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On C-injective generalized hyper S-acts

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Abstract. This paper explores generalized hyper S-acts (GHS-acts) over a hypermonoid S as generalizations of monoid acts within the context of algebraic hyperstructures. Specifically, we extend the definition of C-injectivity to GHS-acts and investigate their internal and homological properties. It is established that to determine the GHS-injectivity of GHS-acts with a fixed element, we only need to consider the inclusions of cyclic GHS-subacts into indecomposable ones. Additionally, we introduce the concepts of semi-injectivity and semi-C-injectivity and give some characterizations of these types of injectivity for quotients of S_S. It is demonstrated that, in contrast to the case of ordinary acts over monoids, cyclic GHS-acts are not necessarily a quotient of S_S, and injectivity and semi-injectivity do not coincide in the category of GHS-acts with a fixed element. Among other things, we also show that all pure GHS-acts are injective if and only if all pure cyclic GHS-acts are C-injective.

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1 Introduction and Preliminaries

Algebraic hyperstructures, initiated by Marty [3] in 1934, serve as generalizations of classical algebraic structures. An algebraic hyperstructure consists of a set with operations in which the composition of two elements is a non-empty set. A natural generalization of the concept of actions of monoids on sets involves hyperactions of monoids on sets, which were introduced in [4] and [6]. This idea was further developed in [5, 7, 8], by introducing hypermonoids and their hyperactions on sets. In keeping with the notation used in [7], we refer to this generalization as a generalized hyper S-act, or briefly GHS-act, where S is a hypermonoid. An S-act A is called C-injective, as studied in [9], if it is injective relative to all inclusions with cyclic domains. In this paper, we extend the definition of C-injectivity to GHS-acts and examine some of their internal and homological properties. For the necessary background information about S-acts and their properties, one may consult [2].

Let A be a set. We denote the set of all subsets of A by $\mathcal{P}(A)$ and put $\mathcal{P}^*(A) := \mathcal{P}(A)\backslash\{\emptyset\}$. Let S be a non-empty set and $\circ : S \times S \to \mathcal{P}^*(S)$ be a map, called a *hyperoperation*, satisfying the following conditions:

(i) For any $s, t, r \in S$, $s \circ (t \circ r) = (s \circ t) \circ r$ (where $T \circ s = \bigcup_{t \in T} t \circ s$ and $s \circ T = \bigcup_{t\in T} s \circ t$ for any $T \subseteq S$ and $s \in S$).

(ii) There exists an element $e \in S$ with $s \circ e = e \circ s$ and $s \in s \circ e$ for any $s \in S$.

Then $(S, \circ)$ is called a *hypermonoid*. In the second condition, the element e is called an *identity element* of S.

Let $(S, \circ)$ be a hypermonoid and K be a non-empty subset of S. Then K is called a *right ideal* of S, if $K \circ S = \bigcup_{s\in S} K \circ s \subseteq K$. Let X be a non-empty set and $* : X \times S \to \mathcal{P}^*(X)$ satisfy the following conditions:

(i) For any $x \in X$ and $s, t \in S$, $x * (s \circ t) = (x * s) * t$ (where $y * T = \bigcup_{t\in T} y * t$ and $Y * t = \bigcup_{y\in Y} y * t$ for any $y \in X$, $t \in S$, $T \subseteq S$ and $Y \subseteq X$).

(ii) $x \in x * e$ for any $x \in X$.

Then $(X_S, *)$ (or simply, X_S) is called a (*right*) *generalized hyper S-act* (or simply, a *GHS-act*). Let X_S be a GHS-act and $Y \subseteq X$. Whenever $Y * S \subseteq Y$, then $(Y_S, *)$ is a GHS-act called a *GHS-subact* of X_S, where by $Y * T$ we mean the set $\bigcup_{y\in Y} y * T$ for any $T \subseteq S$. Clearly, the GHS-subacts of S_S are precisely the right ideals of S. The GHS-act X_S is called *finitely generated* if there exists a finite subset Y of X with $X_S = Y * S$. Moreover, if Y is a singleton, then X_S is called *cyclic*. An element $x \in X_S$ is called a *fixed element* if $x * s = \{x\}$ for any $s \in S$.

Let $(S, \circ)$ be a hypermonoid. An identity element $e \in S$ is called *pure* whenever $s \circ e = e \circ s = \{s\}$ for any $s \in S$. Let S have a pure identity element e and X_S be a GHS-act with $x * e = \{x\}$ for any $x \in X$. In this case, we call X_S a pure GHS-act. In the sequel, any hypermonoid has a (unique) pure identity element denoted by 1.

