s)_{s\in M}$, where for $s \in M$,

$$\begin{aligned}\epsilon_X(g)_s :&\mathcal{O}(Hom_{\mathbf{Frm}}(L,\Omega)) \to \Omega\\ &(U_{a_m})_{m\in M} \mapsto \{t \in M : t(sg) \in U_{a_t}\} = \{t \in M : tsg(a_t) = M\}.\end{aligned}$$

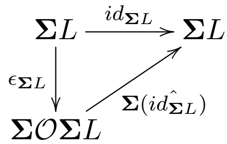
Now $\epsilon_{\mathbf{\Sigma}L}$ is one-one, for, if $g_1, g_2 \in Hom_{\mathbf{Frm}}(L,\Omega)$ are such that $\epsilon_{\mathbf{\Sigma}L}(g_1) = \epsilon_{\mathbf{\Sigma}L}(g_2)$, then for each $s \in M$, $\epsilon_{\mathbf{\Sigma}L}(g_1)_s = \epsilon_{\mathbf{\Sigma}L}(g_2)_s$ and, in particular, $\epsilon_{\mathbf{\Sigma}L}(g_1)_e = \epsilon_{\mathbf{\Sigma}L}(g_2)_e$, which gives $\epsilon_{\mathbf{\Sigma}L}(g_1)_e(U_{ma})_{m\in M} = \epsilon_{\mathbf{\Sigma}L}(g_2)_e(U_{ma})_{m\in M}$, for every $a \in L$, that is

$$\{t \in M : tg_1(a) = M\} = \{t \in M : tg_2(a) = M\},$$

since A has trivial action. Therefore, $g_1(a) = g_2(a)$, for every $a \in L$. □

Theorem 4.15. *If L is an M-frame with trivial action, then $\mathbf{\Sigma}L$ is a retract of $\mathbf{\Sigma}\mathcal{O}\mathbf{\Sigma}L$ as M-topological spaces.*

Proof. Consider $\hat{id_{\mathbf{\Sigma}L}} : L \to \mathcal{O}\mathbf{\Sigma}L$. Its definition in this case turns to $(\hat{id_{\mathbf{\Sigma}(L)}}(a))_m = \mathbf{\Sigma}_a$, for all $m \in M$, and so $\hat{id_{\mathbf{\Sigma}L}}$ is well-defined. Also, it is clearly an M-frame map, by Lemma 4.8. Now, $\mathbf{\Sigma}(\hat{id_{\mathbf{\Sigma}L}})$ is an M-continuous map which makes the following triangle commutative and hence is a left inverse of $\epsilon_{\mathbf{\Sigma}L}$:



□

5 M-sober topological spaces

In this section, we study the counterpart of the notion of sober spaces for M-topological spaces.

Definition 5.1. We say that an M-topological space X is M-sober if ϵ_X is an isomorphism in M-**Top**.

Lemma 5.2. *For any M-topological space X, ϵ_X is an M-topological isomorphism if and only if it is an isomorphism of M-sets.*

Proof. To prove the not clear part, assume tha ϵ_X is an isomorphism of M-sets. By Theorem 4.3, ϵ_X is M-continuous. To prove that ϵ_X^{-1} is M-continuous, it is enough to show that

$$\epsilon_X(U) = \mathbf{\Sigma}_{(U)_{m\in M}}, \quad \forall U \in \mathcal{O}(X).$$

Let $\epsilon_X(x)$ be a member of $\epsilon_X(U)$, where $x \in U$. Then, for each $s \in M$ we have

$$(\epsilon_X(x))_s : ((U)_{m\in M}) \rightsquigarrow \{t \in M : tsx \in U\} = M.$$

Therefore, $\epsilon_X(x)$ is a member of $\mathbf{\Sigma}_{(U)_{m\in M}}$. Conversely, taking $(f_s)_{s\in M} \in \mathbf{\Sigma}_{(U)_{m\in M}}$, since $\epsilon_X : X \to \mathbf{\Sigma}\mathcal{O}(X)$ is a bijection, there exists a unique $x \in X$ such that $(f_s)_{s\in M} = \epsilon_X(x)$. Therefore,

$$\{t \in M : tx \in U\} = (\epsilon_X(x)_e)((U)_{m\in M}) = f_e((U)_{m\in M}) = M,$$

and so $x \in U$, and $(f_s)_{s\in M} \in \epsilon_X(U)$. □

Corollary 5.3. *An M-topological space X is M-sober if and only if ϵ_X is a bijection.*