Let $\mathcal{F} = \{X^i_S\}_{i\in I}$ be a family of GHS-acts. Assume that $X_S = \prod_{i\in I} X^i_S = \{(x_i)_{i\in I} : \forall\ i \in I,\ x_i \in X_i\}$, $Y_S = \coprod_{i\in I} X^i_S = \dot{\bigcup}_{i\in I} X^i_S = \bigcup_{i\in I} X^i_S \times \{i\}$, and for any $i \in I$, $*_i$ is the action of S on X^i_S. For any $(x_i)_{i\in I} \in X_S$, $j \in I$, $y_j \in X^j_S$ and $s \in S$, define $(x_i)_{i\in I} * s = \{(a_i)_{i\in I} : \forall\ i \in I,\ a_i \in x_i *_i s\}, (y_j, j) *' s = \{(z, j) : z \in y_j *_j s\}$. Then X_S and Y_S are two GHS-acts called the *product* and the *coproduct* of the family $\mathcal{F}$, respectively.

Let X_S and Y_S be two GHS-acts and $\phi : X_S \to Y_S$ be a map. Then by $\phi(x * s)$ we mean $\{\phi(z) : z \in x * s\}$ for any $x \in X$ and $s \in S$. A map $\phi : X_S \to Y_S$ is called a *GHS-homomorphism* if $\phi(x * s) = \phi(x) * s$ for any $x \in X$ and $s \in S$. A one to one GHS-homomorphism is called a *GHS-monomorphism* and a bijective GHS-homomorphism whose inverse is also a GHS-homomorphism is called a *GHS-isomorphism.* Let X_S be a GHS-act and Y_S be a GHS-subact of X_S. A GHS-homomorphism $f : X_S \to Y_S$ is called a *retraction* whenever $f|_Y = \mathrm{id}_Y$. In this case, Y_S is called a *retract* of X_S. Let X_S be a GHS-act and μ be a relation on X_S. Assume that $X_1, X_2 \subseteq X_S$. Then by $X_1\,\mu\,X_2$ we mean for any $x_1 \in X_1$, there exists an element $x_2 \in X_2$ with $x_1\,\mu\,x_2$ and vice versa. An equivalence relation μ on X_S is called a *congruence* on X_S whenever $x\,\mu\,y$ implies $(x * s)\,\mu\,(y * s)$ for any $x, y \in X_S$ and $s \in S$. Let μ be a congruence on X_S. Then we denote the equivalence class including $x \in X_S$ by $\frac{x}{\mu}$ for any $x \in X_S$ and the set of all equivalence classes by $\frac{X_S}{\mu}$. By defining $\frac{x}{\mu} \odot s = \frac{x*s}{\mu}$ for any $\frac{x}{\mu} \in \frac{X_S}{\mu}$ and

$s \in S$, one can see that $(\frac{X_S}{\mu}, \odot)$ is a GHS-act, called the *quotient* GHS-*act* of X_S by μ. Note that $L \subseteq \frac{X_S}{\mu}$ is a GHS-subact of $\frac{X_S}{\mu}$ if and only if there exists a GHS-subact Y_S of X_S, where $Y_S = \{y \in X : [y]_\mu \in L\}$ such that $L = \frac{Y_S}{\mu}$, where $\frac{Y_S}{\mu} = \{[y]_\mu \in \frac{X_S}{\mu} : y \in Y\}$. We denote the trivial congruence $\{\{x\} : x \in X_S\}$ on X_S by Δ_X. Let $\phi : X_S \to Y_S$ be a GHS-homomorphism. The *kernel* of ϕ, denoted by $\mathrm{Ker}(\phi)$, is defined as $\mathrm{Ker}(\phi) = \{(x_1, x_2) \in X_S \times X_S : \phi(x_1) = \phi(x_2)\}$. The *image* of a GHS-homomorphism ϕ, denoted by $\mathrm{Im}(\phi)$, is naturally defined. Clearly, $\mathrm{Ker}(\phi)$ is a congruence on X_S, and $\frac{X_S}{\mathrm{Ker}(\phi)}$ and $\mathrm{Im}(\phi)$ are isomorphic GHS-acts, see [6].

A GHS-act X_S is called *injective* if for any GHS-monomorphism $i : Y_S \hookrightarrow Z_S$ and any GHS-homomorphism $f : Y_S \to X_S$, there exists a GHS-homomorphism $g : Z_S \to X_S$ for which the following diagram commutes:

$$\begin{tikzcd}
Y_S \arrow[r, hook, "i"] \arrow[d, "f"'] & Z_S \arrow[dl, dotted, "g"] \\
X_S &
\end{tikzcd}$$

Given the assumptions mentioned above, X_S is referred to as C-*injective* if Y_S is cyclic. The injectivity of GHS-acts was studied in [5], where it was shown that the retracts of injective GHS-acts are also injective. Additionally, when S has a unique pure identity element, any GHS-act X_S can be essentially embedded in an injective GHS-act known as the *injective hull* of X_S (For the notion of essentiality of S-acts, see [1]). Here we provide an equivalent definition of an injective GHS-act by assuming that Y_S is a GHS-subact of Z_S and i denotes the inclusion. It is worth noting that every injective GHS-act is C-injective. A GHS-act X_S is said to be *decomposable* if there exist two disjoint GHS-subacts Y_S and Z_S of X_S whose union equals X_S, otherwise, it is called *indecomposable*. Any cyclic GHS-act is clearly indecomposable. Furthermore, similarly to ordinary acts over monoids, every GHS-act has a unique decomposition into indecomposable GHS-subacts (see [7]).

The organization of this paper is as follows. First we examine C-

injective GHS-acts and prove that any such act has a fixed element. It is also established that being injective and C-injective for cyclic pure GHS-acts are equivalent. Furthermore, if a GHS-act X_S has a fixed element, it suffices to consider indecomposable GHS-acts Z_S in the definition of C-injectivity. We show that, for a hypermonoid S, all pure GHS-acts are injective if and only if all pure cyclic GHS-acts are C-injective. Moreover, we introduce the notion of semi-injectivity (semi-C-injectivity), that is, injectivity with respect to all inclusions into cyclic GHS-acts (with cyclic domains) and provide some characterizations of such kinds of injectivity for quotients of S_S. We illustrate that cyclic GHS-acts may not always be a quotient of S_S, in contrast to ordinary acts over monoids. Furthermore, we demonstrate that although the Skornjakov criterion holds for GHS-acts with a fixed element (see [5, Theorem 5]), injectivity and semi-injectivity are not equivalent.

2 C-injectivity of GHS-acts

In this section, we introduce the concept of a C-injective GHS-act. We then proceed to examine various elementary and homological properties, as well as certain categorical aspects associated with this concept.

Lemma 2.1. *Let a GHS-act X_S have a fixed element and $f : X_S \to Y_S$ be a GHS-homomorphism. Then Y_S has a fixed element.*

Corollary 2.2. *Any C-injective GHS-act has a fixed element.*

Proof. Let X_S be a C-injective GHS-act and $x \in X_S$. Take $X_S \cup \{\theta\}$ as the S-act with a fixed element θ adjoined to X. Consider the following diagram:

$$\begin{tikzcd} x * S \arrow[r, hook] \arrow[d, hook] & X_S \cup \{\theta\} \\ X_S & \end{tikzcd}$$

Since X_S is C-injective, there exists a GHS-homomorphism $g : X_S \cup \{\theta\} \to X_S$ making the above diagram commutative. Now, it follows from Lemma 2.1 that X_S has a fixed element. $\square$

The following theorem states a relationship between injectivity and C-injectivity.

Theorem 2.3. *A cyclic pure GHS-act X_S is C-injective if and only if it is injective.*

Proof. Assume that X_S is a cyclic pure C-injective GHS-act. By [5, Proposition 4.2], X_S is embedded in an injective GHS-act Z_S. Using the assumption, there exists a GHS-homomorphism $g : Z_S \to X_S$ such that $g|_{X_S} = id_{X_S}$. So X_S is a retract of the injective GHS-act Z_S, which gives that X_S is injective. $\square$

The next result provides an equivalent condition to C-injectivity of GHS-acts.

Theorem 2.4. *Let X_S be a GHS-act. Then X_S is C-injective if and only if X_S has a fixed element and for any indecomposable GHS-act Z_S and any cyclic GHS-subact Y_S of Z_S, any GHS-homomorphism $f : Y_S \to X_S$ is extended to a GHS-homomorphism $g : Z_S \to X_S$.*

Proof. In view of Corollary 2.2, it suffices to prove the sufficiency. Consider a GHS-act X_S with a fixed element θ satisfying the property mentioned in the hypothesis. To prove that X_S is C-injective, let Y_S be a cyclic GHS-subact of a GHS-act Z_S. Assume that $Z_S = \bigcup_{i\in I} Z_S^i$ is the (unique) decomposition of Z_S into its indecomposable GHS-subacts Z_S^i. One can see that there exists $i_0 \in I$ with $Y_S \subseteq Z_S^{i_0}$. So there exists a GHS-homomorphism $g_{i_0} : Z_S^{i_0} \to X_S$ such that $g_{i_0}|_{X_S} = f$. Define the map $g : Z_S \to X_S$ as follows:

$$g(x) = \begin{cases} g_{i_0}(x) & x \in Z_S^{i_0}, \\ \theta & x \notin Z_S^{i_0}. \end{cases}$$

Hence, g is a GHS-homomorphism extending f, as desired. $\square$

Note that for a family $\{X_S^i\}_{i\in I}$ of GHS-acts in which X_S^i has a fixed element θ_i for any $i \in I$, $\lambda_j : X_S^j \to \prod_{i\in I} X_S^i$, mapping any $x_j \in X_S^j$ to $\{y_i\}_{i\in I}$ with $y_j = x_j$ and $y_i = \theta_i$ for any $i \neq j$, is a GHS-homomorphism.

The following is devoted to study the behaviour of C-injectivity with respect to the product and coproduct of a family of GHS-acts.