Recall that in the classical frame theory there is a one to one correspondence between frame homomorphisms $f : L \to 2$ and completely prime filters of L. Then, applying this correspondence, it is obtained that X is a sober space if and only if for every completely prime filter P of $O(X)$, there exists a unique $x \in X$ such that $P = \{U \in O(X) : x \in U\}$.

In the following, we consider the counterpart of the above result for M-frames. First, we define the notion of a completely prime filter for an M-frame.

Recall that, a completely prime filter of a frame L is a non-empty subset F of L which is upward closed and also closed under binary meets. Also if $\bigvee_{i\in I} a_i \in F$ then there exists $i \in I$ such that $a_i \in F$.

Definition 5.4. Let L be an M-frame. By a *completely prime internal filter* of L, we mean a member $(F_s)_{s\in M}$ of the power object Ω^L such that each F_s is a completely prime filter of L.

Lemma 5.5. *There is a one-one correspondence between* $[L, \Omega]$ *and completely prime internal filters of* L.

Proof. Taking $(f_s)_{s\in M} \in [L, \Omega]$, we see that $(f_s^{-1}\{M\})_{s\in M}$ is a completely prime internal filter of L. This is proved similar to the case of frames. □

Now, applying the above lemma and the fact that $(\epsilon_X(x)_s)_{s\in M} \in [\mathcal{O}(X), \Omega]$, we get the following fact.

Lemma 5.6. *For each M-topological space X and $x \in X$, $((\epsilon_X(x)_s)^{-1}\{M\})_{s\in M}$ is a completely prime internal filter of $\mathcal{O}(X)$.*

Theorem 5.7. *An M-topological space X is M-sober if and only if the completely prime internal filters of $\mathcal{O}(X)$ are of the form $(F_s)_{s\in M}$, where for each $s \in M$, $F_s = (\epsilon_X(x)_s)^{-1}\{M\}$, for a unique $x \in X$.*

Proof. Let X be an M-sober M-topological space, and $(F_s)_{s\in M}$ be a completely prime internal filter of $\mathcal{O}(X)$. Define $(g_s)_{s\in M} \in \mathbf{\Sigma}(\mathcal{O}(X)) = [\mathcal{O}(X), \Omega]$ by $g_s((U_m)_{m\in M}) = \{t \in M : t(U_m)_{m\in M} \in F_{ts}\}$, for each $s \in M$. It is easily seen that $(g_s)_{s\in M} \in \Omega^{\mathcal{O}(X)}$. Also, it is straightforward to show that each $g_s : \mathcal{O}(X) \to \Omega$ is a frame homomorphism. Now, since X is M-sober, $\epsilon_X : X \to \mathbf{\Sigma}\mathcal{O}(X)$ is a bijection, and hence there exists a unique $x \in X$ such that $\epsilon_X(x) = (g_s)_{s\in M}$. So, for each $s \in M$,

$$(\epsilon_X(x)_s)^{-1}(\{M\}) = g_s^{-1}\{M\} = \{(U_m)_{m\in M} \in \mathcal{O}(X) : (U_m)_{m\in M} \in F_s\} = F_s.$$

Conversely, applying Corollary 5.3, we prove that if the completely prime M-filters of an M-topological space X are of the above form, then ϵ_X is a bijection. It is one-one, for, if $x, y \in X$ are such that $\epsilon_X(x) = \epsilon_X(y)$, then for each $s \in M$, $(\epsilon_X(x)_s)^{-1}\{M\} = (\epsilon_X(y)_s)^{-1}\{M\}$. But, $\epsilon_X(x)$ is a completely prime M-filter of $\mathcal{O}(X)$, and so by the hypothesis, x should be equal to y. Also, ϵ_X is surjective, for, if $(F_s)_{s\in M} \in \mathbf{\Sigma}\mathcal{O}(X)$, then for each $s \in M$, $(F_s)^{-1}\{M\}$ is a completely prime M-filter of $\mathcal{O}(X)$, and so, by the assumption, there exists $x \in X$ such that $\epsilon_X(x)_s = F_s$. Therefore, $\epsilon_X(x) = (F_s)_{s\in M}$. □

Recall that in the classical case, considering $\mathbf{2}$ as the two element frame, $\Sigma\mathbf{2}$ is the one element topological space $\mathbf{1}$. In the following, we consider the case when $\mathbf{\Sigma}\Omega$ is the one element M-topological space $\mathbf{1}$.

Lemma 5.8. *For any monoid M, $\mathbf{\Sigma}L_M = \mathbf{1}$ if and only if $\mathbf{1}$ is an M-sober space.*

Proof. Since $\mathcal{O}(\mathbf{1}) = \Omega = L_M$, we get that $\mathbf{\Sigma}\mathcal{O}(\mathbf{1}) = \mathbf{\Sigma}L_M$. Therefore, $\mathbf{\Sigma}L_M = \mathbf{1}$ if and only if $\mathbf{\Sigma}\mathcal{O}(\mathbf{1}) = \mathbf{1}$ if and only if $\mathbf{1}$ is M-sober. □

Theorem 5.9. *$Z\mathbf{\Sigma}L_M = \mathbf{\Sigma}L_M$ if and only if $\mathbf{\Sigma}L_M = \mathbf{1}$ if and only if $\mathbf{1}$ is M-sober.*

Proof. First, we notice that if F is a completely prime filter of L_M, then $(C_s)_{s\in M}$ defined by

$$C_s = \begin{cases} F & s \in I \\ \{M\} & \text{otherwise,} \end{cases}$$

where I is an arbitrary left ideal of M, is a completely prime internal filter of L_M. Therefore, L_M is non-trivial if and only if $\mathbf{\Sigma}L_M$ has elements with non-equal components if and only if $Z\mathbf{\Sigma}L_M = \mathbf{\Sigma}L_M$. The second equivalence in the theorem is true by the above lemma. □

6 M-spatial frames

In this section, we study the counterpart of the notion of spatial frame for M-frames. Recall that a frame L is called spatial if the unit of the adjunction between Σ and O is a frame isomorphism. But, as we saw in the last section, for the case of frames and topological spaces in M-**Set**, the adjunction does not necessarily exist. Therefore, we consider this notion for the case that $\mathbf{\Sigma}$ and $\mathcal{O}$ form an adjoint pair with ϵ as the unit. Let us study the counit of this adjunction. Then the counit $\eta_L : L \to \mathcal{O}\mathbf{\Sigma}L$, for an M-frame L, makes the following triangle commutative:

$$\begin{array}{ccc} \mathbf{\Sigma}L & \xrightarrow{\epsilon_{\mathbf{\Sigma}L}} & \mathbf{\Sigma}\mathcal{O}\mathbf{\Sigma}L \\ & {\scriptstyle id_{\mathbf{\Sigma}L}}\searrow & \downarrow{\scriptstyle \mathbf{\Sigma}\eta_L} \\ & & \mathbf{\Sigma}L \end{array}$$

So, by recalling Remark 4.5, we get

$$(\hat{id_{\mathbf{\Sigma}L}})_m = (\eta_L(a))_m = \{f \in \mathbf{\Sigma}L : f_e(ma) = M\} = \mathbf{\Sigma}_{ma}.$$

for each $a \in L$ and $m \in M$.