Theorem 2.5. *Let $\{X_S^i\}_{i\in I}$ be a family of GHS-acts and $X_S = \prod_{i\in I} X_S^i$ and $Y_S = \coprod_{i\in I} X_S^i$. Then*

(i) *X_S is C-injective if and only if X_S^i is C-injective for any $i \in I$.*

(ii) *Y_S is C-injective and X_S^i has a fixed element for any $i \in I$ if and only if X_S^i is C-injective for any $i \in I$.*

Theorem 2.6. *Let S be a hypermonoid. Then the following are equivalent:*

(i) *All pure GHS-acts are C-injective.*

(ii) *All cyclic pure GHS-acts are C-injective.*

(iii) *All indecomposable pure GHS-acts are C-injective.*

(iv) *All cyclic pure GHS-acts are injective.*

Proof. (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i) are trivial.

(ii) $\Rightarrow$ (iii) Let X_S be an indecomposable pure GHS-act. Consider the following diagram:

$$\begin{tikzcd} Y_S \arrow[r, hook] \arrow[d, "f"'] & Z_S \\ X_S & \end{tikzcd}$$

where Y_S is a cyclic GHS-act. Note that $f(Y_S)$ is a cyclic GHS-subact of X_S. So there exists a GHS-homomorphism $g : Z_S \to f(Y_S)$ extending f.

(iii) $\Rightarrow$ (iv) This follows from Theorem 2.3 and the fact that all cyclic GHS-acts are indecomposable. $\square$

3 Semi-injectivity and semi-C-injectivity

In this section, we introduce the notions of semi-injectivity and semi-C-injectivity and show that they are actually different. Then we give some characterization results for being such kinds of injectivity of quotients of S_S, where S is a hypermonoid.

Remark 3.1. As we know, any cyclic S-act is isomorphic to a quotient of S_S, where S is a monoid. However, there are some cyclic GHS-acts over a hypermonoid S isomorphic to no quotient of S_S. For instance, assume that the hypermonoid $S = \{1, s, t\}$ and the GHS-act $X_S = \{x_1, x_2, x_3\}$ are defined as the following tables:

$\circ$	1	s	t
1	$\{1\}$	$\{s\}$	$\{t\}$
s	$\{s\}$	$\{s\}$	$\{s,t\}$
t	$\{t\}$	$\{s\}$	$\{t\}$

$*$	1	s	t
x_1	$\{x_1\}$	$\{x_1, x_2\}$	$\{x_1\}$
x_2	$\{x_2\}$	$\{x_2\}$	$\{x_2\}$
x_3	$\{x_3\}$	X	X

Then X_S is cyclic but isomorphic to no quotient of S_S. For this, it suffices to show that X_S and S_S are not isomorphic since the cardinalities of the other quotients of S_S are less than 3. On the contrary, assume that $\phi : S_S \to X_S$ is a GHS-isomorphism. Note that both X_S and S_S have only one generator. Hence, $\phi(1) = x_3$, which implies $\{\phi(s)\} = \phi(1 \circ s) = \phi(1) * s = x_3 * s = X$, which is a contradiction.

Based on Remark 3.1, we present the following definition:

Definition 3.2. GHS-act X_S is called *semi-injective* whenever for any congruence μ on S_S, any GHS-subact Y_S of $\frac{S_S}{\mu}$ and any GHS-homomorphisms $f : Y_S \to X_S$, there exists a GHS-homomorphism $g : \frac{S_S}{\mu} \to X_S$ extending f. Further, if Y_S is cyclic, then X_S is called *semi-C-injective*. These notions can also be defined for acts over monoids.

Obviously, any semi-injective GHS-act is semi-C-injective. The following example shows that the converse is not true in general.

Example 3.3. Let $S = \{1, s, t\}$. Define $\circ$ on S as follows:

$$x \circ y = \{x\}, \quad 1 \circ x = x \circ 1 = \{x\}, \quad \forall x, y \in \{s, t\}.$$

Then S is a hypermonoid. Obviously, $K = \{s,t\}$ is a right ideal of S. We claim that K_S is C-injective and then semi-C-injective but not semi-injective. Note that the identity map id : $K_S \to K_S$ has no extension to S_S, since the image of any GHS-homomorphism from S_S to K_S must be cyclic. This follows that K_S is not semi-injective. Now assume that Z_S is a GHS-act and Y_S is a cyclic GHS-subact of Z_S. Moreover, assume that $f : Y_S \to K_S$ is a GHS-homomorphism. Then $\mathrm{Im}(f)$ is cyclic and hence a singleton. Let $\mathrm{Im}(f) = \{q\}$, where $q \in K_S$. Then the GHS-homomrphism $g : Z_S \to X_S$, by setting $g(m) = q$, is an extension of f. Therefore, K_S is C-injective.

The well-known Skornjakov criterion states that an S-act, for a monoid S, containing a fixed element is injective if and only if it is injective relative to all inclusions into cyclic acts, i.e., semi-injective in our setting. This criterion also holds for GHS-acts with a fixed element (see [5, Theorem 5]). However, the following example shows that semi-injective GHS-acts (with a fixed element) are not necessarily injective.

Example 3.4. Consider the hypermonoid S mentioned in Remark 3.1. Note that S_S is not C-injective and then not injective since it has no fixed element. We show that S_S is semi-injective. All congruences on S_S are the sets $\mu_1 = \{\{1\},\{s\},\{t\}\}, \mu_2 = \{\{1\},\{s,t\}\}, \mu_3 = \{\{1,t\},\{s\}\}, \mu_4 = \{\{1,s,t\}\}$.

Also $K_1 = \{s,t\}$ and $K_2 = S$ are the only right ideals of S. We must show that for any congruence μ on S_S and any right ideal K of S, any GHS-homomorphism $f : \frac{K_S}{\mu} \to S_S$ is extended to a GHS-homomorphism $g : \frac{S_S}{\mu} \to S_S$. If $K = S$, then we can choose $g = f$. Otherwise, assume that $K = K_1$. One can see that the only GHS-homomorphism from K_S to S_S is the inclusion. It follows that there is no GHS-homomorphism from $\frac{K_S}{\mu_2}$ and $\frac{K_S}{\mu_4}$ to S_S. Also, if $\mu = \mu_3$, then $\frac{K_2}{\mu} = \frac{S_S}{\mu}$ established before. It remains to verify the case $K = K_1$ and $\mu = \mu_1$. In this case, f is the inclusion, and it is enough to take g the identity map.

In what follows, we study semi-injectivity and semi-C-injectivity of quotients of S_S for a hypermonoid S. To this end, we list some preliminaries.

Let S be a hypermonoid, K be a right ideal of S and μ and λ be two congruences on S_S. For any $s \in S$, we define $K(s,\mu) = \{a \in S : \frac{s \circ a}{\mu} \subseteq \frac{K_S}{\mu}\}$.

It is easily seen that $K(s,\mu)$ is a right ideal of S. For any $q \in S$, the relation $\mathcal{R}(K,\mu,\lambda,q)$ on S is defined as follows:

$$s\,\mathcal{R}(K,\mu,\lambda,q)\,t \Leftrightarrow K(s,\mu) = K(t,\mu), \forall a \in K(s,\mu), (q \circ s \circ a)\ \lambda\ (q \circ t \circ a),$$

for all $s, t \in S$. Now we have the following:

Lemma 3.5. *The relation $\mathcal{R}(K,\mu,\lambda,q)$ is a congruence on S_S.*

Proof. It is clear that $\mathcal{R}(K,\mu,\lambda,q)$ is an equivalence relation. Denote $\mathcal{R}(K,\mu,\lambda,q)$ briefly by $\mathcal{R}$. Let $x,y,s \in S$ and $\frac{x}{\mathcal{R}} = \frac{y}{\mathcal{R}}$. We must show that $\frac{x\circ s}{\mathcal{R}} = \frac{y\circ s}{\mathcal{R}}$. It follows from $\frac{x}{\mathcal{R}} = \frac{y}{\mathcal{R}}$ that $K(x,\mu) = K(y,\mu)$ and $(q \circ x \circ b)\ \lambda\ (q \circ y \circ b)$ for any $b \in K(x,\mu)$. First, we need to show that $K(x \circ s,\mu) = K(y \circ s,\mu)$. For any $z \in K(x \circ s,\mu)$, we have $\frac{(x\circ s)\circ z}{\mu} \subseteq \frac{K}{\mu}$. Note that $\frac{(x\circ s)\circ z}{\mu} = \frac{x\circ(s\circ z)}{\mu}$, which implies $s \circ z \subseteq K(x,\mu) = K(y,\mu)$. Hence, $\frac{(y\circ s)\circ z}{\mu} = \frac{y\circ(s\circ z)}{\mu} \subseteq \frac{K}{\mu}$. Thus $z \in K(y \circ s,\mu)$, which implies $K(x \circ s,\mu) \subseteq K(y \circ s,\mu)$. Similarly, $K(y \circ s,\mu) \subseteq K(x \circ s,\mu)$ and so $K(x \circ s,\mu) = K(y \circ s,\mu)$. Now let $a \in K(x \circ s,\mu)$. Then $s \circ a \subseteq K(x,\mu)$. This gives that $(q \circ x \circ (s \circ a))\ \lambda\ (q \circ y \circ (s \circ a))$ whence $(q \circ (x \circ s) \circ a)\ \lambda\ (q \circ (y \circ s) \circ a)$. Therefore, $\frac{x\circ s}{\mathcal{R}} = \frac{y\circ s}{\mathcal{R}}$. □

Lemma 3.6. *Let $p, q \in S$, μ and λ be two congruences on S_S, and K be a right ideal of S. If for any $m \in S$, $\frac{m}{\mu} \in \frac{K}{\mu}$ implies $(p \circ m)\ \lambda\ (q \circ m)$, then $\mathcal{R}(K,\mu,\lambda,p) = \mathcal{R}(K,\mu,\lambda,q)$.*