Theorem 6.1. *If $Z\mathbf{\Sigma}L = \mathbf{\Sigma}L$, then η_L is well-defined.*

Proof. Assume that $Z\mathbf{\Sigma}L = \mathbf{\Sigma}L$. To see that η_L is well-defined, we have to prove that the family $(\Sigma_{sa})_{s\in M}$ is a member of $\mathcal{O}\mathbf{\Sigma}L$. First, we note that, by the definition of the M-topology on $\mathbf{\Sigma}L$, Σ_{sa} is an M-open subset, for each $s \in M$. Secondly, for each $a \in L$ and $s, t \in M$ we have $t\Sigma_{sa} \subseteq \Sigma_{tsa}$. This is because, for each $f \in \Sigma_{sa}$, we have $f_e(sa) = M$ and so $(tf)_e(tsa) = f_t(tsa) = tf_e(sa) = M$. Since the action on $\mathbf{\Sigma}L$ is trivial, the latter gives $(tf)_m(tsa) = M$, for all $m \in M$, as required. □

Theorem 6.2. *If M is a commutative monoid, then η_L is well-defined.*

Proof. To see that η_L is well-defined, we should prove that for each $a \in L$ and $s, t \in M$, $t\mathbf{\Sigma}_{sa} \subseteq \mathbf{\Sigma}_{tsa}$. Let $f \in \Sigma_{sa}$. Then for each $m \in M$, we have $f_m(sa) = M$, and so

$$(tf)_m(tsa) = f_{mt}(tsa) = f_{tm}(tsa) = tf_m(sa) = tM = M.$$

□

Remark 6.3. The converse of the above theorem is not necessarily true. In fact, the monoid M_2 is an example of a commutative monoid for which the adjunction does not hold (see Example 4.10) while, by the above lemma, η_L for all M_2-frames L is well-defined and action-preserving.

Definition 6.4. We say that an M-frame L is *M-spatial* if η_L is an M-frame isomorphism.

Lemma 6.5. *If L is an M-spatial frame, then there exists $a \in L$ such that for each $s \in M$,*

$$\mathbf{\Sigma}_{sa} = \mathbf{\Sigma}L \quad \textit{or} \quad \mathbf{\Sigma}_{sa} = \emptyset.$$

Proof. First notice for any ideal I of M, the family $(U_s)_{s\in M}$ defined by $U_s = \mathbf{\Sigma}L$ for $s \in I$, and $U_s = \emptyset$, for $s \in M \setminus I$, is a member of $\mathcal{O}\mathbf{\Sigma}L$. This is because, for each $s \in M$, U_s is an open sub M-set of $\mathbf{\Sigma}L$. Also, for each $s, t \in M$, if $s \in I$, then

$$tU_s = t\mathbf{\Sigma}L \subseteq \mathbf{\Sigma}L = U_{ts}.$$

If $s \notin I$, then

$$tU_s = \emptyset \subseteq U_{ts}.$$

Now, if L is an M-spatial frame, then taking any ideal I of M, since η_L is a bijection, there exists (a unique) $a \in L$ such that for each $s \in M$, $\mathbf{\Sigma}_{sa} = U_s$, and so $\mathbf{\Sigma}_{sa} = \mathbf{\Sigma}L$ or $\mathbf{\Sigma}_{sa} = \emptyset$, as required. □

7 G-topological spaces and G-frames

In this section, we study the results of the paper for the case where M is a group, and through out this section we denote M by G.

Theorem 7.1. *For any G-topological space X, $\mathcal{O}(X)$ is isomorphic to $O(X)$ as frames.*

Proof. Recall from [3] that, since G is a group, each $(U_s)_{s\in M} \in \Omega^X$ is completely determined by U_e ($U_s = sU_e$), and $\Omega^X \cong \mathcal{P}(X)$ with the assignment $(U_s)_{s\in M} \mapsto U_e$. This isomorphism is in fact a frame isomorphism, and the restriction of it gives the required frame isomorphism $\mathcal{O}(X) \cong O(X)$. □

Remark 7.2. Let X be a G-topological space. Then the action on $\mathcal{O}(X)$, as a sub G-set of Ω^X, is trivial. This is because, as the proof of the above lemma shows, every $(U_m)_{m\in M} \in \mathcal{O}(X)$ is completely determined by U_e, in fact $U_s = sU_e$ for all $s \in M$. But, U_e is a sub M-set of X and so $s^{-1}U_e \subseteq U_e$, which gives $U_e = ss^{-1}U_e \subseteq sU_e \subseteq U_e$, for all $s \in M$, and hence $U_s = U_e$. Therefore, for every $s \in M$, $s(U_m)_{m\in M} = s(U_e)_{m\in M} = (U_s)_{m\in M} = (U_e)_{m\in M} = (U_m)_{m\in M}$.