Proof. Let $s\,\mathcal{R}(K,\mu,\lambda,p)\,t$. To show that $s\,\mathcal{R}(K,\mu,\lambda,q)\,t$, it is enough to verify $(q \circ s \circ a)\ \lambda\ (q \circ t \circ a)$ for any $a \in K(s,\mu)$. Let $a \in K(s,\mu) = K(t,\mu)$. Then $\frac{s\circ a}{\mu}, \frac{t\circ a}{\mu} \subseteq \frac{K}{\mu}$. It follows from $s\,\mathcal{R}(K,\mu,\lambda,p)\,t$ that $(p\circ s\circ a)\ \lambda\ (p\circ t\circ a)$. On the other hand, $(p \circ (s \circ a))\ \lambda\ (q \circ (s \circ a))$ and $(p \circ (t \circ a))\ \lambda\ (q \circ (t \circ a))$. Therefore, $(q\circ s\circ a)\ \lambda\ (q\circ t\circ a)$, which implies $\mathcal{R}(K,\mu,\lambda,p) \subseteq \mathcal{R}(K,\mu,\lambda,q)$. The reverse inclusion is established in a similar manner. □

The following theorem determines a relationship between being semi-injective of a quotient of S_S and the form of some certain GHS-homomorphisms.

Theorem 3.7. *Let S be a hypermonoid and λ be a congruence on S_S. Then the following are equivalent*:

(i) $\frac{S_S}{\lambda}$ *is semi-injective* (*semi-C-injective*).

(ii) *For any right ideal K of S and any congruence μ on S_S* (*that $\frac{K_S}{\mu}$ is cyclic*), *if $f : \frac{K_S}{\mu} \to \frac{S_S}{\lambda}$ is a GHS-homomorphism, then there exists $q \in S_S$ such that $f(\frac{m}{\mu}) = \frac{q \circ m}{\lambda}$ for any $\frac{m}{\mu} \in \frac{K_S}{\mu}$ and for any $s, t \in S$, $s\, \mathcal{R}(K, \mu, \lambda, q)\, t$ implies $(q \circ s)\ \lambda\ (q \circ t)$.*

Proof. (i) $\Rightarrow$ (ii) Let K be a right ideal of S and μ be a congruence on S_S (such that $\frac{K_S}{\mu}$ is cyclic). Also let $f : \frac{K_S}{\mu} \to \frac{S_S}{\lambda}$ be a GHS-homomorphism. Consider the following diagram:

$$\begin{tikzcd} \frac{K_S}{\mu} \arrow[r, hook, "i"] \arrow[d, "f"'] & \frac{S_S}{\mu} \\ \frac{S_S}{\lambda} & \end{tikzcd}$$

Using the assumption, there exists a GHS-homomorphism $g : \frac{S_S}{\mu} \to \frac{S_S}{\lambda}$ such that $gi = f$. Assume that $g(\frac{1}{\mu}) = \frac{p_0}{\lambda}$. Then

$$g(\frac{m}{\mu}) = g(\frac{1}{\mu} \odot m) = g(\frac{1}{\mu}) \odot m = \frac{p_0}{\lambda} \odot m = \frac{p_0 \circ m}{\lambda},$$

for any $m \in S$. If $\frac{m}{\mu} \in \frac{K_S}{\mu}$, then

$$f(\frac{m}{\mu}) = (gi)(\frac{m}{\mu}) = \frac{p_0 \circ m}{\lambda}.$$

Now assume that $\rho = \mathcal{R}(K, \mu, \lambda, p_0)$. We define $\alpha : \frac{K_S}{\rho} \to \frac{S_S}{\lambda}$ by setting $\alpha(\frac{m}{\rho}) = \frac{p_0 \circ m}{\lambda}$. Consider $\frac{m_1}{\rho}, \frac{m_2}{\rho} \in \frac{K_S}{\rho}$ with $\frac{m_1}{\rho} = \frac{m_2}{\rho}$. Then $K(m_1, \mu) = K(m_2, \mu)$ and $(p_0 \circ m_1 \circ a)\ \lambda\ (p_0 \circ m_2 \circ a)$ for any $a \in K(m_1, \mu)$. Since $\frac{m_1}{\rho} \in \frac{K_S}{\rho}$, there exists $k \in K$ such that $m_1 \rho k$ and $1 \in K(k, \mu) = K(m_1, \mu)$, which implies $(p_0 \circ m_1)\ \lambda\ (p_0 \circ m_2)$. Therefore, α is well-defined. One can see that α is indeed a GHS-homomorphism.

Let $\frac{m_1}{\mu} = \frac{m_2}{\mu}$. We have

$$s \in K(m_1, \mu) \Leftrightarrow \frac{m_1}{\mu} \odot s \subseteq \frac{K_S}{\mu} \Leftrightarrow \frac{m_2}{\mu} \odot s \subseteq \frac{K_S}{\mu} \Leftrightarrow s \in K(m_2, \mu),$$

which means $K(m_1, \mu) = K(m_2, \mu)$. Take any $a \in K(m_1, \mu)$. Then

$$\begin{aligned} \frac{p_0 \circ m_1 \circ a}{\lambda} &= \frac{p_0 \circ m_1}{\lambda} \odot a = g(\frac{m_1}{\mu}) \odot a = \\ g(\frac{m_2}{\mu}) \odot a &= \frac{p_0 \circ m_2}{\lambda} \odot a = \frac{p_0 \circ m_2 \circ a}{\lambda}, \end{aligned}$$

which gives that $\frac{m_1}{\rho} = \frac{m_2}{\rho}$. Now we show that $\frac{m}{\rho} \in \frac{K_S}{\rho}$ if and only if $\frac{m}{\mu} \in \frac{K_S}{\mu}$. If $\frac{m}{\rho} \in \frac{K_S}{\rho}$, as above, $1 \in K(m, \mu)$, and hence $\frac{m}{\mu} = \frac{m_1 \circ 1}{\mu} \in \frac{K_S}{\mu}$. Let $\frac{m}{\mu} \in \frac{K_S}{\mu}$. Then there exists $k \in K_S$ with $\frac{m}{\mu} = \frac{k}{\mu}$. Thus $\frac{m}{\rho} = \frac{k}{\rho}$, which implies $\frac{m}{\rho} \in \frac{K_S}{\rho}$ ($\frac{K_S}{\rho}$ is cyclic). Now we consider the following diagram:

$$\begin{tikzcd} \frac{K_S}{\rho} \arrow[r, hook, "i"] \arrow[d, "\alpha"'] & \frac{S_S}{\rho} \\ \frac{S_S}{\lambda} & \end{tikzcd}$$

It follows from semi-injectivity (semi-C-injectivity) of $\frac{S_S}{\lambda}$ that there exists a GHS-homomorphism $\beta : \frac{S_S}{\rho} \to \frac{S_S}{\lambda}$ commuting the diagram.

Suppose that $\beta(\frac{1}{\rho}) = \frac{q}{\lambda}$ for some $q \in S_S$. Then $\beta(\frac{m}{\rho}) = \frac{q \circ m}{\lambda}$ for any $\frac{m}{\rho} \in \frac{S_S}{\rho}$. Let $\frac{m}{\mu} \in \frac{K_S}{\mu}$. Then $\frac{m}{\rho} \in \frac{K_S}{\rho}$ and hence

$$f(\frac{m}{\mu}) = \frac{p_0 \circ m}{\lambda} = \alpha(\frac{m}{\rho}) = \beta(\frac{m}{\rho}) = \frac{q \circ m}{\lambda}.$$

Also we get

$$\frac{p_0 \circ m}{\lambda} = \alpha(\frac{m}{\rho}) = \beta(\frac{m}{\rho}) = \frac{q \circ m}{\lambda}.$$

Therefore, by Lemma 3.6, $\mathcal{R}(K, \mu, \lambda, q) = \mathcal{R}(K, \mu, \lambda, p_0) = \rho$. Let $s\, \mathcal{R}(K, \mu, \lambda, q)\, t$. Then $\frac{s}{\rho} = \frac{t}{\rho}$ and hence

$$\frac{q \circ s}{\lambda} = \beta(\frac{1}{\rho} \odot s) = \beta(\frac{1}{\rho} \odot t) = \frac{q \circ t}{\lambda}.$$

(ii) $\Rightarrow$ (i) Let μ be a congruence on S_S. Consider the following diagram:

$$\begin{array}{ccc} \frac{K_S}{\mu} & \xhookrightarrow{\ i\ } & \frac{S_S}{\mu} \\ {\scriptstyle f}\big\downarrow & & \\ \frac{S_S}{\lambda} & & \end{array}$$

where K is a right ideal of S (that $\frac{K_S}{\mu}$ is cyclic). Using the assumption, there exists $q \in S_S$ in such a way that $f(\frac{m}{\mu}) = \frac{q \circ m}{\lambda}$ for any $\frac{m}{\mu} \in \frac{K_S}{\mu}$. Define $g : \frac{S_S}{\mu} \to \frac{S_S}{\lambda}$ by setting $g(\frac{m}{\mu}) = \frac{q \circ m}{\lambda}$. We show that g is well-defined. Let $\frac{m_1}{\mu}, \frac{m_2}{\mu} \in \frac{S_S}{\mu}$ with $\frac{m_1}{\mu} = \frac{m_2}{\mu}$. It suffices to show that $m_1\ \mathcal{R}(K, \mu, \lambda, q)\ m_2$. Note that $K(m_1, \mu) = K(m_2, \mu)$ since $\frac{m_1}{\mu} = \frac{m_2}{\mu}$. Now let $a \in K(m_1, \mu)$. So $\frac{m_1 \circ a}{\mu} = \frac{m_2 \circ a}{\mu}$. It follows from $\frac{m_1 \circ a}{\mu} \subseteq \frac{K_S}{\mu}$ that $\frac{q \circ (m_1 \circ a)}{\lambda} = f(\frac{m_1 \circ a}{\mu}) = f(\frac{m_2 \circ a}{\mu}) = \frac{q \circ (m_2 \circ a)}{\lambda}$. Thus $m_1\ \mathcal{R}(K, \mu, \lambda, q)\ m_2$ and hence $\frac{q \circ m_1}{\lambda} = \frac{q \circ m_2}{\lambda}$. Therefore, g is well-defined. It is easily seen that g is indeed a GHS-homomorphism and $gi = f$. Consequently, $\frac{S_S}{\lambda}$ is semi-injective (semi-C-injective). $\square$