Theorem 7.3. *Let L be a G-frame. Then $\mathbf{\Sigma}L$ is isomorphic to ΣL as sets (not necessarily as topological spaces).*

Proof. Let L be a G-frame. We show that $\mathbf{\Sigma}L \cong Hom_{\mathbf{Frm}}(L, 2)$. Recall from [3] that, since G is a group, each $(f_s)_{s\in M} \in \Omega^L$ is completely determined by f_e (in fact $f_s(a) = sf_e(s^{-1}a)$), and $\Omega^L \cong Hom_{\mathbf{S}et}(L, \Omega)$ with the assignment $(f_s)_{s\in M} \mapsto f_e$. The restriction of this isomorphism gives the required isomorphism $[L, \Omega] \cong Hom_{\mathbf{F}rm}(L, \Omega)$.

Notice that under this isomorphism, an M-open subsets $\mathbf{\Sigma}_a$, for $a \in L$, correspond to the sets of the form

$$\{f \in \Sigma L : f(sa) = G, \ \forall s \in G\}$$

which are not necessarily of the form Σ_a (or even open), unless L has the trivial action. □

It may be interesting to notice that applying the above bijection, and using the action on $[L, \Omega]$, the following action arises on ΣL: $(sg)(a) = g(s^{-1}a)$, for $g \in \Sigma L$, $a \in L$, $s \in M$.

Remark 7.4. If L is a G-frame with the trivial action then, by Remark 4.13, for each $(f_s)_{s\in M}$, we have $f_s = sf_e$, for all $s \in M$. But, since $\Omega = 2$ has the trivial action, we also get that $sf_e = f_e$ and so $f_s = f_e$. Therefore, $\mathbf{\Sigma}L = Z\mathbf{\Sigma}L$.

By the above remark and as a corollary of Theorem 4.11, we have:

Theorem 7.5. *$\mathcal{O}(-) : G-\mathbf{Top} \to G-\mathbf{Frm}^{op}$ is a left adjoint to the functor $Z\mathbf{\Sigma}(-) : G - \mathbf{Frm}^{op} \to G - \mathbf{Top}$.*

In the following, we study G-sober space and G-spatial frames.

Remark 7.6. Notice that, if X is a G-sober space then X has the trivial action. This is because, if X is a G-sober space then $\epsilon_X : X \to \mathbf{\Sigma}\mathcal{O}X$ is an isomorphism in M-**Top**. But, applying Remarks 7.2 and 7.4, the action on $\mathbf{\Sigma}\mathcal{O}(X)$ is also trivial, and so X has the trivial action, as well.

Theorem 7.7. *A G-space X is G-sober if and only if X is a sober topological space.*

Proof. By Remark 7.2, the action on $\mathbf{\Sigma}\mathcal{O}(X)$ is trivial. So, by Remark 7.4, we have $\mathbf{\Sigma}\mathcal{O}(X) \cong \Sigma\mathcal{O}(X)$. This isomorphism is in fact given by $\phi : (f_s)_{s\in M} \mapsto f_e$, and it commutates the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\epsilon_X} & \mathbf{\Sigma}\mathcal{O}X \\ {\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle \phi} \\ X & \xrightarrow[(\epsilon_X(-))_e]{} & \Sigma \mathcal{O}X. \end{array}$$

Thus, ϵ_X is an isomorphism of sets if and only if $(\epsilon_X(-))_e$ is an isomorphism of sets. Therefore, by Corollary 5.3 and its counter part for classical sober spaces, we have X is a G-sober space if and only if X is a sober space. □