Corollary 3.8. *Let S be a hypermonoid. Then the following are equivalent*:

(i) *All quotients of S_S are semi-injective (semi-C-injective).*

(ii) *For any right ideal K of S and any two congruences μ and λ on S_S (that $\frac{K_S}{\mu}$ is cyclic), if $f : \frac{K_S}{\mu} \to \frac{S_S}{\lambda}$ is a GHS-homomorphism, then there exists $q \in S_S$ such that*

(1) $f(\frac{m}{\mu}) = \frac{q \circ m}{\lambda}$ *for any* $\frac{m}{\mu} \in \frac{K_S}{\mu}$,

(2) *for any $s, t \in S$, $s\ \mathcal{R}(K, \mu, \lambda, q)\ t$ implies $(q \circ s)\ \lambda\ (q \circ t)$.*

Let S be a hypermonoid and K be a right ideal of S. Then for any $q \in S$, we define the relation $\rho(K, q)$ on S as follows:

$$\forall s, t \in S,\ s\ \rho(K, q)\ t \Leftrightarrow K_s = K_t \quad \text{and} \quad \forall a \in K_s,\ \ q \circ s \circ a = q \circ t \circ a,$$

where $K_m = \{u \in S : m \circ u \subseteq K\}$ for any $m \in S$.

Corollary 3.9. *Let S be a hypermonoid. Then the following are equivalent*:

(i) S_S *is semi-injective.*

(ii) *For any right ideal* K *of* S*, if* $f : K_S \to S_S$ *is a* GHS*-homomorphism, then there exists* $q \in S_S$ *such that* $f(m) = q \circ m$ *for any* $m \in K$*, and for any* $s, t \in S$*,* $s\ \rho(K, q)\ t$ *implies* $q \circ s = q \circ t$*.*

Proof. (i) $\Rightarrow$ (ii) By Theorem 3.7, it suffices to set $\lambda = \mu = \Delta_S$.

(ii) $\Rightarrow$ (i) It is enough to show that Condition (ii) of Theorem 3.7 is satisfied with the assumption $\lambda = \Delta_S$. Let K be a right ideal of S and $f : \frac{K_S}{\mu} \to S_S$ be a GHS-homomorphism. Let K' be the set of all elements $m \in S_S$ with $\frac{m}{\mu} \in \frac{K_S}{\mu}$. Indeed, $K' = K(1, \mu)$ which is a right ideal of S. Now, define $f' : K'_S \to S_S$ by $f'(m) = f(\frac{m}{\mu})$ for any $m \in K'_S$. Then f' is a GHS-homomorphism. It follows from the assumption that there exists $q \in S_S$ such that $f'(m) = q \circ m$ for any $m \in K'_S$, and $s\ \rho(K', q)\ t$ implies $q \circ s = q \circ t$ for any $s, t \in S$. So $f(\frac{m}{\mu}) = f'(m) = q \circ m$ for any $\frac{m}{\mu} \in \frac{K_S}{\mu}$. Consider any $s, t \in S$ with $s\ \mathcal{R}(K, \mu, \Delta, q)\ t$. We must show that $q \circ s = q \circ t$. It follows from $s\ \mathcal{R}(K, \mu, \lambda, q)\ t$ that $K(s, \mu) = K(t, \mu)$ and $q \circ s \circ a = q \circ t \circ a$ for any $a \in K(s, \mu)$. It suffices to prove that $s\ \rho(K', q)\ t$. For this, first we show that $K'_s = K'_t$. We have

$$\begin{aligned} K'_s = \{u \in S : s \circ u \subseteq K'\} &= \{u \in S : \frac{s \circ u}{\mu} \subseteq \frac{K_S}{\mu}\} \\ &= K(s, \mu) = K(t, \mu) = K'_t. \end{aligned}$$

Thus for any $a \in K'_s = K(s, \mu)$, $q \circ s \circ a = q \circ t \circ a$, which implies $s\ \rho(K', q)\ t$ and hence $q \circ s = q \circ t$ by the assumption. Therefore, S_S is semi-injective. $\square$

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