Corollary 7.8. *If L is a G-frame, then $\mathbf{\Sigma}L$ is a G-sober space.*

Theorem 7.9. *A G-frame L is G-spatial if and only if L is spatial as a frame.*

Proof. Applying Theorems 7.1 and 7.3, for any M-frame L we have $\mathcal{O}\mathbf{\Sigma}L \cong \mathcal{O}\Sigma L$, with the assignment $\psi : (U_s)_{s\in M} \mapsto U_e$. This isomorphism also commutes the following diagram

$$\begin{array}{ccc} L & \xrightarrow{\eta_L} & \mathcal{O}\mathbf{\Sigma}L \\ {\scriptstyle id_L}\downarrow & & \downarrow{\scriptstyle \psi} \\ L & \xrightarrow[(\eta_L(-))_e]{} & \mathcal{O}\Sigma L. \end{array}$$

Therefore, η_L is a frame isomorphism if and only if so is $(\eta_L(-))_e$. This means that L is spatial if and only if L is G-spatial. □

Remark 7.10. A G-spatial G-frame L has just the trivial action. This is because, if L is G-spatial then, since $L \cong \mathcal{O}\mathbf{\Sigma}L$ and by Remark 7.2, the action on L is trivial.

Corollary 7.11. *If X is a G-topological space, then $\mathcal{O}(X)$ is a G-spatial frame.*

As a consequence of the above results, we have

Proposition 7.12. *The restriction of the adjoint pair and $\mathcal{O}$ and $\mathbf{\Sigma}$ provides a dual isomorphism between the categories of G-sober spaces and G-spatial frames.*

We close the paper by finding some conditions related to soberity of $\mathbf{1}$, and spatiality of $\mathcal{O}(\mathbf{1})$ which make a monoid into a group.

Theorem 7.13. *A commutative monoid M is a group if and only if $\mathbf{1}$ is an M-sober space.*

Proof. Let M be a commutative monoid. Then for each $s \in M$, the family $(\uparrow Ms)_{s\in M}$, where

$$\uparrow Ms = \{I \in L_M : I \supseteq Ms\},$$

is a completely prime internal filter of L_M. Now, applying Lemma 5.8, if $\mathbf{1}$ is an M-sober space then $\mathbf{\Sigma} L_M = \mathbf{1}$, and hence $(\uparrow Ms)_{s\in M}$ should be equal to the constant family $(M)_{s\in M}$. This means that $Ms = M$, for all $s \in M$, and so M is a group.

Conversely, if M is a group then $L_M = \mathbf{2}$, and hence $\mathbf{\Sigma} L_M = \mathbf{1}$. Now, by Lemma 5.8, $\mathbf{1}$ is an M-sober space. □

Theorem 7.14. *A monoid M is a group if and only if there exists an M-topological space X such that $\mathcal{O}(X)$ has the trivial action.*

Proof. By Remark 7.2, if M is a group then for each M-topological space X, $\mathcal{O}(X)$ has the trivial action.

To prove the converse, let X be an M-topological space such that $\mathcal{O}(X)$ has the trivial action. Take $f : X \to \mathbf{1}$ to be the unique M-continuous (constant) map. By Theorem 3.7, the functor $\mathcal{O} : M - \mathbf{Top} \to M - \mathbf{Frm}^{\mathbf{op}}$ preserves finite colimits, so it takes the epimorphism f to the monomorphism $\mathcal{O}(f) : \mathcal{O}(1) \to \mathcal{O}(X)$. But, $\mathcal{O}(1) \cong \Omega = L_M$, and so L_M, being isomorphic to a sub of $\mathcal{O}(X)$, has the trivial action. This implies that M is a group, because for each ideal I and $s \in M$, $sI = M$ if and only if $s \in I$. □

Corollary 7.15. *If any M-frame L with the trivial action is M-spatial frame, then M is a group.*

Corollary 7.16. *A monoid M is a group if and only if $\mathbf{2}$ is a spatial M-frame.*

Proof. An element of $\mathbf{\Sigma 2}$ is of the form $(f_s)_{s\in M}$, where each $f_s : \mathbf{2} \to \mathbf{2}$ is a frame homomorphism. But there is only one frame homomorphism from $\mathbf{2}$ to $\mathbf{2}$. Hence $\mathbf{\Sigma 2}$ has only one element. So $\mathcal{O}\mathbf{\Sigma 2}$ is isomorphic to $\mathbf{2}$, and so $\eta_{\mathbf{2}}$ is clearly an isomorphism. □

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